Gershgorin-Brualdi Perturbations and Riccati Equations

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This paper is dedicated to Paul Fuhrmann
on the occasion of his 70th birthday.

Abstract

For uncertain linear systems with complex parameter perturbations of static output feedback type a quadratic Liapunov function of maximal robustness was constructed in [5]. Such Liapunov functions can be used to ensure the stability of uncertain systems under arbitrary nonlinear and time-varying perturbations which are smaller than the stability radius. In this paper we establish analogous results for structured Gershgorin-Brualdi type perturbations of diagonal matrices where all the matrix entries at an arbitrarily prescribed set of positions are independently perturbed. We also derive explicit and computable formulae for the associated μ-values, stability radii and spectral value sets.

1 Introduction

Gershgorin's theorem [4] has found many applications in control theory. For instance, the direct and inverse Nyquist array methods in multivariable feedback design [12] are based upon this result and the related concept of diagonal dominance plays an important role in stability analysis and control of large scale systems [13]. However, refinements of Gershgorin's inclusion theorem, like Bauer's and Brualdi's theorems, have hardly ever been applied in systems theory, even though they may yield less conservative robustness results, for instance, in the stability analysis of composite systems with a given interconnection structure, see [11]. For a detailed and up-to-date survey of the results available in the field of Gershgorin type inclusion theorems from a linear algebraic point of view, the reader is referred to the recent book by R. S. Varga [14]. This book also contains an extensive list of references.

In this paper we will study perturbations of diagonal system matrices where all the entries at an arbitrarily prescribed set of positions are independently perturbed. The structure of the perturbations is defined via a matrix $E = (e_{ij})$ with entries either 1 or 0. If $e_{ij} = 1$ there is a perturbation at the (i, j) position and if $e_{ij} = 0$ there is no perturbation. Such perturbations will be called Gershgorin-Brualdi perturbations. Brualdi was the first to derive refined eigenvalue inclusion regions by taking the zero pattern of the matrix into account. In his paper [3] he used this idea in order to sharpen Gershgorin's and Brauer's previous inclusion theorems [4], [2], see also [9, Ch. 6] and [14, Ch. 2].

In contrast with the above authors we do not deal with the problem of finding an inclusion region for the spectrum of a *given* matrix, but we try to determine the set of eigenvalues of *all* the matrices which are perturbations of a fixed diagonal matrix, with zero perturbation entries at prescribed positions, and where the perturbations are bounded in norm by an arbitrary positive real $\delta > 0$. The problem of determining these so-called *spectral value sets* will be studied in the context of generalized μ-analysis, see [7]. The first aim is to obtain explicit computable formulas for the associated μ-values, stability radii and spectral value sets. Our approach is different from that of the recent article [11] where, more generally, Gershgorin type perturbations of *block-diagonal* matrices and various perturbation norms are considered. Here we study a *specific perturbation norm* for which a surprising variety of concrete results can be obtained. The essential new tool is a parametrized Metzler matrix which permits a computable characterization of the μ-value. This in turn yields explicit formulas for the corresponding stability radii and spectral value sets. Moreover in the case where E is reducible we will see that the problem of computing them can be decomposed into computing stability radii and spectral value sets of subsystems corresponding to its irreducible components.

In the case of full-block perturbations carrying the spectral norm it is known (see [5], [6] and Section 2) that the stability radius can be characterized by means of a parametrized Riccati equation. This makes it possible to construct a quadratic Liapunov function of maximal robustness and is a key for obtaining tight robustness results with respect to *time-varying* and *nonlinear* full-block perturbations. We will show that, although there are subtle differences, similar results can be obtained for arbitrary Gershgorin-Brualdi perturbation structures. To the best of our knowledge Liapunov functions of maximal robustness have not been constructed for highly structured perturbations before. The problem of constructing such Liapunov functions in the context of μ-analysis has been stated as an open problem in [11]. This problem is solved here for the special case of Gershgorin-Brualdi perturbations.

We proceed as follows. In Section 2 we recall the definitions of μ-value, stability radius and spectral value sets as well as some known results, especially for the full-block case. We also introduce a parametrized algebraic Riccati equation and state a theorem concerning the existence of Hermitian solutions. In Section 3 we introduce the crucial tools of our analysis, the *parametrized Metzler matrix* and the *singularity parameter* associated with a diagonal matrix and a given Gershgorin-Brualdi perturbation structure. By means of this singularity parameter we characterize the stability radius and spectral value sets. In particular we prove that the real and the complex stability radii coincide for Gershgorin-Brualdi perturbations if the system data are real. In Section 4 we introduce parametrized algebraic Riccati equations and construct quadratic Liapunov functions of maximal robustness for diagonal systems. Then in Section 5 we show how these Liapunov functions can be used to obtain domains of guaranteed attraction for uncertain systems with nonlinear and/or time-varying perturbations. We obtain precise formulas for the complex stability radii of diagonal systems with respect to time-varying and/or nonlinear perturbations of Gershgorin-Brualdi type. Showing that they are equal to each other but, rather surprisingly, not always equal to the complex stability radius for linear time invariant perturbations of the same structure. This is in contrast to the full-block case carrying the spectral norm where all three are equal, see [7, §5.6].

2 Preliminaries

In this section we introduce some basic concepts and fix the notation. The symbols $\mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$ denote the sets of positive integers, real numbers, non-negative real numbers and complex numbers, respectively. By $\mathbb{K}^{n\times m}$ we denote the set of n by m matrices with entries in $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For any $n \in \mathbb{N}$ let $\underline{n} = \{1, \ldots, n\}$. If $A = (a_{ij}) \in \mathbb{C}^{n\times m}$

we define $|A| := (|a_{ij}|)$ and for real matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times m}$ we write $A \le B$ if $a_{ij} \le b_{ij}$ for all $i \in \underline{n}, j \in \underline{m}$. If A is square then $\sigma(A)$, $\rho(A) = \mathbb{C} \setminus \sigma(A)$, $\alpha(A)$ denote its spectrum, its resolvent set and its spectral abscissa, $\alpha(A) = \max\{\mathrm{Re}\, s;\ s \in \sigma(A)\}$. By $\mathbb{L}_{n,l,q}$ we denote the set of triples of matrices (A, B, C) with $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times l}, C \in \mathbb{C}^{q\times n}$, $n, l, q \in \mathbb{N}$. The open left half-plane is denoted by $\mathbb{C}_- = \{s \in \mathbb{C}, \mathrm{Re}\, s < 0\}$ and $A \in \mathbb{C}^{n\times n}$ is called *stable* if $\sigma(A) \subset \mathbb{C}_-$. We use the conventions

$$0^{-1} = \infty, \quad \infty^{-1} = 0, \quad \inf \emptyset = \infty. \tag{1}$$

In the following definitions we suppose that $(A, B, C) \in \mathbb{L}_{n,l,q}$ and $\mathbf{\Delta} \subset \mathbb{K}^{l\times q}$ is a $\mathbb{K}$-linear subspace provided with a norm $\|\cdot\|_{\mathbf{\Delta}}$. For a more detailed account of the definitions and results presented in this section see [7]. We consider perturbations of the form

$$A \rightsquigarrow A_\Delta = A + B\Delta C, \quad \Delta \in \mathbf{\Delta}. \tag{2}$$

Definition 2.1. The μ-value of a matrix $M \in \mathbb{C}^{q\times l}$ (with respect to the normed perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$) is defined by

$$\mu_{\mathbf{\Delta}}(M) := \left[\inf\{\|\Delta\|_{\mathbf{\Delta}}\,;\ \Delta \in \mathbf{\Delta},\ 1 \in \sigma(\Delta M)\,\}\right]^{-1}. \tag{3}$$

Definition 2.2. Given a system $(A, B, C) \in \mathbb{L}_{n,l,q}$ and a perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$, the spectral value set of A for a perturbation level $\delta > 0$, with respect to perturbations of the form (2), is the following subset of the complex plane.

$$\sigma_{\mathbf{\Delta}}(A, B, C; \delta) := \bigcup_{\Delta \in \mathbf{\Delta},\ \|\Delta\|_{\mathbf{\Delta}} < \delta} \sigma(A + B\Delta C). \tag{4}$$

Definition 2.3. Given a system $(A, B, C) \in \mathbb{L}_{n,l,q}$ with $\sigma(A) \subset \mathbb{C}_-$ and a perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$, the stability radius of A with respect to perturbations of the form (2) is defined by

$$r_{\mathbf{\Delta}}(A, B, C) = \inf\{\|\Delta\|_{\mathbf{\Delta}}\,;\ \Delta \in \mathbf{\Delta},\ \sigma(A + B\Delta C) \not\subset \mathbb{C}_-\}. \tag{5}$$

It is easily seen that a minimum in (5) always exists if $r_{\mathbf{\Delta}}(A, B, C)$ is finite. In this case the stability radius is the norm of a smallest perturbation in $\mathbf{\Delta}$ which destabilizes A. Spectral value sets and stability radii can be characterized via μ-values as follows.

Proposition 2.4. *Let $(A, B, C) \in \mathbb{L}_{n,l,q}$ be a given system and $G(s) = C(sI_n - A)^{-1}B$ be the associated transfer function. Then*

$$\begin{aligned} \sigma_{\mathbf{\Delta}}(A, B, C; \delta) &= \sigma(A) \cup \{s \in \rho(A);\ \mu_{\mathbf{\Delta}}(G(s)) > \delta^{-1}\}, \quad \delta > 0; & (6)\\ r_{\mathbf{\Delta}}(A, B, C) &= \left(\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta}}(G(\imath\omega))\right)^{-1} \quad \textit{if } A \textit{ is stable.} & (7) \end{aligned}$$

Specializing to *block-diagonal* and *full-block* perturbations further results are known if the perturbation norm $\|\cdot\|_{\mathbf{\Delta}}$ is an operator norm. Let $\mathbb{C}^l$, $\mathbb{C}^q$ be endowed with arbitrary norms and $\mathbb{C}^{l\times q}$, $\mathbb{C}^{q\times l}$ with the induced operator norms $\|\cdot\|_{\mathcal{L}(\mathbb{C}^q,\mathbb{C}^l)}$ and $\|\cdot\|_{\mathcal{L}(\mathbb{C}^l,\mathbb{C}^q)}$. In the case where $\mathbb{C}^l$, $\mathbb{C}^q$ are provided with 2-norms, we write $\|\cdot\|_2$ for both vector norms and the corresponding operator norm. Suppose $\mathbf{\Delta} \subset \mathbb{C}^{l\times q}$ is the vector space of block-diagonal perturbations of the following form provided with the corresponding operator norm

$$\mathbf{\Delta} = \{\mathrm{diag}\,(\Delta_1, \ldots, \Delta_N); \Delta_i \in \mathbb{C}^{l_i\times q_i},\ i \in \underline{N}\}, \quad \|\cdot\|_{\mathbf{\Delta}} = \|\cdot\|_{\mathcal{L}(\mathbb{C}^q,\mathbb{C}^l)} \tag{8}$$

where $N \geq 1$, $l_i \geq 1$, $q_i \geq 1$, $i \in \underline{N}$ are given such that $\sum_{i=1}^N l_i = l$, $\sum_{i=1}^N q_i = q$. Then estimates of the associated μ-values, spectral value sets and stability radii can be obtained by the following scaling method. For any scaling vector $\gamma = (\gamma_1, \ldots, \gamma_N)$ where $\gamma_i > 0$, $i \in \underline{N}$ we set $R_\gamma = \mathrm{diag}(\gamma_1 I_{q_1}, \ldots, \gamma_N I_{q_N})$ and $L_\gamma = \mathrm{diag}\,(\gamma_1 I_{l_1}, \ldots, \gamma_N I_{l_N})$. Then $L_\gamma \Delta R_\gamma^{-1} = \Delta$ for all $\Delta \in \mathbf{\Delta}$ and this fact implies, see [7, §4.4], that

$$\mu_{\mathbf{\Delta}}(G) \leq \|R_\gamma G L_\gamma^{-1}\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)}, \quad G \in \mathbb{C}^{q \times l}. \tag{9}$$

As a consequence we have, for every scaling vector $\gamma = (\gamma_1, \ldots, \gamma_N) > 0$,

$$\sigma_{\mathbf{\Delta}}(A, B, C; \delta) \subset \sigma(A) \cup \{s \in \rho(A); \|R_\gamma G(s) L_\gamma^{-1}\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)} > \delta^{-1}\}, \tag{10}$$

$$r_{\mathbf{\Delta}}(A, B, C) \geq \left(\sup_{\omega \in \mathbb{R}} \|R_\gamma G(\imath\omega) L_\gamma^{-1}\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)} \right)^{-1}. \tag{11}$$

In the *full-block case* where $\mathbf{\Delta} = \mathbb{K}^{l \times q}$ precise formulas are obtained without any scaling. For this case the spectral value sets and stability radii are denoted by $\sigma_{\mathbb{K}}(A, B, C; \delta)$ and $r_{\mathbb{K}}(A, B, C)$, respectively, and are called the *complex* and *real* spectral value sets and stability radii according to whether $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. For the complex case ($\mathbf{\Delta} = \mathbb{C}^{l \times q}$, $\|\cdot\|_{\mathbf{\Delta}} = \|\cdot\|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C}^l)}$), one has $\mu_{\mathbb{C}^{l \times q}}(G(s)) = \|G(s)\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)}$ and therefore Proposition 2.4 implies

$$\sigma_{\mathbb{C}}(A, B, C; \delta) = \sigma(A) \cup \{s \in \rho(A); \|G(s)\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)} > \delta^{-1}\}; \tag{12}$$

$$r_{\mathbb{C}}(A, B, C) = \left(\max_{\omega \in \mathbb{R}} \|G(\imath\omega)\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)} \right)^{-1}. \tag{13}$$

In the following we will assume that *both $\mathbb{C}^l$ and $\mathbb{C}^q$ are provided with the 2-norm so that the norms $\|\cdot\|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C}^l)}, \|\cdot\|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^q)}$ are both spectral norms.* Then

$$r_{\mathbb{C}}(A, B, C) = \left(\max_{\omega \in \mathbb{R}} \|G(\imath\omega)\|_2 \right)^{-1} = \|G(\cdot)\|_{H^\infty}^{-1}. \tag{14}$$

Throughout the rest of this paper we will reserve the notation $r_{\mathbb{C}}(A, B, C)$ for the complex stability radius with respect to the spectral norm on $\mathbb{C}^{l \times q}$. For stable A this stability radius can be characterized in terms of the parametrized algebraic Riccati equation

$$PA + A^*P - \rho^2 C^*C - PBB^*P = 0. \tag{15}$$

A solution P of (15) is said to be *stability preserving* if $\sigma(A - BB^*P) \subset \mathbb{C}_-$. The real vector space of all the Hermitian matrices in $\mathbb{C}^{n \times n}$ is denoted by $\mathcal{H}_n(\mathbb{C})$.

Theorem 2.5. *Let $(A, B, C) \in \mathbb{L}_{n,l,q}$ be a given system where A is stable, and let $G(s) = C(sI_n - A)^{-1}B$ be the associated transfer matrix. Then the following statements are equivalent.*

(i) There exists a (unique) stability preserving solution $P_\rho \in \mathcal{H}_n(\mathbb{C})$ of (15) which is maximal.

(ii) $\rho < r_{\mathbb{C}} = r_{\mathbb{C}}(A, B, C)$.

If $\rho = r_{\mathbb{C}}$, then (15) has a unique solution $P_{r_{\mathbb{C}}} \in \mathcal{H}_n(\mathbb{C})$ with $\sigma(A - BB^*P_{r_{\mathbb{C}}}) \subset \overline{\mathbb{C}_-}$ which is maximal. Moreover, if (A, C) is observable, then $P_{r_{\mathbb{C}}}$ is negative definite and $V(x) = -\langle x, P_{r_{\mathbb{C}}} x \rangle$ is a joint quadratic Liapunov function for all perturbed systems

$$\dot{x} = Ax + B\Delta Cx, \quad \Delta \in \mathbb{C}^{l \times q}, \quad \|\Delta\|_2 < r_{\mathbb{C}}(A, B, C). \tag{16}$$

Since there is no joint Liapunov function for all the perturbed systems (16) with $\|\Delta\|_2 < \rho$ if $\rho > r_{\mathbb{C}}(A, B, C)$, we may call the quadratic Liapunov function $V(x)$ given in Theorem 2.5 one of *maximal robustness* for the class of perturbations $\mathbf{\Delta} = \mathbb{C}^{l \times q}$ endowed with the spectral norm.

3 μ-value and stability radii

We analyse perturbations of stable diagonal matrices. Let $a_1, \dots, a_n \in \mathbb{C}_-$ be given and $A = \mathrm{diag}(a_1, \dots, a_n)$. The perturbations of A that we consider will have an arbitrarily pre-specified zero structure. Suppose that $E \in \{0,1\}^{n \times n}$ is a given matrix with entries e_{ij} either 1 or 0 and for each $i \in \underline{n}$, $\mathcal{I}_i = \{j \in \underline{n}\,;\, e_{ij} = 1\}$. We say that $\Delta = (\delta_{ij}) \in \mathbb{C}^{n \times n}$ is of structure E if $e_{ij} = 0$ implies $\delta_{ij} = 0$. Let $\mathbf{\Delta} = \mathbf{\Delta}_E \subset \mathbb{C}^{n \times n}$ be the vector space of all the perturbations $\Delta \in \mathbb{C}^{n \times n}$ of structure E. We provide it with the norm $\|\cdot\|_{\mathbf{\Delta}}$ induced by the operator norm $\|\cdot\|_{2,\infty}$ on $\mathbb{C}^{n \times n}$ given by

$$\|X\|_{2,\infty} = \max_{i \in \underline{n}} \Big(\sum_{j \in \underline{n}} |x_{ij}|^2 \Big)^{1/2}, \quad X = (x_{ij}) \in \mathbb{C}^{n \times n}. \tag{17}$$

Thus $\|\Delta\|_{\mathbf{\Delta}}$ is the operator norm of $\Delta \in \mathbf{\Delta}$ regarded as a map from $\mathbb{C}^n$ with the 2-norm to $\mathbb{C}^n$ with the ∞-norm. The reason for this particular choice of perturbation norm will be made clear in Section 4. In summary, the normed perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$ is defined by

$$\mathbf{\Delta} = \{\Delta = (\delta_{ij}) \in \mathbb{C}^{n \times n}; \delta_{ij} = 0 \text{ if } e_{ij} = 0\}, \; \|\Delta\|_{\mathbf{\Delta}} = \|\Delta\|_{2,\infty} = \max_{i \in \underline{n}} \Big(\sum_{j \in \mathcal{I}_i} |\delta_{ij}|^2 \Big)^{1/2}. \tag{18}$$

In the following we consider additive perturbations of the diagonal matrix A of the following form

$$A \rightsquigarrow A(\Delta) := A + \Delta, \quad \Delta \in \mathbf{\Delta}. \tag{19}$$

Comparing (19) with (2) we see that the structure matrices B, C are both equal to the identity matrix I_n. We denote the spectral value set at uncertainty level δ by $\sigma_{\mathbf{\Delta}}(A; \delta)$ and the stability radius by $r_{\mathbf{\Delta}}(A)$.
For an analysis of the spectral effects of these perturbations we will sometimes make the assumption that the structure matrix E is irreducible. E is said to be *irreducible* if it is not possible to find a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that $PEP^\top = \begin{bmatrix} E^{11} & 0 \\ E^{21} & E^{22} \end{bmatrix}$, with $E^{11} \in \mathbb{R}^{n_1 \times n_1}$, $E^{22} \in \mathbb{R}^{(n-n_1) \times (n-n_1)}$ for some $n_1 \in [1, n)$. The irreducibility of E has a nice graph theoretical interpretation. Let $\mathcal{G}(E)$ be the directed graph corresponding to the matrix E, see [9]. Then the matrix E is irreducible if and only if $\mathcal{G}(E)$ is strongly connected, see [1, §2.2].
If E is reducible, it is known (see [1, §2.3]) that E can be reduced by simultaneous row and column permutations to block triangular form

$$PEP^\top = \begin{bmatrix} E_{11} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{bmatrix}, \qquad (P \text{ a permutation matrix}) \tag{20}$$

where each diagonal block E_{ii} is square and is either irreducible or a 1×1 null matrix. Applying the permutation P to $A + \Delta$ we obtain

$$P(A+\Delta)P^\top = A_P + \Delta_P, \ A_P = PAP^\top = \operatorname{diag}(A_1, \ldots, A_s), \ \Delta_P = P\Delta P^\top = (\Delta_{kl})_{k,l \in \underline{s}} \quad (21)$$

where the $A_k \in \mathbb{R}^{n_k \times n_k}$ are diagonal matrices of the same size as the E_{kk} and Δ_P is a block triangular matrix with blocks $\Delta_{kl} \in \mathbb{C}^{n_k \times n_l}$ of the same size as the E_{kl}, $k, l \in \underline{s}$. Clearly the diagonal elements of A_P are a permutation of the diagonal elements of A. Moreover, the blocks Δ_{kl} are of the structure E_{kl} in the sense that if an entry $E_{kl}(i,j)$ of E_{kl} is zero then $\Delta_{kl}(i,j) = 0$. For any given pair $(k,l) \in \underline{s} \times \underline{s}$, let $\mathbf{\Delta}_{kl}$ denote the set of all blocks $\Delta_{kl} \in \mathbb{C}^{n_k \times n_l}$ of structure E_{kl}. We provide $\mathbb{C}^{n_k \times n_l}$ and $\mathbf{\Delta}_{kl} \subset \mathbb{C}^{n_k \times n_l}$ with the norm

$$\|Y\|_{2,\infty} = \max_{i \in \underline{n_k}} \Big(\sum_{j \in \underline{n_l}} |y_{ij}|^2 \Big)^{1/2}, \quad Y = (y_{ij}) \in \mathbb{C}^{n_k \times n_l}. \quad (22)$$

Lemma 3.1. *Suppose that E is reduced to block triangular form as in* (20) *by a permutation matrix P, and A_P, Δ_P are defined by* (21)*. Then, for any $\delta > 0$*

$$\sigma_{\mathbf{\Delta}}(A) = \bigcup_{\Delta \in \mathbf{\Delta}, \, \|\Delta\|_{\mathbf{\Delta}} < \delta} \sigma(A + \Delta) = \bigcup_{k \in \underline{s}} \ \bigcup_{\substack{\Delta_{kk} \in \mathbf{\Delta}_{kk}, \\ \|\Delta_{kk}\|_{2,\infty} < \delta}} \sigma(A_k + \Delta_{kk}) = \bigcup_{k \in \underline{s}} \sigma_{\mathbf{\Delta}_{kk}}(A_k; \delta). \quad (23)$$

Moreover

$$r_{\mathbf{\Delta}}(A) = \min_{k \in \underline{s}} r_{\mathbf{\Delta}_{kk}}(A_k). \quad (24)$$

Proof: Suppose $\lambda \in \sigma(A + \Delta)$ for some $\Delta \in \mathbf{\Delta}$, $\|\Delta\|_{\mathbf{\Delta}} < \delta$. Then $\lambda \in \sigma(A_P + \Delta_P)$ and since $A_P + \Delta_P$ is block triangular with diagonal blocks $A_i + \Delta_{ii}$, there exists $k \in \underline{s}$ such that $\lambda \in \sigma(A_k + \Delta_{kk})$. By (18) and (22) we have $\|\Delta_{kk}\|_{2,\infty} \leq \|\Delta\|_{\mathbf{\Delta}}$. This proves $\subset$ in (23).
Now suppose that $\lambda \in \sigma(A_k + \Delta_{kk})$ for some $k \in \underline{s}$ and $\Delta_{kk} \in \mathbf{\Delta}_{kk}$ with $\|\Delta_{kk}\|_{2,\infty} < \delta$. Let $\Delta_P = (\Delta_{kl})_{k,l \in \underline{s}}$ be the block matrix obtained from Δ_{kk} by setting all the other blocks Δ_{hl}, $(h,l) \neq (k,k)$, equal to zero. Then $\Delta_P \in P\mathbf{\Delta}P^\top$, $\|\Delta_P\|_{2,\infty} = \|\Delta_{kk}\|_{2,\infty} < \delta$ and $\lambda \in \sigma(A_k + \Delta_{kk}) \subset \sigma(A_P + \Delta_P)$ and so $\lambda \in \sigma(A + \Delta)$ where $\Delta := P^\top \Delta_P P \in \mathbf{\Delta}$, $\|\Delta\|_{\mathbf{\Delta}} = \|\Delta_P\|_{2,\infty} < \delta$. This concludes the proof of (23).
Now suppose $\delta > r_{\mathbf{\Delta}_{kk}}(A_k)$ where $k \in \underline{s}$, then $\sigma_{\mathbf{\Delta}_{kk}}(A_k; \delta) \not\subset \mathbb{C}_-$ and hence by (23) $\sigma_{\mathbf{\Delta}}(A; \delta) \not\subset \mathbb{C}_-$. So $\delta \geq r_{\mathbf{\Delta}}(A)$ and therefore $r_{\mathbf{\Delta}_{kk}}(A_k) \geq r_{\mathbf{\Delta}}(A)$ for all $k \in \underline{s}$. Conversely if $\delta > r_{\mathbf{\Delta}}(A)$, then $\sigma_{\mathbf{\Delta}}(A; \delta) \not\subset \mathbb{C}_-$ and hence by (23) there exists $k \in \underline{s}$ such that $\sigma_{\mathbf{\Delta}_{kk}}(A_k; \delta) \not\subset \mathbb{C}_-$ and so $\delta \geq r_{\mathbf{\Delta}_{kk}}(A_k)$. Since for every $\delta > r_{\mathbf{\Delta}}(A)$ such an inequality holds for some $k \in \underline{s}$ we conclude that there exists $\hat{k} \in \underline{s}$ such that $r_{\mathbf{\Delta}}(A) \geq r_{\mathbf{\Delta}_{\hat{k}\hat{k}}}(A_{\hat{k}})$. This proves (24). □

Remark 3.2. If E_{kk} is a 1×1 null matrix then $A_k = [a_i]$ for some $i \in \underline{n}$ and $\mathbf{\Delta}_{kk} = \{[0]\}$. It follows that a_i is a fixed eigenvalue of all perturbed matrices $A + \Delta$, $\Delta \in \mathbf{\Delta}$ and $r_{\mathbf{\Delta}_{kk}}(A_k) = \infty$. □

Lemma 3.1 shows that the analysis of the spectral perturbation problem under consideration can be reduced to the case where E is irreducible.
In order to determine the stability radius $r_{\mathbf{\Delta}}(A)$ of A with respect to perturbations of the form (19) we will make use of specially structured Metzler matrices.

Definition 3.3. Given any diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{C}^{n\times n}$ and $E \in \{0,1\}^{n\times n}$ we define the associated *parametrized Metzler matrix* $M(\rho, D, E)$, $\rho \geq 0$ by

$$M(\rho, D, E) := -|D|^2 + \rho^2 E \in \mathbb{R}^{n\times n}, \quad \rho \geq 0. \tag{25}$$

The *singularity parameter* of the parametrized Metzler matrix is defined by

$$r(D, E) := \inf\{\rho \in \mathbb{R}_+; \det M(\rho, D, E) = 0\}. \tag{26}$$

Remark 3.4. If $r(D, E) < \infty$, the "inf" in (26) may be replaced by "min", see Lemma 3.5. We have $r(D, E) = 0$ if and only if D is singular. If $0 < r(D, E) < \infty$ then $r(D, E)$ can be computed by resolving a generalized eigenvalue problem. In fact, we have for $\rho > 0$

$$\det(\rho^2 E - |D|^2) = 0 \;\Leftrightarrow\; \det(E - \rho^{-2}|D|^2) = 0, \tag{27}$$

and so $r(D, E) = 1/\sqrt{\lambda_{\max}}$ where $\lambda_{\max} > 0$ is the dominant eigenvalue of the regular matrix pencil $E - \lambda|D|^2$ (or, equivalently, of the non-negative matrix $E|D|^{-2}$). □

The following lemma is a consequence of Perron-Frobenius theory, see [9, Ch. 8], [1].

Lemma 3.5. *Let $D, D' \in \mathbb{C}^{n\times n}$ be diagonal matrices and $E, E' \in \{0,1\}^{n\times n}$. Then*

*(i) $\alpha(M(\rho, D, E)) \in \sigma(M(\rho, D, E))$ and there exists a non-negative eigenvector z of $M(\rho, D, E)$ corresponding to the eigenvalue $\alpha(M(\rho, D, E))$ (*Perron vector*). If $M(\rho, D, E)$ is irreducible, the Perron vector is uniquely determined modulo multiplication by a positive scalar and all its coordinates are positive.*

(ii) The function $\rho \mapsto \alpha(M(\rho, D, E))$ is continuous and non-decreasing on $\mathbb{R}_+$. If E is irreducible, this function is strictly increasing.

(iii) If $\lambda \neq \alpha(M(\rho, D, E))$ is any other eigenvalue of $M(\rho, D, E)$ then $\operatorname{Re}\lambda < \alpha(M(\rho, D, E))$.

(iv) $r(D, E) = \inf\{\rho \in \mathbb{R}_+; \alpha(M(\rho, D, E)) = 0\} = \inf\{\rho \in \mathbb{R}_+; \alpha(M(\rho, D, E)) \geq 0\}$. If $r(D, E) < \infty$ the "inf" in these equalities may be replaced by "min". If additionally E is irreducible, then $\rho = r(D, E)$ is the unique zero of the function $\rho \mapsto \alpha(M(\rho, D, E))$.

(v) If there exists $z \in \mathbb{R}^n_+$, $z \neq 0$ such that $M(\rho, D, E)z \geq 0$ then $\alpha(M(\rho, D, E)) \geq 0$ and $\rho \geq r(D, E)$.

(vi) If there exists $z > 0$ such that $z^\top M(\rho, D, E) \leq \beta z^\top$ (respectively $z^\top M(\rho, D, E) < \beta z^\top$) then $\alpha(M(\rho, D, E)) \leq \beta$ (respectively $\alpha(M(\rho, D, E)) < \beta$).

(vii) If $|D| \leq |D'|$ then $\alpha(M(\rho, D, E)) \geq \alpha(M(\rho, D', E))$ and $r(D, E) \leq r(D', E)$.

(viii) If $E \leq E'$ then $\alpha(M(\rho, D, E)) \leq \alpha(M(\rho, D, E'))$ and $r(D, E) \geq r(D, E')$.

Proof: (i) and (iii) follow from the Perron-Frobenius theory of non-negative matrices.
(ii) The function $\rho \mapsto \alpha(M(\rho, D, E))$ is continuous by the continuity of the spectrum and of the map $\rho \mapsto M(\rho, D, E)$. It is non-decreasing since the map $M \mapsto \alpha(M)$ is non-decreasing on the set of Metzler matrices [8, Lemma 2] and $\rho \leq \rho'$ implies $M(\rho, D, E) \leq M(\rho', D, E)$. Now suppose additionally that E is irreducible and $\rho < \rho'$. Then $M := M(\rho, D, E) \leq M' := M(\rho', D, E)$, $M \neq M'$ and M, M' are irreducible. Hence $\alpha(M) < \alpha(M')$ follows from Corollary 1.3.29 in [1].

(iv) It follows from (i) and (26) that $r(D,E) \leq \inf\{\rho \in \mathbb{R}_+; \alpha(M(\rho,D,E)) = 0\}$. On the other hand, suppose $\det M(\rho,D,E) = 0$ for some $\rho \geq 0$. Then $\alpha(M(\rho,D,E)) \geq 0$ and by (ii) there exists a smallest $\rho_0 \in [0,\rho]$ such that $\alpha(M(\rho_0,D,E)) = 0$. This proves $r(D,E) \geq \rho_0 = \min\{\rho \in \mathbb{R}_+; \alpha(M(\rho,D,E)) = 0\} = \min\{\rho \in \mathbb{R}_+; \alpha(M(\rho,D,E)) \geq 0\}$. Hence the two equalities in the first statement of (iv) hold. The second statement follows from the continuity statement in (ii), and the final statement in (iv) follows from the final statement in (ii).
(v) Suppose $M(\rho,D,E)z \geq 0$ for some $z \in \mathbb{R}^n_+$, $z \neq 0$. There exists $t \geq 0$ such that $M(\rho,D,E) + tI_n \geq 0$. Since $(M(\rho,D,E) + tI_n)z \geq tz$ it follows from [9, Thm. 8.3.2] that $\alpha(M(\rho,D,E) + tI_n) = \varrho(M(\rho,D,E) + tI_n) \geq t$. Hence $\alpha(M(\rho,D,E)) \geq 0$ and so $\rho \geq r(D,E)$ by (iv).
(vi) The first statement can be derived in a similar way as Theorem 2.1.11 in [1]. The second statement follows from the first.
(vii) $|D| \leq |D'|$ implies $M(\rho,D,E) \geq M(\rho,D',E)$ and so $\alpha(M(\rho,D,E)) \geq \alpha(M(\rho,D',E))$ by the monotonicity of $\alpha(\cdot)$ on the set of Metzler matrices. The inequality $r(D,E) \leq r(D',E)$ now follows from (iv).
(viii) is proved in a similar way. $\square$

Example 3.6. Let $n \geq 2$ and let $D = \operatorname{diag}(d_1,\ldots,d_n) \in \mathbb{C}^{n\times n}$ be an arbitrary nonsingular diagonal matrix. We consider *full-block* perturbations described by the structure matrix E_{full} whose entries are all equal to one and off-diagonal perturbations described by the structure matrix $E_{off} := E_{full} - I_n$.
(i) Let $E = E_{full}$. Then all the off-diagonal entries of $M(\rho,D,E_{full})$ are of the form ρ^2 whereas the diagonal entries are of the form $-|d_i|^2+\rho^2$, $i \in \underline{n}$. Note that E is irreducible. Hence $\rho = r(D,E_{full})$ is characterized by the property that the system of equations $M(\rho,D,E_{full})x = 0$, has a strictly positive solution vector $x > 0$, i.e. there exist $x_i > 0$ such that

$$-|d_i|^2 x_i + \sum_{j=1}^n \rho^2 x_j = 0, \quad i \in \underline{n}. \tag{28}$$

Additionally we may impose the condition $\sum_{j=1}^n x_j = 1$. The unique solution of (28) is then given by $x_i = \rho^2/|d_i|^2$, $i \in \underline{n}$ where $\rho = r(D,E_{full})$ is determined by $\sum_{j=1}^n \rho^2/|d_j|^2 = 1$. Hence

$$r(D,E_{full}) = \left[\sum_{j=1}^n \frac{1}{|d_j|^2}\right]^{-1/2}. \tag{29}$$

(ii) If $E = E_{off}$ then the associated parametrized Metzler matrix is of the form

$$M(\rho,D,E_{off}) = \begin{bmatrix} -|d_1|^2 & \rho^2 & \rho^2 & . & . & . & \rho^2 \\ \rho^2 & -|d_2|^2 & \rho^2 & . & . & . & \rho^2 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho^2 & \rho^2 & . & . & . & \rho^2 & -|d_n|^2 \end{bmatrix}, \quad \rho \geq 0. \tag{30}$$

Again $M(\rho,D,E_{off})$ is irreducible and so $\rho = r(D,E_{off})$ is characterized by the property that the system of equations $M(\rho,D,E_{off})x = 0$, has a strictly positive normalized solution vector $x > 0$, i.e. $x = (x_i)$ satisfies

$$-|d_i|^2 x_i + \sum_{j=1}^n \rho^2 x_j - \rho^2 x_i = 0, \quad i \in \underline{n} \quad \text{and} \quad \sum_{j=1}^n x_j = 1.$$

This system of equations is equivalent to

$$x_i = \frac{\rho^2}{|d_i|^2 + \rho^2}, \quad i \in \underline{n} \quad \text{and} \quad \sum_{j=1}^n x_j = 1. \tag{31}$$

Hence $\rho = r(D, E_{off})$ if and only if

$$\rho^2 \sum_{j\in\underline{n}} (|d_j|^2 + \rho^2)^{-1} = 1. \tag{32}$$

Since $d_j \neq 0$ and $\rho \mapsto \rho^2/(|d_j|^2 + \rho^2)$ is strictly increasing from 0 to 1 as ρ goes from 0 to ∞ and $n \geq 2$, there exists a unique positive solution $\rho = \hat{\rho}$ satisfying this equation. Hence $r(D, E_{off}) = \hat{\rho}$. It follows from (29) and (32) that $r(D, E_{off}) > r(D, E_{full})$. $\square$

Proposition 3.7. *Let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix, $E \in \{0, 1\}^{n\times n}$ and P be a permutation matrix such that $PEP^\top$ is of the form (20) and $PDP^\top = \mathrm{diag}(D_1, \dots, D_s)$ where D_i has the same size as E_{ii} for $i \in \underline{s}$. Then*

(i) The function $\rho \mapsto \alpha(M(\rho, D, E))$ on $\mathbb{R}_+$ is the maximum of the s functions $\rho \mapsto \alpha(M(\rho, D_i, E_{ii}))$ and we have

$$r(D, E) = \min_{i\in\underline{s}} r(D_i, E_{ii}). \tag{33}$$

(ii) Each function $\rho \mapsto \alpha(M(\rho, D_i, E_{ii}))$ is analytic on $\mathbb{R}_+$ and either constant or strictly increasing.

(iii) There exists $\rho_0 \geq 0$ such that $\rho \mapsto \alpha(M(\rho, D, E))$ is strictly increasing on $[\rho_0, \infty)$ and constant on $[0, \rho_0]$.

Proof: (i) Since

$$PM(\rho, D, E)P^\top = M(\rho, PDP^\top, PEP^\top), \quad \alpha(M(\rho, D, E)) = \alpha(M(\rho, PDP^\top, PEP^\top))$$

we may assume that E is already of the form (20) (hence $P = I_n$). Then $\alpha(M(\rho, D, E)) = \max_{i\in\underline{s}} \alpha(M(\rho, D_i, E_{ii}))$ follows from the fact that $M(\rho, D, E)$ is block triangular with diagonal blocks $M(\rho, D_i, E_{ii})$, $i \in \underline{s}$. Hence (i) is a consequence of Lemma 3.5 (iv).
(ii) If E_{ii} is the 1×1 zero matrix then $D_i = [d_j]$ for some $j \in \underline{n}$ and $\rho \mapsto \alpha(M(\rho, D_i, E_{ii})) = -|d_j|^2$ is constant. Otherwise E_{ii} is irreducible and so $M(\rho, D_i, E_{ii})$ is irreducible. But then $\alpha(M(\rho, D_i, E_{ii}))$ is a *simple* eigenvalue of $M(\rho, D_i, E_{ii})$ for all $\rho \in \mathbb{R}_+$, and it is known that simple eigenvalues depend analytically on the matrix parameters, see [7]. As a consequence $\rho \mapsto \alpha(M(\rho, D_i, E_{ii}))$ is analytic on $\mathbb{R}_+$. Moreover, it is strictly increasing by Lemma 3.5 (ii).
(iii) First suppose that there exists $\rho > 0$ such that $\alpha(M(\rho, D, E)) = \alpha(M(0, D, E)) = -|d_k|^2$ for some $k \in \underline{n}$. Let $\rho_0 = \sup\{\rho \in \mathbb{R}_+; \alpha(M(\rho, D, E)) = -|d_k|^2\}$. Then $\alpha(M(\rho, D, E)) = -|d_k|^2$ for $\rho \in [0, \rho_0)$. Now let $\rho_0 < \rho_1 < \rho_2 < \infty$ then $\alpha(M(\rho_1, D, E)) = \alpha(M(\rho_1, D_i, E_{ii})) > -|d_k|^2 \geq \alpha(M(0, D_i, E_{ii}))$ for some $i \in \underline{n}$. Hence $\rho \mapsto \alpha(M(\rho, D_i, E_{ii}))$ is not constant and so $\alpha(M(\rho_2, D, E)) \geq \alpha(M(\rho_2, D_i, E_{ii})) > \alpha(M(\rho_1, D_i, E_{ii})) = \alpha(M(\rho_1, D, E))$ by (ii). This shows that $\rho \mapsto \alpha(M(\rho, D, E))$ is strictly increasing on (ρ_0, ∞). On the other hand it follows from the definition of ρ_0 and the continuity of $\rho \mapsto \alpha(M(\rho, D, E))$ that $\alpha(M(\rho_0, D, E)) = -|d_k|^2 < \alpha(M(\rho_1, D, E))$ for every $\rho_1 > \rho_0$. This concludes the proof. $\square$

Remark 3.8. Since $E \geq 0$ we have $\alpha(E) \geq 0$. If $\alpha(E) > 0$ then $\rho \mapsto \alpha(M(\rho, D, E))$ tends to ∞ as $\rho \to \infty$ and so, in particular, $r(D, E) < \infty$. In fact, if we choose $\beta < 0$ such that $\beta I_n \leq -|D|^2$ then $\beta I_n + \rho^2 E \leq M(\rho, D, E))$ and hence $\beta + \rho^2\alpha(E) \leq \alpha(M(\rho, D, E))$ by monotonicity of the spectral abscissa on the convex cone of Metzler matrices. This proves the assertion. Conversely, if $\alpha(E) = 0$ then the function $\rho \mapsto \alpha(M(\rho, D, E))$ is constant on $\mathbb{R}_+$ and $r(D, E) = \infty$. In fact, setting $a := \alpha(-|D|^2) = -\min_{i\in\underline{n}} |d_i|^2$ we obtain $-|D|^2 \leq aI_n$ and so

$$a = \alpha(-|D|^2) \leq \alpha(M(\rho, D, E)) \leq \alpha(aI_n + \rho^2 E) \leq a + \rho^2\alpha(E) = a.$$

The equality $\alpha(E) = 0$ has a nice graph theoretic interpretation. Let $\mathcal{G}(E)$ be the directed graph corresponding to the matrix E and let $\mathcal{Z}(E)$ be the set of cycles of the directed graph $\mathcal{G}(E)$. Then $\alpha(E) = 0$ (or, equivalently, $r(D, E) = \infty$) if and only if $\mathcal{Z}(E) = \emptyset$, see [10, §5.7]. Consequently, if E is irreducible then $\alpha(E) > 0$. $\square$

The following example illustrates some of the previous results.

Example 3.9. Let

$$D = \mathrm{diag}(-10, -8, -6, -4, -2, -2, -4, -6, -8, -10)$$

and define the 10×10 symmetric structure matrix $E_k = (e_{ij}^{(k)})$ by setting $e_{ij}^{(k)} = 0$ if $|i-j| < k$ and $e_{ij}^{(k)} = 1$ otherwise. So E_0 is the full-block 10×10 structure matrix, E_1 is the off-diagonal structure matrix and E_9 is the structure matrix whose only non-zero entries are in the south-western and north-eastern corners. Making use of the graph theoretic interpretation in Remark 3.8 and Lemma 3.5 (vii) we obtain

$$\alpha(E_0) \geq \alpha(E_1) \geq \ldots \geq \alpha(E_9) > 0.$$

The following figures show graphs of the functions $f_k : \rho \mapsto \alpha(M(\rho, D, E_k))$ on the intervals $[0, 2]$ and $[0, 10]$. The functions f_0 (full-block), f_1 (off-diagonal), f_2, f_4 and f_6 are shown over the interval $[0, 2]$ in Figure 1. The graphs of f_k, $k = 0, 1, 2$ cut the zero line within the interval $[0, 2]$ so that their singularity parameter $r(D, E_k)$, $k = 0, 1, 2$ is smaller than 2. The functions f_4 grows more slowly and f_6 is constant on the interval $[0, 2]$ so that the corresponding singularity parameters $r(D, E_k) > 2$ for $k = 4, 6$. Figure 2 shows (in a decreasing order) the graphs of f_k, $k = 4, 6, 8, 9$. We see that the functions f_k get smaller as the k increase, i.e. as the structure matrices E_k decrease.

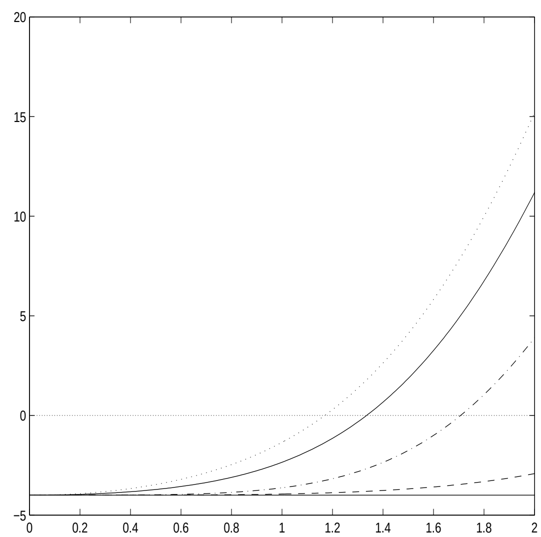

Figure 1: f_0, f_1, f_2, f_4, f_6

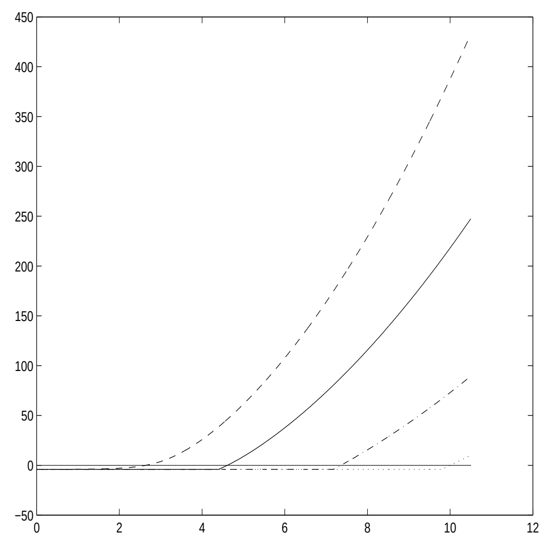

Figure 2: f_4, f_6, f_8, f_9

The functions f_k, $k = 4, 6, 8, 9$ are constant on ever larger intervals and they have corner points on the real axis where they are not differentiable. It can be seen from Figure 2 that $r(D, E_4) < 3$, $r(D, E_6) < 5$, $r(D, E_8) < 8$, $r(D, E_9) \approx 10$. One can show that in fact $r(D, E_9) = 10$. $\square$

In the proof of the following theorem we will make use of the *Hadamard product* of matrices. Given $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{m\times n}$ the Hadamard product of A and B, denoted by $A \circ B$, is defined by $A \circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{m\times n}$.

Theorem 3.10. *For every diagonal matrix* $D = \operatorname{diag}(d_1, ..., d_n) \in \mathbb{C}^{n\times n}$ *and* $E \in \{0,1\}^{n\times n}$, $\mathbf{\Delta} = \mathbf{\Delta}_E$ *we have*

$$\inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta} \text{ and } \det(D+\Delta) = 0\} = r(D,E). \tag{34}$$

If additionally D *is real then* $\inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta} \cap \mathbb{R}^{n\times n} \text{ and } \det(D+\Delta) = 0\} = r(D,E)$.

Proof: Recall that $\mathcal{I}_i = \{j \in \underline{n};\ e_{ij} = 1\}$ and suppose $(D+\Delta)x = 0$ for some $\Delta = (\delta_{ij}) \in \mathbf{\Delta}$, $x = (x_1, ..., x_n)^\top \in \mathbb{C}^n$, $x \neq 0$. If Δ^i denotes the i^{th} row of Δ and E^i the i^{th} row of E, then $\Delta^i x = -d_i x_i$ and $E^i y = \sum_{j\in\mathcal{I}_i} y_j$ for every $y = (y_j) \in \mathbb{C}^n$, $i \in \underline{n}$. Moreover, for any vector $y = (y_j) \in \mathbb{C}^n$ the Hadamard product $\tilde{y} := (E^i)^\top \circ y$ has the coordinates

$$\tilde{y}_j = \begin{cases} y_j & \text{if } j \in \mathcal{I}_i \\ 0 & \text{otherwise} \end{cases}, \quad j \in \underline{n}.$$

Since the product $\Delta^i y$ only depends upon the coordinates y_j of y with $j \in \mathcal{I}_i$ (the others being multiplied by zero) we obtain

$$\Delta^i y = \Delta^i((E^i)^\top \circ y) = \sum_{j\in\mathcal{I}_i} \delta_{ij} y_j, \quad y = (y_j) \in \mathbb{C}^n. \tag{35}$$

Now define $z = (|x_j|^2)$ and set $\rho = \|\Delta\|_{\mathbf{\Delta}}$. Then by (35) and the Cauchy-Schwarz inequality

$$\rho^2 E^i z \geq \|\Delta^i\|_2^2 \sum_{j\in\mathcal{I}_i} |x_j|^2 \geq |\Delta^i((E^i)^\top \circ x)|^2 = |\Delta^i x|^2 = |d_i|^2\,|x_i|^2 = |d_i|^2 z_i, \quad i \in \underline{n}.$$

By (25) this implies $M(\rho, D, E)z \geq 0$ for $z = (|x_i|^2)$ and hence $\rho \geq r(D,E)$ by Lemma 3.5. If $\hat{\rho}$ denotes the left hand side of (34) we conclude that $\hat{\rho} \geq r(D,E)$.
To prove the converse inequality, let $r := r(D,E) < \infty$. Then $\alpha(M(r,D,E)) = 0$ by Lemma 3.5 and there exists $z \in \mathbb{R}^n_+$, $z \neq 0$ such that $M(r,D,E)z = 0$. Let $x = (x_i)$ where $x_i = \sqrt{z_i} \geq 0$ and define

$$x^i = (E^i)^\top \circ x, \quad i \in \underline{n}.$$

If $x^i = 0$ choose $\Delta^i = (0, \ldots, 0) \in \mathbb{R}^{1\times n}$ and if $x^i \neq 0$ let $\Delta^i \in \mathbb{R}^{1\times n}$ be a row vector of norm $\|\Delta^i\|_2 = r$ aligned with $x^i \geq 0$. Then $(x^i)_j = 0$ and $(\Delta^i)_j = 0$ if $j \notin \mathcal{I}_i$, and

$$\Delta^i \geq 0, \qquad \Delta^i x^i = r\,\|x^i\|_2 = r\,\Big(\sum_{j\in\mathcal{I}_i} x_j^2\Big)^{1/2}, \quad i \in \underline{n}.$$

Moreover, since $M(r,D,E)z = 0$ we have

$$|d_i|^2 x_i^2 = r^2 E^i z = r^2 E^i (x_j^2)_{j\in\underline{n}} = r^2 \sum_{j\in\mathcal{I}_i} x_j^2 = r^2 \|x^i\|_2^2 = (\Delta^i x^i)^2, \quad i \in \underline{n}.$$

But this implies $\Delta^i x^i = |d_i| x_i$ for $i \in \underline{n}$ and therefore the matrix $\hat{\Delta}$ with rows $\hat{\Delta}^i = -(d_i/|d_i|)\Delta^i$, $i \in \underline{n}$ satisfies

$$\hat{\Delta} \in \mathbf{\Delta},\ \|\hat{\Delta}\|_{\mathbf{\Delta}} = \max_{i\in\underline{n}} \|\Delta^i\|_2 = r(D,E) \text{ and } \hat{\Delta}^i x = \hat{\Delta}^i x^i = -(d_i/|d_i|)\Delta^i x^i = -d_i x_i,\ i \in \underline{n}.$$

It follows that $(D+\hat{\Delta})x = 0$, hence $\hat{\rho} \leq r(D,E)$. This concludes the proof of (34).
Now note that $\hat{\Delta}$ is real if D is real. Hence in this case

$$\inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta} \cap \mathbb{R}^{n\times n} \text{ and } \det(D+\Delta) = 0\} \leq r(D,E).$$

But always

$$\inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta}\cap\mathbb{R}^{n\times n} \text{ and } \det(D+\Delta)=0\} \geq \inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta} \text{ and } \det(D+\Delta)=0\},$$

and so the second statement of the theorem follows from (34). □

Remark 3.11.
1. The above proof is constructive in that it shows how to construct a perturbation $\Delta \in \mathbf{\Delta}$ of minimum norm $\|\Delta\|_{\mathbf{\Delta}}$ such that $D+\Delta$ is singular, see Example 4.9
2. In the full-block case $E = E_{full}$ where $\mathbf{\Delta} = \mathbb{C}^{n\times n}$, the equality (34) implies that the distance of D from the set of singular matrices with respect to the norm $\|\cdot\|_{\mathbf{\Delta}}$ (18) is $r(D, E_{full})$. If $\mathbf{\Delta} = \mathbb{C}^{n\times n}$ is endowed with the *spectral norm*, the distance of a matrix $M \in$ from singularity is $\sigma_{\min}(M)$, and hence in the diagonal case ($M = D$), $\sigma_{\min}(D) = \min_{i\in\underline{n}} |d_i|$. This is in contrast to the result given in (29) for the norm (18).
3. Given the perturbation set $\mathbf{\Delta}$ and $\lambda \in \mathbb{C}$, it is of interest to determine a matrix $\Delta \in \mathbf{\Delta}$ of minimum norm such that $\lambda \in \sigma(\Delta)$. This problem can be solved by applying the above theorem to $D = -\lambda I_n$.

□

Corollary 3.12. *Under the assumptions of Theorem 3.10*

1. *If $D = \mathrm{diag}(d_1, ..., d_n) \in \mathbb{C}^{n\times n}$ is invertible, the μ-value of D with respect to the perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$ is given by*
$$\mu_{\mathbf{\Delta}}(D) = \left(r(D^{-1}, E)\right)^{-1}. \tag{36}$$
2. *For every $\delta > 0$ the spectral value set of any diagonal matrix $A = \mathrm{diag}(a_1, ..., a_n) \in \mathbb{C}^{n\times n}$ at level δ with respect to the perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$ is given by*
$$\sigma_{\mathbf{\Delta}}(A;\delta) = \bigcup_{\Delta\in\mathbf{\Delta},\ \|\Delta\|_{\mathbf{\Delta}}<\delta} \sigma(A+\Delta) = \{\lambda \in \mathbb{C};\ r(\lambda I_n - A, E) < \delta\}. \tag{37}$$
3. *The stability radius of a stable matrix $A = \mathrm{diag}(a_1, ..., a_n) \in \mathbb{C}^{n\times n}$ with respect to perturbations of the form (2) is given by*
$$r_{\mathbf{\Delta}}(A) = \inf_{\omega\in\mathbb{R}} r(A - \imath\omega I_n, E). \tag{38}$$
4. *If $A = \mathrm{diag}(a_1, ..., a_n) \in \mathbb{R}^{n\times n}$ is a stable real matrix and $\mathbf{\Delta}_{\mathbb{R}} := \mathbf{\Delta} \cap \mathbb{R}^{n\times n}$ then*
$$r_{\mathbf{\Delta}}(A) = r_{\mathbf{\Delta}_{\mathbb{R}}}(A) = r(A, E). \tag{39}$$

Proof: 1. By Theorem 3.10, the singularity parameter $r(D^{-1}, E)$ is equal to

$$\begin{aligned} \inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta},\ \det(D^{-1}+\Delta)=0\} &= \inf\{\|\Delta\|_{\mathbf{\Delta}};\ \Delta \in \mathbf{\Delta},\ \det(I_n + \Delta D)=0\} \\ &= [\mu_{\mathbf{\Delta}}(D)]^{-1} \end{aligned}$$

and this proves (36).
2. Since $\lambda \in \sigma(A+\Delta)$ holds if and only if $\det((\lambda I_n - A) - \Delta)) = 0$, we obtain from (34) that $\lambda \in \sigma_{\mathbf{\Delta}}(A;\delta)$ if and ony if $r(\lambda I_n - A, E) < \delta$. This proves (37).
3. By (7) and (36) we have

$$r_{\mathbf{\Delta}}(A) = \left(\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta}}((\imath\omega I_n - A)^{-1})\right)^{-1} = \inf_{\omega\in\mathbb{R}} \left[\mu_{\mathbf{\Delta}}((\imath\omega I_n - A)^{-1})\right]^{-1} = \inf_{\omega\in\mathbb{R}} r(A - \imath\omega I_n, E).$$

4. Suppose $A = \mathrm{diag}(a_1, ..., a_n)$ is real and stable. By (5) and Theorem 3.10 we have

$$r_{\mathbf{\Delta}}(A) \leq r_{\mathbf{\Delta}_{\mathbb{R}}}(A) \leq \inf\{\|\Delta\|_{\mathbf{\Delta}}\,;\, \Delta \in \mathbf{\Delta}_{\mathbb{R}},\ 0 \in \sigma(A+\Delta)\} = r(A, E).$$

On the other hand we have $|A| \leq |A - \imath\omega I_n|$ for all $\omega \in \mathbb{R}$ and so by (38) and Lemma 3.5

$$r_{\mathbf{\Delta}}(A) = \inf_{\omega\in\mathbb{R}} r(A - \imath\omega I_n, E) = r(A, E).$$

This concludes the proof. □

As an illustration we apply the previous corollary in order to determine the stability radii of a simple two dimensional example with respect to both full-block and off-diagonal perturbations.

Example 3.13. Suppose

$$A = \begin{bmatrix} -1+\imath & 0 \\ 0 & -2+\imath \end{bmatrix}, \quad E_{off} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$M(\rho, A - \imath\omega I_2, E_{off}) = \begin{bmatrix} -1-(1-\omega)^2 & \rho^2 \\ \rho^2 & -4-(1-\omega)^2 \end{bmatrix}.$$

So $\det M(\rho, A - \imath\omega I_2, E_{off}) = 0$ if and only if $\rho^4 = [1 + (1-\omega)^2][4 + (1-\omega)^2]$. The RHS is minimized when $\omega = 1$ and hence we have $r_{\mathbf{\Delta}_{off}}(A) = \sqrt{2}$ by (38) and (26).

For the full-block case $E_{full} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and by (29) we have

$$r_{\mathbf{\Delta}_{full}}(A) = \inf_{\omega\in\mathbb{R}} \left[\frac{1}{1+(1-\omega)^2} + \frac{1}{4+(1-\omega)^2}\right]^{-1/2}.$$

Again the optimizing ω is 1, so that $r_{\mathbf{\Delta}_{full}}(A) = 2/\sqrt{5} < \sqrt{2} = r_{\mathbf{\Delta}_{off}}(A)$. In this full-block case if the perturbation space is normed with the spectral norm then by (14)

$$r_{\mathbb{C}}(A, I_2, I_2)^2 = \min_{\omega\in\mathbb{R}} \sigma_{\min}(\imath\omega I_2 - A)^2 = \min_{\omega\in\mathbb{R}} \min\{1 + (1-\omega)^2, 4 + (1-\omega)^2\} = 1.$$

The two stability radii $r_{\mathbf{\Delta}_{full}}(A)$ and $r_{\mathbb{C}}(A, I_2, I_2)$ give rise to sets of stability preserving Δ's which are quite different. For our norm (18) the set is $S_{2,\infty} = \{\Delta \in \mathbb{C}^{2\times 2}; \|\Delta\|_{2,\infty} < 2/\sqrt{5}\}$ whereas for the spectral norm the set is $S_2 = \{\Delta \in \mathbb{C}^{2\times 2}; \sigma_{\max}(\Delta) < 1\}$ and $S_{2,\infty}$ neither contains nor is contained in S_2. □

4 Riccati equation

In this section we will associate with every normed perturbation space $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$ of the form (18) a normed space $(\tilde{\mathbf{\Delta}}, \|\cdot\|_{\tilde{\mathbf{\Delta}}})$ of *block-diagonal* perturbations. We then use the scaling technique described in Section 2 in order to characterize $r_{\mathbf{\Delta}}(A)$ for real diagonal A by a parametrized algebraic Riccati equation. Throughout this section we assume

$$A = \mathrm{diag}(a_1, \ldots, a_n),\ a_i \in \mathbb{C}_-,\ i \in \underline{n} \quad \text{and} \quad E = (e_{ij}) \in \{0,1\}^{n\times n},\ E \neq 0. \tag{40}$$

Let $\mathbf{\Delta}$ and $\|\cdot\|_{\mathbf{\Delta}}$ be as in (18) and define

$$\tilde{E} = (\tilde{e}_{ij}) = \begin{bmatrix} u_{q_1} & 0 & 0 & \cdots & \cdot & 0 \\ 0 & u_{q_2} & 0 & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdots & 0 & u_{q_n} \end{bmatrix}, \tag{41}$$

where u_k denotes the row vector in $\mathbb{R}^{1\times k}$ whose coordinates are all equal to 1 and q_i is the number of non-zero entries in the i^{th} row E^i of E. So $\tilde{E} \in \{0,1\}^{n\times q}$ is of block-diagonal structure (see (8)) with $N = n$, $l_i = 1$ and $l = \sum_{i=1}^n l_i = n$, $q = \sum_{i=1}^n q_i$. If $q_i = 0$ the corresponding block u_{q_i} is absent in $\tilde{E}$ and the i^{th} row of $\tilde{E}$ contains only zeros. Since $E \neq 0$, we have $q \geq 1$. Let $(\tilde{\boldsymbol{\Delta}}, \|\cdot\|_{\tilde{\boldsymbol{\Delta}}})$ be the normed linear space of block-diagonal matrices given by

$$\tilde{\boldsymbol{\Delta}} = \{\tilde{\Delta} = (\tilde{\delta}_{ij}) \in \mathbb{C}^{n\times q}; \tilde{\delta}_{ij} = 0 \text{ if } \tilde{e}_{ij} = 0\}, \quad \|\tilde{\Delta}\|_{\tilde{\boldsymbol{\Delta}}} = \|\tilde{\Delta}\|_2, \;\; \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}. \tag{42}$$

Suppose that for each $i \in \underline{n}$, $j_{i1} < j_{i2} < ... < j_{iq_i}$, are from the left to the right, the positions of the non-zero elements in the i^{th} row, E^i, of E. Then $u_{q_i} = (e_{ij_{i1}}, e_{ij_{i2}}, \ldots, e_{ij_{iq_i}})$. Given $\Delta = (\delta_{ij}) \in \boldsymbol{\Delta}$ we define $\tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}$ by

$$\tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}^1 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & \tilde{\Delta}^2 & 0 & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdots & 0 & \tilde{\Delta}^n \end{bmatrix}, \qquad \tilde{\Delta}^i = (\delta_{ij_{i1}}, \delta_{ij_{i2}}, \ldots, \delta_{ij_{iq_i}}). \tag{43}$$

If $q_i = 0$ then $\tilde{\Delta}^i$ is considered as a 1×0 matrix so that the block $\tilde{\Delta}^i$ is absent in row i and the i^{th} row of $\tilde{\Delta}$ consists only of zeros. Note that the spectral norm $\|\cdot\|_{\tilde{\boldsymbol{\Delta}}} = \|\cdot\|_2$ on $\tilde{\boldsymbol{\Delta}}$ is given by

$$\|\tilde{\Delta}\|_{\tilde{\boldsymbol{\Delta}}} = \max_{i\in\underline{n}} \Big(\sum_{j\in\mathcal{I}_i} |\delta_{ij}|^2 \Big)^{1/2} = \|\Delta\|_{\boldsymbol{\Delta}}, \quad \Delta = (\delta_{ij}) \in \boldsymbol{\Delta}. \tag{44}$$

Hence $\Delta \mapsto \tilde{\Delta}$ is an isometric isomorphism of the linear perturbation space $(\boldsymbol{\Delta}, \|\cdot\|_{\boldsymbol{\Delta}})$ defined in (18) onto the linear perturbation space $(\tilde{\boldsymbol{\Delta}}, \|\cdot\|_{\tilde{\boldsymbol{\Delta}}})$ introduced in (42). This fact motivated our choice of the norm $\|\cdot\|_{\boldsymbol{\Delta}}$ on $\boldsymbol{\Delta}$. Let

$$C^i = \begin{bmatrix} e_{j_{i1}}^\top \\ e_{j_{i2}}^\top \\ \vdots \\ e_{j_{iq_i}}^\top \end{bmatrix} \in \mathbb{R}^{q_i\times n}, \qquad C = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^n \end{bmatrix} \in \mathbb{R}^{q\times n} \tag{45}$$

where $e_1, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$. If $q_i = 0$ the matrix C^i is void and the corresponding block is absent in C. The n columns of C (respectively C_i, $i \in \underline{n}$) are mutually orthogonal and so $C^\top C$ (respectively $C_i^\top C_i$, $i \in \underline{n}$) are diagonal matrices in $\mathbb{R}^{n\times n}$. We have

$$u_{q_i} C^i = E^i \quad \text{and} \quad \tilde{\Delta}^i C^i = (\delta_{ij_{i1}}, \delta_{ij_{i2}}, \ldots, \delta_{ij_{iq_i}}) \begin{bmatrix} e_{j_{i1}}^\top \\ e_{j_{i2}}^\top \\ \vdots \\ e_{j_{iq_i}}^\top \end{bmatrix} = \Delta^i, \quad \Delta \in \boldsymbol{\Delta},\; i \in \underline{n},\; q_i \neq 0. \tag{46}$$

Hence

$$\tilde{\Delta} C = \Delta \quad \text{and} \quad A + \Delta = A + \tilde{\Delta} C. \tag{47}$$

The transfer function associated with the triple $(A,\, I_n,\, C)$ is

$$G(s) = \begin{bmatrix} C^1 \\ \vdots \\ C^n \end{bmatrix} \operatorname{diag}\left((s-a_1)^{-1}, ..., (s-a_n)^{-1}\right) \in \mathbb{C}^{q\times n}(s). \tag{48}$$

So each of the q rows of $G(s)$ has exactly one entry different from zero and this is of the form $(s-a_j)^{-1}$ for some $j \in \underline{n}$. For every $s \in \mathbb{C} \setminus \{a_1, \dots, a_n\}$ the columns of $G(s)$ are mutually orthogonal.

Lemma 4.1. *Let $G \in \mathbb{C}^{q\times n}$ be a matrix with mutually orthogonal columns $g^1, \dots, g^n$. Then the induced norm of G with respect to the Euclidean norms on $\mathbb{C}^q$, $\mathbb{C}^n$ is given by*

$$\|G\|_2 = \max_{j\in\underline{n}} \|g^j\|_2. \tag{49}$$

Proof: Obviously $\|G\|_2 \geq \max_{j\in\underline{n}} \|g^j\|_2$. Hence the formula follows from

$$x = (x_j) \in \mathbb{C}^n,\ \|x\|_2 = 1 \ \Rightarrow\ \|Gx\|_2^2 = \sum_{j\in\underline{n}} \|x_j g^j\|_2^2 \leq \sum_{j\in\underline{n}} |x_j|^2 \max_{j\in\underline{n}} \|g^j\|_2^2 = \max_{j\in\underline{n}} \|g^j\|_2^2.$$

□

We will now apply the scaling technique described in §2 to the set $\tilde{\mathbf{\Delta}}$ of block-diagonal perturbations (43). Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be any row vector with positive coordinates. Such vectors will be called *scaling vectors* in the sequel. With $\gamma > 0$ we associate the scaling matrices $L_\gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ and $R_\gamma = \operatorname{diag}(\gamma_1 I_{q_1}, \dots, \gamma_n I_{q_n})$ (the block $\gamma_i I_{q_i}$ is missing if $q_i = 0$). The *scaled output matrix* is defined by $C_\gamma = R_\gamma C L_\gamma^{-1}$ and the transfer function matrix of the scaled triple (A, I_n, C_γ) is given by

$$\begin{aligned} G_\gamma(s) &= C_\gamma \operatorname{diag}\left((s-a_1)^{-1}, ..., (s-a_n)^{-1}\right) \\ &= \begin{bmatrix} \gamma_1 C^1 \\ \cdot \\ \cdot \\ \gamma_n C^n \end{bmatrix} \operatorname{diag}\left(\gamma_1^{-1}(s-a_1)^{-1}, ..., \gamma_n^{-1}(s-a_n)^{-1}\right) = R_\gamma G(s) L_\gamma^{-1}. \end{aligned} \tag{50}$$

C_γ has the same form as C with each non-zero element in the j^{th} column of C^i replaced by γ_i/γ_j, $i,j \in \underline{n}$. Since $L_\gamma \tilde{\Delta} R_\gamma^{-1} = \tilde{\Delta}$ for all $\tilde{\Delta} \in \tilde{\mathbf{\Delta}}$ we have by (47)

$$\tilde{\Delta} C_\gamma L_\gamma = L_\gamma \tilde{\Delta} R_\gamma^{-1} R_\gamma C L_\gamma^{-1} L_\gamma = L_\gamma \tilde{\Delta} C = L_\gamma \Delta, \quad \Delta \in \mathbf{\Delta}. \tag{51}$$

Now let $z = L_\gamma x$, then the perturbed system $\dot{x} = (A+\Delta)x$ is transformed into

$$\dot{z} = L_\gamma(A+\Delta)x = L_\gamma(A+\tilde{\Delta}C)L_\gamma^{-1} z = Az + \tilde{\Delta} R_\gamma C L_\gamma^{-1} z = (A + \tilde{\Delta} C_\gamma) z\,. \tag{52}$$

So $\sigma(A+\Delta) = \sigma(A+\tilde{\Delta}C_\gamma)$. The Riccati equation associated with the triple (A, I_n, C_γ) is given by (15) with $B = I_n, C = C_\gamma$, i.e.

$$PA + A^*P - \rho^2 C_\gamma^\top C_\gamma - P^2 = 0. \tag{53}$$

Since $\sigma(A+\Delta) = \sigma(A+\tilde{\Delta}C_\gamma)$ and (44) holds, the following theorem is a direct consequence of Theorem 2.5.

Theorem 4.2. *Suppose that A and E are given as in* (40), $(\mathbf{\Delta}, \|\cdot\|_{\mathbf{\Delta}})$ *is defined by* (18) *and $\gamma_i > 0$, $i \in \underline{n}$. Then $\sigma(A+\Delta) \subset \mathbb{C}_-$ for all $\Delta \in \mathbf{\Delta}$ with $\|\Delta\|_{\mathbf{\Delta}} < \rho_\gamma$ where*

$$\rho_\gamma = \rho_\gamma(A,E) = \left[\max_{\omega\in\mathbb{R}} \|G_\gamma(\imath\omega)\|_2\right]^{-1}. \tag{54}$$

Moreover if the pair (A, C_γ) is observable and P_{ρ_γ} is the maximal solution of (53) *with $\rho = \rho_\gamma$, then $V_{\rho,\gamma}(x) = -\langle L_\gamma x, P_{\rho_\gamma} L_\gamma x\rangle$ is a joint Liapunov function for all perturbed systems $\dot{x} = (A+\Delta)x$ with $\|\Delta\|_{\mathbf{\Delta}} < \rho_\gamma$.*

Remark 4.3. (i) The maximal solution P_{ρ_γ} of (53) is negative semi-definite, however if (A, C_γ) is an observable pair it is negative definite, see Section 2. By the Hautus criterion (A, C_γ) is not observable if C_γ contains a zero column. Conversely, if C_γ does not contain a zero column it follows from the fact that the column vectors of C_γ are mutually orthogonal that $\operatorname{rank} C_\gamma = n$ and hence (A, C_γ) is observable. By its definition C_γ does not contain a zero column if and only if C has this property and it follows from (45) that C has a zero column if and only if E has a zero column. Hence (A, C_γ) *is observable (for any row vector $\gamma > 0$) if and only if E does not contain a zero column.*
(ii) It is shown in Theorem 5.1 (ii) of the next section that the function $V_\gamma(x) = -\langle L_\gamma x, (\operatorname{Re} A)\, L_\gamma x\rangle$ is a joint Liapunov function for all perturbed systems $\dot{x} = (A+\Delta)x$ satisfying $\|\Delta\|_{\mathbf{\Delta}} < \rho_\gamma$ without the requirement that (A, C_γ) is an observable pair. □

It follows from the first statement in Theorem 4.2 and the definition of the stability radius $r_{\mathbf{\Delta}}(A)$ that $\rho_\gamma \le r_{\mathbf{\Delta}}(A)$. But in general we will have $\rho_\gamma < r_{\mathbf{\Delta}}(A)$. It is an interesting question whether $\hat{\rho}(A,E) = \sup_{\gamma\in(0,\infty)^n} \rho_\gamma(A,E)$ satisfies $\hat{\rho}(A,E) = r_{\mathbf{\Delta}}(A)$ and whether there exists a scaling vector $\hat{\gamma} \in (0,\infty)^n$ such that

$$\rho_{\hat{\gamma}}(A,E) = \hat{\rho}(A,E). \tag{55}$$

Lemma 4.4. *Let $A(\imath\omega) = (\imath\omega I_n - A)$, $\rho \ge 0$, $\gamma = (\gamma_1, \ldots, \gamma_n) > 0$, $\gamma^2 := (\gamma_1^2, \ldots, \gamma_n^2)$. Then*

$$G_\gamma(\imath\omega)^* G_\gamma(\imath\omega) = L_\gamma^{-2} |A(\imath\omega)|^{-2} \operatorname{diag}\left((\gamma^2 E)_1, \ldots, (\gamma^2 E)_n\right), \tag{56}$$

$$\rho_\gamma(A,E)^{-2} = \rho_\gamma(\operatorname{Re} A, E)^{-2} = \max_{i\in\underline{n}} (\gamma^2 E)_i / (\gamma_i^2 |\operatorname{Re} a_i|^2). \tag{57}$$

In particular, $\rho_\gamma(A,E) = \infty$ if and only if $E = 0$.

Proof: Since $(\imath\omega I_n - A)^{-1}$, $C^\top R_\gamma^2 C$ and L_γ^{-1} are all diagonal, these matrices commute and we have

$$G_\gamma(\imath\omega)^* G_\gamma(\imath\omega) = (\imath\omega I_n - A)^{*-1} L_\gamma^{-1} C^\top R_\gamma^2 C L_\gamma^{-1} (\imath\omega I_n - A)^{-1} = L_\gamma^{-2} |A(\imath\omega)|^{-2} C^\top R_\gamma^2 C.$$

Now

$$\gamma^2 \tilde{E} = (\gamma_1^2, \ldots, \gamma_1^2, \gamma_2^2, \ldots, \gamma_2^2, \ldots, \gamma_n^2, \ldots, \gamma_n^2) \in \mathbb{R}^{1\times qn},$$

where for each $i \in \underline{n}$ the number of γ_i's which occur on the RHS is q_i. Hence $u_n C^\top R_\gamma^2 = \gamma^2 \tilde{E}$ where $u_n = (1, 1, \ldots, 1) \in \mathbb{R}^{1\times n}$. But by (41) and (45) we have $\tilde{E} C = E$. Thus

$$u_n C^\top R_\gamma^2 C = \gamma^2 \tilde{E} C = \gamma^2 E,$$

and since $C^\top R_\gamma^2 C$ is diagonal it follows that

$$C^\top R_\gamma^2 C = \operatorname{diag}\left((\gamma^2 E)_1, \ldots, (\gamma^2 E)_n\right). \tag{58}$$

This proves (56). Now

$$\begin{aligned}
&\rho_\gamma(A,E)^{-2}\\
&= \max_{\omega\in\mathbb{R}} \|G_\gamma(\imath\omega)\|_2^2 = \max_{\omega\in\mathbb{R}} \lambda_{\max}\left(G_\gamma(\imath\omega)^* G_\gamma(\imath\omega)\right) = \max_{\omega\in\mathbb{R}} \max_{i\in\underline{n}} \gamma_i^{-2}|\imath\omega - a_i|^{-2}(\gamma^2 E)_i\\
&= \max_{i\in\underline{n}} \max_{\omega\in\mathbb{R}} \gamma_i^{-2}|\imath\omega - a_i|^{-2}(\gamma^2 E)_i = \max_{i\in\underline{n}} \gamma_i^{-2}|\operatorname{Re} a_i|^{-2}(\gamma^2 E)_i = \rho_\gamma(\operatorname{Re} A, E)^{-2}.
\end{aligned}$$

Here the last equality follows by applying the same calculations as before to $\operatorname{Re} A$ instead of A. This concludes the proof. □

For $\rho > 0$ we obtain the following equivalences from (57).

$$\begin{aligned}
\rho_\gamma(A,E) \ge \rho \Leftrightarrow \max_{i\in\underline{n}} (\gamma^2 E)_i/(\gamma_i^2|\operatorname{Re} a_i|^2) \le \rho^{-2} \Leftrightarrow \forall i \in \underline{n} : \rho^2(\gamma^2 E)_i \le \gamma_i^2|\operatorname{Re} a_i|^2\\
\Leftrightarrow \gamma^2 M(\rho, \operatorname{Re} A, E) = \gamma^2(\rho^2 E - |\operatorname{Re} A|^2) \le 0.
\end{aligned} \tag{59}$$

Moreover, (57), Theorem 4.2 and Corollary 3.12 imply together that

$$\hat\rho(A,E) = \hat\rho(\operatorname{Re} A, E) \le r_{\mathbf{\Delta}}(\operatorname{Re} A) = r(\operatorname{Re} A, E). \tag{60}$$

In the next two lemmas we will assume that A is real. This simplifies the formulas and is not restrictive because we can later derive results for the complex case from the real case via (57), see Theorem 4.7.

Lemma 4.5. *Suppose that A and E are given as in (40), A is real and $r(A,E) < \infty$. Then, for any scaling vector $\gamma > 0$*

$$\rho_\gamma(A,E) = r(A,E) \Leftrightarrow \rho_\gamma(A,E) \ge r(A,E) \Leftrightarrow \gamma^2 M(r(A,E), A, E) \le 0. \tag{61}$$

If E is irreducible then

$$\rho_\gamma(A,E) = r(A,E) \Leftrightarrow \gamma^2 M(r(A,E), A, E) = 0 \tag{62}$$

and there always exists a scaling vector $\hat\gamma > 0$ such that

$$\hat\rho(A,E) = \rho_{\hat\gamma}(A,E) = |a_i|^2\hat\gamma_i^2/\left(\hat\gamma^2 E\right)_i = r(A,E), \quad i \in \underline{n}. \tag{63}$$

Proof: The first equivalence in (61) follows from (60). The second equivalence is obtained from (59) by setting $\rho = r(A,E)$.
To prove the second statement of the lemma, let $r = r(A,E)$ and suppose that E is irreducible with $\gamma^2 M(r,A,E) \le 0$ but $\gamma^2 M(r,A,E) \ne 0$. Choose $\beta > 0$ such that $M := M(r,A,E) + \beta I_n \ge 0$. Since M is irreducible we have $(I_n + M)^{n-1} > 0$ (see Theorem 1.3 in [1]). $(I_n + M)^{n-1}$ commutes with $M(r,A,E)$ and therefore

$$\gamma^2(I_n + M)^{n-1} M(r,A,E) = \gamma^2 M(r,A,E)(I_n + M)^{n-1} < 0$$

that is, $\tilde\gamma^2 M(r,A,E) < 0$ with $\tilde\gamma^2 := \gamma^2(I_n + M)^{n-1} > 0$. But this implies, by Lemma 3.5 (vi), the strict inequality $\alpha(M(r,A,E)) < 0$ and so, by Lemma 3.5 (ii),(iv) we have $r < r(A,E)$. Hence we obtain a contradiction and conclude that

$$\gamma^2 M(r,A,E) \le 0 \quad \Rightarrow \quad \gamma^2 M(r,A,E) = 0 \quad \text{(if } E \text{ is irreducible)}. \tag{64}$$

The converse implication is trivial.
To prove the last statement it suffices by (60) and (57) to show $r(A,E) = \rho_{\hat\gamma}(A,E)$ for some scaling row vector $\hat\gamma > 0$. But the existence of such a scaling vector follows from (62) since by irreducibility there exists a left eigenvector $\hat\gamma^2 = (\hat\gamma_1^2, \ldots, \hat\gamma_n^2) > 0$ of $M(r,A,E)$ corresponding to the eigenvalue $\alpha(M(r,A,E)) = 0$, see Lemma 3.5 (i). □

Lemma 4.6. *Suppose that A and E are given as in* (40) *and A is real. Then, for every $\varepsilon > 0$ there exists a scaling row vector $\gamma > 0$ such that*

$$\rho_\gamma(A,E)^{-2} = \max_{i\in\underline{n}} \frac{\left(\gamma^2 E\right)_i}{\gamma_i^2|a_i|^2} \leq r(A,E)^{-2} + \varepsilon. \tag{65}$$

Proof: Let $\varepsilon > 0$ be arbitrary. Without restriction of generality we may assume that E is of the form (20) where each diagonal block $E_{ii} \in \{0,1\}^{\nu_i \times \nu_i}$ is square and is either irreducible or a 1×1 null matrix.
If $s = 1$ then E is irreducible since $E \neq 0$ by assumption (40). (65) holds with equality for $\varepsilon = 0$. If $s > 1$ we prove the lemma by induction. Suppose it has been proved for $s = 1, \dots, \ell - 1$ for some $\ell \geq 2$ and E is of the form (20) with $s = \ell$. E can be written as a triangular 2×2 block matrix

$$E = \begin{bmatrix} E^{11} & 0_{\nu_1 \times (n-\nu_1)} \\ E^{21} & E^{22} \end{bmatrix} \text{ where } E^{11} = E_{11},\ E^{21} = \begin{bmatrix} E_{21} \\ \vdots \\ E_{\ell 1} \end{bmatrix},\ E^{22} = \begin{bmatrix} E_{22} & 0 & \cdots & 0 \\ E_{32} & E_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{\ell 2} & E_{\ell 3} & \cdots & E_{\ell\ell} \end{bmatrix}. \tag{66}$$

We write A in compatible form $A = \operatorname{diag}(A^{11}, A^{22})$ where $A^{11} = \operatorname{diag}(a_{11}, \dots a_{\nu_1\nu_1})$ and $A^{22} = \operatorname{diag}(a_{\nu_1+1,\nu_1+1}, \dots, a_{nn})$. Let

$$r = r(A,E),\ \ r_1 = r(A^{11}, E^{11}),\ \ r_2 = r(A^{22}, E^{22}),\ \ I_1 = \{1, \dots, \nu_1\},\ \ I_2 = \{\nu_1 + 1, \dots, n\},$$

then $I_1 \cup I_2 = \underline{n}$ and $r = \min\{r_1, r_2\}$ by (33). E^{11} is either irreducible or $E^{11} = [0] \in \mathbb{R}^{1\times 1}$. In the first case there exists by (63) a scaling vector $\hat\gamma = (\hat\gamma_1, \dots, \hat\gamma_{\nu_1}) > 0$ such that

$$\rho_{\hat\gamma}(A^{11}, E^{11})^{-2} = \left(\hat\gamma^2 E^{11}\right)_i / (\hat\gamma_i^2 |a_i|^2) = r(A^{11}, E^{11})^{-2} = r_1^{-2}, \quad i \in I_1. \tag{67}$$

If $E^{11} = [0] \in \mathbb{R}^{1\times 1}$ then $\nu_1 = 1$, $r(A^{11}, E^{11})^{-2} = r_1^{-2} = 0$ (see Remark 3.2) and by (57) $\rho_{\hat\gamma}(A^{11}, E^{11})^{-2} = \left(\hat\gamma^2 E^{11}\right)_1 / (\hat\gamma_1^2 |a_1|^2) = 0$ for any $\hat\gamma = (\hat\gamma_1) > 0$. Hence again (67) holds. On the other hand there exists by assumption of induction a scaling vector $\tilde\gamma = (\tilde\gamma_{\nu_1+1}, \dots, \tilde\gamma_n) > 0$ such that

$$\rho_{\tilde\gamma}(A^{22}, E^{22})^{-2} = \max_{i\in I_2} \frac{\left(\tilde\gamma^2 E^{22}\right)_i}{\tilde\gamma_i^2 |a_i|^2} \leq r(A^{22}, E^{22})^{-2} + \varepsilon = r_2^{-2} + \varepsilon. \tag{68}$$

Now define

$$\gamma(\alpha) = (\hat\gamma, \alpha\tilde\gamma) = (\hat\gamma_1, \dots, \hat\gamma_{\nu_1}, \alpha\tilde\gamma_{\nu_1+1}, \dots, \alpha\tilde\gamma_n) \in \mathbb{R}^{1\times n}, \quad \alpha > 0.$$

Then

$$\gamma(\alpha)^2 E = \left(\hat\gamma^2 E^{11} + \alpha^2 \tilde\gamma^2 E^{21}, \alpha^2 \tilde\gamma^2 E^{22}\right)$$

and so by Lemma 4.4, (67) and (68)

$$\begin{aligned} \rho_{\gamma(\alpha)}(A,E)^{-2} &= \max_{i\in\underline{n}} \frac{\left(\gamma(\alpha)^2 E\right)_i}{\gamma(\alpha)_i^2 |a_i|^2} = \max\left\{ \max_{i\in I_1} \frac{\left(\hat\gamma^2 E^{11}\right)_i + \alpha^2 \left(\tilde\gamma^2 E^{21}\right)_i}{\hat\gamma_i^2 |a_i|^2}, \max_{i\in I_2} \frac{\left(\tilde\gamma^2 E^{22}\right)_i}{\tilde\gamma_i^2 |a_i|^2} \right\} \\ &\leq \max\left\{ r_1^{-2} + \alpha^2 \max_{i\in I_1} \frac{\left(\tilde\gamma^2 E^{21}\right)_i}{\hat\gamma_i^2 |a_i|^2}, r_2^{-2} + \varepsilon \right\}. \end{aligned}$$

Since $r^{-2} = \max\{r_1^{-2}, r_2^{-2}\}$ it follows that $\rho_{\gamma(\alpha)}(A,E)^{-2} \leq r^{-2} + \varepsilon$ for $\alpha > 0$ sufficiently small. This proves the lemma for $s = \ell$. $\square$

Theorem 4.7. *Suppose that A and E are given as in* (40), $(\boldsymbol{\Delta}, \|\cdot\|_{\boldsymbol{\Delta}})$ *is defined by* (18) *and $\boldsymbol{\Delta}_{\mathbb{R}} = \boldsymbol{\Delta} \cap \mathbb{R}^{n\times n}$ is provided with the restriction of the norm $\|\cdot\|_{\boldsymbol{\Delta}}$ to $\boldsymbol{\Delta}_{\mathbb{R}}$. Then*

$$\hat{\rho}(A, E) = \hat{\rho}(\operatorname{Re} A, E) = r(\operatorname{Re} A, E) = r_{\boldsymbol{\Delta}}(\operatorname{Re} A) = r_{\boldsymbol{\Delta}_{\mathbb{R}}}(\operatorname{Re} A). \tag{69}$$

Proof: The first equality in (69) follows from Lemma 4.4. The last two equalities in (69) follow from (39). Hence, by (60) it only remains to show $r(\operatorname{Re} A, E) \le \hat{\rho}(\operatorname{Re} A, E)$. But this follows directly from Lemma 4.6. □

Remark 4.8. (i) Suppose A is real then by the previous theorem we have $\hat{\rho}(A, E) = r_{\boldsymbol{\Delta}}(A)$. For every $\rho' < r_{\boldsymbol{\Delta}}(A)$ there exists a scaling vector $\gamma > 0$ such that $\rho_\gamma > \rho'$. Then by Theorem 5.1 $V_\gamma(x) = -\langle L_\gamma x, A\, L_\gamma x\rangle$ is a *joint quadratic Liapunov function* for all perturbed systems

$$\dot{x} = Ax + \Delta x, \quad \Delta \in \boldsymbol{\Delta}, \ \|\Delta\|_{\boldsymbol{\Delta}} < \rho_\gamma. \tag{70}$$

Now suppose that there exists a scaling vector $\hat{\gamma} > 0$ such that $\rho_{\hat{\gamma}} = \hat{\rho}(A, E)$. Then $V_{\hat{\gamma}}(x) = -\langle L_{\hat{\gamma}} x, A\, L_{\hat{\gamma}} x\rangle$ is a *joint quadratic Liapunov function of maximal robustness* in the sense that there does not exist a positive definite quadratic function $\tilde{V}(x)$ which is a joint Liapunov function for a larger set of perturbed systems $\dot{x} = Ax + \Delta x$, $\Delta \in \boldsymbol{\Delta}$, $\|\Delta\|_{\boldsymbol{\Delta}} < \rho$ with $\rho > \rho_{\hat{\gamma}} = r_{\boldsymbol{\Delta}}(A)$.

(ii) In both the real and complex cases the maximal Hermitian solution $P_{\rho_\gamma(A,E)}$ of (53) is diagonal (and hence real). To see this note that since (57) holds,

$$p_i = \operatorname{Re} a_i + \sqrt{(\operatorname{Re} a_i)^2 - \rho_\gamma(A, E)^2 \gamma_i^{-2} (\gamma^2 E)_i}, \quad i \in \underline{n} \tag{71}$$

is well defined and satisfies

$$2(\operatorname{Re} a_i) p_i - \rho_\gamma(A, E)^2 \gamma_i^{-2} (\gamma^2 E)_i - p_i^2 = 0, \quad i \in \underline{n}. \tag{72}$$

Hence by (58) $P = \operatorname{diag}(p_1, \ldots, p_n)$ solves (53) with $\rho = \rho_\gamma(A, E)$. Moreover

$$a_i - p_i = \imath \operatorname{Im} a_i - \sqrt{(\operatorname{Re} a_i)^2 - \rho_\gamma(A, E)^2 \gamma_i^{-2} (\gamma^2 E)_i}, \quad i \in \underline{n},$$

and since $(\operatorname{Re} a_i)^2 - \rho_\gamma(A, E)^2 \gamma_i^{-2} (\gamma^2 E)_i \ge 0$, $i \in \underline{n}$ we have $\sigma(A - P) \subset \overline{\mathbb{C}_-}$. So $P = P_{\rho_\gamma(A,E)}$ is the maximal solution of (53). Note also that $P_{\rho_\gamma(A,E)} \succeq \operatorname{Re} A$.
Now suppose that E is irreducible. Then, by (57) and Lemma 4.5, there exists a scaling vector $\hat{\gamma} > 0$ such that $\rho_{\hat{\gamma}}(A, E)^2 = r(\operatorname{Re} A, E)^2 = (\operatorname{Re} a_i)^2 \hat{\gamma}_i^2 / \left(\hat{\gamma}^2 E\right)_i$ for $i \in \underline{n}$, and so we have $p_i = \operatorname{Re} a_i$, $i \in \underline{n}$ by formula (71) and hence $P_{r(\operatorname{Re} A, E)} = \operatorname{Re} A$. It follows that if E is irreducible and A is real, then $V_{\hat{\gamma}}(x) = -\langle L_{\hat{\gamma}} x, A\, L_{\hat{\gamma}} x\rangle$ is a joint quadratic Liapunov function of maximal robustness as described in (i). □

The following example shows that in the complex case the stability radius $r_{\boldsymbol{\Delta}}(A)$ and $\hat{\rho}(A, E)$ may be different.

Example 4.9. Consider

$$A = \begin{bmatrix} -1 + \imath & 0 \\ 0 & -1 + 3\imath \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then for every $\omega \in \mathbb{R}$ by (29)

$$\begin{aligned} r(A - \imath\omega I_2, E)^2 &= \left[\frac{1}{|-1 + (1-\omega)\imath|^2} + \frac{1}{|-1 + (3-\omega)\imath|^2} \right]^{-1} = \frac{[1 + (1-\omega)^2][1 + (3-\omega)^2]}{2 + (1-\omega)^2 + (3-\omega)^2} \\ &= \frac{4 + \tau^4}{4 + 2\tau^2} \end{aligned}$$

where $\tau = 2 - \omega$. The RHS is minimized at $\hat{\tau}^2 = 2(\sqrt{2} - 1)$ and an easy calculation yields by (38) $r_{\boldsymbol{\Delta}}(A)^2 = 2(\sqrt{2} - 1) = \hat{\tau}^2$. Again by (29), $r(\operatorname{Re} A, E)^2 = [1 + 1]^{-1}$. Hence $\hat{\rho}(A, E)^2 = r(\operatorname{Re} A, E)^2 = 1/2 < 2(\sqrt{2} - 1) = r_{\boldsymbol{\Delta}}(A)^2$.

Following the proof of Theorem 3.10 a destabilizing perturbation of minimum norm can be constructed as follows. Let $r_{\mathbf{\Delta}} := r_{\mathbf{\Delta}}(A) = \sqrt{2(\sqrt{2}-1)}$, $z = (z_1, z_2)^\top \in \mathbb{C}^2$ and $\hat{\omega} = 2 - \hat{\tau}$, then

$$(M(r_{\mathbf{\Delta}}, A - \imath\hat{\omega}I_2, E)z)_1 = (r_{\mathbf{\Delta}}^2 - (1 + (1-\hat{\omega})^2)z_1 + r_{\mathbf{\Delta}}^2 z_2 = -2(1 - r_{\mathbf{\Delta}})z_1 + r_{\mathbf{\Delta}}^2 z_2.$$

Hence for $z \in \mathbb{C}^2$ with $z_1 + z_2 = 1$ we have $M(r_{\mathbf{\Delta}}, A - \imath\hat{\omega}I_2, E)z = 0$ if and only if

$$z_1 = \frac{r_{\mathbf{\Delta}}^2}{r_{\mathbf{\Delta}}^2 - 2r_{\mathbf{\Delta}} + 2} > 0, \qquad z_2 = \frac{2(1 - r_{\mathbf{\Delta}})}{r_{\mathbf{\Delta}}^2 - 2r_{\mathbf{\Delta}} + 2} > 0.$$

We now set $x_1 = \sqrt{z_1}$, $x_2 = \sqrt{z_2}$ and choose Δ^i, $i = 1, 2$ to be a row vector of norm $r_{\mathbf{\Delta}}$ aligned with $x^i = x$. Hence $\Delta = r_{\mathbf{\Delta}} \begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix}$. Finally we obtain a destabilizing $\hat{\Delta}$ by multiplying Δ by $-\operatorname{diag}(-1+\imath(1-\hat{\omega})/|-1+\imath(1-\hat{\omega})|, -1+\imath(3-\hat{\omega})/|-1+\imath(3-\hat{\omega})|)$, see the proof of Theorem 3.10. So

$$\hat{\Delta} = -r_{\mathbf{\Delta}} \operatorname{diag}\left(\frac{-1 + (r_{\mathbf{\Delta}} - 1)\imath)}{(1 + (r_{\mathbf{\Delta}} - 1)^2)^{1/2}}, \frac{-1 + (r_{\mathbf{\Delta}} + 1)\imath}{(1 + (r_{\mathbf{\Delta}} + 1)^2)^{1/2}} \right) \begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix}.$$

To see that this is indeed (marginally) destabilizing, note that the first component of $(A + \hat{\Delta})x$ is

$$(-1+\imath)x_1 - r_{\mathbf{\Delta}} \frac{-1 + (r_{\mathbf{\Delta}} - 1)\imath}{(r_{\mathbf{\Delta}}^2 - 2r_{\mathbf{\Delta}} + 2)^{1/2}}(x_1^2 + x_2^2) = (-1 + \imath + 1 - (r_{\mathbf{\Delta}} - 1)\imath)x_1 = \imath\,\hat{\omega}x_1,$$

and similarly the second component of $(A + \hat{\Delta})x$ is $\imath\,\hat{\omega}x_2$, so that $\imath\,\hat{\omega} \in \sigma(A + \hat{\Delta})$. □

We will now investigate under which conditions there exists an *optimal* scaling vector $\hat{\gamma} > 0$, i.e. such that (55) is satisfied.

Theorem 4.10. *Suppose that A and E are given as in* (40) *and $r(A, E) < \infty$. Then the following conditions are equivalent:*

(i) There exists a scaling vector $\gamma > 0$ such that $\rho_\gamma(A, E) = \hat{\rho}(A, E)$ $(= r(\operatorname{Re} A, E))$.

(ii) Let P be a permutation matrix such that $PEP^\top$ is of the form (20) *and $PAP^\top = \operatorname{diag}(A_1, \ldots, A_s)$ is partitioned in a compatible way as in* (21)*. Then, for every $k \in \underline{s}$,*

$$r(\operatorname{Re} A_k, E_{kk}) = r(\operatorname{Re} A, E) \quad \Rightarrow \quad E_{ik} = 0 \text{ for all } i = k+1, \ldots, s. \tag{73}$$

Proof: Since $\rho_\gamma(A, E) = \rho_\gamma(\operatorname{Re} A, E)$ for all γ by (57), it suffices to prove (i) ⇔ (ii) for real A.
(i) ⇒ (ii): Suppose that $\gamma > 0$ satisfies $\rho_\gamma(A, E) = r := r(A, E)$ and let P be a permutation matrix such that $PEP^\top$ is of the form (20) and $PAP^\top = \operatorname{diag}(A_1, \ldots, A_s)$. If we partition $\tilde{\gamma} = \gamma P^\top > 0$ in a compatible way, $\tilde{\gamma} = (\tilde{\gamma}^1, \ldots, \tilde{\gamma}^s)$ where $\tilde{\gamma}^i \in \mathbb{R}^{1 \times n_i}$ then by Lemma 4.5

$$\tilde{\gamma}^{k2}(r^2 E_{kk} - A_k) + \sum_{i=k+1}^{s} \tilde{\gamma}^{i2} r^2 E_{ik} = (\tilde{\gamma}^2 PM(r, A, E)P^\top)_k = (\gamma^2 M(r, A, E)P^\top)_k \leq 0,$$

and hence $\tilde{\gamma}^{k2}(r^2 E_{kk} - A_k) \leq 0$ for all $k \in \underline{s}$. Now assume that $r(A_k, E_{kk}) = r(A, E)$ for some $k \in \underline{s}$. Then $E_{kk} \neq [0]$ and is therefore irreducible. It follows from (64) that $\tilde{\gamma}^{k2}(r^2 E_{kk} - A_k) = 0$, hence $\sum_{i=k+1}^{s} \tilde{\gamma}^{i2} r^2 E_{ik} = 0$ and this proves $E_{ik} = 0$ for $i = k+1, \ldots, s$.
(ii) ⇒ (i) is proved by induction on s. If $s = 1$, then since $r(A, E) < \infty$, E is irreducible and (i) holds by Lemma 4.5. Now suppose the implication (ii) ⇒ (i) has been proved for $s = 1, \ldots, \ell - 1$ for some $\ell \geq 2$ and E is of the form (20) with $s = \ell$. Assume that (ii) holds. We use the notation of the proof of Lemma 4.6 and write E as a 2×2 block matrix

as in (66).
First suppose $r_1 := r(A^{11}, E^{11}) = r(A, E) < \infty$. Then E^{11} is irreducible and there exists a scaling row vector $\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_{\nu_1}) > 0$ such that $\hat{\gamma}^2 M(r_1, A^{11}, E^{11}) = 0$, see Lemma 4.5. By (33) either $r(A^{22}, E^{22}) = r(A, E)$ or $r(A^{22}, E^{22}) > r(A, E)$. If $r_2 := r(A^{22}, E^{22}) = r(A, E)$ then (A^{22}, E^{22}) satisfies condition (ii) with (A, E) replaced by (A^{22}, E^{22}) and so there exists, by assumption of induction, a scaling row vector $\tilde{\gamma} = (\gamma_{\nu_1+1}, \ldots, \gamma_n) > 0$ such that $\tilde{\gamma}^2 M(r_2, A^{22}, E^{22}) \leq 0$ (see (62)). Since $E_{i1} = 0$ for $i = 2, \ldots, \ell$ it follows that $\gamma = (\hat{\gamma}, \tilde{\gamma})$ satisfies $\gamma^2 M(r(A, E), A, E) \leq 0$. If $r(A^{22}, E^{22}) > r(A, E)$ then we may apply Lemma 4.6 to conclude that there exists $\tilde{\gamma} = (\gamma_{\nu_1+1}, \ldots, \gamma_n) > 0$ such that $\rho_{\tilde{\gamma}}(A^{22}, E^{22}) \geq r(A, E)$, i.e. $\tilde{\gamma}^2 M(r(A, E), A^{22}, E^{22}) \leq 0$ by Lemma 4.5. Again we obtain $\gamma^2 M(r(A, E), A, E) \leq 0$ for $\gamma = (\hat{\gamma}, \tilde{\gamma})$.
Finally suppose that $r(A^{11}, E^{11}) > r(A, E)$. Then there exists by Lemma 4.5 a scaling vector $\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_{\nu_1}) > 0$ such that $\hat{\gamma}^2 M(r(A, E), A^{11}, E^{11}) \leq \hat{\gamma}^2 M(r(A^{11}, E^{11}), A^{11}, E^{11}) = 0$. Moreover, $r(A^{22}, E^{22}) = r(A, E)$ by (33). Hence (A^{22}, E^{22}) satisfies the condition of (ii) with (A, E) replaced by (A^{22}, E^{22}) and so there exists $\tilde{\gamma} = (\gamma_{\nu_1+1}, \ldots, \gamma_n) > 0$ such that $\tilde{\gamma}^2 M(r(A, E), A^{22}, E^{22}) \leq 0$. We conclude that in all three cases $\gamma = (\hat{\gamma}, \tilde{\gamma}) > 0$ satisfies $\gamma^2 M(r(A, E), A, E) \leq 0$. By (61) this implies $\rho_\gamma(A, E) = r(A, E)$. □

Remark 4.11. It follows from the previous theorem and (61) that there exists a scaling vector $\gamma > 0$ such that $\gamma^2 M(r(\operatorname{Re} A, E), \operatorname{Re} A, E) \leq 0$ if and only if (73) is satisfied. Using similar methods it can be proved that there exists a scaling vector $\gamma > 0$ such that $\gamma^2 M(r(\operatorname{Re} A, E), \operatorname{Re} A, E) = 0$ if and only if the following equivalence holds instead of (73):

$$r(\operatorname{Re} A_k, E_{kk}) = r(\operatorname{Re} A, E) \quad \Leftrightarrow \quad E_{ik} = 0 \text{ for all } i = k+1, \ldots, s. \tag{74}$$

Note that if (74) holds then necessarily $r(\operatorname{Re} A_s, E_{ss}) = r(\operatorname{Re} A, E)$. □

Example 4.12. Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A = \operatorname{diag}(a_1, a_2, a_3), \quad a_i < 0.$$

E is of the form (20) with $s = 3$. Using the notation of Theorem 4.10 we have $(A_i, E_{ii}) = (a_i, 1)$, $i = 1, 2, 3$ and the corresponding singularity parameters $r_i = r(A_i, E_{ii})$ are

$$r_1 = |a_1|, \; r_2 = |a_2|, \; r_3 = |a_3|, \quad \text{and so} \quad r := r(A, E) = \min\{|a_1|, |a_2|, |a_3|\} > 0.$$

The associated Metzler matrix for $\rho = r$ is

$$M(r, A, E) = \begin{bmatrix} r^2 - |a_1|^2 & 0 & 0 \\ r^2 & r^2 - |a_2|^2 & 0 \\ r^2 & 0 & r^2 - |a_3|^2 \end{bmatrix}.$$

First consider the case where $|a_1| \leq \min\{|a_2|, |a_3|\}$, hence $r = r_1 = |a_1|$. In this case the condition (73) is not satisfied and therefore, by Theorem 4.10, $\rho_\gamma(A, E) < r(A, E)$ for all scaling vectors $\gamma = (\gamma_1, \gamma_2, \gamma_3) > 0$. In fact, in this case there exists no $\gamma > 0$ such that $\gamma^2 M(|a_1|, A, E) \leq 0$ since $(\gamma^2 M(r, A, E))_1 = |a_1|^2(\gamma_2^2 + \gamma_3^2) > 0$ and this shows $\rho_\gamma(A, E) < r(A, E)$ for all $\gamma > 0$ by (61).
Now consider the case where $|a_2| < \min\{|a_1|, |a_3|\}$. Then $r = r_2 = |a_2|$ and we see that condition (73) is satisfied. By Theorem 4.10 there exists $\hat{\gamma} > 0$ such that $\rho_{\hat{\gamma}}(A, E) = r(A, E)$ or, equivalently, $\hat{\gamma}^2 M(|a_2|, A, E) \leq 0$. In fact, since $|a_2|^2 - |a_1|^2 < 0$ and $|a_2|^2 - |a_3|^2 < 0$, every scaling vector $\gamma = (1, \gamma_2, \gamma_3) > 0$ with γ_2, γ_3 sufficiently small satisfies

$$(1, \gamma_2^2, \gamma_3^2) M(|a_2|, A, E) = (1, \gamma_2^2, \gamma_3^2) \begin{bmatrix} |a_2|^2 - |a_1|^2 & 0 & 0 \\ |a_2|^2 & 0 & 0 \\ |a_2|^2 & 0 & |a_2|^2 - |a_3|^2 \end{bmatrix} \leq 0.$$

Note, however, that in this case the stronger condition (74) is not satisfied and in fact, there clearly does not exist a scaling vector $\gamma > 0$ satisfying $\gamma^2 M(|a_2|, A, E) = 0$.
Finally consider the case $|a_3| \leq \min\{|a_1|, |a_2|\}$. Then $r = r_3 = |a_3|$ and we see that condition (73) is satisfied if and only if $|a_3| < |a_1|$. Moreover (74) is satisfied if and only if $|a_3| = |a_2| < |a_1|$. In this latter case it is easily verified that there exists $\hat{\gamma} > 0$ such that $\rho_{\hat{\gamma}}(A, E) = r(A, E)$ and $\hat{\gamma}^2 M(|a_3|, A, E) = 0$: It suffices to choose $\hat{\gamma} = (\gamma_1, 1, 1) > 0$ such that $\gamma_1^2(|a_3|^2 - |a_1|^2) + 2|a_3|^2 = 0$. □

5 Nonlinear and/or time-varying perturbations

Throughout this section we suppose that A and E are given as in (40) and Ω is an open neighbourhood of 0 in $\mathbb{C}^n$. We consider nonlinear time-varying perturbations of $\dot{x} = Ax$ of the form

$$\dot{x} = Ax + \Delta(x,t)x, \quad \Delta(\cdot,\cdot) \in \mathbf{\Delta}_{nt}(\Omega). \tag{75}$$

Here $\mathbf{\Delta}_{nt}(\Omega)$ is the vector space of all bounded $\Delta(\cdot,\cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{C}^{n\times n}$ with entries

$$\delta_{ij}(x,t) = 0 \text{ if } e_{ij} = 0, \ \ (x,t) \in \Omega \times \mathbb{R}_+$$

such that $\Delta(x,\cdot) : \mathbb{R}_+ \to \mathbb{C}^{n\times n}$ is measurable for each $x \in \Omega$, $\Delta(\cdot,t) : \Omega \to \mathbb{C}^{n\times n}$ is continuous for each fixed $t \in \mathbb{R}_+$, and for each compact subset $K \times I \subset \Omega \times \mathbb{R}_+$ there exists an integrable $k(\cdot) : I \to \mathbb{R}_+$ such that

$$\|\Delta(x,t)x - \Delta(\hat{x},t)\hat{x}\|_2 \leq k(t)\|x - \hat{x}\|_2, \quad (x,t),\, (\hat{x},t) \in K \times I.$$

The norm on $\mathbf{\Delta}_{nt}(\Omega)$ is taken to be

$$\|\Delta(\cdot,\cdot)\| = \left[\sup_{x\in\Omega, t\geq 0} \max_{i\in\underline{n}} \sum_{j\in\underline{n}} |\delta_{ij}(x,t)|^2\right]^{1/2}, \quad \Delta(\cdot,\cdot) = (\delta_{ij}(\cdot,\cdot)) \in \mathbf{\Delta}_{nt}(\Omega). \tag{76}$$

By Caratheodory's Theorem for every $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$, there exists a unique solution $x(t) = x(t; t_0, x^0)$ of (75) with $x(t_0) = x^0$ on some maximal semi-open interval $[t_0, t_+(t_0, x^0))$ where $t_+(t_0, x^0) > t_0$, see [7, Thm. 2.1.14]. In the following theorem we will see that $t_+(t_0, x^0) = \infty$ if $\|\Delta(\cdot,\cdot)\| < r(\operatorname{Re} A, E)$ and x^0 is sufficiently close to the equilibrium state $\overline{x} = 0$. For simplicity, we call the nonlinear system (75) *(uniformly) asymptotically stable* if $\overline{x} = 0$ is a (uniformly) asymptotically stable equilibrium point of the system (75).

Theorem 5.1. *Suppose* (40). *Then the following statements hold.*

(i) The nonlinear system (75) *is uniformly asymptotically stable for all* $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ *satisfying*
$$\|\Delta(\cdot,\cdot)\| < r(\operatorname{Re} A, E). \tag{77}$$

(ii) For every scaling vector $\gamma > 0$*, the quadratic fuction* $V_\gamma(x) = -\langle L_\gamma x, (\operatorname{Re} A) L_\gamma x\rangle$ *is a joint strict Liapunov function at* $\overline{x} = 0$ *for all perturbed systems* (75) *with* $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ *satisfying* $\|\Delta(\cdot,\cdot)\| < \rho_\gamma(A, E)$.

(iii) If additionally $\delta > 0$ *is chosen so that* $D_\delta = \{x \in \mathbb{C}^n\,; -\langle L_\gamma x, (\operatorname{Re} A) L_\gamma x\rangle < \delta\} \subset \Omega$, *then* D_δ *is a joint domain of attraction of the equilibrium point* $\overline{x} = 0$ *for all the systems* (75) *with* $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$, $\|\Delta(\cdot,\cdot)\| < \rho_\gamma(A, E)$.

Proof: We begin with the proof of (ii). For any $\Delta(\cdot,\cdot) \in \mathbf{\Delta}_{nt}(\Omega)$ define

$$\tilde{\Delta}(x,t) = \begin{bmatrix} \tilde{\Delta}^1(x,t) & 0 & 0 & \cdots & \cdot & 0 \\ 0 & \tilde{\Delta}^2(x,t) & 0 & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdots & 0 & \tilde{\Delta}^n(x,t) \end{bmatrix}, \quad (x,t) \in \Omega \times \mathbb{R}_+,$$

where $\tilde{\Delta}^i(x,t) \in \mathbb{C}^{1\times q_i}$, $i \in \underline{n}$ is the i^{th} row of $\Delta(x,t)$ with the zero elements $\delta_{ij}(x,t)$, $j \notin \mathcal{I}_i$ removed. We provide the vector space $\tilde{\mathbf{\Delta}}_{nt}(\Omega)$ of all these $\tilde{\Delta}(\cdot,\cdot)$ with the norm

$$\|\tilde{\Delta}(\cdot,\cdot)\|_{\tilde{\mathbf{\Delta}}_{nt}} := \sup_{x\in\Omega, t\geq 0} \|\tilde{\Delta}(x,t)\|_2 = \|\Delta(\cdot,\cdot)\|, \quad \tilde{\Delta}(\cdot,\cdot) \in \tilde{\mathbf{\Delta}}_{nt}(\Omega). \tag{78}$$

For every $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ and each $(x,t) \in \Omega \times \mathbb{R}_+$ we have $\Delta(x,t) \in \mathbf{\Delta}$ and the (isometric) isomorphism $\Delta \mapsto \tilde{\Delta}$ defined by (43) maps $\Delta(x,t)$ onto $\tilde{\Delta}(x,t) \in \tilde{\mathbf{\Delta}}$. Hence by (51)

$$\tilde{\Delta}(x,t) C_\gamma L_\gamma = L_\gamma \Delta(x,t), \qquad \Delta(\cdot,\cdot) \in \mathbf{\Delta}_{nt}(\Omega),\ (x,t) \in \Omega \times \mathbb{R}_+. \tag{79}$$

Now consider the time-invariant positive definite quadratic function $V_\gamma(x) = -\langle L_\gamma x, PL_\gamma x\rangle$ on $\mathbb{C}^n$ where $\gamma > 0$ is any scaling vector and $P := \operatorname{Re} A$. Since the diagonal matrices A, P, L_γ commute, the derivative of V_γ along the solutions of (75) is given by

$$\begin{aligned} \dot{V}_\gamma(x) &= -\left[\langle L_\gamma(Ax + \Delta(x,t)x), PL_\gamma x\rangle + \langle L_\gamma x, PL_\gamma(Ax + \Delta(x,t)x)\rangle\right] \\ &= -\langle (PA + A^*P) L_\gamma x, L_\gamma x\rangle - 2\operatorname{Re}\langle L_\gamma \Delta(x,t)x, PL_\gamma x\rangle, \ \ (x,t) \in \Omega \times \mathbb{R}_+. \end{aligned} \tag{80}$$

Now $P = \operatorname{Re} A$ satisfies the algebraic Riccati equation

$$PA + A^*P - (\operatorname{Re} A)^2 - P^2 = 0. \tag{81}$$

Therefore, applying (79),

$$\begin{aligned} \dot{V}_\gamma(x) &= -\|(\operatorname{Re} A)L_\gamma x\|_2^2 - \|(\operatorname{Re} A)L_\gamma x\|_2^2 - 2\operatorname{Re}\langle \tilde{\Delta}(x,t) C_\gamma L_\gamma x, (\operatorname{Re} A) L_\gamma x\rangle \\ &= -\|\tilde{\Delta}(x,t)C_\gamma L_\gamma x + (\operatorname{Re} A)L_\gamma x\|_2^2 - \left[\|(\operatorname{Re} A)L_\gamma x\|_2^2 - \|\tilde{\Delta}(x,t)C_\gamma L_\gamma x\|_2^2\right] \\ &\leq -\left[\|(\operatorname{Re} A)L_\gamma x\|_2^2 - \|\tilde{\Delta}(\cdot,\cdot)\|_{\tilde{\mathbf{\Delta}}_{nt}}^2 \|C_\gamma L_\gamma x\|_2^2\right] \end{aligned} \tag{82}$$

Since the diagonal matrices L_γ and $C^\top R_\gamma^2 C$ commute, we have by (58)

$$C_\gamma^\top C_\gamma = L_\gamma^{-1} C^\top R_\gamma^2 C L_\gamma^{-1} = \operatorname{diag}((\gamma^2 E)_1/\gamma_1^2, \ldots, (\gamma^2 E)_n/\gamma_n^2). \tag{83}$$

By (57) the i^{th} diagonal entries of the matrix $C_\gamma^\top C_\gamma$ satisfy $(\gamma^2 E)_i/\gamma_i^2 \leq \rho_\gamma^{-2}(\operatorname{Re} a_i)^2$ where $\rho_\gamma = \rho_\gamma(A,E)$, and hence

$$\rho_\gamma^{-2}(\operatorname{Re} A)^2 \geq C_\gamma^\top C_\gamma. \tag{84}$$

Now suppose that $\rho < \rho_\gamma$ and $\|\Delta(\cdot,\cdot)\| = \|\tilde{\Delta}(\cdot,\cdot)\|_{\tilde{\mathbf{\Delta}}_{nt}} \leq \rho$ and set

$$a_{\max} := \max_{i\in\underline{n}} |\operatorname{Re} a_i|\,,\ a_{\min} := \min_{i\in\underline{n}} |\operatorname{Re} a_i|\,,\ \gamma_{\max} := \max_{i\in\underline{n}} \gamma_i\,,\ \gamma_{\min} := \min_{i\in\underline{n}} \gamma_i. \tag{85}$$

Then (82) and (84) imply

$$\begin{aligned} \dot{V}_\gamma(x) &\leq -\left[\|(\operatorname{Re} A)L_\gamma x\|_2^2 - \|\tilde{\Delta}(\cdot,\cdot)\|_{\tilde{\mathbf{\Delta}}_{nt}}^2 \rho_\gamma^{-2} \|(\operatorname{Re} A)L_\gamma x\|_2^2\right] \\ &= -\left(1 - \|\tilde{\Delta}(\cdot,\cdot)\|_{\tilde{\mathbf{\Delta}}_{nt}}^2/\rho_\gamma^2\right) \|(\operatorname{Re} A)L_\gamma x\|_2^2 \\ &\leq -(1 - \rho^2/\rho_\gamma^2)\, a_{\min}^2\, \|L_\gamma x\|_2^2 \leq -(1 - \rho^2/\rho_\gamma^2)\, a_{\min}^2\, \gamma_{\min}^2 \|x\|_2^2,\ (x,t) \in \Omega \times \mathbb{R}_+. \end{aligned} \tag{86}$$

Hence V_γ is a strict Liapunov function for the system (75) on $\Omega \times \mathbb{R}_+$ at the origin (see Def. 3.2.16 in [7]). This concludes the proof of (ii).
(i) follows from (ii) by Theorem 4.7 and Liapunov's Stability Theorem, see [7, Thm. 3.2.17].
(iii) D_δ is the union of all sublevel sets $D_{\delta'} = \{x \in \mathbb{C}^n\,; V_\gamma(x) < \delta'\}$, $\delta' \in (0, \delta)$ and these sublevel sets have compact closures $\overline{D_{\delta'}} \subset \Omega$. This implies (iii) by [7, Thm.3.2.17]. □

Remark 5.2. If E does not contain a zero column and P_{ρ_γ} is the maximal solution of (53) with $\rho = \rho_\gamma$, then we could have used the quadratic function $V_{\rho,\gamma}(x) = -\langle L_\gamma x, P_{\rho_\gamma} L_\gamma x\rangle$ to prove the above theorem. In fact in a way similar to that by which (86) was obtained one can show

$$\dot{V}_{\rho,\gamma}(x) \le -(\rho_\gamma^2 - \|\tilde{\Delta}(\cdot,\cdot)\|^2_{\tilde{\mathbf{\Delta}}_{nt}})\|C_\gamma L_\gamma x\|_2^2. \tag{87}$$

Then since $C_\gamma^\top C_\gamma \succ 0$, $V_{\rho,\gamma}$ is a joint strict Liapunov function at the origin for all perturbed systems (75) with $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ satisfying $\|\Delta(\cdot,\cdot)\| < \rho_\gamma(A, E)$. Now by Remark 4.8 (ii) we have $-\operatorname{Re} A \succeq -P_{\rho_\gamma}$ and hence, in general, the set $D_{\rho,\delta} = \{x \in \mathbb{C}^n\,; -\langle L_\gamma x, P_{\rho_\gamma} L_\gamma x\rangle < \delta\} \subset \Omega$, will be a larger domain of attraction of the equilibrium point $\overline{x} = 0$ than the domain D_δ given in the theorem. □

Example 5.3. In this example we consider a real A and illustrate the three steps: (a) find $r(A, E)$, (b) find an optimal γ such that $\rho_\gamma(A, E) = r(A, E)$ is satisfied, (c) find δ required to calculate a domain of attraction, D_δ. Consider

$$A = \operatorname{diag}(-\sqrt{3/2}, -\sqrt{2}), \quad E = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Omega = (-1, +1) \times (-1, +1).$$

Then $M(\rho, A, E) = \begin{bmatrix} \rho^2 - 3/2 & \rho^2 \\ \rho^2 & -2 \end{bmatrix}$, so $\det M(\rho, A, E) = 0$ if $\rho^4 + 2\rho^2 - 3 = 0$ and hence $r(A, E) = 1$.

Now $\gamma^2 M(r(A, E), A, E) = (\gamma_1^2, \gamma_2^2)\begin{bmatrix} -1/2 & 1 \\ 1 & -2 \end{bmatrix} = 0$ provided $\gamma_1^2 : \gamma_2^2 = 2 : 1$. Hence choosing $\gamma_1 = \sqrt{2}$, $\gamma_2 = 1$, we obtain the Liapunov function

$$V_\gamma(x) = -\langle L_\gamma x, P L_\gamma x\rangle = \sqrt{6}x_1^2 + \sqrt{2}x_2^2 \quad \text{where} \quad P = A = \operatorname{diag}(-\sqrt{3/2}, -\sqrt{2}).$$

In order to ensure $D_\delta = \{(x_1, x_2); \sqrt{6}x_1^2 + \sqrt{2}x_2^2 < \delta\} \subset (-1, +1) \times (-1, +1)$, we may take $\delta = \sqrt{2}$. We conclude that $D_{\sqrt{2}} = \{(x_1, x_2); \sqrt{3}x_1^2 + x_2^2 < 1\}$ will be a joint domain of attraction for the origin with respect to all the uniformly asymptotically stable nonlinear systems

$$\dot{x}_1 = -(\sqrt{3/2})x_1 + \delta_{11}(x,t)x_1 + \delta_{12}(x,t)x_2, \qquad \dot{x}_2 = \delta_{21}(x,t)x_1 - (\sqrt{2})x_2$$

where

$$\Delta \in \mathbf{\Delta}_{nt}(\Omega) \quad \text{and} \quad \sup_{x \in (-1,+1)^2,\, t \ge 0} \max\{|\delta_{11}(x,t)|^2 + |\delta_{12}(x,t)|^2, |\delta_{21}(x,t)|^2\} < 1.$$

□

We now examine whether or not (77) is a tight robustness estimate. In order to do this we introduce a stability radius with respect to time-varying linear perturbations. Consider the following time-varying linear system

$$\dot{x}(t) = (A + \Delta(t))x(t), \quad \Delta(\cdot) \in \mathbf{\Delta}_{tv} \tag{88}$$

where $\mathbf{\Delta}_{tv}$ is the vector space of all bounded measurable matrix functions $\Delta(\cdot) : \mathbb{R}_+ \to \mathbb{C}^{n \times n}$ of structure E, i.e. satisfying

$$\Delta(t) = (\delta_{ij}(t)) \in \mathbb{C}^{n \times n}, \quad \delta_{ij}(t) = 0 \text{ for all } t \ge 0 \quad \text{if} \quad e_{ij} = 0,$$

endowed with the norm

$$\|\Delta(\cdot)\|_{\mathbf{\Delta}_{tv}} = \sup_{t\geq 0} \max_{i\in \underline{n}} \Big(\sum_{j\in\underline{n}} |\delta_{ij}(t)|^2 \Big)^{1/2}, \quad \Delta \in \mathbf{\Delta}_{tv}. \tag{89}$$

Note that, with the obvious embeddings $\mathbf{\Delta} \subset \mathbf{\Delta}_{tv} \subset \mathbf{\Delta}_{nt}$ where $\mathbf{\Delta}_{nt} := \mathbf{\Delta}_{nt}(\mathbb{C}^n)$, the norm $\|\cdot\|_{\mathbf{\Delta}_{tv}}$ is the restriction of the norm $\|\cdot\|$ to $\mathbf{\Delta}_{tv}$ and the norm $\|\cdot\|_{\mathbf{\Delta}}$ is the restriction of the norm $\|\cdot\|_{\mathbf{\Delta}_{tv}}$ to $\mathbf{\Delta}$.

Definition 5.4. Given $A \in \mathbb{C}^{n\times n}$ the stability radius of A with respect to complex time-varying linear (respectively nonlinear) perturbations $\Delta(\cdot) \in \mathbf{\Delta}_{tv}$ (respectively $\Delta(\cdot,\cdot) \in \mathbf{\Delta}_{nt}$) are defined by

$$r_{\mathbf{\Delta}_{tv}}(A) = \inf\{\|\Delta(\cdot)\|_{\mathbf{\Delta}_{tv}}\,;\ \Delta(\cdot) \in \mathbf{\Delta}_{tv} \text{ and (88) is not asymptotically stable}\},$$
$$r_{\mathbf{\Delta}_{nt}}(A) = \inf\{\|\Delta(\cdot,\cdot)\|\,;\ \Delta(\cdot,\cdot) \in \mathbf{\Delta}_{nt} \text{ and (75) is not asymptotically stable}\}.$$

We have the following result.

Theorem 5.5. *Suppose* (40)*, then*

$$r_{\mathbf{\Delta}_{nt}}(A) = r_{\mathbf{\Delta}_{tv}}(A) = r(\operatorname{Re} A, E). \tag{90}$$

If the diagonal matrix A is real then

$$r_{\mathbf{\Delta}_{nt}}(A) = r_{\mathbf{\Delta}_{tv}}(A) = r_{\mathbf{\Delta}}(A) = r(A, E) \tag{91}$$

Proof: It follows from the definitions and the isometric embeddings $\mathbf{\Delta} \subset \mathbf{\Delta}_{tv} \subset \mathbf{\Delta}_{nt}$ that $r_{\mathbf{\Delta}_{nt}}(A) \leq r_{\mathbf{\Delta}_{tv}}(A) \leq r_{\mathbf{\Delta}}(A)$. On the other hand we obtain from Theorem 5.1(i) and Corollary 3.12 that $r_{\mathbf{\Delta}_{nt}}(A) \geq r(A, E) = r_{\mathbf{\Delta}}(A)$ if A is real. This proves (91).
Now consider the case where A is complex. By Theorem 5.1 with $\Omega = \mathbb{C}^n$ we have $r_{\mathbf{\Delta}_{nt}} \geq r(\operatorname{Re} A, E)$. Conversely, suppose $\rho > r(\operatorname{Re} A, E)$. By Corollary 3.12 applied to $\operatorname{Re} A$ there exists $\Delta^\rho \in \mathbf{\Delta}$, $\Delta^\rho = (\delta^\rho_{ij})$ with $\|\Delta^\rho\|_{\mathbf{\Delta}} < \rho$ such that $\sigma(\operatorname{Re} A + \Delta^\rho) \not\subset \mathbb{C}_-$. Let $z(\cdot)$ be a solution of

$$\dot z(t) = (\operatorname{Re} A + \Delta^\rho) z(t)$$

such that $z(t)$ does not converge to zero as $t \to \infty$. Since A is diagonal, $e^{\imath \operatorname{Im} At}$ and $\operatorname{Re} A$ commute for all $t \geq 0$. Therefore $x(t) := e^{\imath \operatorname{Im} At} z(t)$ is a solution of

$$\begin{aligned}\dot x(t) &= \imath(\operatorname{Im} A)x(t) + e^{\imath \operatorname{Im} At}\dot z(t) = \imath(\operatorname{Im} A)x(t) + e^{\imath \operatorname{Im} At}\left[\operatorname{Re} A + \Delta^\rho\right] z(t)\\ &= \imath(\operatorname{Im} A)x(t) + (\operatorname{Re} A)x(t) + e^{\imath \operatorname{Im} At}\Delta^\rho e^{-\imath \operatorname{Im} At}x(t) = Ax(t) + \Delta(t)x(t)\end{aligned}$$

where $\Delta(t) = e^{\imath \operatorname{Im} At}\Delta^\rho e^{-\imath \operatorname{Im} At}$. Since $\|x(t)\|_2 = \|z(t)\|_2$ for $t \geq 0$, $x(t)$ does not converge to zero as $t \to \infty$. Thus $\dot x(t) = (A + \Delta(t))x(t)$ is not asymptotically stable and since $\|\Delta(\cdot)\|_{\mathbf{\Delta}_{tv}} = \|\Delta^\rho\|_{\mathbf{\Delta}} < \rho$, we obtain $r_{\mathbf{\Delta}_{tv}}(A) \leq r(\operatorname{Re} A, E)$. □

Remark 5.6. (i) The above theorem shows that the robustness bound (77) is tight. Moreover, if there is a scaling vector $\gamma > 0$ satisfying $\rho_\gamma = r(\operatorname{Re} A, E)$ then $V_\gamma(x) = -\langle L_\gamma x, (\operatorname{Re} A)L_\gamma x\rangle$ is a strict Liapunov function *of maximal robustness* for the uncertain system (75): $V_\gamma(x)$ is a joint strict Liapunov function for every system (75) with $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ satisfying $\|\Delta(\cdot,\cdot)\| < r(\operatorname{Re} A, E)$, but there does not exist a joint strict Liapunov function for all systems (75) with $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ satisfying $\|\Delta(\cdot,\cdot)\| < \rho$ if $\rho > r(\operatorname{Re} A, E)$.
(ii) In [7] (Theorem 5.6.14) it has been shown that the complex stability radius with respect to the spectral norm is invariant under extensions of the perturbation class to time-varying and nonlinear perturbations. In other words, if for a given $\rho > 0$, a stable system $\dot x = Ax$, $A \in \mathbb{C}^{n\times n}$ cannot be

destabilized by a time-invariant linear perturbation $\Delta \in \mathbb{C}^{n\times n}$ (i. e. $E = E_{full}$) of spectral norm $\|\Delta\|_2 < \rho$, there does not exist a time-varying nonlinear perturbation $\Delta(\cdot,\cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{C}^{n\times n}$ of norm $< \rho$ which destabilizes the system. In contrast, Example 4.9 and Theorem 5.5 show that in the present set-up this is not the case, even if A is diagonal (but not real). The reason for this apparent discrepancy is that in [7] the norms placed on the perturbation spaces are based on the spectral norm, and not those given by (18), (76) and (89). On the other hand we have seen in the proof of Theorem 5.1 that each time-varying nonlinear perturbation $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ can be expressed in the form $\Delta(\cdot,\cdot) = \tilde{\Delta}(\cdot,\cdot)C$ where $\tilde{\Delta} \in \tilde{\mathbf{\Delta}}_{nt}(\Omega)$ is block-diagonal and has *spectral norm* $\|\tilde{\Delta}(\cdot,\cdot)\|_{\tilde{\mathbf{\Delta}}_{nt}} = \|\Delta(\cdot,\cdot)\|_{\mathbf{\Delta}_{nt}}$, cf. (78). Summarizing we conclude: The invariance of the stability radius under extensions of the perturbation class to time-varying and nonlinear perturbations which was established in [7] (Theorem 5.6.14) for complex full block perturbations does not hold, in general, if the spectral norm is replaced by a different norm. Nor does it hold if the complex full block perturbations are replaced by complex block-diagonal perturbations even if these are measured by the spectral norm. In particular, the invariance of the stability radius under extensions of the perturbation class to time-varying and nonlinear perturbations does not hold, in general, in the context of μ-analysis.
(iii) In general, it is very difficult to determine the stability radius with respect to *time-varying* linear perturbations even for modest dimensions, see [15], [7, 5.6]. (90) is the only explicit and easily computable formula for $r_{\mathbf{\Delta}_{tv}}$ we know in the non-trivial case where $r_{\mathbf{\Delta}_{tv}}$ is strictly smaller than $r_{\mathbf{\Delta}}$. □

In the following we present an example where the stability radii of a complex matrix with respect to time-invariant linear perturbations $\Delta \in \mathbb{C}^{n\times n}$ and with respect to time-varying linear perturbations $\Delta(\cdot) : \mathbb{R}_+ \to \mathbb{C}^{n\times n}$ are different. We also illustrate how to construct a minimum norm time-varying destabilizing perturbation.

Example 5.7. Consider

$$A = \begin{bmatrix} -1+\imath & 0 \\ 0 & -1+3\imath \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We have seen in Example 4.9 that $r^2 = r(\operatorname{Re} A, E)^2 = 1/2$ and that the stability radius with respect to time-invariant perturbations $A \rightsquigarrow A + \Delta$, $\Delta \in \mathbb{C}^{n\times n}$ is $r_{\mathbf{\Delta}} = \sqrt{2(\sqrt{2}-1)} > r$. We will now show how to construct a time-varying destabilizing perturbation $\hat{\Delta}(\cdot) : \mathbb{R}_+ \to \mathbb{C}^{n\times n}$ of norm $r = \sqrt{1/2} < r_{\mathbf{\Delta}}$. It is easily verified that $M(r, \operatorname{Re} A, E)z = 0$ for $z = (1/2, 1/2)^\top$. We set $x_1 = \sqrt{z_1} = 1/\sqrt{2}$, $x_2 = \sqrt{z_2} = 1/\sqrt{2}$ and choose Δ^i, $i = 1, 2$ to be row vectors of norm $r = 1/\sqrt{2}$ aligned with x. Then $\Delta = \begin{bmatrix} \Delta^1 \\ \Delta^2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Following the procedure in the proof of Theorem 5.5 we obtain the minimum norm time-varying destabilizing $\hat{\Delta}(t)$ by pre-multiplying Δ by $e^{\imath \operatorname{Im} At}$ and post-multiplying with $e^{-\imath \operatorname{Im} At}$, i.e.

$$\hat{\Delta}(t) = \operatorname{diag}(e^{\imath t}, e^{3\imath t}) \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \operatorname{diag}(e^{-\imath t}, e^{-3\imath t}) = \begin{bmatrix} 1/2 & (1/2)e^{-2\imath t} \\ (1/2)e^{2\imath t} & 1/2 \end{bmatrix}.$$

In fact, a short calculation shows that $x(t) = e^{\imath \operatorname{Im} At} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution of $\dot{x}(t) = (A + \hat{\Delta}(t))x(t)$. So the perturbed system is not asymptotically stable. □

If we slightly strengthen the conditions in Theorem 5.1 we are able to obtain uniform exponential bounds for the solutions of (75). In [7] we introduced the notion of (M, β)-stability for a linear system. The following generalizes this definition to nonlinear systems.

Definition 5.8. Suppose $M \geq 1$, $\beta < 0$. A nonlinear system of the form (75) with $\Delta \in \mathbf{\Delta}_{nt}(\Omega)$ is said to be (M, β)-stable on a neighbourhood $\Omega_0 \subset \Omega$ of the origin if the solution $x(t; t_0, x^0)$ exists on $[t_0, \infty)$ for all $(t_0, x^0) \in \mathbb{R}_+ \times \Omega_0$ and

$$\|x(t; t_0, x^0)\|_2 \leq M e^{\beta(t-t_0)} \|x^0\|_2 \quad \text{for all} \quad (t_0, x^0) \in \mathbb{R}_+ \times \Omega_0, \;\; t \geq t_0.$$

(M, β)-stability combines a guarantee of a specific decay rate (given by $-\beta \geq 0$) and a specific bound for the transient behaviour (given by M).

Corollary 5.9. *Assume* (40) *and suppose* $\rho < r(\operatorname{Re} A, E)$*. Then there exists a scaling vector* $\gamma > 0$ *such that* $\rho_\gamma(A, E) > \rho$*. If* $\delta > 0$ *is chosen such that*

$$D_\delta = \{x \in \mathbb{C}^n \, ; -\langle L_\gamma x, (\operatorname{Re} A) L_\gamma x \rangle < \delta\} \subset \Omega,$$

then all the nonlinear systems of the form (75) *with* $\Delta \in \boldsymbol{\Delta}_{nt}(\Omega)$, $\|\Delta\| \leq \rho$ *are* (M, β)*-stable on* D_δ*, where (with the notation* (85)*)*

$$M = \frac{\gamma_{\max}}{\gamma_{\min}} \sqrt{\frac{a_{\max}}{a_{\min}}}, \qquad \beta = -\frac{(1 - \rho^2/\rho_\gamma(A, E)^2)\, a_{\min}^2}{2a_{\max}}.$$

Proof: Since $\rho < r := r(\operatorname{Re} A, E)$ there exists by Theorem 4.7 a scaling vector $\gamma > 0$ such that $\rho < \rho_\gamma = \rho_\gamma(A, E)$. Suppose that $\delta > 0$ is chosen such that $D_\delta \subset \Omega$, $V_\gamma(\cdot)$ is as in Theorem 5.1, $\Delta(\cdot, \cdot) \in \boldsymbol{\Delta}_{nt}(\Omega)$ and $\|\Delta(\cdot, \cdot)\| \leq \rho$. We use the notation in (85). By (86) we have

$$\dot{V}_\gamma(x) \leq -(1 - \rho^2/\rho_\gamma^2)\, a_{\min}^2 \, \|L_\gamma x\|_2^2, \quad (x, t) \in D_\delta \times \mathbb{R}_+.$$

But

$$a_{\min} \|L_\gamma x\|_2^2 \leq V_\gamma(x) \leq a_{\max} \|L_\gamma x\|_2^2, \quad x \in \mathbb{C}^n \tag{92}$$

and so for $(x^0, t_0) \in D_\delta \times \mathbb{R}_+$,

$$\dot{V}_\gamma(x(t; t_0, x^0)) \leq -[(1 - \rho^2/\rho_\gamma^2)\, a_{\min}^2/a_{\max}]\, V_\gamma(x(t; t_0, x^0)), \quad t \geq t_0$$

(Note that $x(t; t_0, x^0)$ exists for all $t \geq t_0$ by Theorem 5.1(iii)). Hence, making use of (92),

$$\begin{aligned} V_\gamma(x(t; t_0, x^0)) &\leq e^{-[(1-\rho^2/\rho_\gamma^2)\, a_{\min}^2/a_{\max}](t-t_0)}\, V_\gamma(x^0) \\ &\leq a_{\max} e^{-[(1-\rho^2/\rho_\gamma^2)\, a_{\min}^2/a_{\max}](t-t_0)}\, \|L_\gamma x^0\|_2^2, \quad t \geq t_0. \end{aligned}$$

Using again (92) we see that

$$\gamma_{\min}^2 \|x(t; t_0, x^0)\|_2^2 \leq \|L_\gamma x(t; t_0, x^0)\|_2^2 \leq \frac{a_{\max}}{a_{\min}} e^{-[(1-\rho^2/\rho_\gamma^2)\, a_{\min}^2/a_{\max}](t-t_0)}\, \|L_\gamma x^0\|_2^2.$$

Since $\|L_\gamma x^0\|_2^2 \leq \gamma_{\max}^2 \|x^0\|_2^2$ it follows that

$$\|x(t; t_0, x^0)\|_2^2 \leq (\gamma_{\max}^2 a_{\max})/(\gamma_{\min}^2 a_{\min})\, e^{-[(1-\rho^2/\rho_\gamma^2)\, a_{\min}^2/a_{\max}](t-t_0)}\, \|x^0\|_2^2 \tag{93}$$

for all $(x^0, t_0) \in D_\delta \times \mathbb{R}_+$, $t \geq t_0$. □

Remark 5.10. Once again if E does not have a zero column it is possible to use the Liapunov function $V_{\rho,\gamma}$ (see Remark 5.2) to obtain formulas for M, β which complement those in the above corollary. Suppose $P_{\rho_\gamma} = \operatorname{diag}(p_1, ..., p_n)$ is the maximal solution of (53) with $\rho = \rho_\gamma$, $p_{\min} = \min_{i \in \underline{n}} |p_i|$, $p_{\max} = \max_{i \in \underline{n}} |p_i|$, and σ_γ is the lowest singular value of C_γ. Then, given any $\rho < \rho_\gamma$, we obtain from (87) that $V_{\rho,\gamma}$ and the derivative of $V_{\rho,\gamma}$ along solutions of (75) satisfy, for every $\Delta \in \boldsymbol{\Delta}_{nt}(\Omega)$ with $\|\Delta(\cdot, \cdot)\| \leq \rho$, the inequalities

$$\dot{V}_{\rho,\gamma}(x) \leq -(\rho_\gamma^2 - \rho^2)\sigma_\gamma^2 \|L_\gamma x\|_2^2, \quad p_{\min} \|L_\gamma x\|_2^2 \leq V_{\rho,\gamma}(x) \leq p_{\max} \|L_\gamma x\|_2^2.$$

Following the same argument as in the proof of Corollary 5.9 leads to an analogous statement as in Corollary 5.9 with

$$M = \frac{\gamma_{\max}}{\gamma_{\min}} \sqrt{\frac{p_{\max}}{p_{\min}}}, \qquad \beta = -\frac{(\rho_\gamma^2 - \rho^2)\, \sigma_\gamma^2}{2p_{\max}}.$$

□

Acknowledgement: We thank Prof. Bernd Tibken (U. Wuppertal, Germany) for pointing out to us that $r(A, E)$ can be calculated as the solution of a generalized eigenvalue problem, see Remark 3.4.

References

[1] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*, volume 9 of *Classics in Applied Mathematics*. SIAM Publications, Philadelphia, PA, 1994.

[2] A. Brauer. Limits for the characteristic roots of a matrix II. *Duke Math. J.*, 14:21–26, 1947.

[3] R. A. Brualdi. Matrices, eigenvalues, and directed graphs. *Linear Multilinear Algebra*, 11(2):143–165, 1982.

[4] S. A. Gershgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Izvestia Akad. Nauk SSSR, Ser. Fis-Mat.*, 6:749–754, 1931.

[5] D. Hinrichsen and A. J. Pritchard. Stability radius for structured perturbations and the algebraic Riccati equation. *Syst. Control Lett.*, 8:105–113, 1986.

[6] D. Hinrichsen and A. J. Pritchard. Destabilization by output feedback. *Differ. Integral Equ.*, 5(2):357–386, 1992.

[7] D. Hinrichsen and A. J. Pritchard. *Mathematical Systems Theory I. Modelling, State Space Analysis, Stability and Robustness*. Springer-Verlag, Berlin, 2005.

[8] D. Hinrichsen and N. K. Son. Robust stability of positive continuous time systems. *Numer. Functional Anal. Optim.*, 17:649–659, 1996.

[9] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.

[10] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.

[11] M. Karow, D. Hinrichsen, and A. J. Pritchard. Interconnected systems with uncertain couplings: Explicit formulae for μ-values, spectral value sets and stability radii. *SIAM J. Control Optim.*, 45:856–884, 2006.

[12] J. M. Maciejowski. *Multivariable Feedback Design*. Electronic Systems Engineering. Addison-Wesley, 1989.

[13] D. D. Šiljak. *Large Scale Dynamic Systems. Stability and Structure*. Series in System Science and Engineering. North-Holland, New York, 1978.

[14] R. S. Varga. *Geršgorin and His Circles*, volume 36 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2004.

[15] F. Wirth. On the calculation of time-varying stability radii. *Internat. J. Robust Nonlinear Control*, 8(12):1043–1058, 1998.