Stability Estimates for the Helmholtz Inverse Problem and Seismic Reconstruction

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Workshop on Inverse Problems for PDEs

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Universität Bremen

Zentrum für Technomathematik

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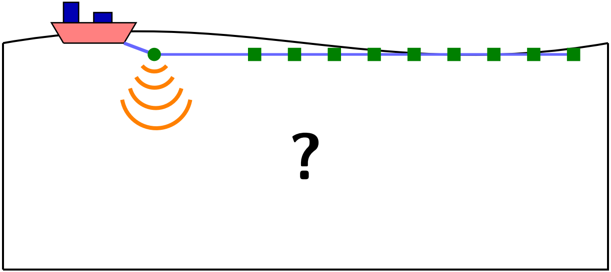
Inverse problem to image the subsurface

We aim at the reconstruction of subsurface area of Earth from the observation of waves propagation; Geophysical context.



Overview

Plan

Helmholtz equation with Dirichlet BC

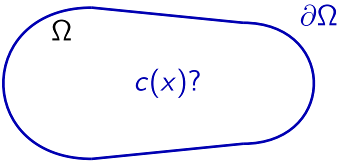
$$\begin{cases} \left(-\dfrac{\omega^2}{c(x)^2} - \Delta\right) u(x,\omega) = 0 & \text{on } \Omega, \\ u(x,\omega) = g(x,\omega) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

- $c(x)$ the wavespeed
- $u(x,\omega)$ the wavefield
- ω the angular frequency
- $q = \omega^2/c(x)^2$ the potential

Helmholtz equation with Dirichlet BC

$$\begin{cases} \left(-\dfrac{\omega^2}{c(x)^2} - \Delta\right) u(x,\omega) = 0 & \text{on } \Omega, \\ u(x,\omega) = g(x,\omega) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

- Considering subsurface area and unknown wavespeed $c(x)$
- From observation of the wavefield u over the boundary (possibly partial), the inverse problem aims the recovery of c



Data: Dirichlet-to-Neumann map

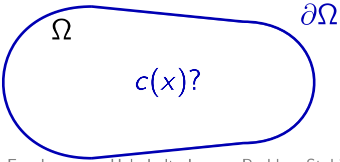
$$\begin{cases} \left(-\dfrac{\omega^2}{c^2} - \Delta\right) u = 0 & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The data: DtoN map Λ: $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$:

$$\Lambda_q : g \longrightarrow \frac{\partial u}{\partial \nu} |_{\partial\Omega}$$

The Forward operator F: $L^2(\Omega) \to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$:

$$F_\omega(c^{-2}) = \Lambda_q.$$



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J. Sylvester and G. Uhlmann
A global uniqueness theorem for an inverse boundary value problem
Annals of Mathematics 1987

A. I. Nachman
Reconstructions from boundary measurements
Annals of Mathematics 1988

A. I. Nachman
Global uniqueness for a two-dimensional inverse boundary problem
Annals of Mathematics 1996

Plan

2. Stability constant formulation
   - Conditional stability
   - Bounds of the stability constant

Stability of the Helmholtz Inverse Problem

$$\|c_1^{-2} - c_2^{-2}\| \leq \mathcal{C}\left(\|F(c_1^{-2}) - F(c_2^{-2})\|\right)$$

G. Alessandrini
Stable determination of conductivity by boundary measurement
Applicable Analysis 1988

N. Mandache
Exponential instability in an inverse problem for Schrödinger equation
Inverse Problems 2001

Stability of the Helmholtz Inverse Problem

$$\|\underbrace{c_1^{-2}}_{\text{target}} - \underbrace{c_2^{-2}}_{\text{initial model}}\| \leq \mathcal{C}\left(\overbrace{\|\underbrace{F(c_1^{-2})}_{\text{observation}} - \underbrace{F(c_2^{-2})}_{\text{simulation}}\|}^{\delta}\right)$$

- Stability associate data and model correspondence
- Reconstruction is based on the iterative minimization of the difference between observation and simulation using an initial model.

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- Stability associate data and model correspondence
- $\mathcal{C}(\delta) \leq C\big(\log(1+\delta^{-1})\big)^{-\alpha}$

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Stable determination of conductivity by boundary measurement
Applicable Analysis 1988

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Inverse Problems 2001

Conditional Lipschitz stability: assumptions

- $c(x)$ is bounded $B_1 \leq c^{-2}(x) \leq B_2$ in Ω
- $c(x)$ has a piecewise constant representation of size N

$$c(x)^{-2} = \sum_{k=1}^{N} c_k \chi_k(x)$$

- Ω has Lipschitz boundary

$$\|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} \leq \mathcal{C} \, \|F(c_1^{-2}) - F(c_2^{-2})\| \qquad (2)$$

G. Alessandrini and S. Vessella
Lipschitz stability for the inverse conductivity problem
Advances in Applied Mathematics 2005

E. Beretta, M. V. de Hoop, F. and O. Scherzer
Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates. 2016

Formulation

The stability constant is bounded

$$\frac{1}{4\omega^2} e^{K_1 N^{1/5}} \leq \mathcal{C} \leq \frac{1}{\omega^2} e^{\left(K(1+\omega^2 B_2) N^{4/7}\right)} \tag{3}$$

- depends on the partitioning N and the frequency ω
- grows exponentially with N

E. Beretta, M. V. de Hoop, F. and O. Scherzer
Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates 2016

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- depends on the partitioning N and the frequency ω
- grows exponentially with N
- related with the lower bound of the Fréchet derivative

$$\frac{1}{\min \|DF[c_2]\delta c_2\|}$$

E. Beretta, M. V. de Hoop, F. and O. Scherzer
Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates 2016

Numerical estimates

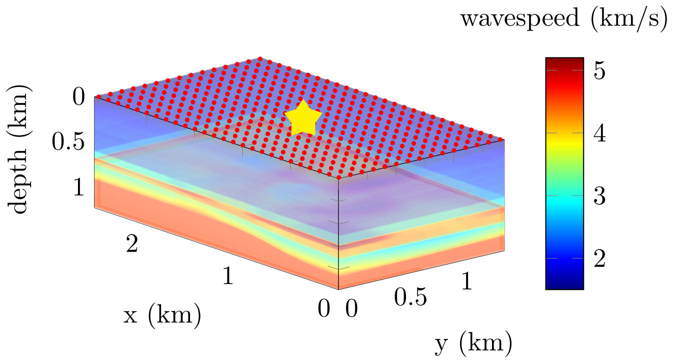
The objective is to

- Verify the accuracy of the stability constant bounds on a realistic example
- Compare with estimates provided with the Fréchet derivative
- Design a multi-level scheme for model reconstruction

Plan

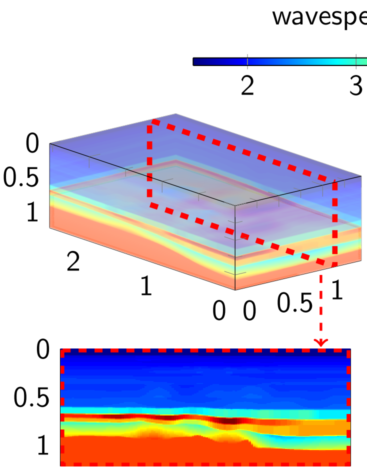
3. Numerical estimates
   - Geophysical situation
   - Numerical stability constant estimates

Numerical acquisition



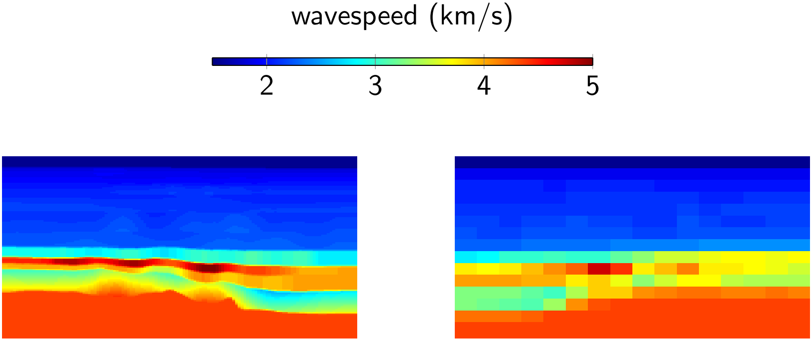
- $\left(-\frac{\omega^2}{c^2}-\Delta\right)u = 0$ on Ω, $\qquad u = g$ on $\partial\Omega$
- Data are $\frac{\partial u}{\partial \nu}$ at the receivers location
- 3D domain $2.54 \times 1.44 \times 1.22$km.

Wavespeed models



(a) 'True' wavespeed (c_1)   (b) Starting wavespeed (c_2)

Wavespeed models



(a) $N = 1\ 527\ 168$ (b) $N = 2\ 880$

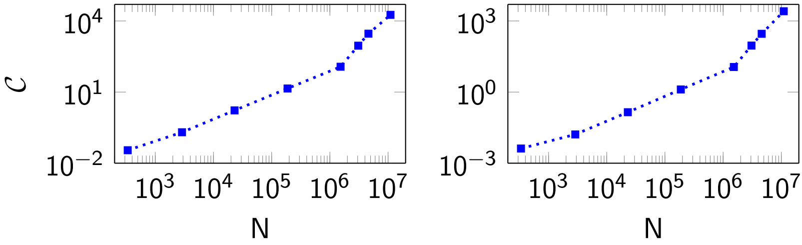
Direct estimates

- 'Direct' numerical estimates of the stability constant from

$$\|c_1^{-2} - c_2^{-2}\| \leq \mathcal{C} \|F(c_1^{-2}) - F(c_2^{-2})\|$$

Direct estimates

Stability constant estimates with N

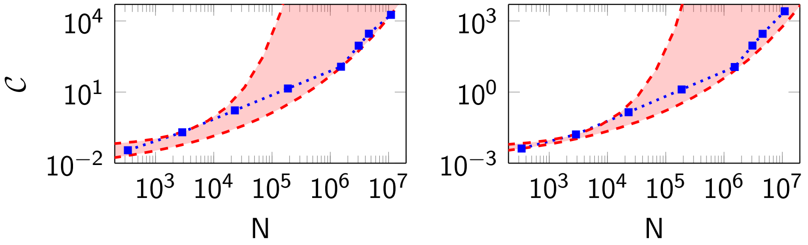


(a) 5Hz (b) 10Hz

$$\|c_1^{-2} - c_2^{-2}\| \leq \mathcal{C} \|F(c_1^{-2}) - F(c_2^{-2})\|$$

Direct estimates

Stability constant estimates with N

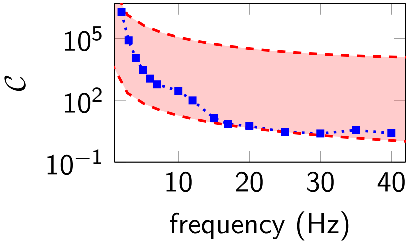


(a) 5Hz (b) 10Hz

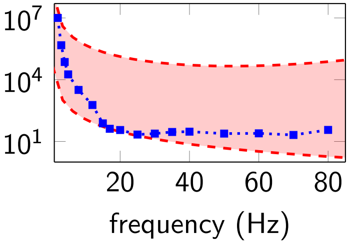
$$\frac{1}{4\omega^2} e^{K_1 N^{1/5}} \leq \mathcal{C} \leq \frac{1}{\omega^2} e^{(K(1+\omega^2 B_2) N^{4/7})}$$

Direct estimates

Stability constant estimates with ω



(a) $N = 4\ 597\ 248$ domains



(b) $N = 11\ 003\ 850$ domains

$$\frac{1}{4\omega^2} e^{K_1 N^{1/5}} \leq \mathcal{C} \leq \frac{1}{\omega^2} e^{\left(K(1+\omega^2 B_2) N^{4/7}\right)}$$

Fréchet derivative estimates

- c_1 is unknown in realistic case
- DF is obtained from the Gauss Newton Hessian as $DF^* DF = H^{GN}$
- Lower bound of DF approximated with $\sqrt{\min(sv(H^{GN}))}$

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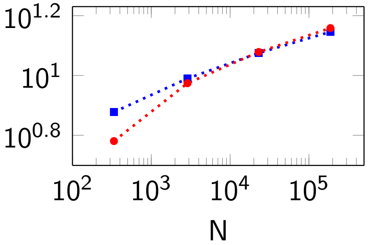
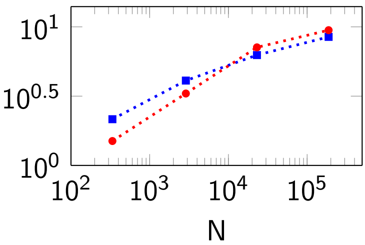


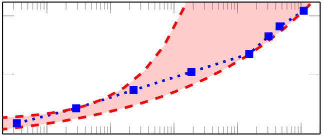
(a) 2Hz



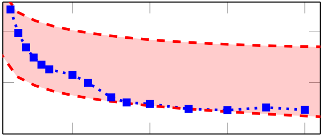
(b) 10Hz

Direct estimates (···■···) and lower bound of the Fréchet derivative (···●···)

Multi-level FWI



(a) N



(b) ω

- The stability constant $\nearrow$ with N
- The stability constant $\searrow$ with ω

Multi-level FWI

(a) N (b) ω

- The stability constant $\nearrow$ with N
- The stability constant $\searrow$ with ω

- Low frequency should use low scale (N) $\Rightarrow$ low resolution
- High frequency can use high scale (N) $\Rightarrow$ high resolution

Plan

Numerical reconstruction

- Iterative minimization of the residuals, c_2 updates

$$\mathcal{J}(c_2) = \frac{1}{2}\|F(c_1^{-2}) - F(c_2^{-2})\|^2$$

- Newton like methods
- Low frequency should use low scale (N) $\Rightarrow$ low resolution
- High frequency can use high scale (N) $\Rightarrow$ high resolution

Numerical reconstruction

- Iterative minimization of the residuals, c_2 updates

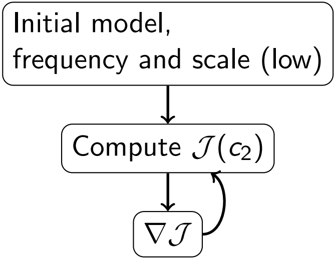
$$\mathcal{J}(c_2) = \frac{1}{2}\|F(c_1^{-2}) - F(c_2^{-2})\|^2$$

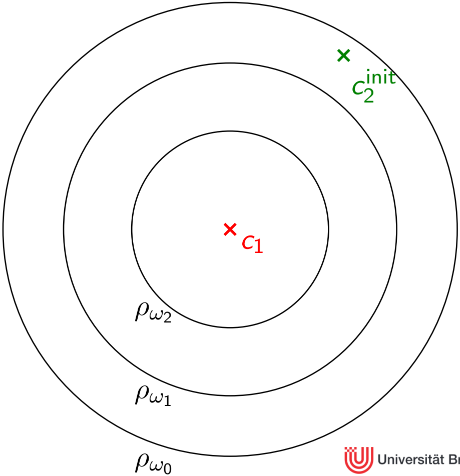
- Newton like methods
- Low frequency should use low scale (N) $\Rightarrow$ low resolution
- High frequency can use high scale (N) $\Rightarrow$ high resolution
- The radius of convergence $\searrow$ with ω
  - low frequency can use initial models with no information
  - high frequency requires initial information

M. V. de Hoop, L. Qiu and O. Scherzer
An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints
Numerische Mathematik 2013

Multi-level algorithm

- low frequency $\Rightarrow$ starting models and low scale
- high frequency $\Rightarrow$ requires information but high scale





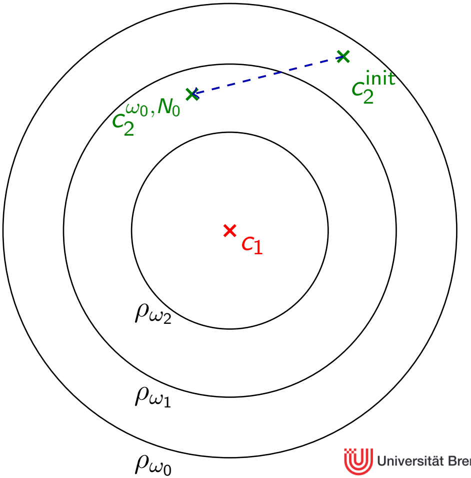
Multi-level algorithm

- low frequency $\Rightarrow$ starting models and low scale
- high frequency $\Rightarrow$ requires information but high scale

Initial model, frequency and scale (low)

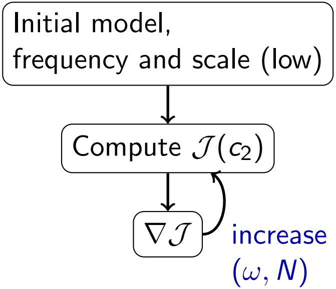
Compute $\mathcal{J}(c_2)$

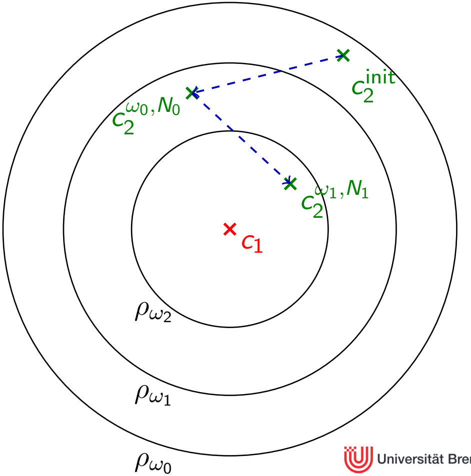
$\nabla\mathcal{J}$



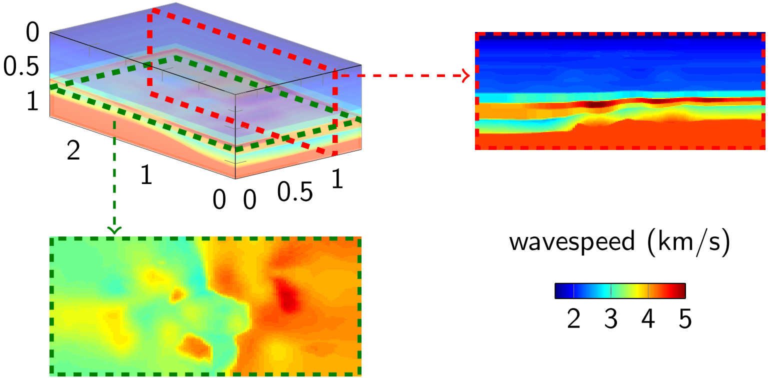
Multi-level algorithm

- low frequency $\Rightarrow$ starting models and low scale
- high frequency $\Rightarrow$ requires information but high scale



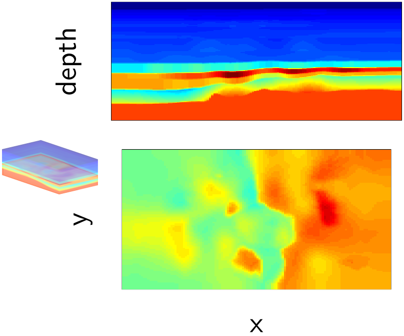


Multi-level 3D reconstruction

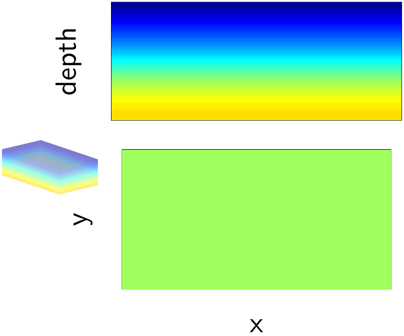


$N = 2\ 015\ 232$ domains

Multi-level 3D reconstruction

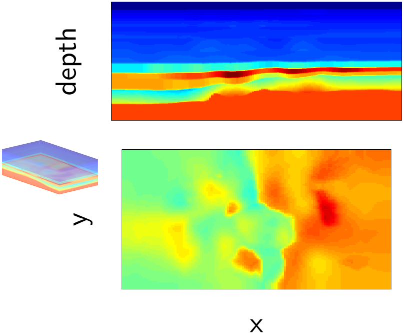


(a) True

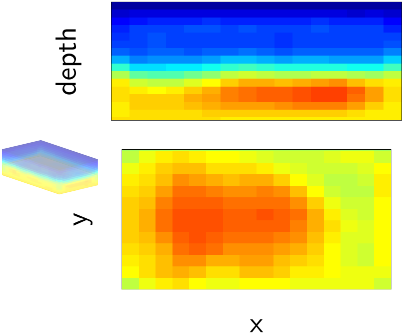


(b) Starting

Multi-level 3D reconstruction

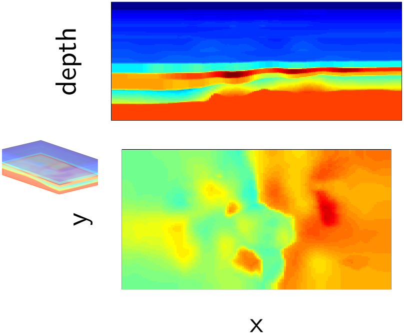


(a) True ($N = 2\ 015\ 232$)

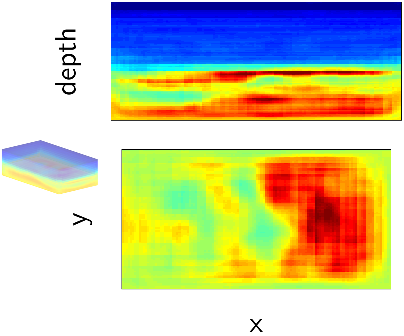


(b) 3Hz ($N = 2\ 880$)

Multi-level 3D reconstruction



(a) True ($N = 2\ 015\ 232$)



(b) 15Hz ($N = 2\ 015\ 232$)

Conclusion 1/3

- Numerical stability constant estimates fits the bounds (DF)
- Multi-level inversion pairing frequency and partitioning
- Elastic extension (numerical)?

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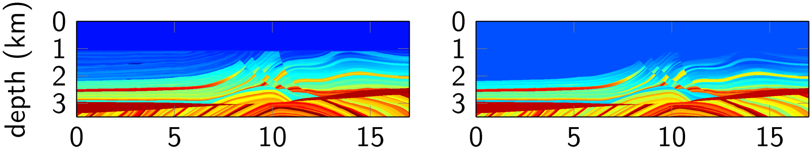
$$-\rho\omega^2 u - \nabla(\lambda\nabla\cdot u) - \nabla\cdot\left(\mu\left[\nabla u + (\nabla u)^T\right]\right) = 0 \qquad (5)$$

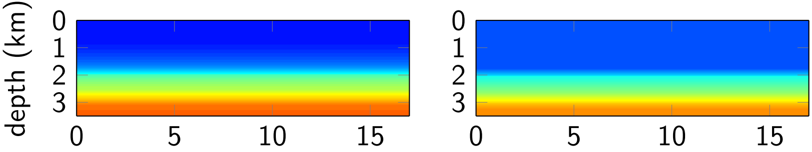
- THREE parameters to invert

E. Beretta, M. V. de Hoop, E. Francini, S. Vessella and J. Zhai
Uniqueness and Lipschitz stability of an inverse boundary value problem for time-harmonic elastic waves
2014

Conclusion 2/3: elastic multi parameter inversion

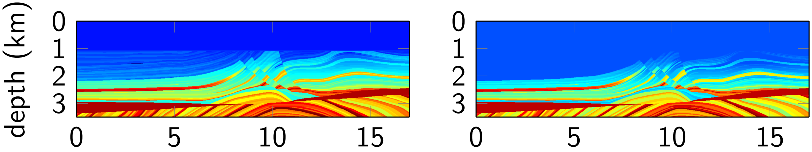


(a) P-wavespeed (b) S-wavespeed



Starting models, 17×3.5 km

Conclusion 2/3: elastic multi parameter inversion



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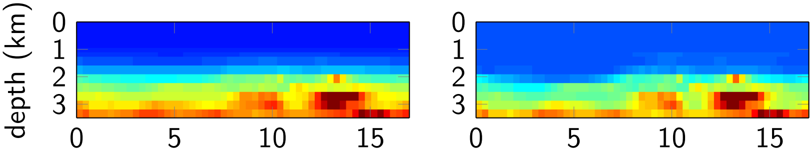
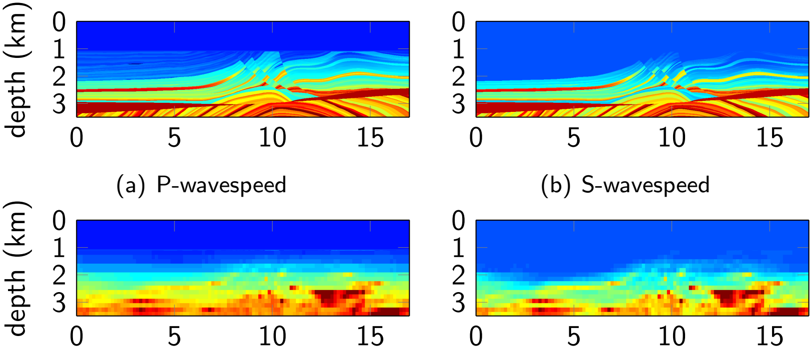


Reconstruction after 1Hz, $N = 530$ domains; 17×3.5 km

Conclusion 2/3: elastic multi parameter inversion

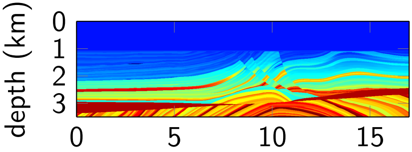


(a) P-wavespeed

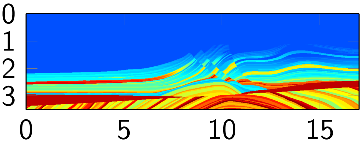
(b) S-wavespeed

Reconstruction after 2Hz, $N = 2\ 226$ domains; 17×3.5 km

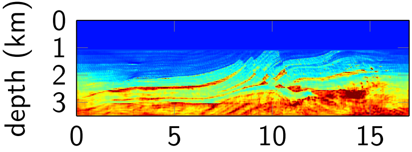
Conclusion 2/3: elastic multi parameter inversion

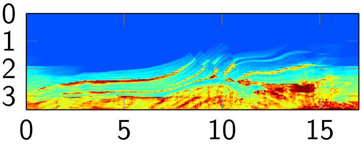


(a) P-wavespeed



(b) S-wavespeed





Reconstruction after 10Hz, $N = 597\ 051$ domains; 17×3.5 km

Conclusion 3/3

- Numerical stability constant estimates with bounds (DF)
- Multi-level inversion pairing frequency and partitioning
- Numerical extension for elastic
- Computational optimization methods
- Link between frequency and scale
- Improve frequency dependency of the bounds?
- Numerical inversion for TI wave equation

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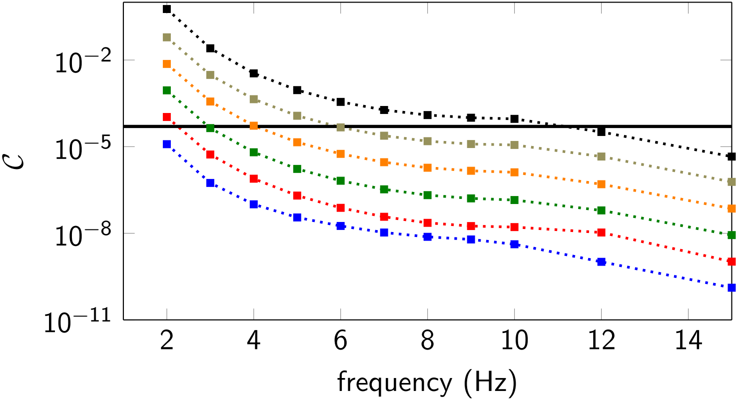
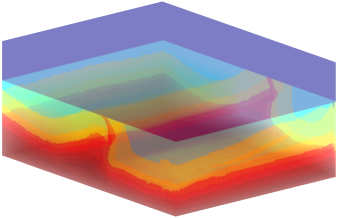
THANK YOU

Appendix

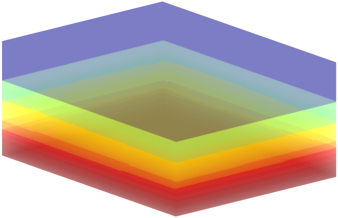
Stability with frequency for different N



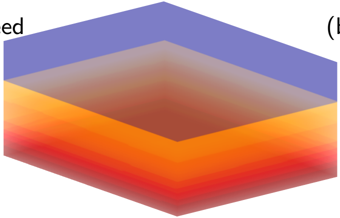
3D models



(a) P-wavespeed



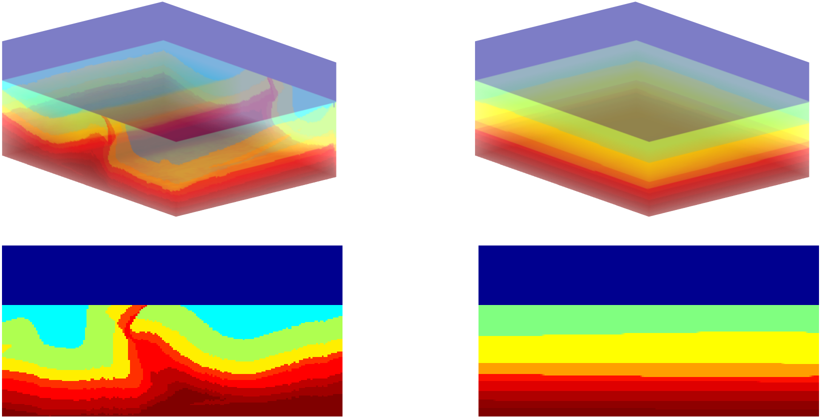
(b) S-wavespeed



(c) Density

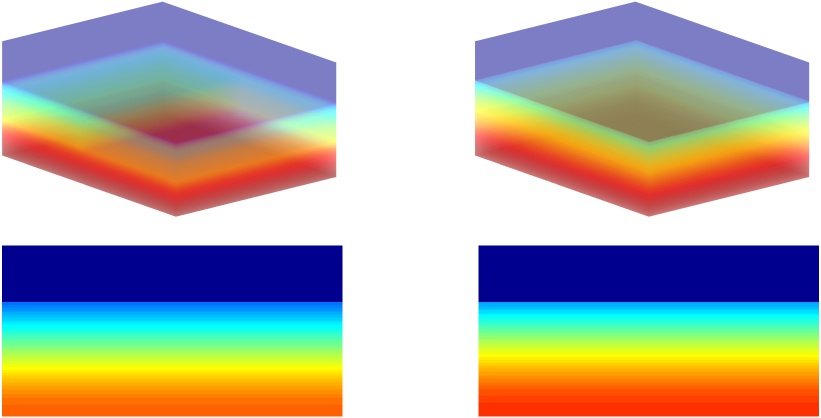
3D test-case 1.8km x 1.4km x 1.2km

3D models



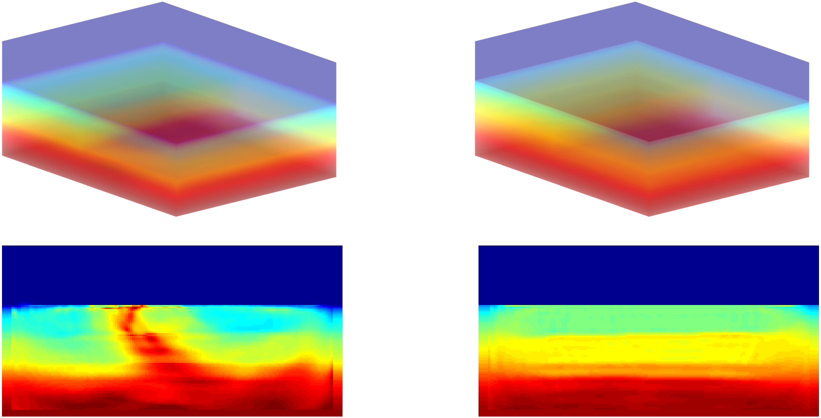
True models, 1.8km x 1.4km x 1.2km P- and S-wavespeed

3D models



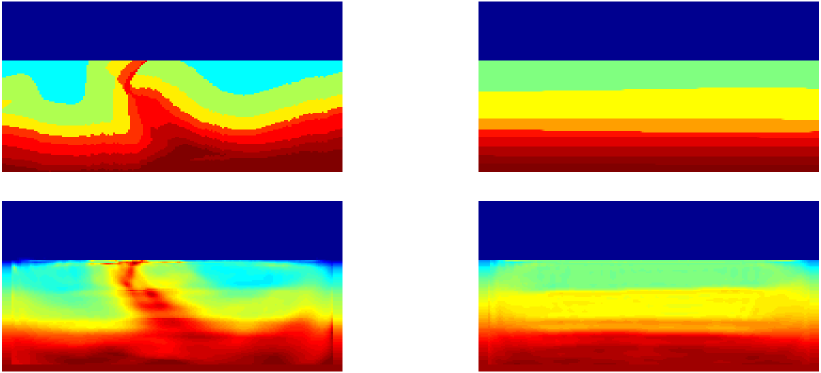
Starting models, 1.8km x 1.4km x 1.2km P- and S-wavespeed

3D models



Final reconstruction after 14Hz iterations, P- and S-wavespeed

3D models



True models and final reconstruction after 14Hz iterations, P- and S-wavespeed

Theory

- Complex Geometrical Optics Solutions for the wave equation are of form

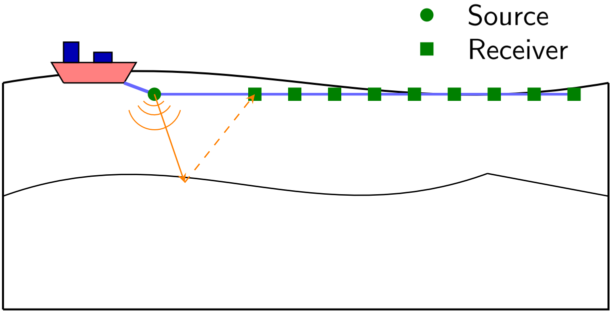
$$u(x) = e^{\mathrm{i}x\cdot\zeta}(1 + R(x))$$

J. Sylvester and G. Uhlmann
A global uniqueness theorem for an inverse boundary value problem
Annals of Mathematics 1987

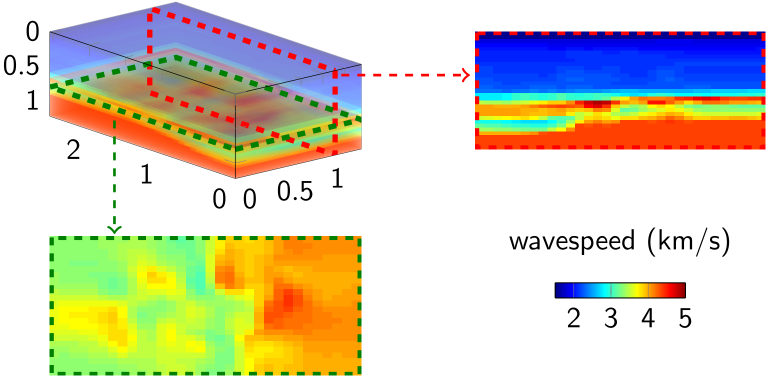
J. Feldman, M. Salo and G. Uhlmann
The Calderón Problem - An Introduction to Inverse Problems

Geophysical situation

- 3D acoustic domain
- Sources and receivers are located at the surface
- Sources are airguns ; Receivers are hydrophones
- Partial data



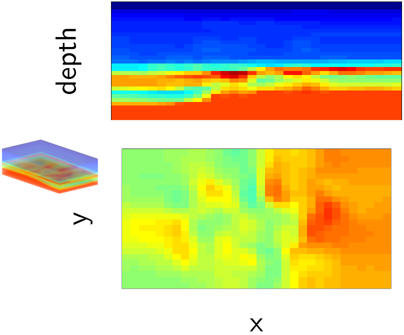
Single frequency reconstruction



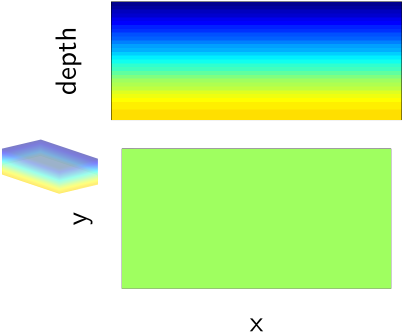
$N = 23\ 040$ domains for the piecewise constant decomposition

Single frequency reconstruction

Iterations using only 7Hz frequency data, $N = 23\ 040$ domains



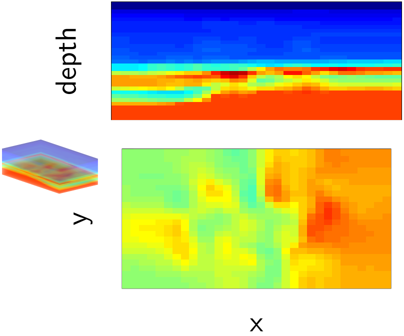
(a) True



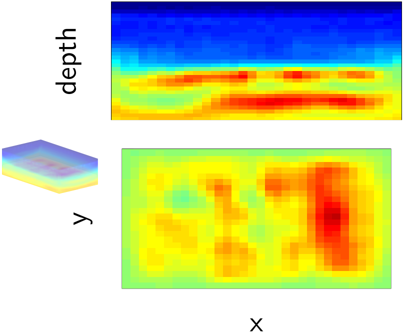
(b) Starting

Single frequency reconstruction

Iterations using only 7Hz frequency data, $N = 23\ 040$ domains



(a) True



(b) Reconstruction