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Stability radii of higher order positive difference systems

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Abstract

In this paper we study stability radii of positive polynomial matrices under affine perturbations of the coefficient matrices. It is shown that the real and complex stability radii coincide. Moreover, explicit formulas are derived for these stability radii and illustrated by some examples.

Keywords: Positive system; Polynomial matrix; Robust stability; Stability radius

1. Introduction

A dynamical system with state space $\mathbb{R}^n$ is called *positive* if any trajectory of the system starting at an initial state in the positive orthant $\mathbb{R}^n_+$ remains forever in $\mathbb{R}^n_+$. Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative. This model class is used in many areas such as economics, populations dynamics and ecology, see [2,9]. The mathematical theory of positive systems is based on the theory of nonnegative matrices founded by Perron and Frobenius. As references we mention [1,3].

In this paper we will study robust stability of discrete time linear systems described by higher order difference equations of the form

$$A_\nu y(t+\nu) = A_{\nu-1} y(t+\nu-1) + \cdots + A_1 y(t+1) + A_0 y(t), \qquad t \in \mathbb{N}, \tag{1}$$

where we suppose $\det A_\nu \neq 0$. Such a system is called *positive* if every solution $y(\cdot)$ whose first ν values $y(0), \ldots, y(\nu-1)$ lie in the positive orthant, remains there for all later times. It is easily seen that this condition is equivalent to the nonnegativity of the matrices $A_\nu^{-1}A_0, \ldots, A_\nu^{-1}A_{\nu-1}$. The above difference equation is called asymptotically stable if all the solutions $y(t)$ of (1) tend to zero as $t \to \infty$. An equivalent condition is that the associated polynomial matrix $P(z) = A_\nu z^\nu - A_{\nu-1} z^{\nu-1} \cdots - A_0$ is *Schur stable* in the sense that all the roots of $p(z) = \det P(z)$ lie in the complex open unit disk.

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Now assume that the coefficient matrices A_i, $i \in \underline{v} := \{0, 1, \dots, v\}$ are subjected to affine perturbations of the form

$$A_i \rightsquigarrow A_i(\Delta) = A_i + D_i \Delta_i E_i, \quad i \in \underline{v}, \tag{2}$$

where the matrices D_i, E_i (determining the structure of the perturbations) are given and the matrices $\Delta_i \in \mathbb{C}^{l_i \times q_i}$ are unknown disturbance matrices. The object of this paper is to determine the stability radii [5] of positive higher order difference equations (1), under perturbations of the above form with nonnegative structure matrices D_i, E_i. The complex (resp. real) stability radius of the system (1) under perturbations of the form (2) is by definition the supremal value of $r > 0$ such that all the difference equations with perturbed coefficients $A_i(\Delta)$ are stable whenever the overall perturbation $\Delta = (\Delta_0, \dots, \Delta_v)$ is of norm $\gamma(\Delta) = \sum_{i=0}^{v} \|\Delta_i\| < r$ (and real).

It is known that for positive state-space systems which are perturbed by affine perturbations with nonnegative structure matrices, the real and complex stability radii coincide and can be computed by a simple formula [6]. In the present paper we will extend these results to higher order difference equations of the above form. Since the modelling of discrete time systems often results in higher order difference or differential equations [12,14], it is useful to have results available which can be directly applied without prior transformation into state-space form. Such results have recently been systematically derived in the context of behavioral systems theory [11]. Stability criteria for higher order difference and differential equations can be found in [13].

The complex stability radius of such systems was first investigated in [10] under the condition that the leading coefficient matrix A_v is the identity and remains unperturbed (monic case). Real and complex stability radii of matrix polynomials $P(z)$ under *unstructured* perturbations of *all* the coefficient matrices including the leading one have recently been analyzed in [4]. In the present paper we will consider *structured* perturbations of all the coefficient matrices, but will mainly deal with the special case of positive systems. For these systems it will be possible to derive computable formulae for arbitrary affine perturbation structures for which in the general case only estimates can be derived by techniques of μ-analysis, even for state-space systems, see [5]. The formulae derived in this paper extend some of the results in [6] from state-space systems to higher order difference equations. We proceed as follows.

After recalling some preliminary results on nonnegative matrices in the next section, we first deal with arbitrary asymptotically stable difference equations of form (1) and derive a computable formula for their complex stability radii under structured perturbations of the coefficient matrices (Section 3). This formula generalizes the main result in [10] to matrix polynomials with perturbed leading coefficient matrix. In Section 4 we specialize to the positive case and show that for arbitrary affine perturbations with nonnegative structure matrices the real and the complex stability radii coincide and can be computed by simple formulae. The results are illustrated by two examples.

2. Preliminaries

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and m, n be positive integers. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real $m \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geqslant B$ means $a_{ij} \geqslant b_{ij}$ for $i = 1, \dots, m$, $j = 1, \dots, n$. The set of all nonnegative $m \times n$-matrices is denoted by $\mathbb{R}_+^{m \times n}$. If $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{m \times n}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. For arbitrary $A, B \in \mathbb{C}^{m \times n}$

$$|A + B| \leqslant |A| + |B|, \quad |AB| \leqslant |A|\,|B|. \tag{3}$$

For any matrix $A \in \mathbb{K}^{n \times n}$ the *spectral radius* of A is denote by $\rho(A) = \max\{|\lambda|; \lambda \in \sigma(A)\}$, where $\sigma(A) := \{z \in \mathbb{C};\ \det(zI_n - A) = 0\}$ is the set of all eigenvalues of A. The spectral radius has the following monotonicity property, see e.g. [6].

$$\forall C \in \mathbb{C}^{n\times n},$$
$$B \in \mathbb{R}_+^{n\times n}\colon\ |C| \leqslant B \Rightarrow \rho(C) \leqslant \rho(|C|) \leqslant \rho(B). \tag{4}$$

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in \mathbb{K}^n$. Every p-norm on $\mathbb{K}^n, 1 \leqslant p \leqslant \infty$, is monotonic. Throughout the paper, the norm $\|M\|$ of a matrix $M \in \mathbb{K}^{m\times n}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1}\|My\|$ where $\mathbb{K}^n$ and $\mathbb{K}^m$ are provided with given *monotonic* vector norms. The operator norm $\|\cdot\|$ then has the following monotonicity property, see e.g. [6],

$$P \in \mathbb{K}^{m\times n},\ Q \in \mathbb{R}_+^{m\times n}, \quad |P| \leqslant Q \Rightarrow \|P\| \leqslant \|\,|P|\,\| \leqslant \|Q\|. \tag{5}$$

For later use, we summarize some basic properties of nonnegative matrices which will be used in the sequel (see [1,6,9]).

Theorem 2.1. *Let $A \in \mathbb{R}_+^{n\times n}$, $t \in \mathbb{R}$. Then*

(i) (*Perron–Frobenius*) *$\rho(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \geqslant 0$, $x \neq 0$ such that $Ax = \rho(A)x$.*
(ii) *Given $\alpha \in \mathbb{R}, \alpha > 0$, there exists a nonzero vector $x \geqslant 0$ such that $Ax \geqslant \alpha x$ if and only if $\rho(A) \geqslant \alpha$.*
(iii) *$(tI_n - A)^{-1}$ exists and is nonnegative if and only if $t > \rho(A)$.*

3. Complex stability radii of high order difference equations

Consider the ν-th order linear difference equation of form (1) where $A_0, A_1, \ldots, A_\nu \in \mathbb{R}^{n\times n}$ are given matrices. With this equation, we associate the polynomial matrix

$$P(z) := A_\nu z^\nu - A_{\nu-1}z^{\nu-1} - \cdots - A_0. \tag{6}$$

Denote by $\sigma(P(\cdot))$ the set of all roots of the characteristic polynomial $\det P(z)$ of (1), that is $\sigma(P(\cdot)) = \{\lambda \in \mathbb{C}; \det P(\lambda) = 0\}$, and let $\rho(P(\cdot)) := \sup\{|\lambda|;\ \lambda \in \sigma(P(\cdot))\}$. Then, $\sigma(P(\cdot))$ and $\rho(P(\cdot))$ are called the *spectrum and spectral radius* of the polynomial matrix $P(\cdot)$, respectively. If $\det P(z) \equiv 0$ then $\sigma(P(\cdot)) = \mathbb{C}$, otherwise the spectrum of $P(\cdot)$ is a finite subset of $\mathbb{C}$ consisting of at most deg $\det P(z)$ "eigenvalues" of (1).

Definition 3.1. A polynomial matrix $P(\cdot)$ of form (6) is called Schur stable if $\sigma(P(\cdot)) \subset \mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$, or equivalently $\rho(P(\cdot)) < 1$.

Remark 3.2. If $\det P(z) \not\equiv 0$ the difference equation (1) has a finite dimensional solution set (i.e. defines an autonomous behaviour, in terms of behavioural system theory [11]) and can be transformed equivalently into a first order difference equation $x(t+1) = Ax(t)$ (see e.g. [14]). It is known that $\lambda \in \sigma(A)$ if and only if $\det P(\lambda) = 0$ (see [13]). Hence the state-space system described by $x(t+1) = Ax(t)$ is asymptotically stable if and only if the polynomial matrix $P(z)$ is Schur stable. If $\det A_\nu \neq 0$, an equivalent state-space system is easily determined by introducing the state vector $x(t) = [y(t-v+1)^{\mathrm{T}}, y(t-v+2)^{\mathrm{T}}, \ldots, y(t)^{\mathrm{T}}]^{\mathrm{T}}$:

$$x(t+1) = Ax(t), \quad t \in \mathbb{N};$$

$$A = \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ .. & .. & .. & \ldots & .. & .. \\ 0 & 0 & 0 & \ldots & 0 & I_n \\ A_v^{-1}A_{v-1} & A_v^{-1}A_{v-2} & .. & \ldots & .. & A_v^{-1}A_0 \end{bmatrix}. \tag{7}$$

If $\det A_v = 0$, the construction of an equivalent state-space system is more complicated (see [14]).

We now assume that the polynomial matrix (6) is Schur stable and the coefficient matrices $A_0, A_1, \ldots, A_v$ are subjected to structured perturbations of the output feedback type (see [5])

$$A_i \rightsquigarrow A_i(\Delta) := A_i + D_i \Delta_i E_i, \qquad \Delta_i \in \mathbb{C}^{l_i \times q_i}, \tag{8}$$

where $D_i \in \mathbb{R}^{n \times l_i}, E_i \in \mathbb{R}^{q_i \times n}, i \in \underline{v}$ are given matrices defining the *structure of the perturbations* and $\Delta_i, i \in \underline{v}$ are unknown disturbance matrices.

Remark 3.3. (i) The perturbations of form (8) can be described as a *blockdiagonal* perturbation of the coefficient vector

$$[A_0, \ldots, A_v] \rightsquigarrow [A_0, \ldots, A_v] + [D_0, \ldots, D_v] \mathrm{diag}(\Delta_0, \ldots, \Delta_v)\, \mathrm{diag}(E_0, \ldots, E_v).$$

However, not every linear perturbation of the matrices A_i can be represented in this way, see [10]. In the next section we will consider arbitrary affine parameter perturbations under a positivity condition.

(ii) It should be noted that the perturbation of the leading coefficient matrix A_v leads in general to a nonlinear (rational) dependency of the system matrix A (7) on the disturbance parameters (the entries of $\Delta_i \in \mathbb{C}^{l_i \times q_i}$). Choosing the structure matrices D_i and E_i appropriately, a variety of rational parameter dependencies of A can be modeled in this way.

The linear space $\boldsymbol{\Delta}_{\mathbb{K}} = \mathbb{K}^{l_0 \times q_0} \times \cdots \times \mathbb{K}^{l_v \times q_v}$ of all $\Delta = (\Delta_0, \ldots, \Delta_v)$, with $\Delta_i \in \mathbb{K}^{l_i \times q_i}$ is endowed with the norm $\gamma(\Delta) = \gamma(\Delta_0, \ldots, \Delta_v) = \sum_{i=0}^{v} \|\Delta_i\|$, where the norms $\|\Delta_i\|$ are operator norms on $\mathbb{K}^{l_i \times q_i}$, induced by given monotonic vector norms on the spaces $\mathbb{K}^{l_i}, \mathbb{K}^{q_i}, i \in \underline{v}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Denote the perturbed polynomial matrices by

$$\begin{aligned} P_\Delta(s) := {} & (A_v + D_v \Delta_v E_v)s^v - (A_{v-1} + D_{v-1}\Delta_{v-1}E_{v-1})s^{v-1} \\ & - \cdots - (A_0 + D_0 \Delta_0 E_0). \end{aligned} \tag{9}$$

Throughout the paper, we set $\inf \emptyset = \infty, 0^{-1} = \infty, \infty^{-1} = 0$.

Definition 3.4. Let the polynomial matrix (6) be Schur stable. Then the complex stability radius of the polynomial matrix (6) with respect to perturbations of the form (8) is defined by

$$r_{\mathbb{C}} = \inf\{\gamma(\Delta_0, \ldots, \Delta_v);\ \forall i \in \underline{v} : \Delta_i \in \mathbb{C}^{l_i \times q_i}, \quad \rho(P_\Delta(\cdot)) \geqslant 1\}. \tag{10}$$

If the disturbance matrices Δ_i in (10) are restricted to the real spaces $\mathbb{R}^{l_i \times q_i}, i \in \underline{v}$, then we obtain the real stability radius $r_{\mathbb{R}}$.

We first analyse under which condition $r_{\mathbb{K}} = 0$.

Proposition 3.5. *Let $P(z)$ be Schur stable and $p_\Delta(z) := \det P_\Delta(z)$, $p(z) := \det P(z)$. Then*

(i) $r_{\mathbb{K}} = 0 \Leftrightarrow \exists \Delta \in \boldsymbol{\Delta}_{\mathbb{K}};\ \deg p_\Delta(z) > \deg p(z)$.

(ii) *If the perturbation* (8) *are unstructured, i.e* $D_i = E_i = I_n$ *for* $i \in \underline{v}$, *then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = 0 \Leftrightarrow \det A_v = 0.$$

Proof. (i) To prove the implication $\Leftarrow$, let

$$p_\Delta(z) = \det P_\Delta(z) = a_{nv}(\Delta) z^{nv} + a_{nv-1}(\Delta) z^{nv-1} + \cdots + a_0(\Delta)$$

and suppose that $\tilde{\Delta} = (\tilde{\Delta}_0, \ldots, \tilde{\Delta}_v) \in \boldsymbol{\Delta}_{\mathbb{K}}$ is chosen such that

$$p_{\tilde{\Delta}}(z) = a_\beta(\tilde{\Delta}) z^\beta + a_{\beta-1}(\tilde{\Delta}) z^{\beta-1} + \cdots + a_0(\tilde{\Delta}), \qquad a_\beta(\tilde{\Delta}) \neq 0$$

is a perturbed polynomial of maximum degree among all $p_\Delta(z), \Delta \in \boldsymbol{\Delta}_{\mathbb{K}}$. By assumption $\alpha := \deg p(z) < \deg p_{\tilde{\Delta}}(z) = \beta$. Note that $a_\beta(\Delta)$ is a multivariate polynomial of the elements of the matrices $\Delta_0, \Delta_1, \ldots, \Delta_v$. Since $a_\beta(\tilde{\Delta}) \neq 0$, it is not identically zero. Hence for every $\varepsilon > 0$, there exists $\Delta(\varepsilon) = (\Delta_0(\varepsilon), \ldots, \Delta_v(\varepsilon)) \in \boldsymbol{\Delta}_{\mathbb{K}}$ such that $a_\beta(\Delta(\varepsilon)) \neq 0$ and $\gamma(\Delta(\varepsilon)) < \varepsilon$. Consider the following polynomials

$$q(s) := a_\alpha(0) s^{\beta-\alpha} + a_{\alpha-1}(0) s^{\beta-\alpha+1} + \cdots + a_0(0) s^\beta,$$

$$q_\varepsilon(s) := a_\beta(\Delta(\varepsilon)) + a_{\beta-1}(\Delta(\varepsilon)) s + \cdots + a_0(\Delta(\varepsilon)) s^\beta.$$

Since $a_i(\Delta(\varepsilon)) \to a_i(0)$ for $i \in \underline{\beta}$ as $\varepsilon \to 0$ and $q(0)=0$, it follows from Rouché's Theorem that the polynomials $q_\varepsilon(s)$ have zeros $s_\varepsilon \neq 0$ satisfying $s_\varepsilon \to 0$ as $\varepsilon \to 0$. Then, $z_\varepsilon = 1/s_\varepsilon$ is a zero of $p_{\Delta(\varepsilon)}(z)$ and so the perturbations $\Delta(\varepsilon)$ are destabilizing if ε is sufficiently close to 0. It follows that $r_{\mathbb{K}} < \varepsilon$ for every $\varepsilon > 0$, i.e. $r_{\mathbb{K}} = 0$.

Conversely, assume that $\deg p_\Delta(z) \leqslant \deg p(z)$ for all $\Delta \in \boldsymbol{\Delta}_{\mathbb{K}}$. Since the coefficients $a_i(\Delta), i \in \underline{v}$ are continuous in Δ, it follows from the continuity property of polynomial roots and the Schur stability of $P(z)$ that $r_{\mathbb{K}} > 0$, see e.g. [11, Theorem 7.3.5].

(ii) Suppose that $D_i = E_i = I_n, i \in \underline{v}$. From the explicit expression for the determinant we see that $\deg p_\Delta(z) \leqslant vn$ and the coefficient of z^{vn} in the expansion of $\det P_\Delta(z) = p_\Delta(z)$ is $\det(A_v + \Delta_v)$ for all $\Delta \in \boldsymbol{\Delta}_{\mathbb{K}}$. Hence there exists $\Delta \in \boldsymbol{\Delta}_{\mathbb{K}}$ such that $\deg p_\Delta(z) > \deg p(z)$ if and only if $\det A_v = 0$. Thus (ii) is a direct consequence of (i). □

In the rest of this paper, *we always assume that A_v is a regular matrix.*

Remark 3.6. The assumption $\det A_v \neq 0$ is quite restrictive for structured perturbations. We believe that for perturbation of the form (8), robust stability criteria can be obtained under the weaker assumption $\det P(z) \not\equiv 0$. However, such an analysis will require more subtle algebraic considerations connecting algebraic systems theory with perturbation theory and robust stability analysis.

With any given polynomial matrix (6) and perturbation structure (8) we associate a partitioned transfer matrix $G(z) = (G_{ij}(z))_{i,j \in \underline{v}}$ defined by

$$G_{ij}(z) := E_i P(z)^{-1} D_j \in \mathbb{C}^{q_i \times l_j}(z), \qquad i, j \in \underline{v}. \tag{11}$$

Lemma 3.7. *Let the polynomial matrix* (6) *be Schur stable. Then, for every $z_0 \in \mathbb{C}, |z_0| = 1$, there exists a perturbation $\Delta := (\Delta_0, \ldots, \Delta_v) \in \boldsymbol{\Delta}_{\mathbb{C}}$ such that*

$$\gamma(\Delta_0, \ldots, \Delta_v) = \frac{1}{\max_{i \in \underline{v}} \|G_{ii}(z_0)\|} \quad \text{and} \quad \det P_\Delta(z_0) = 0. \tag{12}$$

Moreover, if $G_{ii}(1) \in \mathbb{R}_+^{q_i \times l_i}$ *for every* $i \in \underline{v}$ *then there exists a real perturbation* Δ *satisfying* (12) *with* $z_0 = 1$.

Proof. Given $z_0 \in \mathbb{C}, |z_0| = 1$, suppose that $\max_{i \in \underline{v}} \|G_{ii}(z_0)\| = \|G_{i_0 i_0}(z_0)\|$ for some $i_0 \in \underline{v}$. By the definition of $\|G_{i_0 i_0}(z_0)\|$, there exists a vector $u_0 \in \mathbb{C}^{l_{i_0}}, \|u_0\| = 1$ such that $\|G_{i_0 i_0}(z_0)u_0\| = \|G_{i_0 i_0}(z_0)\|$. Then, by the Hahn–Banach Theorem, there exists $y_0^* \in (\mathbb{C}^{q_{i_0}})^*, \|y_0^*\| = 1$ satisfying $y_0^* G_{i_0 i_0}(z_0)u_0 = \|G_{i_0 i_0}(z_0)u_0\|$. Define $\tilde{\Delta}_{i_0} := \|G_{i_0 i_0}(z_0)\|^{-1} u_0 y_0^* \in \mathbb{C}^{l_{i_0} \times q_{i_0}}$. It is clear that $\|\tilde{\Delta}_{i_0}\| = \|G_{i_0 i_0}(z_0)\|^{-1}$ and $\tilde{\Delta}_{i_0} G_{i_0 i_0}(z_0)u_0 = u_0$. Setting $x_0 := P(z_0)^{-1} D_{i_0} u_0$, we have $\tilde{\Delta}_{i_0} E_{i_0} x_0 = \tilde{\Delta}_{i_0} G_{i_0 i_0}(z_0) u_0 = u_0$. Thus $x_0 \neq 0$ and $P(z_0)x_0 = D_{i_0} u_0 = D_{i_0} \tilde{\Delta}_{i_0} E_{i_0} x_0$, i.e. $(P(z_0) - D_{i_0} \tilde{\Delta}_{i_0} E_{i_0}) x_0 = 0$. If $i_0 = v$, we set $\Delta_i = 0$ for $i \in \{0, 1, \ldots, v-1\}$ and $\Delta_v := -\tilde{\Delta}_v z_0^{-v}$. Otherwise, we set $\Delta_i = 0, i \neq i_0$ and $\Delta_{i_0} = \tilde{\Delta}_{i_0} z_0^{-i_0}$. Then, we get $P_\Delta(z_0)x_0 = 0$, where $\Delta = (\Delta_0, \ldots, \Delta_v)$. Therefore $\Delta = (\Delta_0, \ldots, \Delta_v)$ satisfies (12).

If $G_{i_0 i_0}(z_0) \in \mathbb{R}_+^{q_{i_0} \times l_{i_0}}$ for $z_0 = 1$ then we have $\|G_{i_0 i_0}(1)\| = \max_{u \in \mathbb{R}_+^{l_{i_0}}, \|u\|=1} \|G_{i_0 i_0}(1)u\|$, see [7]. Thus we can choose $u_0 \in \mathbb{R}_+^{l_{i_0}}$ such that $\|u_0\| = 1$ and $\|G_{i_0 i_0}(1)u_0\| = \|G_{i_0 i_0}(1)\|$. Since $G_{i_0 i_0}(1)u_0 \geqslant 0$ there exists by a theorem of Krein and Rutman [8] a positive linear form $y_0^* \in (\mathbb{C}^{q_{i_0}})^*$ of dual norm $\|y_0^*\| = 1$ such that $y_0^* G_{i_0 i_0}(1)u_0 = \|G_{i_0 i_0}(1)u_0\|$. So the perturbation $\tilde{\Delta}_{i_0}$ constructed as above is nonnegative and the proof is complete. □

Using this lemma we obtain the following estimates for the complex stability radius.

Theorem 3.8. *Let* $G(z) = (G_{ij}(z))_{i,j \in \underline{v}}$ *be the transfer matrix associated with the polynomial matrix* (6) *and the perturbation structure* (8). *If* $P(z)$ *is Schur stable then*

$$\frac{1}{\max_{z \in \mathbb{C}, |z|=1} \max\{\|G_{ij}(z)\|; i,j \in \underline{v}\}} \leqslant r_{\mathbb{C}} \leqslant \frac{1}{\max_{z \in \mathbb{C}, |z|=1} \max\{\|G_{ii}(z)\|; i \in \underline{v}\}}. \tag{13}$$

In particular, if $D_i = D_j$ *(or* $E_i = E_j$*) for all* $i, j \in \underline{v}$ *then*

$$r_{\mathbb{C}} = \frac{1}{\max_{z \in \mathbb{C}, |z|=1} \max\{\|G_{ii}(z)\|; i \in \underline{v}\}}. \tag{14}$$

Proof. Let $\Delta = (\Delta_0, \ldots, \Delta_v)$ be a destabilizing disturbance of minimum norm. By continuity of the map $\Delta \mapsto \rho(P_\Delta(\cdot))$ it follows that $\rho(P_\Delta(\cdot)) = 1$. Hence there exist a complex number $\lambda, |\lambda| = 1$ and a non-zero vector $x \in \mathbb{C}^n$ such that

$$P_\Delta(\lambda)x = \left((A_v + D_v \Delta_v E_v)\lambda^v - \sum_{i=0}^{v-1} (A_i + D_i \Delta_i E_i)\lambda^i \right) x$$
$$= 0.$$

Since the polynomial matrix (6) is stable, we get

$$P(\lambda)^{-1} \left(-D_v \Delta_v E_v \lambda^v + \sum_{i=0}^{v-1} (D_i \Delta_i E_i)\lambda^i \right) x = x.$$

Let $k \in \underline{v}$ be an index such that $\|E_k x\| = \max\{\|E_i x\|, i \in \underline{v}\}$ then the last equality implies $E_k x \neq 0$. Multiplying the last equation with E_k from the left and taking norms, we obtain

$$\sum_{j=0}^{v} \|G_{kj}(\lambda)\| \|\Delta_j\| \|E_j x\| \geqslant \|E_k x\|.$$

Hence

$$\gamma(\Delta_0, \dots, \Delta_v) = \sum_{j=0}^{v} \|\Delta_j\| \geqslant \frac{1}{\max_{z\in\mathbb{C},|z|=1} \max\{\|G_{ij}(z)\|; i,j \in \underline{v}\}},$$

and this proves the first inequality in (13). The second inequality in (13) follows from Lemma 3.7 and the definition of $r_{\mathbb{C}}$. Finally (14) follows from (13) and the fact that if $D_i = D_j$ (resp. $E_i = E_j$) for all $i,j \in \underline{v}$ then all the block entries $G_{ij}(z)$ of $G(z)$ in the same block row (resp. block column) are identical. □

4. Stability radii of positive polynomial matrices

We now restrict ourselves to *positive* polynomial matrices (6) (with regular A_v).

Definition 4.1. The polynomial matrix (6) with $\det A_v \neq 0$ is called positive if the matrices $A_v^{-1}A_{v-1}, \dots, A_v^{-1}A_0$ are nonnegative.

Remark 4.2. From the above definition, it follows that $P(z)$ in (6) with $\det A_v \neq 0$ is a positive polynomial matrix if and only if the associated discrete state-space system of the form (7) is positive.

It is clear from the Definition 3.4, that the stability radii of the polynomial matrix (6) with respect to perturbations of the type (8) satisfy

$$r_{\mathbb{C}} \leqslant r_{\mathbb{R}}. \tag{15}$$

We will show that, for positive polynomial matrices, equality in (15) holds and moreover, these stability radii are easily computed.

We need the following property of the spectral radius of positive polynomial matrices.

Lemma 4.3. *Let the polynomial matrix* (6) *be stable and positive. Then,*

$$\rho(A_v^{-1}A_{v-1} + A_v^{-1}A_{v-2} + \cdots + A_v^{-1}A_0) < 1. \tag{16}$$

Proof. The proof is based on (4) and is omitted here. □

We are now in a position to derive the first main result of the paper.

Theorem 4.4. *Let the polynomial matrix* (6) *be stable and positive. Assume that the matrices A_i are subjected to parameter perturbations of form* (8), *where $A_v^{-1}D_i \in \mathbb{R}_+^{n\times l_i}, E_i \in \mathbb{R}_+^{q_i\times n}, i \in \underline{v}$. If $D_i = D_j$ (or $E_i = E_j$) for all $i,j \in \underline{v}$, then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = \frac{1}{\max_{i\in\underline{v}}\|E_i(A_v - A_{v-1} - \cdots - A_0)^{-1}D_i\|}. \tag{17}$$

Proof. Since the matrices $A_v^{-1}A_j, j \in \underline{v}$ are nonnegative matrices, we have for all $z \in \mathbb{C}, |z| = 1$

$$|A_v^{-1}A_{v-1}z^{v-1} + A_v^{-1}A_{v-2}z^{v-2} + \cdots + A_v^{-1}A_0| \leqslant A_v^{-1}A_{v-1} + A_v^{-1}A_{v-2} + \cdots + A_v^{-1}A_0$$

so that by (4) and Lemma 4.3 for all $z \in \mathbb{C}$, $|z|=1$

$$\rho(A_\nu^{-1}A_{\nu-1}z^{\nu-1} + A_\nu^{-1}A_{\nu-2}z^{\nu-2} + \cdots + A_\nu^{-1}A_0) \quad < 1.$$

As a consequence, we obtain from (6) the following expansion for $P(z)^{-1}$ on the unit circle in $\mathbb{C}$

$$P(z)^{-1} = \left(\sum_{k=0}^{\infty} \frac{(A_\nu^{-1}A_{\nu-1}z^{\nu-1} + A_\nu^{-1}A_{\nu-2}z^{\nu-2} + \cdots + A_\nu^{-1}A_0)^k}{z^{\nu(k+1)}} \right) A_\nu^{-1}.$$

Hence, on the unit circle

$$|P(z)^{-1}A_\nu| \leqslant \sum_{k=0}^{\infty} (A_\nu^{-1}A_{\nu-1} + A_\nu^{-1}A_{\nu-2} + \cdots + A_\nu^{-1}A_0)^k$$
$$= P(1)^{-1}A_\nu. \tag{18}$$

It follows from (3) that for $z \in \mathbb{C}$, $|z|=1$

$$\begin{aligned} |G_{ii}(z)| &= |E_i P(z)^{-1} D_i| \\ &= |E_i P(z)^{-1} A_\nu A_\nu^{-1} D_i| \\ &\leqslant E_i P(1)^{-1} D_i = G_{ii}(1). \end{aligned} \tag{19}$$

Hence by the monotonicity property (5) we get $\|G_{ii}(z)\| \leqslant \|G_{ii}(1)\|$ for all $z \in \mathbb{C}, |z|=1$, and from this we conclude by (14) that

$$r_{\mathbb{C}} = \left[\max_{i \in \underline{\nu}} \|G_{ii}(1)\| \right]^{-1}. \tag{20}$$

On the other hand, (19) implies that $G_{ii}(1) \in \mathbb{R}_+^{l_i \times q_i}$ for every $i \in \underline{\nu}$ so that by definition of the real stability radius $r_{\mathbb{R}}$ and Lemma 3.7 we have $r_{\mathbb{R}} \leqslant [\max_{i \in \underline{\nu}} \|G_{ii}(1)\|]^{-1}$. This together with (20) proves the equality (17). $\square$

We now turn to a different perturbation structure and assume that the polynomial matrix (6) is subjected to perturbations of the following kind:

$$A_i \rightsquigarrow A_i(\delta) := A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij}, \tag{21}$$

where $B_{ij} \in \mathbb{C}^{n \times n}$, $i \in \underline{\nu}$, $j = 1, 2, \ldots, N$ are given matrices and $\delta_{ij} \in \mathbb{C}$ are unknown scalar parameters. This is a very flexible perturbation structure since arbitrary additive linear perturbations of the A_i can be represented in this way. The size of a disturbance $\delta = (\delta_{ij}) \in \mathbb{C}^{\nu \times N}$ is measured by the norm $\|\delta\|_\infty = \max_{i \in \underline{\nu}, 1 \leqslant j \leqslant N} |\delta_{ij}|$ and the corresponding perturbed matrix polynomial is denoted by

$$P_\delta(z) := \left(A_\nu + \sum_{j=1}^{N} \delta_{\nu j} B_{\nu j} \right) z^\nu \quad - \sum_{i=0}^{\nu-1} \left(A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij} \right) z^i. \tag{22}$$

Definition 4.5. Let the polynomial matrix (6) be Schur stable. Then, the complex stability radius of the polynomial matrix (6) with respect to perturbations of the form (21) is defined by

$$r_{\mathbb{C}}^a = \inf\{\|\delta\|_\infty; \delta = (\delta_{ij}) \in \mathbb{C}^{p \times N}, \ \rho(P_\delta(\cdot)) \geqslant 1\}. \tag{23}$$

If the scalar disturbances δ_{ij} in (23) are restricted to be real, then we obtain the real stability radius $r_{\mathbb{R}}^a$.

In order to represent perturbations (21) in output feedback form with blockdiagonal disturbance matrices (as usual in μ-analysis) we introduce the following:

$$D_i := [B_{i1} \cdots B_{iN}] \in \mathbb{C}^{n\times nN}, \quad E_i := [I_n \ldots I_n]^\top \in \mathbb{C}^{nN\times n},$$
$$\mathscr{D}_i := \{\mathrm{diag}(\delta_{i1} I_n, \ldots, \delta_{iN} I_n); \delta_{ij} \in \mathbb{C}, j \in N_i\} \subset \mathbb{C}^{nN\times nN}, \ i \in \underline{v}. \tag{24}$$

Then perturbations (21) can be represented in the blockdiagonal form

$$A_i \rightsquigarrow A_i(\delta) = A_i + D_i \varDelta_i(\delta) E_i,$$
$$\varDelta_i(\delta) = \mathrm{diag}(\delta_{i1} I_n, \ldots, \delta_{iN} I_n) \in \mathscr{D}_i. \tag{25}$$

Now define

$$D(z) := [D_v z^v, D_{v-1} z^{v-1}, \ldots, D_0] \in \mathbb{C}^{n\times(v+1)nN}, \quad E := \begin{bmatrix} E_v \\ \vdots \\ E_0 \end{bmatrix} \in \mathbb{C}^{(v+1)nN\times n} \tag{26}$$

and let $\mathscr{D}$ be the set of all diagonal disturbance matrices of the form

$$\varDelta(\delta) := \mathrm{diag}(\varDelta_v(\delta), -\varDelta_{v-1}(\delta), \ldots, -\varDelta_0(\delta)), \quad \varDelta_i(\delta) \in \mathscr{D}_i, \quad i \in \underline{v}.$$

Then, we have $D(z)\varDelta(\delta)E = D_v \varDelta_v(\delta) E_v z^v - \sum_{i=0}^{v-1} D_i \varDelta_i(\delta) E_i z^i$, and so by (22)

$$P_\delta(z) = P(z) + D(z)\varDelta(\delta)E, \quad \varDelta(\delta) \in \mathscr{D}. \tag{27}$$

Thus, according to Definition 4.5, the complex and real stability radii of the polynomial matrix (6) subjected to the affine parameter perturbations (21) can be rewritten as

$$r^a_{\mathbb{K}} = \inf\{\|\varDelta(\delta)\|;\ \varDelta(\delta) \in \mathscr{D}_{\mathbb{K}}, \quad \rho(P(\cdot) + D(\cdot)\varDelta(\delta)E) \geqslant 1\}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \tag{28}$$

where $\mathscr{D}_{\mathbb{C}} := \mathscr{D}, \mathscr{D}_{\mathbb{R}} := \mathscr{D} \cap \mathbb{R}^{(v+1)nN\times(v+1)nN}$ are provided with the perturbation norm $\|\varDelta(\delta)\| = \|\delta\|_\infty$.

Theorem 4.6. *Suppose* (6) *is Schur stable, positive and subjected to affine perturbations of the form* (21), *where* $A_v^{-1} B_{ij} \in \mathbb{R}_+^{n\times n}, i \in \underline{v}, \ 1 \leqslant j \leqslant N$. *If* $B := \sum_{i=0}^{v} \sum_{j=1}^{N} B_{ij}$ *then*

$$r^a_{\mathbb{C}} = r^a_{\mathbb{R}} = 1/\rho(P(1)^{-1}B). \tag{29}$$

Proof. Setting $I := I_{(v+1)nN\times(v+1)nN}$, we obtain from (28)

$$r^a_{\mathbb{R}} = \inf\{\|\varDelta(\delta)\|; \varDelta(\delta) \in \mathscr{D}_{\mathbb{R}},\ \exists z \in \mathbb{C} |z| \geqslant 1 \text{ and } \det(I + EP(z)^{-1} D(z)\varDelta(\delta)) = 0\},$$

where $E, D(z)$ are defined by (26). Now $EP(1)^{-1}D(1) = EP(1)^{-1}A_v A_v^{-1} D(1) \geqslant 0$ (see (18), hence

$$\begin{aligned} r^a_{\mathbb{C}} &\leqslant r^a_{\mathbb{R}} \\ &\leqslant \inf\{\|\varDelta(\delta)\|;\ \varDelta(\delta) \in \mathscr{D}_{\mathbb{R}}, \quad \det(I + EP(1)^{-1}D(1)\varDelta(\delta)) = 0\} \\ &\leqslant \inf\{|\alpha|;\ \alpha \in \mathbb{R},\ \det(I + \alpha EP(1)^{-1}D(1)) = 0\} \\ &= 1/\rho(EP(1)^{-1}D(1)). \end{aligned} \tag{30}$$

On the other hand, we have by (28) and the continuity of the spectrum (see the proof of Theorem 3.8)

$$r_{\mathbb{C}}^{a} = \inf_{z\in\mathbb{C},|z|=1} \inf\{\|\Delta(\delta)\|;\ \Delta(\delta)\in\mathscr{D},\quad \det(I_n + P(z)^{-1}D(z)\Delta(\delta)E)=0\}. \tag{31}$$

Now let $\Delta(\delta)\in\mathscr{D}$ be any perturbation such that $\det(I+P(z)^{-1}D(z)\Delta(\delta)E)=0$ for some $z\in\mathbb{C}, |z|=1$. Then $\det(I+EP(z)^{-1}D(z)\Delta(\delta))=0$ and since $\Delta(\delta)$ is diagonal and $\|\Delta(\delta)\|=\|\delta\|_\infty$ we get $\|\Delta(\delta)\|I \geqslant |\Delta(\delta)|$. Hence, by (4)

$$\|\Delta(\delta)\|\rho(|EP(z)^{-1}D(z)|) \geqslant \rho(EP(z)^{-1}D(z)\Delta(\delta)) \geqslant 1. \tag{32}$$

In the proof of Theorem 4.4, we have seen that $|P(z)^{-1}A_\nu| \leqslant P(1)^{-1}A_\nu$ for every $z\in\mathbb{C}, |z|=1$ (see (18)). Moreover, since $A_\nu^{-1}B_{ij}\geqslant 0$ for all $i\in\underline{\nu},\ 1\leqslant j\leqslant N$ by assumption, we have $|A_\nu^{-1}D(z)| = A_\nu^{-1}D(1)$ on the unit circle. It follows that $\rho(|EP^{-1}(z)D(z)|)\leqslant\rho(EP(1)^{-1}D(1))$ for every $z\in\mathbb{C}, |z|=1$. So by (31) and (32)

$$r_{\mathbb{C}}^{a} \geqslant \inf_{z\in\mathbb{C},|z|=1} \frac{1}{\rho(|EP(z)^{-1}D(z)|)} = \frac{1}{\rho(EP(1)^{-1}D(1))}. \tag{33}$$

Combining (30) and (33) we obtain, making use of [7, Theorem 1.3.20],

$$r_{\mathbb{C}}^{a} = r_{\mathbb{R}}^{a} = \frac{1}{\rho(EP(1)^{-1}D(1))} = \frac{1}{\rho(P(1)^{-1}D(1)E)}.$$

Since $D(1)E=\sum_{i=0}^{\nu}\sum_{j=1}^{N}B_{ij}=B$ by (26) and (24), formula (29) follows. □

We conclude the paper by two examples illustrating Theorems 4.4 and 4.6.

Example 4.7. Consider the linear difference system

$$A_2 y(t+2) = A_1 y(t+1) + A_0 y(t), \qquad t\in\mathbb{N},$$

where

$$A_2=\begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix},\quad A_2^{-1}=\begin{bmatrix}1 & -1/2\\ 0 & -1/2\end{bmatrix},\quad A_1=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad A_0=\begin{bmatrix}3/4 & 1\\ 0 & -1\end{bmatrix}.$$

The associated polynomial matrix is

$$P(z)=A_2z^2-A_1z-A_0=\begin{bmatrix}z^2-3/4 & -z^2-z-1\\ 0 & -2z^2+1\end{bmatrix}. \tag{34}$$

Clearly $P(z)$ is Schur stable. Moreover, it is easily verified that $A_2^{-1}A_1\geqslant 0$ and $A_2^{-1}A_0\geqslant 0$, so that $P(z)$ is positive. Now suppose that $P(z)$ is perturbed as follows:

$$P_\Delta(z)=\left(A_2+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\Delta_2\right)z^2-\left(A_1+\begin{bmatrix}1\\ 0\end{bmatrix}\Delta_1\right)z-\left(A_0+\begin{bmatrix}1\\ -1\end{bmatrix}\Delta_0\right), \tag{35}$$

where $\Delta_2\in\mathbb{K}^{2\times 2}$ and $\Delta_1,\Delta_0\in\mathbb{K}^{1\times 2}$. The perturbations are of the form (8) with

$$E_0=E_1=E_2=I_2 \quad\text{and}$$

$$D_0=\begin{bmatrix}1\\ -1\end{bmatrix},\quad D_1=\begin{bmatrix}1\\ 0\end{bmatrix},\quad D_2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Since $E_i \geqslant 0$ and $A_2^{-1}D_0 \geqslant 0$, $A_2^{-1}D_1 \geqslant 0$, $A_2^{-1}D_2 \geqslant 0$, the assumptions of Theorem 4.4 are satisfied. We have $G_{ii}(1) = (A_2 - A_1 - A_0)^{-1}D_i$ and $(A_2 - A_1 - A_0)^{-1} = \begin{bmatrix} 4 & -12 \\ 0 & -1 \end{bmatrix}$ so that

$$G_{00}(1) = \begin{bmatrix} 16 \\ 1 \end{bmatrix}, \quad G_{11}(1) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad G_{22}(1) = \begin{bmatrix} 4 & 12 \\ 0 & 1 \end{bmatrix}.$$

If we provide $\mathbb{K}^2$ with the maximum norm and $\mathbb{K}^1$ with $|\cdot|$, we obtain $\|G_{00}(1)\| = 16, \|G_{11}(1)\| = 4, \|G_{22}(1)\| = 16$. Hence $r_{\mathbb{C}} = r_{\mathbb{R}} = 1/16$. If $\|\Delta_0\| + \|\Delta_1\| + \|\Delta_2\| < 1/16$, $P_\Delta(z)$ is Schur stable. On the other hand a destabilizing perturbation $\Delta = (\Delta_0, \Delta_1, \Delta_2)$ of minimal norm $\gamma(\Delta) = 1/16$ can be constructed as in the proof of Lemma 3.7. Here we may choose $i_0 = 0$ or $i_0 = 2$. Let e.g. $i_0 = 2$. Then we may take $u_0 = [1 \;\; 1]^\top$ to obtain $\|G_{22}(1)u_0\| = \|G_{22}(1)\| = 16$. The dual norm on $(\mathbb{K}^2)^* = \mathbb{K}^{1\times 2}$ is the 1-norm. To obtain $y_0^* G_{22}(1)u_0 = \|G_{22}(1)\|$ with $\|y_0^*\|_1 = 1$ we choose $y_0^* = [1 \;\; 0]$. Hence $\Delta = (0, 0, \Delta_2)$ with

$$\Delta_2 = -\|G_{22}(1)\|^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \;\; 0] = -1/16 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

is a minimal norm destabilizing perturbation. In fact, one easily verifies that $\det P_\Delta(1) = 0$. Note that minimum norm destabilization is obtained here by only perturbing the leading coefficient matrix A_2.

In the following example we deal with the same perturbed polynomial as in the previous one, but represent it in form (22) by an appropriate choice of the structure matrices B_{ij}. Since the corresponding perturbation norm is different from $\gamma(\Delta)$ used in the previous example we obtain different stability radii $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a$.

Example 4.8. Let $P(z)$ and $P_\Delta(z)$ be as in (34) and (35), respectively. Writing $\Delta_0 = [\delta_{01} \;\; \delta_{02}]$, $\Delta_1 = [\delta_{11} \;\; \delta_{12}]$, $\Delta_2 = \begin{bmatrix} \delta_{21} & \delta_{22} \\ \delta_{23} & \delta_{24} \end{bmatrix}$ and setting

$$B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{23} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad B_{24} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_{11} = B_{21}, \quad B_{12} = B_{22}, \quad B_{01} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

$$B_{03} = B_{04} = B_{13} = B_{14} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the perturbed polynomial $P_\Delta(z)$ (35) can be represented in the form

$$P_\delta(z) = \left(A_2 + \sum_{j=1}^{4} \delta_{2j} B_{2j} \right) z^2 - \left(A_1 + \sum_{j=1}^{4} \delta_{1j} B_{1j} \right) z - \left(A_0 + \sum_{j=1}^{4} \delta_{0j} B_{0j} \right).$$

It is easily verified that $A_2^{-1}B_{ij} \geqslant 0$ for $i = 0, 1, 2,\ j = 1, \dots, 4$. Hence we can apply Theorem 4.6 and obtain

$$r_{\mathbb{C}}^{a} = r_{\mathbb{R}}^{a} = \rho(P(1)^{-1}B)^{-1},$$

where

$$B = \sum_{i=0}^{2}\sum_{j=1}^{4} B_{ij} = \begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix}.$$

Since

$$P(1)^{-1} = (A_2 - A_1 - A_0)^{-1} = \begin{bmatrix} 4 & -12 \\ 0 & -1 \end{bmatrix}$$

(see Example 4.7) we get

$$r_{\mathbb{C}}^{a} = r_{\mathbb{R}}^{a} = \rho\left(\begin{bmatrix} 36 & 36 \\ 2 & 2 \end{bmatrix}\right) = 1/38.$$

Thus $P_\delta(z)$ is Schur stable if $|\delta_{ij}| < 1/38$ for all $i \in \underline{2},\ j \in \bar{3}$. On the other hand it is easily verified that if we set $\hat{\delta}_{ij} = 1/38$ for all $i \in \underline{2},\ j \in \bar{3}$ then $\hat{\delta} = (\hat{\delta}_{ij})$ is a minimum norm destabilizing perturbation satisfying $\det P_{\hat{\delta}}(1) = 0$.

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