SIAM J. CONTROL OPTIM.
Vol. 45, No. 3, pp. 856–884


# INTERCONNECTED SYSTEMS WITH UNCERTAIN COUPLINGS: EXPLICIT FORMULAE FOR $\mu$-VALUES, SPECTRAL VALUE SETS, AND STABILITY RADII*

MICHAEL KAROW†, DIEDERICH HINRICHSEN‡, AND ANTHONY J. PRITCHARD§

**Abstract.** In this paper we study the variation of the spectrum of block-diagonal systems under perturbations of compatible block structure with fixed zero blocks at arbitrarily prescribed locations ("Gershgorin-type perturbations"). We derive explicit and computable formulae for the associated $\mu$-values. The results are then applied to characterize spectral value sets and stability radii for such perturbed systems. By specializing our results to the scalar diagonal case, the classical eigenvalue inclusion theorems of Gershgorin, Brauer, and Brualdi are obtained as corollaries. Moreover it follows that the inclusion regions of Brauer and Brualdi are optimal for the corresponding perturbation structures.

**Key words.** linear systems, perturbations, spectral value sets, stability radii, $\mu$-values

**AMS subject classifications.** 15A18, 93C05, 93C73, 93D09

**DOI.** 10.1137/050626508

**1. Introduction.** More than 20 years ago, various researchers recognized the importance of block-diagonal perturbations for describing structured uncertainties of interconnected systems where the overall model uncertainty is a consequence of those in its components; see [7] and [23]. Structured singular values ($\mu$-values) were introduced in [7] as a means of analyzing the effect of block-diagonal perturbations. In recent years this concept has proved to be an effective tool in the robustness analysis of systems with structured uncertainties and in the synthesis of robust control systems; see, e.g., [1], [8], [19], [24].

Generalizing the definition in [7], the $\mu$-value of a matrix $M \in \mathbb{C}^{q\times l}$ with respect to a given perturbation set $\mathbf{\Delta} \subset \mathbb{C}^{l\times q}$ and a given norm $\|\cdot\|$ on $\mathbb{C}^{l\times q}$ is the inverse of the smallest $\|\Delta\|$, $\Delta \in \mathbf{\Delta}$, such that 1 is an element of the spectrum of the matrix product $\Delta M$; see [15]. The $\mu$-value is denoted by $\mu_{\mathbf{\Delta}}(M)$. Explicit characterizations of $\mu_{\mathbf{\Delta}}(M)$, $M \in \mathbb{C}^{q\times l}$, have been obtained in the full block case where $\mathbf{\Delta} = \mathbb{C}^{l\times q}$ or $\mathbf{\Delta} = \mathbb{R}^{l\times q}$. For most other perturbation structures, e.g., block-diagonal, computable formulae are not available and so robust analysis/synthesis is usually based on upper bounds for the $\mu$-value; see [18], [19]. In this paper we study the converse of the usual case in that we consider $\mu$-problems where the *nominal matrix* $M$ is block-diagonal and the *perturbations* $\Delta \in \mathbf{\Delta}$ are constrained only by the condition that they have zero blocks at certain fixed locations, e.g., on the diagonal ("Gershgorin type perturbations"). In contrast to the usual case we will be able to derive a number of computable exact formulae for the corresponding $\mu$-values.

These formulae will then be applied to obtain computable characterizations of

---

*Received by the editors March 9, 2005; accepted for publication (in revised form) December 20, 2005; published electronically June 30, 2006.
http://www.siam.org/journals/sicon/45-3/62650.html

†Mathematics Institute, Berlin University of Technology, D-10623 Berlin, Germany (karow@math.TU-Berlin.de).
‡Mathematics Institute, University of Bremen, D-28334 Bremen, Germany (dh@math.uni-bremen.de).
§Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK (ajp@maths.warwick.ac.uk).

*spectral value sets* and *stability radii* of block-diagonal systems under Gershgorin-type perturbations. Our objective is not only to prove new results but also to illustrate, on the methodological side, that the techniques of $\mu$*-analysis* in combination with the concepts of *spectral values sets* and *stability radii* provide powerful tools for the spectral analysis of interconnected systems with uncertain couplings.

*Pseudospectra* (spectral value sets for unstructured complex perturbations) have been applied in various areas of the mathematical sciences, for instance, in numerical analysis [27] and the stability analysis of fluid flows [22], [27]. However, they have not found many applications in systems and control theory. For some papers in this field, see [12], [14], [15]. The *spectral value set* of a matrix $A$ under perturbations $A \rightsquigarrow A_\Delta$, $\Delta \in \mathbf{\Delta}$, consists of all eigenvalues of the perturbed matrices $A_\Delta$ with $\Delta \in \mathbf{\Delta}$ constrained by $\|\Delta\| < \delta$. Here $\delta$ reflects the level of uncertainty of the nominal matrix measured in terms of some norm $\|\cdot\|$. By visualizing spectral value sets as the perturbation level changes, one obtains insight into the mobility of the eigenvalues under the perturbations in question. This is particularly useful for the stability analysis of uncertain linear systems.

A linear system is said to be stable with respect to a given stability region $\mathbb{C}_g$ in the complex plane if all the eigenvalues of the system matrix lie in $\mathbb{C}_g$. The nominal matrix $A$ is regarded as an approximation to a system matrix whose exact value is unknown. If $\sigma(A) \subset \mathbb{C}_g$ and a bound for the level of uncertainty is known, then the exact system matrix will also be stable provided that the associated spectral value set is contained in $\mathbb{C}_g$.

An alternative but related approach is through the concept of a *stability radius* [13], [15]. This is defined to be the smallest perturbation level for which at least one of the perturbed matrices $A_\Delta$ with $\Delta \in \mathbf{\Delta}$, $\|\Delta\| \leq \delta$ becomes unstable. It is therefore a robustness measure of the $\mathbb{C}_g$-stability of the nominal matrix $A$. We will see that spectral value sets and stability radii can be expressed in terms of $\mu$-values (section 2).

In this paper we consider perturbations of the form $A \rightsquigarrow A_\Delta = A + B\Delta C$, where $A, B, C$ are given block-diagonal matrices and $\Delta \in \mathbf{\Delta}$. The perturbed matrices $A_\Delta$ can be viewed as the system matrices of composite systems obtained by the interconnection of subsystems via couplings determined by the $\Delta$'s; see section 3. The overall transfer matrix of the block-diagonal system is the direct sum of the transfer matrices of its subsystems and thus the formulae we obtain for $\mu$-values of block-diagonal matrices can be applied to this transfer matrix to yield computable formulae for the corresponding spectral value sets and stability radii.

In the decentralized control of large scale systems it is common to adopt a *decomposition principle* where the overall system is regarded as the interconnection of decoupled subsystems. For such systems a notion of *connective stability* has been introduced where the decoupled subsystems are assumed to be stable and the system is said to be connectively stable if the overall system is stable for all interconnections in a set $E$ which reflects the size and structure of the interconnections; see [25]. We will see that the results we develop for the stability radii of systems of the form $A_\Delta$ can be used to obtain precise statements for the connective stability of large scale systems.

The organization of the paper is as follows. In section 2 we give definitions of spectral value sets and stability radii and establish their connection to $\mu$-values. In section 3 we introduce the perturbation structures to be considered and interpret them in the context of interconnected systems. Sections 4 and 5 contain the main results of this paper. Here we provide formulae for the computation of $\mu$-values with respect to Gershgorin-type perturbations and apply them to obtain computable characterizations of spectral value sets and stability radii. Two different types of norms will be considered on the perturbation spaces. In section 6 we specialize our results to the full class of all off-diagonal perturbations. Finally in section 7 we relate our results to the classical eigenvalue inclusion theorems of Gershgorin, Brauer and Brualdi; see [16].

**2. The framework.** In this section we introduce some basic concepts and fix the notation. The symbols $\mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$ denote the sets of positive integers, real numbers, nonnegative real numbers, and complex numbers, respectively. For $a \in \mathbb{C}$ the closed disk of radius $r > 0$ in $\mathbb{C}$ is $\mathcal{D}(a, r) = \{s \in \mathbb{C}; |s - a| \leq r\}$. By $\mathbb{K}^{n\times m}$ we denote the set of $n \times m$ matrices with entries in $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Furthermore, $\mathbb{K}^n = \mathbb{K}^{n\times 1}$ is the set of column vectors of length $n$. The transpose of $A \in \mathbb{K}^{n\times m}$ is denoted by $A^\top$. If $A$ is square, then $\sigma(A)$, $\rho(A) = \mathbb{C} \setminus \sigma(A)$, and $\varrho(A)$ denote its spectrum, its resolvent set, and its spectral radius, respectively, $\varrho(A) = \max\{|s|;\, s \in \sigma(A)\}$. We let $L_{n,l,q}$ be the set of triples of matrices $(A, B, C)$ with $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times l}, C \in \mathbb{C}^{q\times n},\ n, l, q \in \mathbb{N}$.

By $\partial\mathcal{S}$ we denote the boundary of the set $\mathcal{S} \subseteq \mathbb{C}$. We use the conventions

$$0^{-1} = \infty, \quad \infty^{-1} = 0, \quad \inf \emptyset = \infty, \tag{1}$$

where $\emptyset$ stands for the empty set. Throughout the paper we will consider the following perturbation structures; see [15].

Definition 2.1. *Let $l, q \in \mathbb{N}$. By $\mathcal{P}_{l,q}$ we denote the set of pairs $(\mathbf{\Delta}, \|\cdot\|)$, where*

- *$\mathbf{\Delta} \neq \{0\}$ is a nonempty closed subset of $\mathbb{C}^{l\times q}$ which is star-shaped with respect to $0$; i.e., $\Delta \in \mathbf{\Delta}$ implies $t\Delta \in \mathbf{\Delta}$ for every $t \in [0, 1]$;*
- *$\|\cdot\|$ is a norm on the real vector space $\mathrm{span}_{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbb{C}^{l\times q}$.*

*By $\mathcal{P}^{\mathbb{C}}_{l,q}$ we denote the set of pairs $(\mathbf{\Delta}, \|\cdot\|)$, where*

- *$\mathbf{\Delta} \neq \{0\}$ is a nonempty closed subset of $\mathbb{C}^{l\times q}$ which satisfies $\mathbb{C}\mathbf{\Delta} = \mathbf{\Delta}$; i.e., $\Delta \in \mathbf{\Delta}$ implies that $s\Delta \in \mathbf{\Delta}$ for every $s \in \mathbb{C}$;*
- *$\|\cdot\|$ is a norm on the complex vector space $\mathrm{span}_{\mathbb{C}}\mathbf{\Delta} \subseteq \mathbb{C}^{l\times q}$.*

*The pairs $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}_{l,q}$ are called* perturbation structures*, and the pairs $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}^{\mathbb{C}}_{l,q}$ are called* complex perturbation structures.

By definition we have $\mathcal{P}^{\mathbb{C}}_{l,q} \subset \mathcal{P}_{l,q}$. Given any triple $(A, B, C) \in L_{n,l,q}$ and a perturbation structure $(\mathbf{\Delta}, \|\cdot\|)$, we consider perturbations of $A$ of the form

$$A \rightsquigarrow A_\Delta = A + B\Delta C, \quad \Delta \in \mathbf{\Delta}. \tag{2}$$

Definition 2.2. *Let $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}_{l,q}$ be a perturbation structure. The spectral value set of the triple $(A, B, C) \in L_{n,l,q}$ with respect to $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}_{l,q}$ and perturbation level $\delta > 0$ is the following subset of the complex plane:*

$$\begin{aligned} \sigma_{\mathbf{\Delta}}(A, B, C; \delta) &:= \bigcup_{\Delta \in \mathbf{\Delta},\ \|\Delta\| < \delta} \sigma(A + B\Delta C) \\ &= \{s \in \mathbb{C};\ \exists\, \Delta \in \mathbf{\Delta} : \|\Delta\| < \delta,\ \textit{and}\ \det(sI_n - (A + B\Delta C)) = 0\}. \end{aligned} \tag{3}$$

Thus the spectral value set $\sigma_{\mathbf{\Delta}}(A, B, C; \delta)$ is the union of all the spectra of the perturbed matrices $A_\Delta$ where $\Delta \in \mathbf{\Delta}$, $\|\Delta\| < \delta$. The assumption that the perturbation class $\mathbf{\Delta}$ is star-shaped with respect to 0 guarantees that each connected component of $\sigma_{\mathbf{\Delta}}(A, B, C; \delta)$ contains an eigenvalue of $A$.

A concept closely related to the notion of *spectral value set* is that of *stability radius*. It presupposes that a stability region $\mathbb{C}_g \subset \mathbb{C}$ is given and measures the robustness of $\mathbb{C}_g$-stability of a matrix $A$ with respect to perturbations of the form (2).

DEFINITION 2.3. *Let $\mathbb{C}_g$ be a nonempty open subset of $\mathbb{C}$. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be $\mathbb{C}_g$-stable if $\sigma(A) \subset \mathbb{C}_g$. The $\mathbb{C}_g$-stability radius of $(A, B, C) \in L_{n,l,q}$ with respect to $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}_{l,q}$ is defined as follows:*

$$\begin{aligned} r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g) &:= \inf\{\|\Delta\|\,;\ \Delta \in \mathbf{\Delta}, A + B\Delta C \text{ is not } \mathbb{C}_g\text{-stable}\} \\ &= \inf\{\|\Delta\|\,;\ \Delta \in \mathbf{\Delta},\ \sigma(A + B\Delta C) \not\subset \mathbb{C}_g\}. \end{aligned} \tag{4}$$

If $A$ is not $\mathbb{C}_g$-stable, then $r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g) = 0$. It is easily seen that a minimum in (4) always exists if $r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g)$ is finite. Obviously,

$$r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g) = \inf\{\delta > 0\,;\ \sigma_{\mathbf{\Delta}}(A, B, C; \delta) \not\subseteq \mathbb{C}_g\}.$$

Next, we give the definition of $\mu$-values.

DEFINITION 2.4. *For $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}_{l,q}$ the corresponding $\mu$-value of $M \in \mathbb{C}^{q\times l}$ is given by*

$$\mu_{\mathbf{\Delta}}(M) := \left[\,\inf\{\|\Delta\|\,;\ \Delta \in \mathbf{\Delta},\ 1 \in \sigma(\Delta M)\,\}\,\right]^{-1}. \tag{5}$$

Note that the set $\mathbf{\Delta}_M = \{\Delta \in \mathbf{\Delta};\ 1 \in \sigma(\Delta M)\}$ is closed and does not contain the zero matrix. Thus a minimum in (5) is attained and is nonzero unless $\mathbf{\Delta}_M = \emptyset$. Hence, with the conventions (1), $\mu_{\mathbf{\Delta}}(M)$ is always well defined and $\mu_{\mathbf{\Delta}}(M) = 0$ if and only if $\mathbf{\Delta}_M = \emptyset$.

The following theorem specifies the relationship between spectral value sets, stability radii, and $\mu$-values.

THEOREM 2.5. *Let $(\mathbf{\Delta}, \|\cdot\|) \in \mathcal{P}_{l,q}$, $(A, B, C) \in L_{n,l,q}$, and $G(s) = C(sI_n - A)^{-1}B$. Then*

$$\mu_{\mathbf{\Delta}}(G(s)) = [\inf\{\|\Delta\| \mid\ \Delta \in \mathbf{\Delta},\ s \in \sigma(A + B\Delta C)\,\}]^{-1}, \quad s \in \rho(A); \tag{6}$$

$$\sigma_{\mathbf{\Delta}}(A, B, C; \delta) = \sigma(A) \cup \{s \in \rho(A);\ \mu_{\mathbf{\Delta}}(G(s)) > \delta^{-1}\}, \quad \delta > 0; \tag{7}$$

$$r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g) = \left(\sup_{s\in\partial\mathbb{C}_g} \mu_{\mathbf{\Delta}}(G(s))\right)^{-1} \quad \text{if } A \text{ is } \mathbb{C}_g\text{-stable}. \tag{8}$$

*Proof.* (6) follows from the definition of $\mu_{\mathbf{\Delta}}(\cdot)$ and the equivalence

$$s \in \sigma(A + B\Delta C) \ \Leftrightarrow\ 1 \in \sigma(\Delta G(s)), \tag{9}$$

which holds for all $s \in \rho(A)$ and all $\Delta \in \mathbb{C}^{l\times q}$; see [13, Proposition 2.3]. Then the characterizations (7), (8) are immediate consequences of (6). □

Theorem 2.5 is the basis for our further development. It shows that spectral value sets and stability radii can be calculated by evaluating the function $s \mapsto \mu_{\mathbf{\Delta}}(G(s))$. For completeness we mention some facts related to the characterization (7). The proofs can be found in [15], [17].

*Remark* 2.6. Let $(\mathbf{\Delta}, \|\cdot\|)\in\mathcal{P}^{\mathbb{C}}_{l,q}$. Then, for any $\delta > 0$,
(i) the sets $\sigma_{\mathbf{\Delta}}(A,B,C;\delta)\setminus\sigma(A) = \{\, s\in\rho(A)\,;\ \mu_{\mathbf{\Delta}}(G(s)) > \delta^{-1}\,\}$ are open;
(ii) the closure of $\sigma_{\mathbf{\Delta}}(A,B,C;\delta)$ is given by

$$\text{cl}\,(\sigma_{\mathbf{\Delta}}(A,B,C;\delta)) = \bigcup_{\substack{\Delta\in\mathbf{\Delta}\\ \|\Delta\|\le\delta}} \sigma(A+B\Delta C) = \sigma(A)\cup\{s\in\rho(A)\,;\mu_{\mathbf{\Delta}}(G(s))\ge\delta^{-1}\}; \tag{10}$$

(iii) the boundary of $\sigma_{\mathbf{\Delta}}(A,B,C;\delta)$ satisfies

$$\partial\sigma_{\mathbf{\Delta}}(A,B,C;\delta)\setminus\sigma(A) = \{\, s\in\rho(A)\,;\ \mu_{\mathbf{\Delta}}(G(s)) = \delta^{-1}\}.$$

Note that these statements do not hold for all perturbation structures $(\mathbf{\Delta},\|\cdot\|)\in\mathcal{P}_{l,q}$.

Next, we give a useful characterization of $\mu_{\mathbf{\Delta}}(\cdot)$ via the spectral radius. It generalizes a result of [19].

LEMMA 2.7. *Let $M\in\mathbb{C}^{q\times l}$ and $(\mathbf{\Delta},\|\cdot\|)\in\mathcal{P}^{\mathbb{C}}_{l,q}$. Then*

$$\mu_{\mathbf{\Delta}}(M) = \max\{\, \varrho(\Delta M)\,;\ \Delta\in\mathbf{\Delta}, \|\Delta\| = 1\,\}. \tag{11}$$

*Suppose that the maximum in* (11) *is nonzero and is attained at $\Delta\in\mathbf{\Delta}$, $\|\Delta\| = 1$. Let $\Delta_1 = s^{-1}\Delta$, where $s\in\sigma(\Delta M)$ and $|s| = \varrho(\Delta M)\neq 0$. Then $\Delta_1\in\mathbf{\Delta}$, $1\in\sigma(\Delta_1 M)$, and $\|\Delta_1\| = \mu_{\mathbf{\Delta}}(M)^{-1}$.*

*Proof.* Let $\varrho_0$ denote the maximum on the right-hand side of (11). For any nonzero $\Delta\in\mathbf{\Delta}$ we have $\varrho(\Delta M) = \|\Delta\|\,\varrho(\frac{\Delta}{\|\Delta\|}M)\le\|\Delta\|\,\varrho_0$. Hence, the condition $1\in\sigma(\Delta M)$ implies that $1\le\|\Delta\|\,\varrho_0$. This yields $\mu_{\mathbf{\Delta}}(M)\le\varrho_0$. Equality holds if $\varrho_0 = 0$. Suppose $\varrho_0\neq 0$. Then the matrix $\Delta_1$ satisfies $\|\Delta_1\| = \varrho_0^{-1}$ and $\|\Delta_1\|\ge\mu_{\mathbf{\Delta}}(M)^{-1}$. Thus $\varrho_0 = \mu_{\mathbf{\Delta}}(M)$. ∎

We now determine $\mu_{\mathbf{\Delta}}(M)$ for the case that $\mathbf{\Delta} = \mathbb{C}^{l\times q}$ and the underlying norm is an operator norm. Let $\|\cdot\|_\alpha, \|\cdot\|_\beta$ be norms on $\mathbb{C}^q$ and $\mathbb{C}^l$, respectively. Then the induced operator norms on $\mathbb{C}^{l\times q}$ (resp., $\mathbb{C}^{q\times l}$) are defined by

$$\|\Delta\|_{\alpha,\beta} = \max_{y\in\mathbb{C}^q\setminus\{0\}}\frac{\|\Delta y\|_\beta}{\|y\|_\alpha},\ \Delta\in\mathbb{C}^{l\times q},\quad\text{and}\quad \|M\|_{\beta,\alpha} = \max_{u\in\mathbb{C}^l\setminus\{0\}}\frac{\|Mu\|_\alpha}{\|u\|_\beta},\ M\in\mathbb{C}^{q\times l}.$$

Recall that, for every $\Delta\in\mathbb{C}^{l\times q}$, there exist $y\in\mathbb{C}^q$, $u\in\mathbb{C}^l$, with $\|y\|_\alpha = \|u\|_\beta^D = 1$ and

$$\|\Delta\|_{\alpha,\beta} = u^\top\Delta y.$$

Here $\|\cdot\|_\beta^D$ denotes the dual of the norm $\|\cdot\|_\beta$,

$$\|u\|_\beta^D = \max_{z\in\mathbb{C}^l\setminus\{0\}}\frac{|u^\top z|}{\|z\|_\beta},\quad u\in\mathbb{C}^l.$$

PROPOSITION 2.8. *Let $\|\cdot\|_\alpha, \|\cdot\|_\beta$ be norms on $\mathbb{C}^q$ and $\mathbb{C}^l$, respectively. Let $\|\cdot\| = \|\cdot\|_{\alpha,\beta}$ be the induced operator norm and $\mathbf{\Delta} := \mathbb{C}^{l\times q}$. Then the following hold:*
(a) *For any $M\in\mathbb{C}^{q\times l}$, $\mu_{\mathbf{\Delta}}(M) = \|M\|_{\beta,\alpha}$.*
(b) *Suppose $M\neq 0$. Let $u\in\mathbb{C}^l$, $y\in\mathbb{C}^q$ be such that $\|u\|_\beta = \|y\|_\alpha^D = 1$ and $y^\top Mu = \|M\|_{\beta,\alpha}$. Then the matrix $\Delta_0 := \|M\|_{\beta,\alpha}^{-1}uy^\top$ satisfies $1\in\sigma(\Delta_0 M)$ and $\|\Delta_0\| = \mu_{\mathbf{\Delta}}(M)^{-1}$.*

*Proof.* If $1 \in \sigma(\Delta M)$, then there is $u \neq 0$ with $u = \Delta M u$. Hence,

$$0 \neq \|u\|_\beta = \|\Delta M u\|_\beta \leq \|\Delta\|_{\alpha,\beta} \, \|M\|_{\beta,\alpha} \, \|u\|_\beta.$$

Thus $1 \leq \|\Delta\|_{\alpha,\beta} \, \|M\|_{\beta,\alpha}$. This implies $\mu_{\mathbf{\Delta}}(M) \leq \|M\|_{\beta,\alpha}$. Equality holds if $M = 0$. Let $M \neq 0$. Then the matrix $\Delta_0$ satisfies $\|\Delta_0\|_{\alpha,\beta} = \|M\|_{\beta,\alpha}^{-1}$. Furthermore, $\Delta_0 M u = u$. Thus $1 \in \sigma(\Delta_0 M)$. It follows that $\mu_{\mathbf{\Delta}}(M) \geq \|\Delta_0\|_{\alpha,\beta}^{-1} = \|M\|_{\beta,\alpha}$. So $\mu_{\mathbf{\Delta}}(M) = \|M\|_{\beta,\alpha} = \|\Delta_0\|_{\alpha,\beta}^{-1}$. □

*Remark* 2.9. Throughout the rest of this paper we consider only complex perturbation structures. There are some results available for real perturbation structures. For example, if $M \in \mathbb{C}^{q\times l}$ and $\mathbf{\Delta} = \mathbb{R}^{l\times q}$, there are formulae for $\mu_{\mathbf{\Delta}}(M)$ (and hence for spectral value sets and stability radii) if $\mathbb{R}^l$ and $\mathbb{R}^q$ are normed with Euclidean norms; see [15], [17], and [21]. In [15] formulae are proved for stability radii of a real diagonal matrix with respect to real off-diagonal perturbations; see Corollary 6.6 for the complex case. Also in [20] a formula is given for the stability radius of real symmetric systems with respect to real symmetric (or diagonal) dynamic perturbations.

*Remark* 2.10. $\sigma_{\mathbf{\Delta}}(A, B, C; \delta)$, $r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g)$, and $\mu_{\mathbf{\Delta}}(G(s))$ depend strongly upon the perturbation norm $\|\cdot\|$ on $\mathbf{\Delta}$. Also the problem of evaluating numerically the formulae in Theorem 2.5 depends strongly on this norm. In a given application one should therefore carefully choose the norm on the perturbation space in such a way that it reflects the parametric uncertainty of the application and is also suitable from a computational viewpoint. In order to provide greater flexibility we have stated the results in the following sections for different classes of norms from which one can choose the most appropriate norm in a given case. In general, the (approximate) computation of $\sigma_{\mathbf{\Delta}}(A, B, C; \delta)$, $r_{\mathbf{\Delta}}(A, B, C; \mathbb{C}_g)$, $\mu_{\mathbf{\Delta}}(G(s))$ is a difficult problem, but under specific conditions efficient algorithms and estimation procedures are available; see, e.g., [5], [1], [18], [9], [26], [11].

**3. Composite systems.** Let us introduce some additional notation. In the following, $q, l$ are finite sequences $q = (q_1, \ldots, q_m)$, $l = (l_1, \ldots, l_m)$. We write $\underline{m} := \{1, 2, \ldots, m\}$ and denote by $\mathbb{C}^{q\times l} := \{[M_{jk}]\,;\, M_{jk} \in \mathbb{C}^{q_j \times l_k} \text{ for } (j,k) \in \underline{m} \times \underline{m}\}$ the set of $m \times m$ block matrices

$$[M_{jk}] = [M_{jk}]_{j\in\underline{m},k\in\underline{m}} = \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix}. \tag{12}$$

The block-diagonal matrix with blocks $M_j \in \mathbb{C}^{q_j \times l_j}$, $j \in \underline{m}$ is denoted by

$$M = \oplus_{j=1}^m M_j := \operatorname{diag}(M_1, \ldots, M_m) = \begin{bmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_m \end{bmatrix} \in \mathbb{C}^{q\times l}.$$

For any index set $\mathcal{I} \subseteq \underline{m} \times \underline{m}$ we denote by $\mathbf{\Delta}_{\mathcal{I},q,l}$ the set of block matrices $\Delta$ of the form

$$\Delta = [\Delta_{jk}] := \begin{bmatrix} \Delta_{11} & \ldots & \Delta_{1m} \\ \vdots & & \vdots \\ \Delta_{m1} & \ldots & \Delta_{mm} \end{bmatrix}, \quad \Delta_{jk} \in \mathbb{C}^{l_j \times q_k} \text{ and } \Delta_{jk} = 0 \text{ if } (j,k) \notin \mathcal{I}. \tag{13}$$

Given $(A_j, B_j, C_j) \in L_{n_j,l_j,q_j}$, $j \in \underline{m}$, the object of this paper is to study the variation of the spectrum of the block-diagonal matrix $A = \oplus_{j=1}^m A_j$ under perturbations of the form

$$A \rightsquigarrow A_\Delta := A + B\Delta C, \quad \Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}, \tag{14}$$

where $B$, $C$ are the block-diagonal matrices $B = \oplus_{j=1}^m B_j$, $C = \oplus_{j=1}^m C_j$.

The matrices $A_\Delta$ have the following system theoretic interpretation. Consider the system

$$\Sigma: \qquad \dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{15}$$

which is the direct sum of the $m$ subsystems

$$\Sigma_j: \qquad \dot{x}_j(t) = A_j x_j(t) + B_j u_j(t), \qquad y_j(t) = C_j x_j(t), \qquad j \in \underline{m}. \tag{16}$$

The transfer matrix of $\Sigma$ is the direct sum of the transfer matrices of these subsystems:

$$G(s) = C(sI - A)^{-1}B = \oplus_{j=1}^m G_j(s), \qquad G_j(s) := C_j(sI_{n_j} - A_j)^{-1}B_j, \;\; j \in \underline{m}. \tag{17}$$

Introducing the couplings

$$u_j(t) = \sum_{k \in \underline{m},\, (j,k) \in \mathcal{I}} \Delta_{jk}\, y_k(t), \quad j \in \underline{m}, \tag{18}$$

one obtains the composite system

$$\Sigma_\Delta: \qquad \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} = (A + B\Delta C) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = A_\Delta \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}. \tag{19}$$

Thus the perturbed system $\Sigma_\Delta$ with system matrix $A_\Delta$ can be viewed as the composite system obtained by interconnecting the subsystems $\Sigma_j$ via the couplings (18) defined by the perturbation blocks $\Delta_{jk}$. The unperturbed ("nominal") system $\Sigma_0: \; \dot{x} = Ax$ obtained by setting $\Delta = 0$ is simply the direct sum of the subsystems $\dot{x}_j = A_j x_j$.

The pairs $(j,k) \in \mathcal{I}$ can be regarded as the oriented edges of a directed graph $\Gamma(\underline{m}, \mathcal{I})$ whose vertices are the numbers $1, \ldots, m$. This is illustrated in Example 3.1 for the case where $m = 3$. Observe that in the directed graph the endpoint of the edge $(j,k)$ is the first component, $j$.[1] This orientation reflects the interconnection structure (18).

*Example* 3.1. Consider the index set $\mathcal{I} = \{(1,2), (1,3), (2,1), (3,2), (3,3)\}$. Then the matrices $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$ take the form

$$\Delta = \begin{bmatrix} 0 & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & 0 & 0 \\ 0 & \Delta_{32} & \Delta_{33} \end{bmatrix}.$$

The directed graph $\Gamma(\underline{3}, \mathcal{I})$ and the block diagram of the closed loop system (19) are shown in Figure 1.

[1] Note that this is the reverse of standard notation in graph theory. We have used our notation to be in harmony with the system theoretic interpretation.

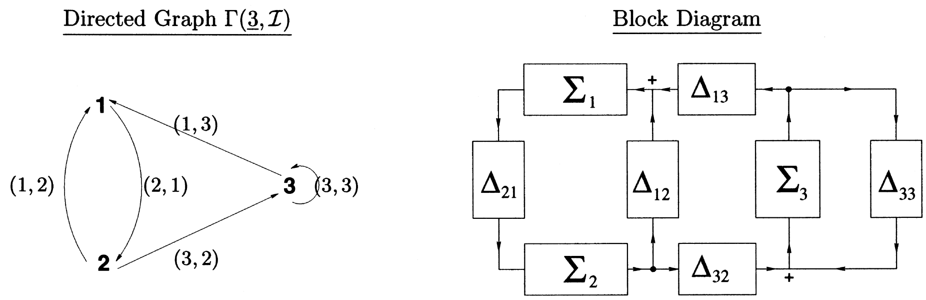



FIG. 1. *Composite system.*

Applying Theorem 2.5, the spectral value sets and stability radii of the system $(A, B, C)$ under perturbations of the form (14) are given by

$$\sigma_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A, B, C; \delta) = \sigma(A) \cup \{s \in \rho(A)\,;\, \mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(G(s)) > \delta^{-1}\} \tag{20}$$

and

$$r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A, B, C; \mathbb{C}_g) = \left( \sup_{s \in \partial \mathbb{C}_g} \mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(G(s)) \right)^{-1}. \tag{21}$$

In order to determine the spectral value sets and stability radii via (20), (21) we need to study the $\mu$-values of block-diagonal matrices $M = \oplus_{j=1}^{m} M_j$ with respect to perturbations $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$. Note that this is just the inverse situation of traditional $\mu$-analysis where block-diagonal perturbations of arbitrary matrices are considered; see [7]. Applying Proposition 2.7, we obtain

$$\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) = \max_{\substack{\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l} \\ \|\Delta\|=1}} \varrho(\Delta M), \quad M \in \mathbb{C}^{q \times l}. \tag{22}$$

The size of the perturbations $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$ will be measured by two types of norms: a weighted maximum of the nonzero block norms $\|\Delta_{jk}\|$, $(j,k) \in \mathcal{I}$, and mixed operator norms of the overall matrix $\Delta$. In the next two sections we derive formulae for $\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M)$ with respect to these types of norms.

*Remark* 3.2. A composite system $\Sigma$ of the form (15) which is the direct sum of subsystems $\Sigma_i$ of the form (16) is said to be *connectively stable* with respect to a given set of interconnections $E$ (possibly time-varying and/or nonlinear) if $\sigma(A_j) \subset \mathbb{C}_-$, $j \in \underline{m}$, and the origin of the interconnected system obtained from the block-diagonal system $\Sigma$ by the feedback $u(t) = e(t, y(t))$ is globally asymptotically stable for all $e \in E$; see [25]. In the literature many different methods have been put forward for obtaining sufficient criteria of connective stability based on knowledge of the subsystems $\Sigma_i$ and their interconnection structure. Input-output and passivity methods have been used, but the most popular seem to be Liapunov methods; see [25]. The advantage of these methods is that time-varying and nonlinear interconnections can be considered. However, the robustness results obtained in this way are in general quite conservative. This is in contrast with the full block case (where $\mathbf{\Delta}_{\mathcal{I},q,l} = \mathbb{C}^{l \times q}$; i.e., $\mathcal{I} = \underline{m} \times \underline{m}$). In this case a quadratic Liapunov function of optimal robustness can be constructed which secures asymptotic stability for all time-varying nonlinearities with gain strictly smaller than $r_{\mathbb{C}^{l \times q}}(A, B, C; \mathbb{C}_-)$; see [13] and [15, section 5.6].

*It remains an open problem to determine those perturbation structures* $\mathbf{\Delta}_{\mathcal{I},q,l}$ *for which it is possible to construct a joint quadratic Liapunov function for all perturbed systems* $\Sigma_\Delta$, $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$, $\|\Delta\| < r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A,B,C;\mathbb{C}_-)$. Such a Liapunov function would secure the connective stability for all time-varying nonlinearities with gain strictly smaller than $r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A,B,C;\mathbb{C}_-)$.

**4. Weighted maximum norms.** We consider the same basic framework as that in section 3. Let $\|\cdot\|_{\alpha_j}$ be a norm on $\mathbb{C}^{q_j}$ and $\|\cdot\|_{\beta_k}$ be a norm on $\mathbb{C}^{l_k}$. We assume that we are given a nonnegative weight matrix $R = [r_{jk}] \in \mathbb{R}_+^{m\times m}$ and introduce the index set

$$\mathcal{I} = \mathcal{I}_R := \{(j,k) \in \underline{m} \times \underline{m}\,;\, r_{jk} > 0\}. \tag{23}$$

With these data we associate a normed perturbation space $(\mathbf{\Delta}_{\mathcal{I},q,l}, \|\cdot\|)$, where (see (13))

$$\mathbf{\Delta}_{\mathcal{I},q,l} = \{[\Delta_{jk}]\,;\, \Delta_{jk} \in \mathbb{C}^{l_j \times q_k} \text{ for } j,k \in \underline{m} \text{ and } \Delta_{jk} = 0 \text{ if } (j,k) \notin \mathcal{I}\} \tag{24}$$

and $\|\cdot\|$ is the weighted maximum norm

$$\|\Delta\| := \max_{(j,k)\in\mathcal{I}} r_{jk}^{-1} \|\Delta_{jk}\|_{\alpha_k,\beta_j}, \quad \Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}. \tag{25}$$

Note that the following equivalence holds for $\Delta \in \mathbb{C}^{l\times q}$:

$$(\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l} \text{ and } \|\Delta\| \le 1) \quad \Leftrightarrow \quad \|\Delta_{jk}\|_{\alpha_k,\beta_j} \le r_{jk} \text{ for all } (j,k) \in \underline{m} \times \underline{m}. \tag{26}$$

In this section we determine the $\mu$-value of block-diagonal matrices with respect to the perturbation structure $(\mathbf{\Delta}_{\mathcal{I},q,l}, \|\cdot\|)$ and apply it to obtain formulae for spectral value sets and stability radii. We will make use of the following well-known results from the theory of nonnegative matrices; see [2], [10], [16].

($\varrho$1) If $A \in \mathbb{R}^{n\times n}$ is nonnegative, the spectral radius $\varrho(A)$ is an eigenvalue of $A$ and there exists a nonnegative eigenvector corresponding to $\varrho(A)$.

($\varrho$2) Let $A_1, A_2 \in \mathbb{R}^{n\times n}$. If $0 \le A_1 \le A_2$, then $\varrho(A_1) \le \varrho(A_2)$.

($\varrho$3) If $\alpha v \le Av$ and $v \ge 0$, $v \ne 0$, then $\alpha \le \varrho(A)$. If $Av \le \beta v$ and $v_i > 0$ for $i \in \underline{n}$, then $\varrho(A) \le \beta$.

The next lemma is a consequence of ($\varrho$3).

LEMMA 4.1. *Let* $Y_{jk} \in \mathbb{C}^{l_j \times l_k}$, $j,k \in \underline{m}$, *and* $\|\cdot\|_{\beta_j}$ *be a norm on* $\mathbb{C}^{l_j}$. *Then*

$$\varrho\left(\begin{bmatrix} Y_{11} & \dots & Y_{1m} \\ \vdots & & \vdots \\ Y_{m1} & \dots & Y_{mm}\end{bmatrix}\right) \le \varrho\left(\begin{bmatrix} \|Y_{11}\|_{\beta_1,\beta_1} & \dots & \|Y_{1m}\|_{\beta_m,\beta_1} \\ \vdots & & \vdots \\ \|Y_{m1}\|_{\beta_1,\beta_m} & \dots & \|Y_{mm}\|_{\beta_m,\beta_m}\end{bmatrix}\right). \tag{27}$$

*Proof.* Let $\lambda \in \mathbb{C}$ be an eigenvalue of the block matrix $[Y_{jk}]$, i.e.,

$$\begin{bmatrix} Y_{11} & \dots & Y_{1m} \\ \vdots & & \vdots \\ Y_{m1} & \dots & Y_{mm}\end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \qquad x_j \in \mathbb{C}^{l_j}, \quad j \in \underline{m},$$

and $x_j \ne 0$ for at least one $j$. Then $\lambda x_j = \sum_{k=1}^m Y_{jk} x_k$ for all $j \in \underline{m}$. This implies

$$|\lambda|\, \|x_j\|_{\beta_j} \le \sum_{k=1}^m \|Y_{jk}\|_{\beta_k,\beta_j}\, \|x_k\|_{\beta_k}, \qquad j \in \underline{m}.$$

Setting $A = (\|Y_{jk}\|_{\beta_k,\beta_j})_{j,k\in\underline{m}}$ and $v = (v_j)_{j\in\underline{m}}$, where $v_j = \|x_j\|_{\beta_j}$, we have

$$|\lambda|\, v_j \leq (Av)_j, \qquad j\in\underline{m}.$$

Hence $|\lambda| \leq \varrho(A)$ by ($\varrho$3), and this implies (27). ∎

We associate with any given block matrix $M = [M_{jk}] \in \mathbb{C}^{q\times l}$ of the form (12) the following nonnegative $m \times m$ matrix of block norms:

$$\tilde{M} = \begin{bmatrix} \|M_{11}\|_{\beta_1,\alpha_1} & \dots & \|M_{1m}\|_{\beta_m,\alpha_1} \\ \vdots & & \vdots \\ \|M_{m1}\|_{\beta_1,\alpha_m} & \dots & \|M_{mm}\|_{\beta_m,\alpha_m} \end{bmatrix}. \tag{28}$$

We now prove the main result of this section.

THEOREM 4.2. *Suppose* $R = [r_{jk}]\in\mathbb{R}^{m\times m}$ *is a nonnegative matrix and* $\mathcal{I} = \mathcal{I}_R$ *is given by* (23). *If* $M = [M_{jk}] \in \mathbb{C}^{q\times l}$ *and* $\tilde{M} = (\|M_{jk}\|_{\beta_k,\alpha_j})_{j,k\in\underline{m}}$ *is the associated matrix of block norms, then, with respect to the norm* (25),

$$\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) \leq \varrho(R\,\tilde{M}). \tag{29}$$

*Equality holds in* (29) *if* $M = \oplus_{j=1}^m M_j$, $M_j\in\mathbb{C}^{q_j\times l_j}$, $j\in\underline{m}$, *is block-diagonal, viz.*

$$\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) = \varrho(R\,\mathrm{diag}(\|M_1\|_{\beta_1,\alpha_1},\dots,\|M_m\|_{\beta_m,\alpha_m})). \tag{30}$$

*Proof.* We have already seen that by Lemma 2.7,

$$\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) = \max_{\substack{\Delta\in\mathbf{\Delta}_{\mathcal{I},q,l}\\ \|\Delta\|=1}} \varrho(\Delta\, M)\,. \tag{31}$$

Moreover, if $(\Delta M)_{jk}$ is the $(j,k)$-entry of the $m\times m$ block matrix $\Delta M \in \mathbb{C}^{l\times l}$, then

(32)
$$\|(\Delta M)_{jk}\|_{\beta_k,\beta_j} = \left\|\sum_{i=1}^m \Delta_{ji}M_{ik}\right\|_{\beta_k,\beta_j} \leq \sum_{i=1}^m \|\Delta_{ji}\|_{\alpha_i,\beta_j}\, \|M_{ik}\|_{\beta_k,\alpha_i} = (\tilde{\Delta}\tilde{M})_{jk},\ j,k\in\underline{m},$$

where $\tilde{\Delta} := [\|\Delta_{jk}\|_{\alpha_k,\beta_j}]_{j,k\in\underline{m}}$. Now let $\Delta = [\Delta_{jk}] \in \mathbf{\Delta}_{\mathcal{I},q,l}$ with $\|\Delta\| \leq 1$. Then $\tilde{\Delta} \leq R$ by (26) and

$$\begin{aligned} \varrho(\Delta M) &\leq \varrho([\,\|\,(\Delta M)_{jk}\,\|_{\beta_k,\beta_j}\,]) && \text{(by Lemma 4.1)}\\ &\leq \varrho(\tilde{\Delta}\tilde{M}) && \text{(by (32) and property ($\varrho$2))}\\ &\leq \varrho(R\tilde{M}) && \text{(by (26) and property ($\varrho$2)).} \end{aligned}$$

Thus,

$$\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) = \max_{\substack{\Delta\in\mathbf{\Delta}_{\mathcal{I},q,l}\\ \|\Delta\|=1}} \varrho(\Delta\, M)\, \leq \varrho(R\tilde{M}).$$

It remains to show that the latter inequality is actually an equality if $M = \oplus_{j=1}^m M_j$ is block-diagonal. For $k\in\underline{m}$ let $y_k\in\mathbb{C}^{q_k}$ and $u_k\in\mathbb{C}^{l_k}$ be such that $\|u_k\|_{\beta_k} = \|y_k\|_{\alpha_k}^D = 1$ and $y_k^\top M_k u_k = \|M_k\|_{\beta_k,\alpha_k}$. Let $\Delta^0 = [\Delta^0_{jk}]$, where $\Delta^0_{jk} = r_{jk}u_j y_k^\top$. Then $\Delta^0\in\mathbf{\Delta}_{\mathcal{I},q,l}$ and $\|\Delta^0\| = 1$. Since $R\tilde{M} = R\ \mathrm{diag}(\|M_1\|_{\beta_1,\alpha_1},\dots,\|M_m\|_{\beta_m,\alpha_m}) \in$

$\mathbb{R}^{m\times m}$ is nonnegative, there is a nonnegative vector $\xi = [\xi_1,\dots,\xi_m]^\top \in \mathbb{R}^m$ such that $R\tilde{M}\xi = \varrho(R\tilde{M})\xi$. Define $w = [\xi_1 u_1^\top,\dots,\xi_m u_m^\top]^\top$. Then a straightforward computation yields $\Delta^0 (\oplus_{k=1}^m M_k)\, w = \varrho(R\tilde{M})w$. Thus $\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) \geq \varrho(\Delta^0 (\oplus_{k=1}^m M_k)) \geq \varrho(R\tilde{M})$, and the proof is complete. ∎

For the case where all the $\alpha_j$ and $\beta_k$ are 2-norms, a characterization of $\mu_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M)$ is given in Part I of [6] which shows that the $\mu$-value can be obtained as the solution of a smooth constrained optimization problem. Associated computational aspects are discussed in Part II of [6].

We will now apply the above theorem to determine the spectral value sets and stability radii of block-diagonal matrices $A$ with respect to perturbations of the form (14). Let $\mathcal{B}_R(\delta)$ denote the open ball with radius $\delta > 0$ about the origin in the perturbation space $\mathbf{\Delta}_{\mathcal{I},q,l}$ provided with the norm (25),

$$\mathcal{B}_R(\delta) = \{\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l};\, \|\Delta\| < \delta\}. \tag{33}$$

It follows from (26) that $\mathcal{B}_R(\delta)$ is the set of block matrices $\Delta = [\Delta_{jk}]$ satisfying

$$\|\Delta_{jk}\|_{\alpha_k,\beta_j} < \delta\, r_{jk}, \quad (j,k)\in\mathcal{I} = \mathcal{I}_R, \quad \|\Delta_{jk}\|_{\alpha_k,\beta_j} = 0 \quad \text{otherwise.}$$

COROLLARY 4.3. *Suppose $R = [r_{jk}]\in\mathbb{R}^{m\times m}$ is a nonnegative matrix and $\mathcal{I} = \mathcal{I}_R$ is given by* (23). *Let $(A_j, B_j, C_j) \in L_{n_j,l_j,q_j}$, $j \in \underline{m}$, and consider perturbations* (14) *of the block-diagonal matrix A. If $\mathbf{\Delta}_{\mathcal{I},q,l}$ is provided with the norm* (25) *and $G_j(s)$ is defined by* (17)*, then the following hold:*

(a) *The spectral value set $\sigma_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A, B, C; \delta)$ is given by*

$$\bigcup_{\Delta\in\mathcal{B}_R(\delta)} \sigma(A_\Delta) = \sigma(A) \cup \left\{ s\in\rho(A)\,;\, \varrho(R\,\mathrm{diag}(\|G_1(s)\|_{\beta_1,\alpha_1},\dots,\|G_m(s)\|_{\beta_m,\alpha_m})) > \delta^{-1} \right\}.$$

(b) *Let $\mathbb{C}_g$ be an open subset of $\mathbb{C}$ and suppose $A_1,\dots,A_m$ are $\mathbb{C}_g$-stable (i.e., $\sigma(A) \subset \mathbb{C}_g$). Then the stability radius is given by*

$$r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A, B, C; \mathbb{C}_g) = \left( \sup_{s\in\partial\mathbb{C}_g} \varrho(R\,\mathrm{diag}(\|G_1(s)\|_{\beta_1,\alpha_1},\dots,\|G_m(s)\|_{\beta_m,\alpha_m})) \right)^{-1}. \tag{34}$$

*Proof.* Applying Theorem 4.2, (a) follows directly from (20) and (b) follows from (21). ∎

We conclude this section by specializing the previous results to the scalar *diagonal* case where $A = \mathrm{diag}(a_1,\dots,a_n)$ is perturbed to $A_\Delta = A + \Delta$ with

$$\Delta \in \mathbf{\Delta}_{\mathcal{I}} := \{\Delta \in \mathbb{C}^{n\times n}\,;\, \Delta_{jk} = 0 \text{ if } r_{jk} = 0\}. \tag{35}$$

Here $R = (r_{jk})_{j,k\in\underline{n}}$ is a given nonnegative $n \times n$ matrix, and the perturbation space $\mathbf{\Delta}_{\mathcal{I}}$ is provided with the norm

$$\|\Delta\| = \max_{(j,k)\in\mathcal{I}} r_{jk}^{-1}|\Delta_{jk}|, \ \ \Delta \in \mathbf{\Delta}_{\mathcal{I}}, \quad \text{where } \mathcal{I} := \mathcal{I}_R = \{(j,k)\in\underline{n}\times\underline{n}\,;\, r_{jk} > 0\}. \tag{36}$$

This can be subsumed into the above framework by setting $m = n$, $l_j = q_j = 1$ for $j \in \underline{m}$; $(A_j, B_j, C_j) = (a_j, 1, 1)$, $j \in \underline{m}$; and $\|\Delta_{jk}\|_{\alpha_k,\beta_j} = |\Delta_{jk}|$, $j, k \in \underline{m}$. Note that for this special case $G_j(s) = (s - a_j)^{-1}$, $j \in \underline{m}$.

COROLLARY 4.4. *Suppose $R = [r_{jk}]\in\mathbb{R}^{n\times n}$ is a nonnegative matrix with associated index set $\mathcal{I} = \mathcal{I}_R$ defined by* (36) *and normed perturbation space $(\mathbf{\Delta}_{\mathcal{I}}, \|\cdot\|)$ defined by* (35) *and* (36)*. Let $a_1,\dots,a_n \in \mathbb{C}$, $\sigma_0 = \{a_1,\dots,a_n\}$, and set $A_\Delta = \mathrm{diag}(a_1,\dots,a_n) + \Delta$ for arbitrary $\Delta\in\mathbb{C}^{n\times n}$. Then the following hold:*

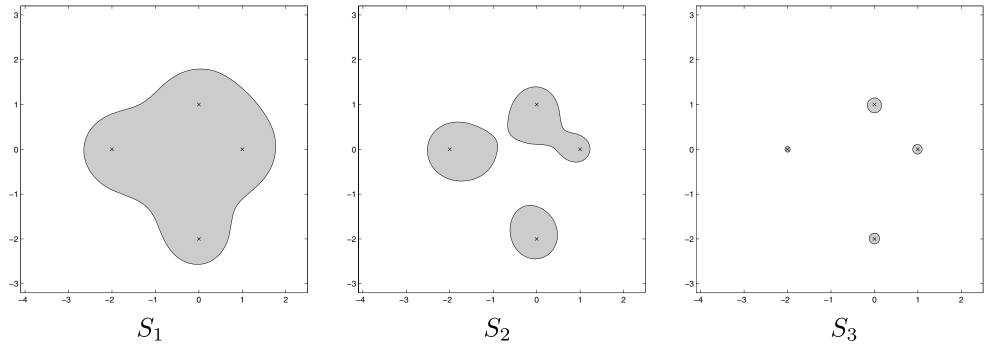



FIG. 2. *The sets $S_j$ defined in* (40).

(a) $$\bigcup_{\Delta\in\mathbb{C}^{n\times n},\,|\Delta|\le R} \sigma(A_\Delta) = \sigma_0 \cup \{ s\in\mathbb{C}\setminus\sigma_0\,;\; \varrho(R\,\mathrm{diag}(|s-a_1|^{-1},\dots,|s-a_n|^{-1})) \ge 1\ \}.$$

(b) *If $\mathbb{C}_g$ is an open subset of $\mathbb{C}$, $\sigma_0\subset\mathbb{C}_g$, then*

(37)
$$r_{\mathbf{\Delta}_{\mathcal{I}_R}}(\mathrm{diag}(a_1,\dots,a_n), I_n, I_n; \mathbb{C}_g) = \left(\sup_{s\in\partial\mathbb{C}_g} \varrho\left(R\,\mathrm{diag}(|s-a_1|^{-1},\dots,|s-a_n|^{-1})\right)\right)^{-1}.$$

(c) *In particular, if $\mathbb{C}_g = \mathbb{C}_- := \{s\in\mathbb{C}\,;\,\Re s<0\}$ and $a_1,\dots,a_n<0$, then*

(38)
$$r_{\mathbf{\Delta}_{\mathcal{I}_R}}(\mathrm{diag}(a_1,\dots,a_n), I_n, I_n; \mathbb{C}_-) = \left(\varrho(R\,\mathrm{diag}(|a_1|^{-1},\dots,|a_n|^{-1}))\right)^{-1}.$$

*Proof.* (a) follows directly from Corollary 4.3(a) since $\|\Delta\|\le 1$ if and only if $\Delta\in\mathcal{B}_R(\delta)$ for all $\delta>1$. Equation (37) is a special case of (34) since $G_j(s) = (s-a_j)^{-1}$. To verify (38) note that by assumption, $a_1,\dots,a_n\in\mathbb{R}$ and so the functions $\omega\mapsto|i\,\omega-a_k|^{-1}$ attain their maxima on $\mathbb{R}$ at $\omega=0$. Hence, the monotonicity property ($\varrho$2) of the spectral radius yields

$$\sup_{s\in i\mathbb{R}} \varrho\left(R\,\mathrm{diag}(|s-a_1|^{-1},\dots,|s-a_n|^{-1})\right) = \varrho\left(R\ \mathrm{diag}(|a_1|^{-1},\dots,|a_n|^{-1})\right).$$

Thus, (38) is a consequence of (37). ◻

*Example* 4.5. Suppose $A = \mathrm{diag}(1, i, -2, -2i)$ and

(39) $$R_1 = \frac{1}{4}\begin{bmatrix}0&6&2&2\\2&0&2&2\\2&2&0&8\\1&1&2&0\end{bmatrix},\quad R_2 = \frac{1}{4}\begin{bmatrix}0&6&0&0\\0&0&2&2\\2&2&0&8\\1&1&2&0\end{bmatrix},\quad R_3 = \frac{1}{4}\begin{bmatrix}0&6&0&0\\0&0&2&2\\0&0&0&8\\1&1&0&0\end{bmatrix}.$$

Figure 2 shows the sets

(40) $$S_j := \bigcup_{\Delta\in\mathbb{C}^{4\times4},\ |\Delta|\le R_j} \sigma(A+\Delta), \qquad j=1,2,3.$$

Note that $R_1$ (resp., $R_2$) is obtained from $R_3$ by replacing all (resp., some) off-diagonal zeros of $R_3$ with 1/2. Since $R_1\ge R_2\ge R_3$, the sets $S_j$ decrease as $j$ varies from 1 to 3. The pictures have been obtained via Corollary 4.4(a).

**5. Mixed operator norms.** We consider the same basic framework as that in the previous two sections. Let $\|\cdot\|_{\mathbb{C}^m}$ be an absolute norm on $\mathbb{C}^m$ which is invariant with respect to a permutation of the coordinates (for instance, a $p$-norm, $1 \le p \le \infty$), and let $\mathcal{N}(\cdot)$ be the induced operator norm on $\mathbb{C}^{m\times m}$. For $j,k \in \underline{m}$ let $\|\cdot\|_{\alpha_j}$ be a norm on $\mathbb{C}^{q_j}$ and let $\|\cdot\|_{\beta_k}$ be a norm on $\mathbb{C}^{l_k}$. Given any index set $\mathcal{I} \subseteq \underline{m} \times \underline{m}$, we define a norm on the perturbation space $\mathbf{\Delta}_{\mathcal{I},q,l}$ (24) by the formula

$$\|\Delta\| := \mathcal{N}\left(\begin{bmatrix} \|\Delta_{11}\|_{\alpha_1,\beta_1} & \dots & \|\Delta_{1m}\|_{\alpha_m,\beta_1} \\ \vdots & & \vdots \\ \|\Delta_{m1}\|_{\alpha_1,\beta_m} & \dots & \|\Delta_{mm}\|_{\alpha_m,\beta_m} \end{bmatrix}\right), \quad \Delta = [\Delta_{jk}] \in \mathbf{\Delta}_{\mathcal{I},q,l}. \tag{41}$$

In this section we derive a formula for the $\mu$-value of *block-diagonal* matrices $M$ with respect to the perturbation space $\mathbf{\Delta}_{\mathcal{I},q,l}$ provided with the norm (41). As a preparation we consider the general case of an arbitrary block matrix $M \in \mathbb{C}^{q\times l}$ of the form (12) and determine an upper bound for $\mu^{\|\cdot\|}_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M)$ [2] in terms of the associated nonnegative $m \times m$ matrix $\tilde{M}$; see (28).

PROPOSITION 5.1. *Let* $M_{jk} \in \mathbb{C}^{q_j \times l_k}$, $j,k \in \underline{m}$, $M = [M_{jk}]$, *and* $\mathcal{I} \subseteq \underline{m} \times \underline{m}$. *Then with respect to the norm* (41),

$$\mu^{\|\cdot\|}_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) \le \mu^{\mathcal{N}}_{\mathbf{\Delta}_{\mathcal{I}}}(\tilde{M}), \tag{42}$$

*where* $\tilde{M}$ *is defined by* (28) *and*

$$\mathbf{\Delta}_{\mathcal{I}} := \{[\delta_{ij}] \in \mathbb{C}^{m\times m}; \delta_{ij} = 0 \text{ for } (i,j) \notin \mathcal{I}\}. \tag{43}$$

*Proof.* To prove (42) it suffices by Lemma 2.7 to show that for each $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$ with $\|\Delta\| = 1$ there exists $\tilde{\Delta} \in \mathbf{\Delta}_{\mathcal{I}}$ such that $\mathcal{N}(\tilde{\Delta}) = 1$ and $\varrho(\Delta M) \le \varrho(\tilde{\Delta}\tilde{M})$. Given any $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$ with $\|\Delta\| = 1$, let $\tilde{\Delta} \in \mathbb{R}^{m\times m}$ be the matrix in the parentheses on the right-hand side of (41). Then $\tilde{\Delta} \in \mathbf{\Delta}_{\mathcal{I}}$ and $\mathcal{N}(\tilde{\Delta}) = \|\Delta\| = 1$ by (41). Let $u = (u^j)_{j\in\underline{m}} \in \oplus_{j=1}^m \mathbb{C}^{l_j}$, $u \ne 0$, and $\lambda \in \mathbb{C}$ be such that $\Delta M u = \lambda u$ and $|\lambda| = \varrho(\Delta M)$. We set $\tilde{u} = (\|u^j\|_{\beta_j})_{j\in\underline{m}}$. If $\Delta M u = ((\Delta M u)^j)_{j\in\underline{m}} \in \oplus_{j=1}^m \mathbb{C}^{l_j}$ is partitioned as $u$, then, for every $j \in \underline{m}$,

$$\begin{aligned} \|(\Delta M u)^j\|_{\beta_j} &= \left\| \sum_{k=1}^m \sum_{i=1}^m \Delta_{ji} M_{ik} u^k \right\|_{\beta_j} \le \sum_{k=1}^m \sum_{i=1}^m \|\Delta_{ji}\|_{\alpha_i,\beta_j} \|M_{ik}\|_{\beta_k,\alpha_i} \|u^k\|_{\beta_k} \\ &= (\tilde{\Delta}\tilde{M}\tilde{u})^j. \end{aligned}$$

It follows that we have the componentwise inequality

$$\tilde{\Delta}\tilde{M}\tilde{u} \ge \left(\|(\Delta M u)^j\|_{\beta_j}\right)_{j\in\underline{m}} = |\lambda|\tilde{u}$$

and so $\varrho(\tilde{\Delta}\tilde{M}) \ge |\lambda| = \varrho(\Delta M)$ by ($\varrho$3). This concludes the proof. ∎

We will now prove that equality holds in (42) if $M = [M_{jk}]$ is block-diagonal, i.e., $M_{jk} = 0$ for $j,k \in \underline{m}$, $j \ne k$. In the proof we will make use of some elementary notions from graph theory [16], [2] which are summarized in the following remark.

[2] Since in this section we will consider $\mu$-values with respect to more than one norm, we use the notation $\mu^{\|\cdot\|}_{\mathbf{\Delta}_{\mathcal{I},q,l}}$ where there may be a risk of confusion.

*Remark* 5.2. A finite sequence $\gamma = (j_1, \dots, j_\ell)$ of integers is said to be a path from $j_1$ to $j_\ell$ in the directed graph $\Gamma(\underline{m}, \mathcal{I})$ if $(j_i, j_{i+1}) \in \mathcal{I}$ for all $i \in \underline{\ell - 1}$. Two nodes $j, k$ of $\Gamma(\underline{m}, \mathcal{I})$ are said to be strongly connected if there exists a path from $j$ to $k$ and a path from $k$ to $j$ in $\Gamma(\underline{m}, \mathcal{I})$. A subset $J \subset \underline{m}$ is said to be strongly connected if any two distinct nodes in $J$ are strongly connected in $\Gamma(\underline{m}, \mathcal{I})$. The maximal strongly connected subsets of $\underline{m}$ are called the strongly connected components of the directed graph $\Gamma(\underline{m}, \mathcal{I})$. They form a partition of $\underline{m}$. A finite sequence $\gamma = (j_1, \dots, j_\ell)$ of mutually distinct integers is said to be a cycle of length $|\gamma| := \ell \geq 1$ of the directed graph $\Gamma(\underline{m}, \mathcal{I})$ if $(j_i, j_{i+1}) \in \mathcal{I}$ for all $i \in \underline{\ell - 1}$ and $(j_\ell, j_1) \in \mathcal{I}$. We will write $j \in \gamma$ if $j = j_i$ for some $i \in \underline{\ell}$. By $\mathcal{Z}(\mathcal{I})$ we denote the set of all cycles in $\Gamma(\underline{m}, \mathcal{I})$. A cycle $\gamma \in \mathcal{Z}(\mathcal{I})$ is said to be nontrivial if $|\gamma| \geq 2$. If for a given $j_0 \in \underline{m}$ there does not exist a nontrivial cycle $\gamma \in \mathcal{Z}(\mathcal{I})$ such that $j_0 \in \gamma$, then $\{j_0\}$ is a strongly connected component of $\Gamma(\underline{m}, \mathcal{I})$.

For any $A = [a_{jk}] \in \mathbb{C}^{m \times m}$ we set $\mathcal{I}_A := \{(j,k) \in \underline{m} \times \underline{m}\,;\, a_{jk} \neq 0\}$. Let $\gamma = (j_1, \dots, j_\ell) \in \mathcal{Z}(\underline{m} \times \underline{m})$. Then the cycle product of $A$ over $\gamma$ is defined as

$$\prod\nolimits_\gamma A := \prod_{i=1}^{\ell} a_{j_i j_{i+1}}, \qquad \text{where } j_{\ell+1} := j_1.$$

Note that if $\gamma = (j)$ is a cycle of length 1, then $\prod_\gamma A = a_{jj}$.

If $A = [a_{jk}], B = [b_{jk}] \in \mathbb{C}^{m \times m}$ we denote by $A \circ B$ the Hadamard product of $A$ and $B$, $A \circ B = [a_{jk} b_{jk}] \in \mathbb{C}^{m \times m}$. For nonnegative matrices the Hadamard product satisfies the following inequality which is a corollary of Theorem 5.7.21 in [16].

LEMMA 5.3. *Let $A, B \in \mathbb{R}_+^{m \times m}$. If $\mathcal{Z}(\mathcal{I}_A) = \emptyset$, then $\varrho(B \circ A) = 0$. Otherwise we have*

$$\varrho(B \circ A) \leq \varrho(B) \max_{\gamma \in \mathcal{Z}(\mathcal{I}_A)} \left( \prod\nolimits_\gamma A \right)^{\frac{1}{|\gamma|}}.$$

Given $(m_1, \dots, m_\ell) \in \mathbb{N}^\ell$, $\ell \geq 1$, and matrices $C_j \in \mathbb{C}^{m_j \times m_{j+1}}$, $j \in \underline{\ell}$, where $m_{\ell+1} := m_1$, the associated block cyclic matrix is defined by

$$Z(C_1, \dots, C_\ell) := \begin{bmatrix} 0 & C_1 & 0 & \cdots & 0 \\ 0 & \ddots & C_2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & C_{\ell-1} \\ C_\ell & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{(\sum_{j=1}^{\ell} m_j) \times (\sum_{j=1}^{\ell} m_j)}.$$

The next result which follows from the Frobenius theorem (see [10, Chapter XIII, section 2]) determines the spectral radius of nonnegative cyclic matrices with scalar blocks.

LEMMA 5.4. *Let $c_1, \dots, c_\ell \geq 0$, $\ell \in \mathbb{N}$. Then the spectrum of the cyclic matrix $Z(c_1, \dots, c_\ell)$ is given by*

$$\sigma\left(Z(c_1, \dots, c_\ell)\right) = \{e^{2\pi i \frac{k-1}{\ell}} \varrho\,;\, k \in \underline{\ell}\}, \tag{44}$$

*where $\varrho = \varrho(Z(c_1, \dots, c_\ell)) = (c_1 c_2 \cdots c_\ell)^{1/\ell}$.*

The following theorem is the main result of this section.

THEOREM 5.5. *Suppose $M_j \in \mathbb{C}^{q_j \times l_j}$ for $j \in \underline{m}$, $M = \oplus_{j=1}^m M_j$, and $\mathcal{I} \subseteq \underline{m} \times \underline{m}$ are given and let $\mathcal{I}_0 := \{(j,k) \in \mathcal{I}\,;\, M_j \neq 0$ and $M_k \neq 0\}$. If $(\mathbf{\Delta}_{\mathcal{I},q,l}, \|\cdot\|)$ is the perturbation structure defined by (24), (41) and $\mathbf{\Delta}_{\mathcal{I}}$ defined by (43) is provided with the norm $\mathcal{N}$, then*

$$\mu^{\|\cdot\|}_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) = \mu^{\mathcal{N}}_{\mathbf{\Delta}_{\mathcal{I}}}(\tilde{M}) = \begin{cases} \max\limits_{\gamma \in \mathcal{Z}(\mathcal{I}_0)} \left(\prod_{j\in\gamma} \|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}} & \text{if } \mathcal{Z}(\mathcal{I}_0) \neq \emptyset, \\ 0 & \text{if } \mathcal{Z}(\mathcal{I}_0) = \emptyset, \end{cases} \tag{45}$$

*where $\tilde{M} = \operatorname{diag}(\|M_1\|_{\beta_1,\alpha_1}, \ldots, \|M_m\|_{\beta_m,\alpha_m})$.*

*Proof.* Let $c$ denote the right-hand-side of (45). We first show that

$$\mu^{\|\cdot\|}_{\mathbf{\Delta}_{\mathcal{I},q,l}}(M) \leq \mu^{\mathcal{N}}_{\mathbf{\Delta}_{\mathcal{I}}}(\tilde{M}) = \max_{\substack{\tilde{\Delta}\in\mathbf{\Delta}_{\mathcal{I}} \\ \mathcal{N}(\tilde{\Delta})=1}} \varrho\left(\tilde{\Delta}\,\tilde{M}\right) \leq c. \tag{46}$$

The first inequality in (46) follows directly from Proposition 5.1. To prove the second inequality in (46) let $E = [e_{jk}] \in \mathbb{R}^{m\times m}$, where $e_{jk} = 1$ if $(j,k) \in \mathcal{I}$ and $e_{jk} = 0$ otherwise. Set

$$A := E\tilde{M} = E\operatorname{diag}(\|M_1\|_{\beta_1,\alpha_1} \ldots \|M_m\|_{\beta_m,\alpha_m}) = [\, e_{jk}\|M_k\|_{\beta_k,\alpha_k}\,] \in \mathbb{R}^{m\times m}_+.$$

Then $\mathcal{I}_A \subseteq \mathcal{I}$, $\mathcal{Z}(\mathcal{I}_A) = \mathcal{Z}(\mathcal{I}_0)$, and we have $\prod_\gamma A = \prod_{j\in\gamma} \|M_j\|_{\beta_j,\alpha_j}$ for every cycle $\gamma \in \mathcal{Z}(\mathcal{I})$. Thus

$$c = \begin{cases} \max\limits_{\gamma\in\mathcal{Z}(\mathcal{I}_A)} \left(\prod_\gamma A\right)^{\frac{1}{|\gamma|}} & \text{if } \mathcal{Z}(\mathcal{I}_A) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

Let $\tilde{\Delta} = [\tilde{\Delta}_{jk}] \in \mathbf{\Delta}_{\mathcal{I}}$ with $\mathcal{N}(\tilde{\Delta}) = 1$. Then $\tilde{\Delta}\tilde{M} = [\tilde{\Delta}_{jk}\, e_{jk}\, \|M_k\|_{\beta_k,\alpha_k}] = \tilde{\Delta} \circ (E\tilde{M}) = \tilde{\Delta} \circ A$, and so by Lemma 5.3 and (47)

$$\varrho(\tilde{\Delta}\tilde{M}) = \varrho(\tilde{\Delta} \circ A) \leq \varrho(\tilde{\Delta})\, c \leq \mathcal{N}(\tilde{\Delta})c = c.$$

This proves (46). If $c = 0$, then equality holds in (46) and hence (45). By Lemma 2.7 it remains to construct, for each cycle $\gamma \in \mathcal{Z}(\mathcal{I}_0)$, a matrix $\Delta^\gamma \in \mathbf{\Delta}_{\mathcal{I},q,l}$ such that $\|\Delta^\gamma\| = 1$ and

$$\varrho\left(\Delta^\gamma \left(\oplus_{j=1}^m M_j\right)\right) \geq \left(\prod_{j\in\gamma} \|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}}. \tag{48}$$

The construction of $\Delta^\gamma$ is as follows. Suppose that $\gamma = (j_1, \ldots, j_\ell)$. For $j \in \underline{m}$ let $u_j \in \mathbb{C}^{\ell_j}$ and $y_j \in \mathbb{C}^{q_j}$ be such that $\|u_j\|_{\beta_j} = \|y_j\|^D_{\alpha_j} = 1$ and $y_j^\top M_j u_j = \|M_j\|_{\beta_j,\alpha_j}$. Let $\Delta^\gamma := [\Delta^\gamma_{jk}]$, where

$$\Delta^\gamma_{jk} := \begin{cases} u_{j_i} y_{j_{i+1}}^\top & \text{if } (j,k) = (j_i, j_{i+1}),\ i \in \underline{\ell - 1}, \\ u_{j_\ell} y_{j_1}^\top & \text{if } (j,k) = (j_\ell, j_1), \\ 0 \in \mathbb{C}^{l_j \times q_k} & \text{otherwise.} \end{cases} \qquad j,k \in \underline{m}.$$

For instance, if $m$=4, then

$$\Delta^{(1,3,4)} = \begin{bmatrix} 0 & 0 & u_1 y_3^\top & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_3 y_4^\top \\ u_4 y_1^\top & 0 & 0 & 0 \end{bmatrix}, \qquad \Delta^{(3,4,2,1)} = \begin{bmatrix} 0 & 0 & u_1 y_3^\top & 0 \\ u_2 y_1^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & u_3 y_4^\top \\ 0 & u_4 y_2^\top & 0 & 0 \end{bmatrix}.$$

We claim that $\Delta^\gamma$ has the required properties. Obviously, $\Delta^\gamma \in \mathbf{\Delta}_{\mathcal{I},q,l}$. To see that $\|\Delta^\gamma\| = 1$, note that $\Delta^\gamma$ contains $\ell \geq 1$ nonzero blocks and in each block row and each block column of $\Delta^\gamma$ there is at most one nonzero block. All nonzero blocks have norm 1. Since the norm $\|\cdot\|_{\mathbb{C}^m}$ is absolute and invariant with respect to a permutation of the coordinates, it follows that

$$\|\Delta^\gamma\| = \mathcal{N}(\,[\,\|\Delta^\gamma_{jk}\|_{\alpha_j,\beta_k}]\,) = 1.$$

Let us show (48). Observe that the principal block submatrix of $\Delta^\gamma M$ corresponding to the block rows and columns with numbers $j_1, \ldots, j_\ell$ is permutation similar to the block cyclic matrix

$$Z(u_{j_1}y_{j_2}^\top M_{j_2}, u_{j_2}y_{j_3}^\top M_{j_3}, \ldots, u_{j_\ell}y_{j_1}^\top M_{j_1}) = Z(u_{j_1}y_{j_2}^\top, u_{j_2}y_{j_3}^\top, \ldots, u_{j_\ell}y_{j_1}^\top)\left(\oplus_{i=1}^{\ell} M_{j_i}\right).$$

By Lemma 5.4 the product $\varrho := (\prod_{j\in\gamma}\|M_j\|_{\beta_j,\alpha_j})^{\frac{1}{|\gamma|}}$ is the spectral radius of the matrix

$$Z = Z(\|M_{j_1}\|_{\beta_{j_1},\alpha_{j_1}}, \ldots, \|M_{j_\ell}\|_{\beta_{j_\ell},\alpha_{j_\ell}}).$$

Let $\xi = [\xi_{j_1}, \ldots, \xi_{j_\ell}]^\top \in \mathbb{R}^\ell_+$ be an eigenvector of $Z$ corresponding to $\varrho$; see ($\varrho$1). Then the following relations hold:

$$\|M_{j_i}\|_{\beta_{j_i},\alpha_{j_i}}\xi_{j_{i+1}} = \left(\prod_{j\in\gamma}\|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}}\xi_{j_i}, \qquad i\in\underline{\ell},\ j_{\ell+1} := j_1. \tag{49}$$

Now set $w := [w_1^\top, \ldots, w_m^\top]^\top$, where

$$w_j = \begin{cases} \xi_{j_{i+1}}u_{j_i} & \text{if } j = j_i,\ i\in\underline{\ell},\ j_{\ell+1} := j_1, \\ 0\in\mathbb{C}^{\ell_j} & \text{otherwise,} \end{cases} \qquad j \in \underline{m},$$

and let $\widehat{w} := [w_{j_1}^\top, w_{j_2}^\top, \ldots, w_{j_\ell}^\top]^\top$. Then using (49), we have

$$\begin{aligned}
&Z(u_{j_1}y_{j_2}^\top, u_{j_2}y_{j_3}^\top, \ldots, u_{j_\ell}y_{j_1}^\top)\left(\oplus_{i=1}^{\ell} M_{j_i}\right)\widehat{w} \\
&\quad = \begin{bmatrix} 0 & u_{j_1}y_{j_2}^\top M_{j_2} & 0 & \ldots & 0 \\ 0 & 0 & u_{j_2}y_{j_3}^\top M_{j_3} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & u_{j_{\ell-1}}y_{j_\ell}^\top M_{j_\ell} \\ u_{j_\ell}y_{j_1}^\top M_{j_1} & 0 & \ldots & \ldots & 0 \end{bmatrix} \begin{bmatrix} \xi_{j_2}u_{j_1} \\ \xi_{j_3}u_{j_2} \\ \vdots \\ \xi_{j_\ell}\,u_{j_{\ell-1}} \\ \xi_{j_1}u_{j_\ell} \end{bmatrix} \\
&\quad = \begin{bmatrix} \|M_{j_2}\|_{\beta_{j_2},\alpha_{j_2}}\,\xi_{j_3}\,u_{j_1} \\ \|M_{j_3}\|_{\beta_{j_3},\alpha_{j_3}}\,\xi_{j_4}\,u_{j_2} \\ \vdots \\ \vdots \\ \|M_{j_\ell}\|_{\beta_{j_\ell},\alpha_{j_\ell}}\,\xi_{j_1}\,u_{j_{\ell-1}} \\ \|M_{j_1}\|_{\beta_{j_1},\alpha_{j_1}}\,\xi_{j_2}\,u_{j_\ell} \end{bmatrix} = \left(\prod_{j\in\gamma}\|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}} \begin{bmatrix} \xi_{j_2}\,u_{j_1} \\ \xi_{j_3}\,u_{j_2} \\ \vdots \\ \vdots \\ \xi_{j_\ell}\,u_{j_{\ell-1}} \\ \xi_{j_1}\,u_{j_\ell} \end{bmatrix} \\
&\quad = \left(\prod_{j\in\gamma}\|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}}\widehat{w}.
\end{aligned}$$

This implies that $\Delta^\gamma\left(\oplus_{j=1}^m M_j\right) w = (\prod_{j\in\gamma}\|M_j\|_{\beta_j,\alpha_j})^{\frac{1}{|\gamma|}}\, w$. Thus (48) holds, and the proof is complete. ∎

As a corollary we obtain the following characterization of spectral value sets and stability radii for the perturbation space $(\mathbf{\Delta}_{\mathcal{I},q,l}, \|\cdot\|)$.

COROLLARY 5.6. *Suppose $(A_j, B_j, C_j) \in L_{n_j,l_j,q_j}$ for $j \in \underline{m}$, $\delta > 0$, $\mathcal{I} \subset \underline{m}\times\underline{m}$ and $\mathbf{\Delta} = \mathbf{\Delta}_{\mathcal{I},q,l}$ is provided with the norm (41). Let $A = \oplus_{j=1}^m A_j$, $B = \oplus_{j=1}^m B_j$, $C = \oplus_{j=1}^m C_j$, and*

$$G_j(s) := C_j(sI_{n_j} - A_j)^{-1}B_j,\ j\in\underline{m},\quad \mathcal{I}_0 := \{(j,k)\in\mathcal{I}\,;\, B_j\neq 0, C_k\neq 0\}.$$

*Then the following hold:*

(a) *If $\mathcal{Z}(\mathcal{I}_0)\neq\emptyset$, the spectral value set of $A$ with respect to perturbations of the form (14) is given by*

$$\bigcup_{\Delta\in\,\mathbf{\Delta}_{\mathcal{I},q,l},\,\|\Delta\|<\delta}\sigma(A_\Delta) = \sigma(A)\cup\left\{s\in\rho(A);\ \max_{\gamma\in\mathcal{Z}(\mathcal{I}_0)}\prod\nolimits_{j\in\gamma}\|G_j(s)\|_{\beta_j,\alpha_j} > \delta^{-|\gamma|}\right\}. \tag{50}$$

*If $\mathcal{Z}(\mathcal{I}_0)=\emptyset$, then all the eigenvalues of $A$ are fixed under perturbations of the form (14); i.e.,*

$$\bigcup_{\Delta\in\mathbf{\Delta}_{\mathcal{I},q,l}}\sigma(A_\Delta) = \sigma(A).$$

(b) *If $j_0\in\underline{m}$, and there does not exist any cycle $\gamma\in\mathcal{Z}(\mathcal{I}_0)$ such that $j_0\in\gamma$, then*

$$\sigma(A_{j_0})\subseteq\sigma(A_\Delta),\quad \Delta\in\mathbf{\Delta}_{\mathcal{I},q,l}.$$

(c) *Let $\mathbb{C}_g$ be an open subset of $\mathbb{C}$ and suppose $A_1,\dots,A_m$ are $\mathbb{C}_g$-stable. Then the stability radius $r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A,B,C;\mathbb{C}_g)$ is given by*

$$r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A,B,C;\mathbb{C}_g) = \begin{cases}\left[\sup_{s\in\partial\mathbb{C}_g}\ \max_{\gamma\in\mathcal{Z}(\mathcal{I}_0)}\left(\prod_{j\in\gamma}\|G_j(s)\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}}\right]^{-1} & \text{if } \mathcal{Z}(\mathcal{I}_0)\neq\emptyset,\\ \infty & \text{if } \mathcal{Z}(\mathcal{I}_0)=\emptyset.\end{cases} \tag{51}$$

*Proof.* (a) Since $G(s) = \oplus_{j=1}^m G_j(s)$ is the transfer function of $(A,B,C)$, (a) is a direct consequence of Theorems 2.5 and 5.5.

(b) Suppose $\Delta\in\mathbf{\Delta}_{\mathcal{I},q,l}$. Since $A_\Delta = [A_{jk}+B_j\Delta_{jk}C_k]$ and $B_j\Delta_{jk}C_k = 0$ if $(j,k)\notin\mathcal{I}$, $B_j=0$, or $C_k=0$, we have $\mathcal{I}_{B\Delta C}\subseteq\mathcal{I}_0$ and $\mathcal{I}_{A_\Delta}\subseteq\mathcal{I}_A\cup\mathcal{I}_0\subseteq\{(k,k)\,;\,k\in\underline{m}\}\cup\mathcal{I}_0$. Now assume that there does not exist any cycle $\gamma\in\mathcal{Z}(\mathcal{I}_0)$ such that $j_0\in\gamma$. Then $(j_0,j_0)\notin\mathcal{I}_0$; hence $B_{j_0}\Delta_{j_0j_0}C_{j_0}=0$, and $\{j_0\}$ is a strongly connected component of the directed graph $\Gamma(\underline{m},\mathcal{I}_{A_\Delta})$. This implies that $A_\Delta$ is permutation similar to a matrix $\tilde{A}$ of block upper triangular form:

$$A_\Delta\sim\tilde{A} = \begin{bmatrix}\tilde{A}_{11} & \cdot & \cdots & \tilde{A}_{1r}\\ & \tilde{A}_{22} & \cdots & \tilde{A}_{2r}\\ & & \ddots & \vdots\\ & & & \tilde{A}_{rr}\end{bmatrix},$$

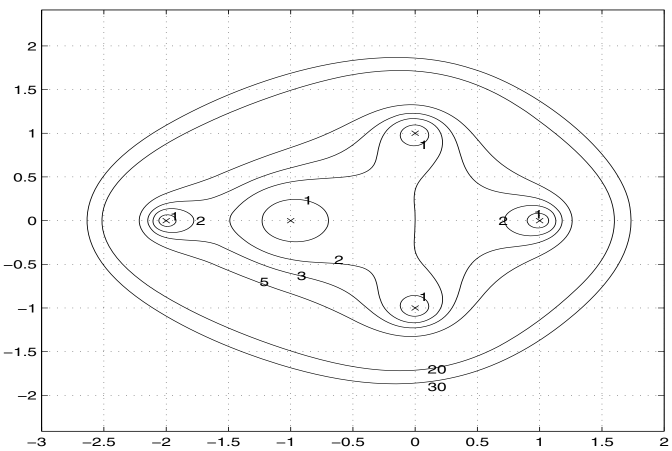



FIG. 3. *The boundaries of the Brualdi sets* $\mathcal{B}(-2,-1,1,-i,i;\delta)$, $\delta = 1,2,3,5,20,30$.

where the diagonal blocks correspond to the connected components of the graph $\Gamma(\underline{m}, \mathcal{I}_{A_\Delta})$; see [2, section 2.3]. Hence $A_{j_0} = A_{j_0} + B_{j_0}\Delta_{j_0 j_0} C_{j_0} = \tilde{A}_{kk}$ for some $k \in \underline{r}$ and this shows that $\sigma(A_{j_0}) \subset \sigma(A_\Delta)$.

(c) If $\mathcal{Z}(\mathcal{I}_0) = \emptyset$, then the last statement of (a) implies $r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A,B,C;\mathbb{C}_g) = \infty$. Now suppose $\mathcal{Z}(\mathcal{I}_0) \neq \emptyset$. By the continuity of the spectrum, $r_{\mathbf{\Delta}_{\mathcal{I},q,l}}(A,B,C;\mathbb{C}_g)$ is the largest value of $\delta$ such that $\sigma(A_\Delta) \cap \partial\mathbb{C}_g = \emptyset$ for all $\Delta \in \mathbf{\Delta}_{\mathcal{I},q,l}$ of norm $\|\Delta\| < \delta$. By (a) this condition is equivalent to

$$\max_{\gamma \in \mathcal{Z}(\mathcal{I}_0)} \prod\nolimits_{j\in\gamma} \|G_j(s)\|_{\beta_j,\alpha_j} \leq \delta^{-|\gamma|}, \quad s \in \partial\mathbb{C}_g,$$

or, equivalently,

$$\sup_{s\in\partial\mathbb{C}_g} \max_{\gamma \in \mathcal{Z}(\mathcal{I}_0)} \left(\prod\nolimits_{j\in\gamma} \|G_j(s)\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}} \leq \delta^{-1}.$$

This concludes the proof of (51). □

We will now specialize the previous result to the case where the blocks are reduced to scalars, i.e., $l_i = q_i = 1$ for $i \in \underline{m}$. In this case a more concrete version of the formula (50) is obtained in which the spectral value sets are expressed as a finite union of sets of the following form:

(52)
$$\mathcal{B}(z_1,\ldots,z_\ell;\delta) := \left\{ s\in\mathbb{C}\,;\, \prod\nolimits_{j\in\underline{\ell}} |s-z_j| \leq \delta \right\}, \qquad \ell\in\mathbb{N},\ z_1,\ldots,z_\ell\in\mathbb{C}, \quad \delta \geq 0.$$

These sets are called *Brualdi sets* in honor of Brualdi who introduced them in [4]. They will be further discussed in what follows. Note that $\mathcal{B}(z_1,\ldots,z_\ell;0) = \{z_1,\ldots,z_\ell\}$ and $\mathcal{B}(z;\delta) = \mathcal{D}(z;\delta)$ is the closed disk of radius $\delta$ about $z$. For an illustration, see Figure 3.

COROLLARY 5.7. *Suppose* $a_j, b_j, c_j \in \mathbb{C}$, $j \in \underline{n}$, $A = \operatorname{diag}(a_1,\ldots,a_n)$, $B = \operatorname{diag}(b_1,\ldots,b_n)$, $C = \operatorname{diag}(c_1,\ldots,c_n)$, $\mathcal{I} \subseteq \underline{n}\times\underline{n}$, *and* $\|\cdot\|_{\mathbb{C}^n}$ *is an arbitrary norm on*

*$\mathbb{C}^n$ with induced operator norm $\mathcal{N}(\cdot)$ on $\mathbb{C}^{n\times n}$. Let*

$$\begin{aligned} \mathbf{\Delta}_{\mathcal{I}} &:= \{\,[\Delta_{jk}]\in\mathbb{C}^{n\times n}\,;\ \Delta_{jk}\in\mathbb{C} \text{ and } \Delta_{jk}=0 \text{ if } (j,k)\notin\mathcal{I}\},\\ \|\Delta\| &:= \mathcal{N}(|\Delta|) = \mathcal{N}\left(\begin{bmatrix} |\Delta_{11}| & \dots & |\Delta_{1n}| \\ \vdots & & \vdots \\ |\Delta_{n1}| & \dots & |\Delta_{nn}| \end{bmatrix}\right), \quad \Delta\in\mathbf{\Delta}_{\mathcal{I}}, \\ A_\Delta &:= A + B\Delta C,\\ \mathcal{I}_0 &:= \{(j,k)\in\mathcal{I}\,;\ b_j c_k \neq 0\}. \end{aligned} \tag{53}$$

*Then the following statements hold.*

(a) *For all $\delta > 0$,*

$$\bigcup_{\substack{\Delta\in\mathbf{\Delta}_{\mathcal{I}}\\ \|\Delta\|\le\delta}} \sigma(A_\Delta) = \{a_1,\dots,a_n\} \cup \bigcup_{(j_1,\dots,j_\ell)\in\mathcal{Z}(\mathcal{I}_0)} \mathcal{B}\left(a_{j_1},\dots,a_{j_\ell}\,;\ \delta^\ell \prod_{i=1}^{\ell} |b_{j_i} c_{j_i}|\right). \tag{54}$$

(b) *Let $j_0\in\underline{n}$ and suppose there does not exist any cycle $\gamma\in\mathcal{Z}(\mathcal{I}_0)$ such that $j_0\in\gamma$. Then $a_{j_0}\in\sigma(A_\Delta)$ for all $\Delta\in\mathbf{\Delta}_{\mathcal{I}}$.*

(c) *If $\mathbb{C}_g$ is an open subset of $\mathbb{C}$, $a_1,\dots,a_n\in\mathbb{C}_g$, then*

$$\begin{aligned} & r_{\mathbf{\Delta}_{\mathcal{I}}}(A,B,C;\mathbb{C}_g) \\ &= \begin{cases} \inf\limits_{s\in\partial\mathbb{C}_g}\ \min\limits_{\gamma\in\mathcal{Z}(\mathcal{I}_0)} \left(\prod_{j\in\gamma}|s-a_j|/|b_j c_j|\right)^{\frac{1}{|\gamma|}} & \text{if } \mathcal{Z}(\mathcal{I}_0)\neq\emptyset,\\ \infty & \text{if } \mathcal{Z}(\mathcal{I}_0)=\emptyset. \end{cases} \end{aligned} \tag{55}$$

*Proof.* (a) If $\mathcal{Z}(\mathcal{I}_0)=\emptyset$, then Corollary 5.6 (a) implies

$$\bigcup_{\Delta\in\mathbf{\Delta}_{\mathcal{I}},\,\|\Delta\|\le\delta} \sigma(A_\Delta) = \{a_1,\dots,a_n\}.$$

Now suppose that $\mathcal{Z}(\mathcal{I}_0)\neq\emptyset$ and let $\delta>0$. Since

$$G(s) = \operatorname{diag}(g_1(s),\dots,g_n(s)), \quad g_j(s) = c_j(s-a_j)^{-1}b_j,\ j\in\underline{n},$$

is the transfer function of the system $(A,B,C)$, we have by (10) and Theorem 5.5 the following equivalences for $s\in\mathbb{C}\setminus\{a_1,\dots,a_n\}$:

$$\begin{aligned} & s\in \bigcup_{\Delta\in\mathbf{\Delta}_{\mathcal{I}},\,\|\Delta\|\le\delta} \sigma(A_\Delta)\\ &\Leftrightarrow \mu^{\|\cdot\|}_{\mathbf{\Delta}_{\mathcal{I}}}(\operatorname{diag}(c_1(s-a_1)^{-1}b_1,\dots,c_n(s-a_n)^{-1}b_n)) \ge \delta^{-1}\\ &\Leftrightarrow \max_{\gamma\in\mathcal{Z}(\mathcal{I}_0)} \left(\prod_{j\in\gamma}|c_j(s-a_j)^{-1}b_j|\right)^{\frac{1}{|\gamma|}} \ge \delta^{-1}\\ &\Leftrightarrow \exists\,\gamma\in\mathcal{Z}(\mathcal{I}_0):\ \left(\prod_{j\in\gamma}|c_j(s-a_j)^{-1}b_j|\right)^{\frac{1}{|\gamma|}} \ge \delta^{-1}\\ &\Leftrightarrow \exists\,\gamma\in\mathcal{Z}(\mathcal{I}_0):\ \prod_{j\in\gamma}|s-a_j| \le \delta^{|\gamma|}\prod_{j\in\gamma}|b_j c_j|. \end{aligned}$$

Hence (54) holds.

(b) is a special case of Corollary 5.6(b).

(c) Since $|g_j(s)|^{-1} = |s-a_j|/|b_j c_j|$ if $b_j c_j\neq 0$, formula (55) is a special case of (51). ∎

*Remark* 5.8. (i) Corollaries 5.6 and 5.7 show that the spectral value sets and stability radii of block-diagonal and diagonal matrices with respect to the normed perturbation structure $(\mathbf{\Delta}_{\mathcal{I},q,l}, \|\cdot\|)$ (see (41)) are independent of the norm $\mathcal{N}$.

(ii) For the special case that $\mathbb{C}_g = \mathbb{C}_-$ and $a_1, \dots, a_n < 0$ it follows from (55) that

$$\begin{aligned} r_{\mathbf{\Delta}_{\mathcal{I}}}(A, B, C; \mathbb{C}_-) &= \min_{\gamma \in \mathcal{Z}(\mathcal{I}_0)} \left( \prod_{j\in\gamma} |a_j|/|b_j c_j| \right)^{\frac{1}{|\gamma|}} \\ &= \min_{\gamma \in \mathcal{Z}(\mathcal{I}_0)} \left( \prod_{j\in\gamma} r_{\mathbb{C}}(a_j, b_j, c_j; \mathbb{C}_-) \right)^{\frac{1}{|\gamma|}}. \end{aligned}$$

By Corollary 5.7 Brualdi sets play a fundamental role in determining the spectral value sets of diagonal matrices with respect to perturbations $\Delta \in \mathbf{\Delta}_{\mathcal{I}}$. We conclude this section with some remarks concerning these sets. Each Brualdi set (52) can be represented as the intersection of a family of sets which are unions of $\ell$ closed disks of centers $z_i$, $i \in \underline{\ell}$. More precisely, we have the following proposition.

PROPOSITION 5.9. *Let $z_1, \dots, z_\ell \in \mathbb{C}$ and $\delta > 0$. Then*

$$\mathcal{B}\,(z_1, \dots, z_\ell; \delta) = \bigcap_{\substack{r_1,\dots,r_\ell > 0, \\ \prod_{j\in\underline{\ell}} r_j = 1}} \left( \bigcup_{j\in\underline{\ell}} \mathcal{D}(z_j; \delta^{\frac{1}{\ell}} r_j) \right). \tag{56}$$

*Proof.* Let $D$ denote the set on the right-hand side of (56). Suppose that $s \notin D$. Then there are $r_1, \dots, r_\ell > 0$ such that $\prod_{j\in\underline{\ell}} r_j = 1$ and $|s - z_j| > \delta^{\frac{1}{\ell}} r_j$ for all $j \in \underline{\ell}$. Multiplying the latter inequalities we obtain that $\prod_{j\in\underline{\ell}} |s - z_j| > \delta$. Thus $s \notin \mathcal{B}\,(z_1, \dots, z_\ell; \delta)$. Hence $\mathcal{B}\,(z_1, \dots, z_\ell; \delta) \subseteq D$. Now suppose that $s \notin \mathcal{B}\,(z_1, \dots, z_\ell; \delta)$; then $\delta_1 := \prod_{j\in\underline{\ell}} |s - z_j| > \delta$. If we define $r_j > 0$ by $|s - z_j| = \delta_1^{\frac{1}{\ell}} r_j$, $j \in \underline{\ell}$, then $s \notin \bigcup_{j\in\underline{\ell}} \mathcal{D}(z_j; \delta^{\frac{1}{\ell}} r_j)$ and $\prod_{j\in\underline{\ell}} r_j = 1$. So $s \notin D$. $\square$

From the relation (56) one can derive an upper bound for the connected components of Brualdi sets.

PROPOSITION 5.10. *Let $z_1, \dots, z_\ell \in \mathbb{C}$ and $\delta > 0$. Then the following hold:*

(a) *Each connected component of $\mathcal{B}(z_1, \dots, z_\ell; \delta)$ contains at least one of the points $z_j$, $j \in \underline{\ell}$.*

(b) *Let $\epsilon > 0$ and suppose that for a given $j \in \underline{\ell}$*

$$\min_{k\in\underline{\ell}, k\neq j} |z_j - z_k| > \delta^{\frac{1}{\ell}} \left( \epsilon + \epsilon^{-\frac{1}{\ell-1}} \right). \tag{57}$$

*Then the connected component $K_j$ of $\mathcal{B}(z_1, \dots, z_\ell; \delta)$ with $z_j \in K_j$ is contained in $\mathcal{D}(z_j; \epsilon\delta^{\frac{1}{\ell}})$.*

*Proof.* (a) Set $f(s) := \prod_{j\in\underline{\ell}} (s - z_j)$. Let $K$ be a connected component of $\mathcal{B}(z_1, \dots, z_\ell; \delta)$. Then $K$ is compact. Hence there exists $s_0 \in K$ such that $|f(s_0)| = \min_{s\in K} |f(s)|$. Let $U$ be an open neighborhood of $s_0$ such that $U \cap \mathcal{B}(z_1, \dots, z_\ell; \delta) = U \cap K$. By the definition of $\mathcal{B}(z_1, \dots, z_\ell; \delta)$ we have $|f(s)| > \delta$ for all $s \in U \setminus K$. Thus $|f(s_0)| = \min_{s\in U} |f(s)|$ and this implies $f(s_0) = 0$ since $f$ is holomorphic and nonconstant. Thus $s_0 = z_j$ for some $j \in \underline{\ell}$.

(b) For $i \in \underline{\ell}$ set

$$r_i = \begin{cases} \epsilon & \text{if } i = j, \\ \epsilon^{-\frac{1}{\ell-1}} & \text{otherwise.} \end{cases}$$

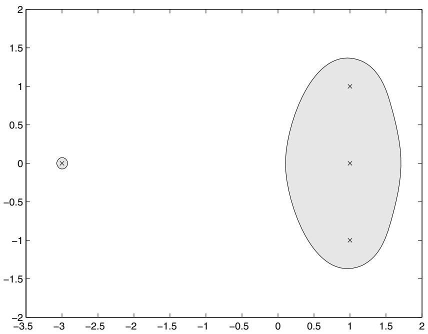



FIG. 4. *The Brualdi set* $\mathcal{B}(-3, 1, 1+\imath, 1-\imath; \delta)$, $\delta = 5$.

Then $\prod_{i\in\underline{\ell}} r_i = 1$ and Proposition 5.9 yields that $\mathcal{B}(z_1, \ldots, z_\ell; \delta) \subseteq \bigcup_{i\in\underline{\ell}} \mathcal{D}(z_i; \delta^{\frac{1}{\ell}} r_i)$. The condition (57) implies that $\mathcal{D}(z_j; \delta^{\frac{1}{\ell}} r_j) \cap \mathcal{D}(z_k; \delta^{\frac{1}{\ell}} r_k) = \emptyset$ for all $k \neq j$. Thus $K_j \subseteq \mathcal{D}(z_j; \delta^{\frac{1}{\ell}} r_j)$. □

Roughly speaking, the above proposition states that if the distance of $z_j \in \mathbb{C}$ from the numbers $z_k \in \mathbb{C}$, $k \neq j$, is large, then the connected component $K_j$ of $\mathcal{B}(z_1, \ldots, z_\ell; \delta)$ is a small set. This is illustrated in Figure 4.

**6. Off-diagonal perturbation structures.** We consider the same basic framework as that in section 5 but now the index set is off-diagonal:

$$\mathcal{I}_{\text{off}} := \{(j,k) \in \underline{m} \times \underline{m}\,;\ j \neq k\}. \tag{58}$$

The corresponding perturbation class $\mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}$ is the set of all $m \times m$ block matrices $\Delta = [\Delta_{jk}]$ such that $\Delta_{jk} \in \mathbb{C}^{l_j \times q_k}$ and $\Delta_{jj} = 0$ for all $j, k \in \underline{m}$. In this section we derive formulae for the corresponding $\mu$-function, spectral value sets, and stability radii.

Recall the following inequality for the geometric mean.

LEMMA 6.1. *Let* $c_1 \geq c_2 \geq \cdots \geq c_\ell \geq 0$. *Then, for all* $k \in \underline{\ell}$, $(\prod_{j=1}^{\ell} c_j)^{\frac{1}{\ell}} \leq (\prod_{j=1}^{k} c_j)^{\frac{1}{k}}$.

For $\mathcal{I} = \mathcal{I}_{\text{off}}$ the following proposition is a special case of Theorem 5.5.

PROPOSITION 6.2. *Let* $M_j \in \mathbb{C}^{q_j \times l_j}$, $j \in \underline{m}$, $M = \bigoplus_{j=1}^{m} M_j$. *Then with respect to the norm* (41),

$$\mu_{\mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,\ell}}(M) = \max_{1\leq j<k\leq m} \sqrt{\|M_j\|_{\beta_j,\alpha_j}\ \|M_k\|_{\beta_k,\alpha_k}}.$$

*Proof.* Each pair $(j,k) \in \underline{m} \times \underline{m}$, $j \neq k$, is a cycle in the graph associated with $\mathcal{I}_{\text{off}}$. Thus

$$\max_{1\leq j<k\leq m} \sqrt{\|M_j\|_{\beta_j,\alpha_j}\ \|M_k\|_{\beta_k,\alpha_k}} \leq \max_{\gamma\in\mathcal{Z}(\mathcal{I}_{\text{off}})} \left(\prod_{j\in\gamma} \|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}}. \tag{59}$$

Each cycle $\gamma \in \mathcal{Z}(\mathcal{I}_{\text{off}})$ has length $|\gamma| \geq 2$. Let $\gamma = (j_1, \ldots, j_\ell)$, and let $j_i, j_r \in \gamma$ be such that $i \neq r$ and $\|M_{j_i}\|_{\beta_{j_i},\alpha_{j_i}} \geq \|M_{j_r}\|_{\beta_{j_r},\alpha_{j_r}} \geq \|M_{j_\nu}\|_{\beta_{j_\nu},\alpha_{j_\nu}}$ for all $\nu \in \underline{\ell} \setminus \{i, r\}$. By Lemma 6.1 we have that $(\prod_{j\in\gamma} \|M_j\|_{\beta_j,\alpha_j})^{\frac{1}{|\gamma|}} \leq \sqrt{\|M_{j_i}\|_{\beta_{j_i},\alpha_{j_i}} \|M_{j_r}\|_{\beta_{j_r},\alpha_{j_r}}}$. Thus, equality holds in (59). Now, the proposition follows from Theorem 5.5. □

An analogous result holds if the underlying norm on $\mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}$ is the operator norm induced by $p$-norms on $\mathbb{C}^{l_1+\cdots+l_m}$ and $\mathbb{C}^{q_1+\cdots+q_m}$. The corresponding $\mu$-function will be denoted by $\mu^{(p)}_{\mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}}(\cdot)$. To prove the result we need the following lemma.

LEMMA 6.3. *Let $m_j, n_j \in \mathbb{N}$, $j = 1, 2$, and suppose $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are absolute norms on $\mathbb{C}^{m_1+m_2}$ and $\mathbb{C}^{n_1+n_2}$, respectively. If $X \in \mathbb{C}^{m_1\times n_2}$, $Y \in \mathbb{C}^{m_2\times n_1}$, and $Z \in \mathbb{C}^{m_2\times n_2}$, then*

$$\left\| \begin{bmatrix} 0 & tX \\ tY & Z \end{bmatrix} \right\|_{\beta,\alpha} \le t \left\| \begin{bmatrix} 0 & X \\ Y & Z \end{bmatrix} \right\|_{\beta,\alpha}, \quad t \ge 1.$$

*Proof.* The function

$$\zeta \mapsto f(\zeta) := \left\| \begin{bmatrix} 0 & X \\ Y & \zeta Z \end{bmatrix} \right\|_{\beta,\alpha}$$

is convex on $\mathbb{R}$, and since $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are absolute, we have that

$$f(-\zeta) = \left\| \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \begin{bmatrix} 0 & X \\ Y & \zeta Z \end{bmatrix} \begin{bmatrix} -I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \right\|_{\beta,\alpha} = f(\zeta)$$

for all $\zeta \in \mathbb{R}$. From this it follows that $f$ is a nondecreasing function on $[0, \infty)$. Thus for all $t \ge 1$,

$$\left\| \begin{bmatrix} 0 & tX \\ tY & Z \end{bmatrix} \right\|_{\beta,\alpha} = tf\left(\frac{1}{t}\right) \le tf(1) = t \left\| \begin{bmatrix} 0 & X \\ Y & Z \end{bmatrix} \right\|_{\beta,\alpha}. \qquad \square$$

In the following theorem $\|\cdot\|_p$ denotes the operator norm induced by $p$-norms on the corresponding vector spaces.

THEOREM 6.4. *Suppose $p \in [1, \infty]$, $M_j \in \mathbb{C}^{q_j\times l_j}$ for $j \in \underline{m}$ and $M = \bigoplus_{j=1}^m M_j$. Then*

$$\mu^{(p)}_{\mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}}\left(\bigoplus_{j=1}^m M_j\right) = \max_{1\le j<k\le m} \sqrt{\|M_j\|_p \, \|M_k\|_p}\,.$$

*Proof.* Without loss of generality we may assume that $\|M_1\|_p \ge \|M_2\|_p \ge \|M_j\|_p$ for $j \ge 3$. Let $\Delta \in \mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}$. Then $\Delta = \left[\begin{smallmatrix} 0 & X \\ Y & Z \end{smallmatrix}\right]$ for some $X \in \mathbb{C}^{l_1\times Q'}, Y \in \mathbb{C}^{L'\times q_1}$, $Z \in \mathbb{C}^{L'\times Q'}$, where $Q' = \sum_{j=2}^m q_j$, $L' = \sum_{j=2}^m l_j$. Suppose first that $M_2 = 0$. Then all eigenvalues of $\Delta M = \left[\begin{smallmatrix} 0 & 0 \\ YM_1 & 0 \end{smallmatrix}\right]$ are zero for all $\Delta \in \mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}$. Consequently, $\mu^{(p)}_{\mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}}(M) = 0 = \sqrt{\|M_1\|_p\|M_2\|_p}$. Suppose now that $M_2 \ne 0$ and let

$$F_t := \operatorname{diag}(tI_{q_1}, I_{Q'}), \qquad G_t := \operatorname{diag}(tI_{l_1}, I_{L'}), \qquad t := \sqrt{\frac{\|M_1\|_p}{\|M_2\|_p}} \ge 1.$$

Then $\|t^{-2}M_1\|_p = \|M_2\|_p$ and therefore

$$\|F_t^{-1}MG_t^{-1}\|_p = \|\operatorname{diag}\left(t^{-2}M_1, M_2, M_3, \ldots, M_m\right)\|_p = \|M_2\|_p.$$

By Lemma 6.3 $\|G_t\Delta F_t\|_p \le t\,\|\Delta\|_p$ for all $\Delta \in \mathbf{\Delta}_{\mathcal{I}_{\text{off}},q,l}$. Suppose that $\|\Delta\|_p = 1$. Then we have

$$\begin{aligned} \varrho(\Delta M) &\le \|G_t\Delta MG_t^{-1}\|_p = \|(G_t\Delta F_t)(F_t^{-1}MG_t^{-1})\|_p \\ &\le \|G_t\Delta F_t\|_p\,\|F_t^{-1}MG_t^{-1}\|_p = \|G_t\Delta F_t\|_p\,\|M_2\|_p \\ &\le \|\Delta\|_p\,t\,\|M_2\|_p = \sqrt{\|M_1\|_p\|M_2\|_p}\,. \end{aligned}$$

Therefore $\mu^{(p)}_{\mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l}}(M) \leq \sqrt{\|M_1\|_p \|M_2\|_p}$. To see that equality holds, let $u_j \in \mathbb{C}^{l_j}$, $y_j \in \mathbb{C}^{q_j}$, $j = 1, 2$, be such that $\|u_j\|_p = \|y_j\|_p^D = 1$ and $y_j^\top M_j u_j = \|M_j\|_p$, $j = 1, 2$. Define

$$\Delta_0 := \operatorname{diag}\left(\begin{bmatrix} 0 & u_1 y_2^\top \\ u_2 y_1^\top & 0 \end{bmatrix}, 0\right) \in \mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l}\ , \qquad u := \begin{bmatrix} \sqrt{\|M_2\|_p}\, u_1 \\ \sqrt{\|M_1\|_p}\, u_2 \\ 0 \end{bmatrix} \in \mathbb{C}^{l_1+\cdots+l_m}.$$

Then $\|\Delta_0\|_p = 1$, and an easy calculation yields $\Delta_0 M u = \sqrt{\|M_1\|_p \|M_2\|_p}\, u$. Thus $\mu^{(p)}_{\mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l}}(M) \geq \varrho(\Delta_0 M) \geq \sqrt{\|M_1\|_p \|M_2\|_p}$, and this concludes the proof. ☐

We have not been able to extend Theorem 6.4 to arbitrary index sets $\mathcal{I} \subseteq \underline{m} \times \underline{m}$ and so we formulate a conjecture in this respect as an open question.

*Open question*: Does the identity

$$\mu^{(p)}_{\mathbf{\Delta}_{\mathcal{I},q,l}}\left(\bigoplus_{j=1}^m M_j\right) = \max_{\gamma \in \mathcal{Z}(\mathcal{I})} \left(\prod_{j\in\gamma} \|M_j\|_p\right)^{\frac{1}{|\gamma|}}$$

hold for arbitrary index sets $\mathcal{I} \subseteq \underline{m} \times \underline{m}$?

As corollaries of Proposition 6.2 and Theorem 6.4 we obtain the following formulae for spectral value sets and stability radii. The first corollary deals with the general block-diagonal case, and the second deals with the (scalar) diagonal case.

COROLLARY 6.5. *Suppose* $(A_j, B_j, C_j) \in L_{n_j,l_j,q_j}$, $j \in \underline{m}$, $A = \oplus_{j=1}^m A_j$, $B = \oplus_{j=1}^m B_j$, $C = \oplus_{j=1}^m C_j$, $\delta > 0$, *and* $\mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l}$ *is provided with the norm* (41) *or with the operator norm induced by some p-norm,* $1 \leq p \leq \infty$.

*Let* $G_j(s) = C_j(sI_{n_j} - A_j)^{-1}B_j$, $j \in \underline{m}$, *and define* $\|G_i(s)\| = \|G_i(s)\|_{\beta_i,\alpha_i}$ *or* $\|G_i(s)\| = \|G_i(s)\|_p$, $i \in \underline{m}$, *respectively. Then the following hold:*

(a) *The spectral value set of A with respect to perturbations of the form*

$$A \rightsquigarrow A_\Delta = A + B\Delta C, \quad \Delta \in \mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l},\ \|\Delta\| < \delta, \tag{60}$$

*is given by*

$$\begin{aligned} &\bigcup_{\substack{\Delta\in\mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l} \\ \|\Delta\|<\delta}} \sigma(A_\Delta) \\ &= \sigma(A) \cup \left\{ s \in \rho(A);\ \max_{1\leq j<k\leq m} \sqrt{\|G_j(s)\|\|G_k(s)\|} > \delta^{-1} \right\}. \end{aligned} \tag{61}$$

(b) *Let* $\mathbb{C}_g$ *be an open subset of* $\mathbb{C}$ *and suppose* $A_1, \ldots, A_m$ *are* $\mathbb{C}_g$*-stable. Then the stability radius is given by*

$$\begin{aligned} r_{\mathbf{\Delta}_{\mathcal{I}_{\rm off},q,l}}(A,B,C;\mathbb{C}_g) &= \left[\sup_{s\in\partial\mathbb{C}_g} \max_{1\leq j<k\leq m} \sqrt{\|G_j(s)\|\|G_k(s)\|}\right]^{-1} \\ &\geq \min_{1\leq j<k\leq m} \sqrt{r_{\mathbb{C}}(A_j,B_j,C_j;\mathbb{C}_g)\, r_{\mathbb{C}}(A_k,B_k,C_k;\mathbb{C}_g)}. \end{aligned} \tag{62}$$

*Proof.* Making use of Theorem 2.5 and Proposition 6.2 (resp., Theorem 6.4), the corollary can be proved in a way similar to that of Corollary 5.6(a),(c). ☐

COROLLARY 6.6. *Suppose $a_j, b_j, c_j \in \mathbb{C}$, $j \in \underline{n}$, $A = \operatorname{diag}(a_1, \ldots, a_n)$, $B = \operatorname{diag}(b_1, \ldots, b_n)$, $C = \operatorname{diag}(c_1, \ldots, c_n)$, and $\mathcal{N}(\cdot)$ is an arbitrary operator norm on $\mathbb{C}^{n\times n}$. Let*

$$\mathbf{\Delta}_{\mathcal{I}_{\text{off}}} := \left\{[\Delta_{jk}]\in\mathbb{C}^{n\times n}\,;\ \Delta_{11} = \cdots = \Delta_{nn} = 0\right\}$$

*be provided with the norm $\|\cdot\| = \|\cdot\|_p$, $1 \le p \le \infty$, or*

$$\|\Delta\| := \mathcal{N}(|\Delta|) = \mathcal{N}\left(\begin{bmatrix} |\Delta_{11}| & \ldots & |\Delta_{1n}| \\ \vdots & & \vdots \\ |\Delta_{n1}| & \ldots & |\Delta_{nn}| \end{bmatrix}\right), \quad \Delta \in \mathbf{\Delta}_{\mathcal{I}_{\text{off}}}, \tag{63}$$

$$A_\Delta := A + B\Delta C.$$

*If $A = \operatorname{diag}(a_1, \ldots, a_n)$, $B = \operatorname{diag}(b_1, \ldots, b_n)$, $C = \operatorname{diag}(c_1, \ldots, c_n)$, then the following statements hold:*

(a) *For all $\delta > 0$,*

$$\bigcup_{\Delta\in\,\mathbf{\Delta}_{\mathcal{I}_{\text{off}}},\,\|\Delta\|\le\delta} \sigma(A_\Delta) = \Big\{s \in \mathbb{C};\ \min_{1\le j<k\le n} |s-a_j||s-a_k| \le \delta^2 |b_j c_j b_k c_k|\Big\}. \tag{64}$$

(b) *If $\mathbb{C}_g$ is an open subset of $\mathbb{C}$ and $a_1, \ldots, a_n \in \mathbb{C}_g$, then*

$$r_{\mathbf{\Delta}_{\mathcal{I}_{\text{off}}}}(A, B, C; \mathbb{C}_g) = \inf_{s\in\partial\mathbb{C}_g} \min_{1\le j<k\le n} \left(\frac{|s-a_j|}{|b_j c_j|}\frac{|s-a_k|}{|b_k c_k|}\right)^{1/2}. \tag{65}$$

*Proof.* Making use of Theorem 2.5 and Proposition 6.2 (resp., Theorem 6.4) the corollary can be proved in a way similar to that of Corollary 5.7. ∎

If $\mathbb{C}_g = \mathbb{C}_-$ and $a_1, \ldots, a_n < 0$, then $|\imath\omega - a_j| \ge |a_j|$ for all $\omega \in \mathbb{R}$, $j \in \underline{n}$, so that (65) implies

$$r_{\mathbf{\Delta}_{\mathcal{I}_{\text{off}}}}(A, B, C; \mathbb{C}_-) = \min_{1\le j<k\le n} \left(\frac{|a_j a_k|}{|b_j c_j b_k c_k|}\right)^{1/2}.$$

**7. Application: Inclusion theorems.** An arbitrary matrix $A = [a_{jk}]\in\mathbb{C}^{n\times n}$ can be represented as a perturbation of the diagonal matrix $D_A = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ by an off-diagonal perturbation matrix $\Delta_A$:

$$A = D_A + \Delta_A, \text{ where } \Delta_A = A - D_A \in \mathbf{\Delta}_{\mathcal{I}_{\text{off}}} := \left\{[\Delta_{jk}]\in\mathbb{C}^{n\times n}\,;\ \Delta_{11} = \cdots = \Delta_{nn} = 0\right\}.$$

Hence, setting

$$\mathcal{I} := \mathcal{I}_A \cap \mathcal{I}_{\text{off}} = \{(j,k) \in \underline{n}\times\underline{n}\,;\ j \ne k \text{ and } a_{jk} \ne 0\}, \tag{66}$$

we have by Remark 2.6(ii) that

$$\sigma(A) \subseteq \bigcup_{\Delta\in\mathbf{\Delta}_{\mathcal{I}},\,\|\Delta\|\le\|\Delta_A\|} \sigma(D_A + \Delta) = \overline{\sigma_{\mathbf{\Delta}_{\mathcal{I}}}(D_A, I_n, I_n; \|\Delta_A\|)}. \tag{67}$$

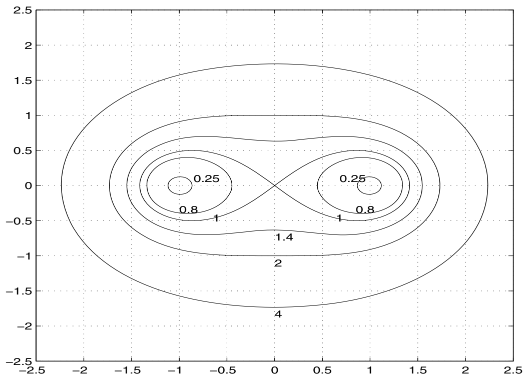



FIG. 5. *The ovals of Cassini* $\mathcal{C}(1,-1;\rho)$, $\rho = 0.25, 0.8, 1, 1.4, 2, 4$.

Applying the previous results about spectral value sets of diagonal matrices, one obtains different estimates for the location of the spectrum of $A$ (depending on whether one chooses the perturbation norm to be (36) or (63)). In this section we recall the classical eigenvalue inclusion theorems of Gershgorin, Brauer, and Brualdi and show how they can be obtained as corollaries of the results in the previous sections.

Gershgorin's theorem states that for all $A = [a_{jk}] \in \mathbb{C}^{n\times n}$

$$\sigma(A) \subset \mathcal{G}_A := \bigcup_{j\in\underline{n}} \mathcal{D}(a_{jj}; R_j(A)), \quad \text{where } R_j(A) := \sum_{k\in\underline{n}, k\neq j} |a_{jk}|. \tag{68}$$

Gershgorin's theorem was improved by Brauer [3]. He used inclusion regions for the eigenvalues of the following type:

$$\mathcal{C}(z_1, z_2; \rho) := \{ s\in\mathbb{C};\, |s - z_1|\,|s - z_2| \le \rho \,\}, \qquad z_1, z_2 \in \mathbb{C},\ \rho \ge 0.$$

The sets $\mathcal{C}(z_1, z_2; \rho)$ and their boundaries are called the *ovals of Cassini*. For an illustration, see Figure 5. Brauer's theorem states that

$$\sigma(A) \subseteq \mathcal{C}_A := \bigcup_{1\le j<k\le n} \mathcal{C}(a_{jj}, a_{kk}; R_j(A)R_k(A)), \quad A = [a_{jk}] \in \mathbb{C}^{n\times n}. \tag{69}$$

A further refinement has been obtained by Brualdi [4] who gave more precise information about the location of the eigenvalues by taking into account the zero structure of $A$. For this he introduced sets of the form (52) which now carry his name. With every matrix $A = [a_{jk}] \in \mathbb{C}^{n\times n}$ we associate the following union of Brualdi sets:

$$\mathcal{B}_A := \bigcup_{(j_1,\ldots,j_\ell)\in\mathcal{Z}(\mathcal{I})} \mathcal{B}\left(a_{j_1j_1}, \ldots, a_{j_\ell j_\ell}; \textstyle\prod_{i=1}^{\ell} R_{j_i}(A)\right), \tag{70}$$

where $R_j$ are as in (68) and $\mathcal{I} := \mathcal{I}_A \cap \mathcal{I}_{\text{off}}$; see (66). Brualdi's theorem states that

$$\sigma(A) \subseteq \mathcal{B}_A \tag{71}$$

provided that each index $j\in\underline{n}$ is contained in some cycle $\gamma\in\mathcal{Z}(\mathcal{I})$. From Corollary 5.7 we obtain the following slight extension of this result.

COROLLARY 7.1. *Let $A\in\mathbb{C}^{n\times n}$ and set*

$$\begin{aligned}\sigma_0(A) &:= \{a_{jj}\,;\, j\in\underline{n} \text{ and } \forall\gamma\in\mathcal{Z}(\mathcal{I}) : j\notin\gamma\}\\ &= \{a_{jj}\,;\, j\in\underline{n} \text{ and } (j\in\gamma\in\mathcal{Z}(\mathcal{I}_A)\Rightarrow\gamma=(j))\}.\end{aligned}$$

*Then $\sigma_0(A)\subseteq\sigma(A)$ and $\sigma(A)\setminus\sigma_0(A)\subseteq\mathcal{B}_A$. In particular, if $\sigma_0(A)=\emptyset$, then $\sigma(A)\subseteq\mathcal{B}_A$.*

*Proof.* If $A\in\mathbb{C}^{n\times n}$ is a diagonal matrix, then there is nothing to prove. Assume that $A$ is nondiagonal and has off-diagonal row sums $R_j(A)$, $j\in\underline{n}$. Set $\tilde\Delta=[\tilde\Delta_{jk}]\in\mathbb{C}^{n\times n}$, where

$$\tilde\Delta_{jk} := \begin{cases} R_j(A)^{-1}a_{jk} & \text{if } j\neq k \text{ and } R_j(A)\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then $A=A_{\tilde\Delta}:=\operatorname{diag}(a_{11},\dots,a_{nn})+\operatorname{diag}(R_1(A),\dots,R_n(A))\,\tilde\Delta$, and $\|\tilde\Delta\|_1=1$. Note that $\|\Delta\|_1=\|\,|\Delta|\,\|_1$ is a norm of the form (63). Furthermore, we have that $\tilde\Delta\in\mathbf{\Delta}_{\mathcal{I}}$, where $\mathcal{I}=\mathcal{I}_A\cap\mathcal{I}_{\text{off}}$. Thus

$$\sigma(A)\subseteq\bigcup_{\Delta\in\mathbf{\Delta}_{\mathcal{I}},\,\|\Delta\|_1\le 1}\sigma(A_\Delta),$$
$$\text{where } A_\Delta:=\operatorname{diag}(a_{11},\dots,a_{nn})+\operatorname{diag}(R_1(A),\dots,R_n(A))\,\Delta.$$

Hence, applying Corollary 5.7 with the norm $\mathcal{N}(\cdot)=\|\cdot\|_1$, $\delta=1$, $b_i=R_i(A)$, and $c_i=1$, $i\in\underline{n}$, we obtain the result: $\sigma_0(A)\subseteq\sigma(A)$ follows from (b) and $\sigma(A)\backslash\sigma_0(A)\subseteq\mathcal{B}_A$ follows from (a). ◻

An equivalent extension of Brualdi's result can be found in [29, Theorem 2.5] (note that the definition of $\mathcal{B}_A$ in [29] is different from ours).

We conclude this paper with a brief discussion of the relationship between the above results of Gershgorin, Brauer, and Brualdi. First note that $\mathcal{B}(z_1;\rho)=\mathcal{D}(z_1;\rho)$ and $\mathcal{B}(z_1,z_2;\rho)=\mathcal{C}(z_1,z_2;\rho)$. The following proposition yields a useful tool for establishing inclusion relations between these sets.

PROPOSITION 7.2. *Let $z_1,\dots,z_\ell\in\mathbb{C}$ and $\rho_1,\dots,\rho_\ell\ge 0$. Then*

$$\mathcal{B}\left(z_1,\dots,z_\ell;\textstyle\prod_{j\in\underline{\ell}}\rho_j\right)\subseteq\displaystyle\bigcup_{k\in\underline{\ell}}\mathcal{B}\left(z_1,\dots,\widehat{z}_k,\dots,z_\ell;\textstyle\prod_{j\in\underline{\ell},\,j\neq k}\rho_j\right),$$

*where* $\mathcal{B}\left(z_1,\dots,\widehat{z}_k,\dots,z_\ell;\prod_{j\in\underline{\ell},\,j\neq k}\rho_j\right)=\left\{s\in\mathbb{C}\,;\ \prod_{j\in\underline{\ell},\,j\neq k}|s-z_j|\le\prod_{j\in\underline{\ell},j\neq k}\rho_j\right\}.$

*Proof.* Suppose that $s\notin\bigcup_{k\in\underline{\ell}}\mathcal{B}(z_1,\dots,\widehat{z}_k,\dots,z_\ell;\prod_{j\in\underline{\ell},\,j\neq k}\rho_j)$. Then we have, for all $k\in\underline{\ell}$, $\prod_{j\in\underline{\ell},\,j\neq k}|s-z_j|>\prod_{j\in\underline{\ell},\,j\neq k}\rho_j$. By multiplying these $\ell$ inequalities we obtain $(\prod_{j\in\underline{\ell}}|s-z_j|)^{\ell-1}>(\prod_{j\in\underline{\ell}}\rho_j)^{\ell-1}$. Thus $s\notin\mathcal{B}(z_1,\dots,z_\ell;\prod_{j\in\underline{\ell}}\rho_j)$. ◻

By induction we obtain from Proposition 7.2 the following corollary.

COROLLARY 7.3. *Let $z_1,\dots,z_\ell\in\mathbb{C}$ and $\rho_1,\dots,\rho_\ell\ge 0$. Then*

$$\mathcal{B}\left(z_1,\dots,z_\ell;\textstyle\prod_{j\in\underline{\ell}}\rho_j\right)\subseteq\displaystyle\bigcup_{1\le j<k\le\ell}\mathcal{C}(z_j,z_k;\rho_j\rho_k)\subseteq\bigcup_{j\in\underline{\ell}}\mathcal{D}(z_j;\rho_j).$$

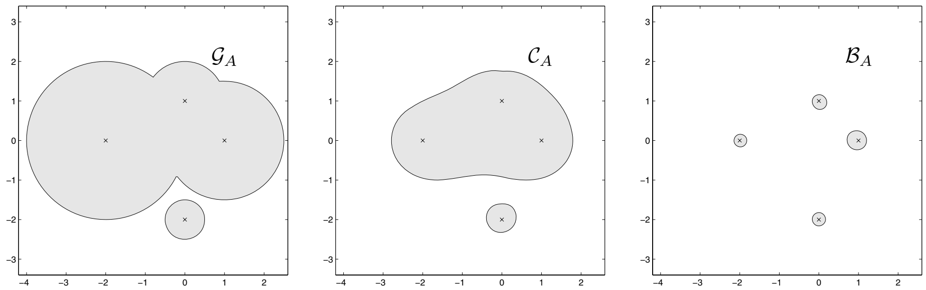



FIG. 6. *Comparison of the regions $\mathcal{G}_A$, $\mathcal{C}_A$, and $\mathcal{B}_A$.*

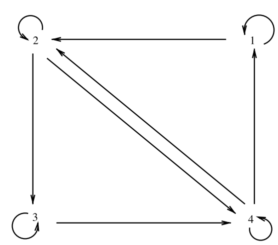
Corollary 7.3 implies that for all $A \in \mathbb{C}^{n\times n}$, $n \geq 2$,

$$\mathcal{B}_A \subseteq \mathcal{C}_A \subseteq \mathcal{G}_A. \tag{72}$$

Thus the theorems of Brauer and Gershgorin are consequences of Corollary 7.1. The first inclusion in (72) has been shown by Varga [28], the second by Brauer [3]. Note that each of the three sets, $\mathcal{B}_A, \mathcal{C}_A, \mathcal{G}_A$, is closed (as a finite union of closed sets).

*Example* 7.4. Consider the following matrix and the corresponding incidence graph:

$$A = \begin{bmatrix} 1 & 1.5 & 0 & 0 \\ 0 & \imath & 0.5 & -0.5\imath \\ 0 & 0 & -2 & 2 \\ 0.25 & 0.25\imath & 0 & -2\imath \end{bmatrix}$$




The matrix $A$ can be represented as a sum of the diagonal matrix $A_0 = \mathrm{diag}(1, \imath, -2, -2\imath)$ and the off-diagonal perturbation $R_3 \in \mathcal{I}_{\rm off}$ defined in Example 4.5. Figure 6 illustrates the eigenvalue inclusion regions for $A$ due to Gershgorin, Brauer, and Brualdi.

The crosses mark the diagonal elements of $A$. Comparing the right-hand figures in Figures 2 and 6, we see that the inclusion provided by Corollary 4.4(a) is somewhat tighter than the estimate provided by Brualdi's theorem; see (71).

The above example should not convey the impression that Brualdi sets are always considerably smaller then the corresponding Brauer sets. In fact, one can easily see that they are equal if all off-diagonal entries of $A$ are nonzero, i.e., $\mathcal{I}_A \supset \mathcal{I}_{\rm off}$. For more details on this, see [29, section 2.3].

Although the theorems of Gershgorin, Brauer, and Brualdi follow directly from our main results, we emphasize that the problems underlying the inclusion theorems and those underlying our results are quite different. The inclusion theorems consider the matrix $A = D_A + \Delta_A$ as *given* and establish *upper bounds* for $\sigma(A)$ viewing $A$ as the result of a (known) off-diagonal perturbation of $D_A$. On the contrary, Corollaries 5.7 and 4.4 provide precise formulae for the union of the spectra of *all* the matrices $A_\Delta = D_A + \Delta$ where $\Delta$ is an arbitrary complex matrix of norm $\leq \delta$ with the zero structure determined by $\mathcal{I}$ (resp., an arbitrary complex matrix satisfying $|\Delta| \le R$). In these corollaries the diagonal matrix $D_A$, the index set $\mathcal{I}$, and the uncertainty level $\delta > 0$ (resp., the diagonal matrix $D_A$ and the nonnegative matrix $R$) are the only data. It follows from Corollary 6.6 that

(73)
$$\sigma(A) \subseteq \bigcup_{\substack{\Delta \in \boldsymbol{\Delta}_{\mathcal{I}_{\text{off}}} \\ \|\Delta\|_1 \le 1}} \sigma(D_A + B\Delta) = \left\{ s \in \mathbb{C};\ \min_{1 \le j < k \le m} |s - a_{jj}||s - a_{kk}| \le R_j(A) R_k(A) \right\} = \mathcal{C}_A,$$

where $B = \operatorname{diag}(R_1(A), \dots, R_n(A))$; see (69). Under the assumptions of Brualdi's theorem we have

(74)
$$\sigma(A) \subseteq \bigcup_{\substack{\Delta \in \boldsymbol{\Delta}_{\mathcal{I}} \\ \|\Delta\|_1 \le 1}} \sigma(D_A + B\Delta) = \bigcup_{(j_1, \dots, j_\ell) \in \mathcal{Z}(\mathcal{I})} \mathcal{B}\left( a_{j_1}, \dots, a_{j_\ell}; \textstyle\prod_{i=1}^{\ell} R_i(A) \right) = \mathcal{B}_A,$$

where $\mathcal{I} := \mathcal{I}_A \cap \mathcal{I}_{\text{off}}$; see the proof of Corollary 7.1, (54), and (70). Hence the upper bounds in the inclusion theorems of Brauer and Brualdi, respectively, are *tight* estimates which cannot be improved if we presuppose as the only a priori knowledge the diagonal of $A$ and the off-diagonal row sums $R_j(A)$ (resp., the diagonal of $A$, the zero pattern of $A$, and the off-diagonal row sums $R_j(A)$). To make this more precise we note that

$$\{D_A + B\Delta;\ \Delta \in \boldsymbol{\Delta}_{\mathcal{I}_{\text{off}}} \text{ and } \|\Delta\|_1 \le 1\}$$

is the set of all matrices $\tilde{A} \in \mathbb{C}^{n \times n}$ with the same diagonal as $A$ and with off-diagonal row sums $R_j(\tilde{A}) \le R_j(A)$. By (73) the Brauer set $\mathcal{C}_A$ is exactly the union of the spectra of all these matrices. Similarly,

$$\{D_A + B\Delta;\ \Delta \in \boldsymbol{\Delta}_{\mathcal{I}} \text{ and } \|\Delta\|_1 \le 1\}$$

is the set of all matrices $\tilde{A} \in \mathbb{C}^{n \times n}$ with the same diagonal as $A$, with row sums $R_j(\tilde{A}) \le R_j(A)$, and with $\mathcal{I}_{\tilde{A}} \subset \mathcal{I}_A$. Under the assumptions of Brualdi's theorem it follows from (74) that the Brualdi set $\mathcal{B}_A$ is exactly the union of the spectra of all these matrices. A more detailed discussion on the sharpness of the Brualdi inclusion theorem can be found in [29, section 2.4].

*Remark* 7.5. In the same way as in the proof of Corollary 5.7 one could derive from Corollaries 5.6 and 6.5 inclusion theorems for the eigenvalues of a block matrix. Such results are obtained in [29] by a different approach from ours.

## REFERENCES

[1] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *$\mu$-Analysis and Synthesis Toolbox. User's Guide*, The MathWorks, South Natick, MA, 1991.

[2] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics Appl. Math. 9, SIAM, Philadelphia, 1994.

[3] A. Brauer, *Limits for the characteristic roots of a matrix.* II, Duke Math. J., 14 (1947), pp. 21–26.

[4] R. A. Brualdi, *Matrices, eigenvalues, and directed graphs*, Linear and Multilinear Algebra, 11 (1982), pp. 143–165.

[5] N. A. Bruinsma and M. Steinbuch, *A fast algorithm to compute the $H^\infty$-norm of a transfer function matrix*, Systems Control Lett., 14 (1990), pp. 287–293.

[6] M. J. Chen, K. N. Fan, and C. N. Nett, *Structured singular values with nondiagonal structures.* I. *Characterizations; Part* II. *Computation*, IEEE Trans. Automat. Control, 41 (1996), pp. 1507–1516.
[7] J. C. Doyle, *Analysis of feedback systems with structured uncertainties*, Proc. IEE-D, 129 (1982), pp. 242–250.
[8] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory*, Texts Appl. Math. 36, Springer-Verlag, New York, 2000.
[9] E. Gallestey, *Computing spectral value sets using the subharmonicity of the norm of rational matrices*, BIT, 38 (1998), pp. 22–33.
[10] F. R. Gantmacher, *The Theory of Matrices*, Vol. 2, Chelsea, New York, 1959.
[11] N. J. Higham and F. Tisseur, *A block algorithm for matrix* 1*-norm estimation, with an application to* 1*-norm pseudospectra*, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 1185–1201.
[12] D. Hinrichsen and B. Kelb, *Spectral value sets: A graphical tool for robustness analysis*, Systems Control Lett., 21 (1993), pp. 127–136.
[13] D. Hinrichsen and A. J. Pritchard, *Real and complex stability radii: A survey*, in Control of Uncertain Systems, D. Hinrichsen and B. Mårtensson, eds., Birkhäuser-Verlag, Basel, Switzerland, 1990, pp. 119–162.
[14] D. Hinrichsen and A. J. Pritchard, *On the robustness of stable discrete-time linear systems*, in New Trends in Systems Theory (Genoa, 1990), Birkhäuser-Verlag, Basel, Switzerland, 1991, pp. 393–400.
[15] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory.* I. *Modelling, State Space Analysis, Stability and Robustness*, Springer-Verlag, Berlin, 2005.
[16] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.
[17] M. Karow, *Geometry of Spectral Value Sets*, Ph.D. thesis, University of Bremen, Bremen, Germany, 2003.
[18] T. C. Lawrence, A. L. Tits, and P. Van Dooren, *A fast algorithm for the computation of an upper bound on the $\mu$-norm*, Automatica, 36 (2000), pp. 449–456.
[19] A. Packard and J. C. Doyle, *The complex structured singular value*, Automatica J. IFAC, 29 (1993), pp. 71–109.
[20] L. Qiu, *On the robustness of symmetric systems*, Systems Control Lett., 27 (1996), pp. 187–190.
[21] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. C. Doyle, *A formula for computation of the real stability radius*, Automatica J. IFAC, 31 (1995), pp. 879–890.
[22] S. C. Reddy, P. J. Schmid, and D. S. Henningson, *Pseudospectra of the Orr–Sommerfeld operator*, SIAM J. Appl. Math., 53 (1993), pp. 15–47.
[23] M. G. Safonov. *Tight bounds on the response of multivariable systems with component uncertainty*, in 16th Annual Allerton Conference on Communication, Control, and Computing, University of Illinois, Urbana, IL, 1978, pp. 451–460.
[24] M. G. Safonov and K.-H. M. Fan, *Special issue: Multivariable stability margin*, Internat. J. Robust Nonlinear Control, 7 (1997), pp. 97–226.
[25] D. D. Šiljak, *Large Scale Dynamic Systems. Stability and Structure*, North–Holland Ser. System Sci. Engrg., North–Holland, New York, 1978.
[26] L. N. Trefethen, *Computation of pseudospectra*, Acta Numer., 8 (1999), pp. 247–295.
[27] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*, Princeton University Press, Princeton, NJ, 2005.
[28] R. S. Varga, *Geršgorin-type eigenvalue inclusion theorems and their sharpness*, Electron. Trans. Numer. Anal., 12 (2001), pp. 113–133.
[29] R. S. Varga, *Geršgorin and His Circles*, Springer Ser. Comput. Math. 36, Springer-Verlag, Berlin, 2004.