

SHAPE AND MATERIAL PARAMETER RECONSTRUCTION OF A SOLID IMMERSED IN A FLUID

Inverse Problems for PDEs, University of Bremen, March of 2016

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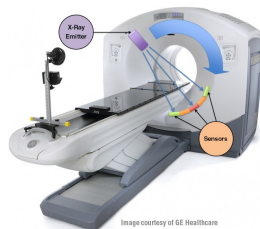
** CSUN & INRIA, Associate Team Magic, USA

Motivation and context

APPLICATIONS



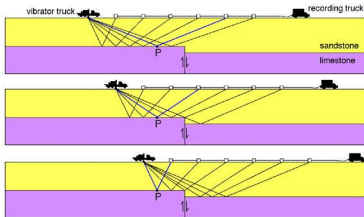
(a) Obstetric ultrasonography



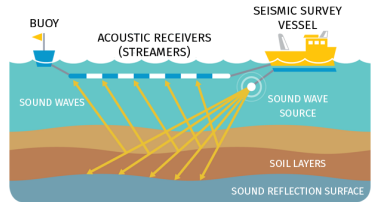
(b) X-ray computed tomography

Motivation and context

APPLICATIONS



(a) Terrestrial seismic survey



(b) Marine seismic survey

Outline

- ▶ Problem Statement

Outline

- ▶ Problem Statement
- ▶ Mathematical Result

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- ▶ Mathematical Result
- ▶ Computational Methodology

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- ▶ Mathematical Result
- ▶ Computational Methodology
- ▶ Numerical Results

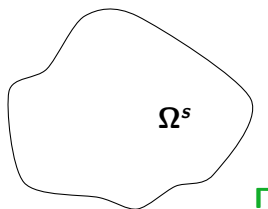
Outline

- ▶ Problem Statement
- ▶ Mathematical Result
- ▶ Computational Methodology
- ▶ Numerical Results
- ▶ Conclusions and Perspectives

Objective

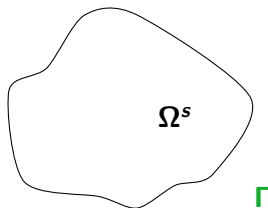
**Recovery of shape and material parameters of
an elastic homogeneous object in a fluid**

Problem statement



Problem statement

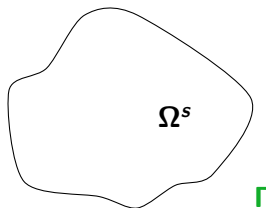
$$\Omega^f = \mathbb{R}^n \setminus \overline{\Omega^s}$$



Problem statement

$$\Omega^f = \mathbb{R}^n \setminus \overline{\Omega^s}$$

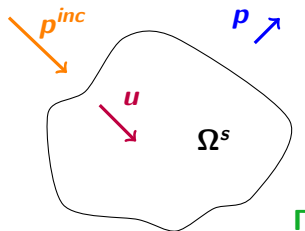
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Problem statement

$$\Omega^f = \mathbb{R}^n \setminus \overline{\Omega^s}$$

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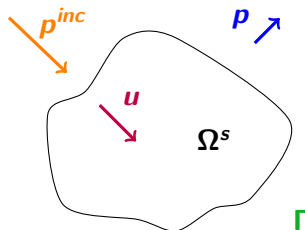


Problem statement

$$\Omega^f = \mathbb{R}^n \setminus \overline{\Omega^s}$$

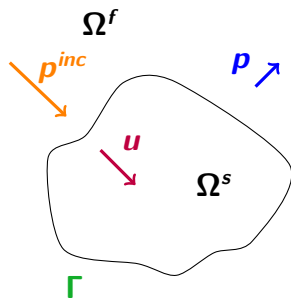
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- Receiver



Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM



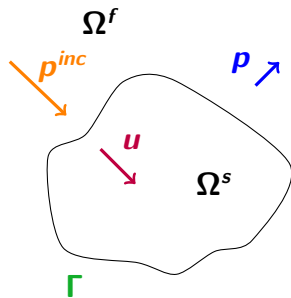
Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM

$$\nabla \cdot \sigma(\mathbf{u}) + \omega^2 \rho_s \mathbf{u} = 0 \quad \text{in } \Omega^s$$

$$\Delta \mathbf{p} + k^2 \mathbf{p} = 0 \quad \text{in } \Omega^f$$

$$k = \left(\frac{\omega}{c_f} \right)$$



Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM

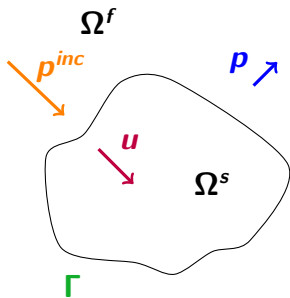
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$$\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} = \frac{\partial \mathbf{p}}{\partial n} + \frac{\partial \mathbf{p}^{inc}}{\partial n} \quad \text{on } \Gamma$$

$$\sigma(\mathbf{u}) \mathbf{n} = -\mathbf{p} \mathbf{n} - \mathbf{p}^{inc} \quad \text{on } \Gamma$$



Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM

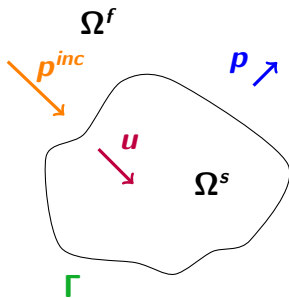
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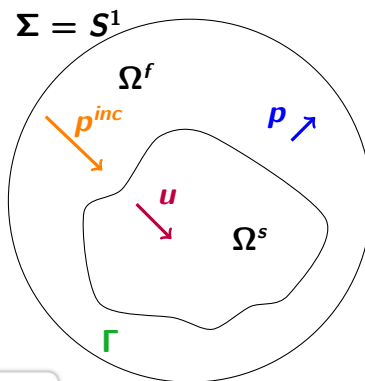
$$\sigma(\mathbf{u}) \mathbf{n} = -\mathbf{p} \mathbf{n} - \mathbf{p}^{inc} \quad \text{on } \Gamma$$



$$\lim_{x \rightarrow +\infty} r \left(\frac{\partial \mathbf{p}}{\partial r} - ik \mathbf{p} \right) = 0 \quad (r = \| \mathbf{x} \|_2)$$

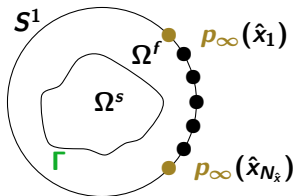
Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM


$$\begin{aligned}\nabla \cdot \sigma(\mathbf{u}) + \omega^2 \rho_s \mathbf{u} &= 0 && \text{in } \Omega^s \\ \Delta \mathbf{p} + k^2 \mathbf{p} &= 0 && \text{in } \Omega^f \\ k &= \left(\frac{\omega}{c_f} \right) \\ \omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} &= \frac{\partial \mathbf{p}}{\partial n} + \frac{\partial \mathbf{p}^{inc}}{\partial n} && \text{on } \Gamma \\ \sigma(\mathbf{u}) \mathbf{n} &= -\mathbf{p} \mathbf{n} - \mathbf{p}^{inc} && \text{on } \Gamma \\ \frac{\partial \mathbf{p}}{\partial r} - ik \mathbf{p} &= 0 && \text{on } \Sigma\end{aligned}$$

Problem statement(3)

THE FAR FIELD PATTERN

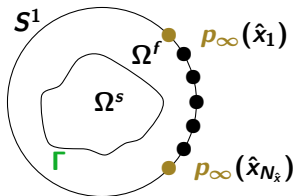


$$p(x) = \frac{e^{ikr}}{\sqrt{r}} \left(p_\infty \left(\frac{x}{r} \right) + \mathcal{O} \left(\frac{x}{r} \right) \right) \\ r = \|x\|_2 \rightarrow \infty$$

$$p_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} \left(e^{ik\hat{x} \cdot y} \frac{\partial p}{\partial n} - \frac{\partial e^{ik\hat{x} \cdot y}}{\partial n} p(y) \right) d\Gamma$$

Problem statement(3)

THE FAR FIELD PATTERN

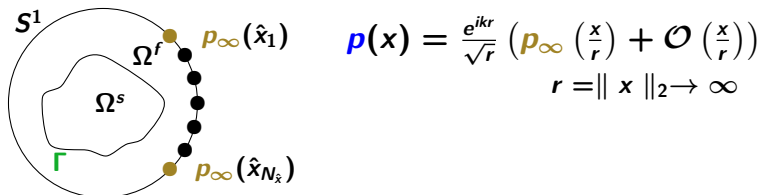


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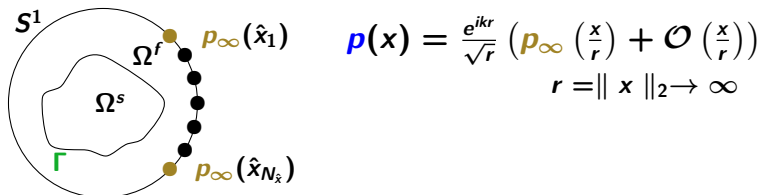
David Colton and Rainer Kress

Inverse Acoustic and Electromagnetic Scattering Theory

Applied Mathematical Sciences

Problem statement(3)

THE FAR FIELD PATTERN



$$p(x) = \frac{e^{ikr}}{\sqrt{r}} \left(p_\infty \left(\frac{x}{r} \right) + \mathcal{O} \left(\frac{x}{r} \right) \right) \\ r = \|x\|_2 \rightarrow \infty$$

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The FFP intensity

$$p^*(\hat{x}) = \bar{p}_\infty(\hat{x}) \cdot p_\infty(\hat{x})$$

Problem statement(4)

- **Direct mapping**

$$\lambda, \mu, \Gamma \longmapsto F(\lambda, \mu, \Gamma) = p_{\infty}$$

Problem statement(4)

► Direct mapping

$$\lambda, \mu, \Gamma \longmapsto F(\lambda, \mu, \Gamma) = p_{\infty}$$

► Inverse Problem

$$(\text{IP}) = \left\{ \begin{array}{l} \text{Given FFP measurements } \tilde{p}_{\infty}(x_j), \\ \text{Find } \lambda, \mu, \Gamma, \text{ such that} \\ F(\lambda, \mu, \Gamma)(x_j) = \tilde{p}_{\infty}(x_j); \quad j = 1, \dots, M \end{array} \right.$$

Problem statement(4)

► Direct mapping

$$\lambda, \mu, \Gamma \mapsto F(\lambda, \mu, \Gamma) = p_\infty$$

$$\lambda, \mu, \Gamma \mapsto A(\lambda, \mu, \Gamma) = \bar{F}(\lambda, \mu, \Gamma) \cdot F(\lambda, \mu, \Gamma) = \bar{p}_\infty p_\infty = p^*$$

► Inverse Problem

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Problem statement(4)

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Problem statement(4)

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► Inverse Problem

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\Rightarrow Ill-Posed & Nonlinear problem

Problem statement(4)

THE NONLINEARITY ISSUE

Find the Lamé coefficients λ, μ and the shape Γ such that

$$\min_{\lambda, \mu, \Gamma} \frac{1}{2} \left\| \sum_{j=1}^{N_x} \mathbf{A}(\lambda, \mu, \Gamma)(\hat{x}_j) - \overline{\tilde{p}_\infty}(\hat{x}_j) \tilde{p}_\infty(\hat{x}_j) \right\|^2$$

THE NONLINEARITY ISSUE

Define

$$J(\lambda, \mu, \Gamma) = \frac{1}{2} \left\| \sum_{j=1}^{N_x} \mathbf{A}(\lambda, \mu, \Gamma)(\hat{x}_j) - \overline{\tilde{p}_\infty}(\hat{x}_j) \tilde{p}_\infty(\hat{x}_j) \right\|^2$$

Find the Lamé coefficients λ, μ and the shape Γ such that

$$\nabla J = 0 \quad \Leftrightarrow \quad \partial_\lambda J = \partial_\mu J = \partial_\Gamma J = 0$$

Problem statement(5)

THE NONLINEARITY ISSUE

The Newton Iteration

$$(\mathbf{A}'_{\lambda}(\hat{x}_j), \mathbf{A}'_{\mu}(\hat{x}_j), \mathbf{A}'_{\Gamma}(\hat{x}_j)) \cdot \begin{pmatrix} \delta_{\lambda} \\ \delta_{\mu} \\ \delta_{\Gamma} \end{pmatrix} = \mathbf{A}(\lambda^n, \mu^n, \Gamma^n)(\hat{x}_j) - \overline{\tilde{p}_{\infty}} \tilde{p}_{\infty}(\hat{x}_j)$$
$$j = 1, \dots, N_x$$

Update the parameters

$$(\lambda^{n+1}, \mu^{n+1}, \Gamma^{n+1}) = (\lambda^n, \mu^n, \Gamma^n) + (\delta_{\lambda}, \delta_{\mu}, \delta_{\Gamma})$$

Problem statement(5)

The Newton Iteration

$$B\delta = f$$

$$B = \begin{pmatrix} \mathbf{A}'_{\lambda}{}^T(\hat{x}_j) \\ \mathbf{A}'_{\mu}{}^T(\hat{x}_j) \\ \mathbf{A}'_{\Gamma}{}^T(\hat{x}_j) \end{pmatrix} (\mathbf{A}'_{\lambda}(\hat{x}_j), \mathbf{A}'_{\mu}(\hat{x}_j), \mathbf{A}'_{\Gamma}(\hat{x}_j)) \quad j = 1, \dots, N_x$$

$$f = \begin{pmatrix} \mathbf{A}'_{\lambda}{}^T(\hat{x}_j) \\ \mathbf{A}'_{\mu}{}^T(\hat{x}_j) \\ \mathbf{A}'_{\Gamma}{}^T(\hat{x}_j) \end{pmatrix} (\mathbf{A}(\lambda^n, \mu^n, \Gamma^n)(\hat{x}_j) - \overline{\tilde{p}_{\infty}} \tilde{p}_{\infty}(\hat{x}_j)) \quad j = 1, \dots, N_x$$

Update the parameters

$$(\lambda^{n+1}, \mu^{n+1}, \Gamma^{n+1}) = (\lambda^n, \mu^n, \Gamma^n) + (\delta_{\lambda}, \delta_{\mu}, \delta_{\Gamma})$$

Problem statement(5)

The Newton Iteration

$$B + \alpha \cdot \delta = (B + \alpha \cdot I) \delta = f$$

$$B = \begin{pmatrix} A'_{\lambda}{}^T(\hat{x}_j) \\ A'_{\mu}{}^T(\hat{x}_j) \\ A'_{\Gamma}{}^T(\hat{x}_j) \end{pmatrix} (A'_{\lambda}(\hat{x}_j), A'_{\mu}(\hat{x}_j), A'_{\Gamma}(\hat{x}_j)) \quad j = 1, \dots, N_x$$

$$f = \begin{pmatrix} A'_{\lambda}{}^T(\hat{x}_j) \\ A'_{\mu}{}^T(\hat{x}_j) \\ A'_{\Gamma}{}^T(\hat{x}_j) \end{pmatrix} (A(\lambda^n, \mu^n, \Gamma^n)(\hat{x}_j) - \overline{\tilde{p}_{\infty}} \tilde{p}_{\infty}(\hat{x}_j)) \quad j = 1, \dots, N_x$$

Update the parameters

$$(\lambda^{n+1}, \mu^{n+1}, \Gamma^{n+1}) = (\lambda^n, \mu^n, \Gamma^n) + (\delta_{\lambda}, \delta_{\mu}, \delta_{\Gamma})$$

Mathematical Result

$$(\mathbf{p}, \mathbf{u}) \in \Omega^f \times \Omega^s$$

- ▶ Lamé coefficients:

- ▶ $\lambda \rightarrow (\mathbf{p}', \mathbf{u}') = (\partial_\lambda \mathbf{p}, \partial_\lambda \mathbf{u})$

- ▶ $\mu \rightarrow (\mathbf{p}', \mathbf{u}') = (\partial_\mu \mathbf{p}, \partial_\mu \mathbf{u})$

- ▶ Shape parameters: $\mathbf{s}_1, \dots, \mathbf{s}_N \rightarrow \Gamma = \Gamma(\mathbf{s}_1, \dots, \mathbf{s}_N)$

- ▶ $\mathbf{s}_i \rightarrow (\mathbf{p}', \mathbf{u}') = \left(\frac{\partial \mathbf{p}}{\partial \Gamma} \frac{\partial \Gamma}{\partial \mathbf{s}_i}, \frac{\partial \mathbf{u}}{\partial \Gamma} \frac{\partial \Gamma}{\partial \mathbf{s}_i} \right) \quad i = 1, \dots, N$

Mathematical Result

Characterization of the Fréchet derivative with respect to Lamé coefficient

$$\nabla \cdot \sigma(\mathbf{u}') + \omega^2 \rho_s \mathbf{u}' = \nabla \cdot (C' \varepsilon(\mathbf{u})) \quad \text{in } \Omega^s$$

$$\Delta \mathbf{p}' + k^2 \mathbf{p}' = 0 \quad \text{in } \Omega^f$$

$$\sigma(\mathbf{u}') \cdot \nu + \mathbf{p}' \cdot \nu = C' \varepsilon(\mathbf{u}) \cdot \nu \quad \text{on } \Gamma$$

$$\omega^2 \rho_s \mathbf{u}' \cdot \nu + \partial_\nu \mathbf{p}' = 0 \quad \text{on } \Gamma$$

$$\partial_\nu \mathbf{p}' - ik \mathbf{p}' = 0 \quad \text{on } \Sigma$$

Characterization of the Fréchet derivative with respect to shape parameters

$$\begin{aligned}
 \nabla \cdot \sigma(\mathbf{u}') + \omega^2 \rho_s \mathbf{u}' &= 0 && \text{in } \Omega^s \\
 \Delta \mathbf{p}' + k^2 \mathbf{p}' &= 0 && \text{in } \Omega^f \\
 \sigma(\mathbf{u}') \cdot \boldsymbol{\nu} + \mathbf{p}' \cdot \boldsymbol{\nu} &= F_j(\mathbf{u}, \mathbf{p}, h_j) && \text{on } \Gamma \\
 \omega^2 \rho_s \mathbf{u}' \cdot \boldsymbol{\nu} - \partial_\nu \mathbf{p}' &= G_j(\mathbf{u}, \mathbf{p}, h_j) && \text{on } \Gamma \\
 \partial_\nu \mathbf{p}' - ik \mathbf{p}' &= 0 && \text{on } \Sigma
 \end{aligned}$$

where, for $\mathbf{p}^T = \mathbf{p} + \mathbf{p}^{inc}$ the function F and G are given by

$$\begin{aligned}
 F_j(\mathbf{u}, \mathbf{p}, h_j) &= -h_j^t \nabla \sigma(\mathbf{u}) \boldsymbol{\nu} - \nabla \mathbf{p}^T \cdot h_j \boldsymbol{\nu} + \sigma(\mathbf{u}) [h_j']^t \boldsymbol{\nu} + \mathbf{p}^T [h_j']^t \boldsymbol{\nu} \\
 G_j(\mathbf{u}, \mathbf{p}, h_j) &= -(\omega^2 \rho_f \nabla \mathbf{u} - \nabla(\nabla \mathbf{p}^T)) h_j \cdot \boldsymbol{\nu} + (\omega^2 \rho_f \mathbf{u} \boldsymbol{\nu} - \nabla \mathbf{p}^T) \cdot [h_j']^t \boldsymbol{\nu}
 \end{aligned}$$

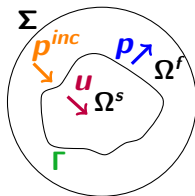


H. Barucq, R. Djellouli, and E. Estecahandy

Efficient DG-like formulation equipped with curved boundary edges for solving elasto-acoustic scattering problems

Solution methodology for the direct elasto-acoustic scattering problem

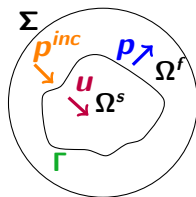
Main features



Solution methodology for the direct elasto-acoustic scattering problem

Main features

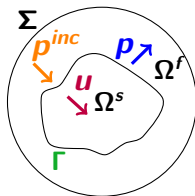
- ▶ A Discontinuous Galerkin method with interior penalty (IPDG)



Solution methodology for the direct elasto-acoustic scattering problem

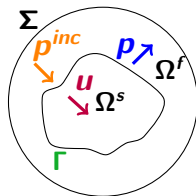
Main features

- ▶ A Discontinuous Galerkin method with interior penalty (IPDG)
- ▶ Higher-order elements



Solution methodology for the direct elasto-acoustic scattering problem

Main features



- ▶ A Discontinuous Galerkin method with interior penalty (IPDG)
- ▶ Higher-order elements
- ▶ Curved edges on the boundaries Γ and Σ

Definitions of errors

► Relative Residual(RR)=
$$\frac{\|\overline{p_\infty} \cdot p_\infty(\lambda, \mu, \Gamma) - \overline{p_\infty} \cdot p_\infty(\lambda^n, \mu^n, \Gamma^n)\|}{\|\overline{p_\infty} \cdot p_\infty(\lambda, \mu, \Gamma)\|}$$

Definitions of errors

- ▶ Relative Residual(RR)=
$$\frac{\|\overline{p_\infty} \cdot p_\infty(\lambda, \mu, \Gamma) - \overline{p_\infty} \cdot p_\infty(\lambda^n, \mu^n, \Gamma^n)\|}{\|\overline{p_\infty} \cdot p_\infty(\lambda, \mu, \Gamma)\|}$$
- ▶ Relative Error of Material Param.(Err_{mat})=
$$\frac{(\|\lambda - \lambda^n\|^2 + \|\mu - \mu^n\|^2)^{\frac{1}{2}}}{(\|\lambda\|^2 + \|\mu\|^2)^{\frac{1}{2}}}$$

Definitions of errors

- ▶ Relative Residual(RR)=
$$\frac{\|\overline{p_\infty} \cdot p_\infty(\lambda, \mu, \Gamma) - \overline{p_\infty} \cdot p_\infty(\lambda^n, \mu^n, \Gamma^n)\|}{\|\overline{p_\infty} \cdot p_\infty(\lambda, \mu, \Gamma)\|}$$
- ▶ Relative Error of Material Param.(Err_{mat})=
$$\frac{(\|\lambda - \lambda^n\|^2 + \|\mu - \mu^n\|^2)^{\frac{1}{2}}}{(\|\lambda\|^2 + \|\mu\|^2)^{\frac{1}{2}}}$$
- ▶ Relative Error of Shape Param.(Err_{shap})=
$$\frac{(\sum_{i=1}^k \|s_i - s_i^n\|^2)^{\frac{1}{2}}}{(\sum_{i=1}^k \|s_i\|^2)^{\frac{1}{2}}}$$

Elliptic coordinate system

s_1, s_2

Elliptic coordinate system

$$s_1, s_2 \rightarrow \Gamma = \{(s_1 \cos \theta_j, s_2 \sin \theta_j), \quad \theta_j \in [0, 2\pi)\}$$

Polygonal parametrization

s_1, s_2, \dots, s_N

Polygonal parametrization

$$s_1, s_2, \dots, s_N \rightarrow X_j = s_j \begin{pmatrix} \cos \theta_j \\ \sin \theta_j \end{pmatrix}, j = 1, \dots, N,$$

$$T_j(t) = (t \quad 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_j \\ X_{j+1} \end{pmatrix}$$

with $X_{N+1} = X_1$, $t \in [0, 1]$

$$\Gamma = \{T_j(t), \quad t \in [0, 1], j = 1, \dots, N\}$$

B-spline representation

s_1, s_2, \dots, s_N

B-spline representation

$$s_1, s_2, \dots, s_N \rightarrow \mathbf{X}_j = s_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}, j = 1, \dots, N,$$

$$T_j(t) = \frac{1}{2} \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X}_{j-1} \\ \mathbf{X}_j \\ \mathbf{X}_{j+1} \end{pmatrix}$$

with $\mathbf{X}_{N+1} = \mathbf{X}_1$ and $\mathbf{X}_0 = \mathbf{X}_N$.

$$\Gamma = \{T_j(t), \quad t \in [0, 1], \quad j = 1, \dots, N\}$$

Fourier Parametrization

$$\Gamma = \{r(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)\}$$

$$\text{where } r^M(\theta) = s_1 + \sum_{k=1}^M s_{2k} \cos(k\theta) + s_{2k+1} \sin(k\theta)$$

$$\Gamma = \left\{ \sum_{k=1}^M s_j \phi_j(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi) \right\}$$

where $\phi_1 = 1$, $\phi_{2k} = \cos(k\theta)$, $\phi_{2k+1} = \sin(k\theta)$ for $k = 1, \dots, M$

Cases depending on Lamé coefficients

Steel

$$\lambda = 9.695 \cdot 10^9 \text{ N/m}^2 \quad \mu = 7.617 \cdot 10^9 \text{ N/m}^2$$

$$\text{Err}_{mat}^0 = 43.59\%$$

$$\text{Err}\lambda^0 = 48.42\%$$

$$\text{Err}\mu^0 = 34.35\%$$

Cases depending on shape parametrization

- ▶ Elliptic coordinate system

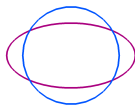


Figure: Ellipse

- ▶ Polygonal shaped objects

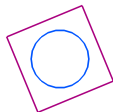


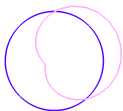
Figure: Square

- B-spline representation

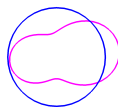


Figure: Rugby

- Fourier Parametrization



(a) Potato



(b) Peanut

Computational Methodology

Algorithm 1

- ▶ Synthetic data, $p_{\infty, meas}$
- ▶ Initialization
 $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$

Computational Methodology

Algorithm 1

- ▶ Synthetic data, $p_{\infty, meas}$
- ▶ Initialization
 $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 $\&\& \quad n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.01

Computational Methodology

Algorithm 1

- ▶ Synthetic data, $p_{\infty, meas}$
- ▶ Initialization
 $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.01
 - ▶ Far Field Pattern $p_{\infty}(\lambda^n, \mu^n, \Gamma^n)$
 - ▶ Fréchet derivative, $\frac{\partial p_{\infty}}{\partial \lambda}, \frac{\partial p_{\infty}}{\partial \mu}, \frac{\partial p_{\infty}}{\partial s_i} \quad i = 1, \dots, r$
 - ▶ Parameters update, $\lambda^{n+1}, \mu^{n+1}, s_1^{n+1}, \dots, s_r^{n+1}$
 - ▶ stagnation = $\frac{p_{\infty}^n - p_{\infty}^{n+1}}{p_{\infty}^n} \cdot 100$
 - ▶ $n = n + 1$

Computational Methodology

Algorithm 1

- ▶ Synthetic data, $p_{\infty, meas}$
- ▶ Initialization
 $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.01
 - ▶ Far Field Pattern $p_{\infty}(\lambda^n, \mu^n, \Gamma^n)$
 - ▶ Fréchet derivative, $\frac{\partial p_{\infty}}{\partial \lambda}, \frac{\partial p_{\infty}}{\partial \mu}, \frac{\partial p_{\infty}}{\partial s_i} \quad i = 1, \dots, r$
 - ▶ Parameters update, $\lambda^{n+1}, \mu^{n+1}, s_1^{n+1}, \dots, s_r^{n+1}$
 - ▶ stagnation = $\frac{p_{\infty}^n - p_{\infty}^{n+1}}{p_{\infty}^n} \cdot 100$
 - ▶ $n = n + 1$
 - ▶ Increase the frequency (*) $p_{\infty, meas}$

Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%

$$s_1, s_2 \rightarrow \Gamma = \{(s_1 \cos \theta, s_2 \sin \theta), \quad \theta \in [0, 2\pi)\}$$

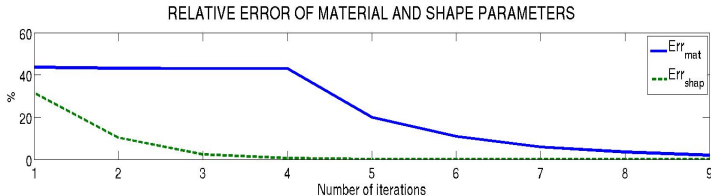
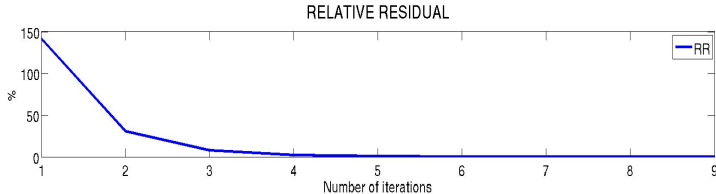


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

$$k_f = 267, \alpha = 0.5 \text{ (after 4iter } \alpha = 5 \cdot 10^{-6})$$

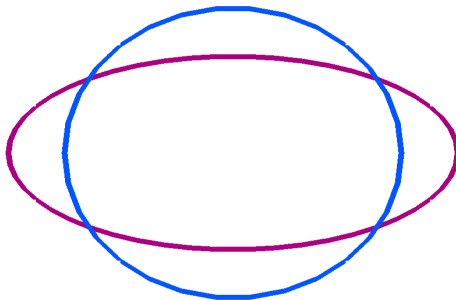
Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%



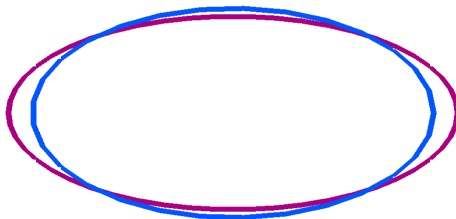
Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 0$



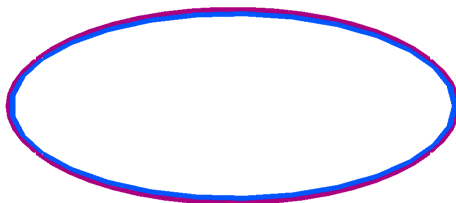
Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 1$



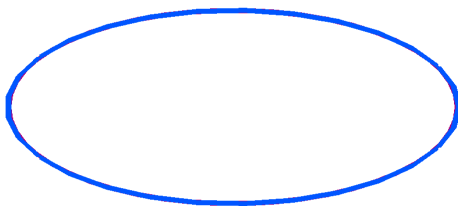
Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 2$



Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 3$



Polygonal parametrization

s_1, s_2, \dots, s_N

Polygonal parametrization

$$s_1, s_2, \dots, s_N \rightarrow X_j = s_j \begin{pmatrix} \cos \theta_j \\ \sin \theta_j \end{pmatrix}, j = 1, \dots, N,$$

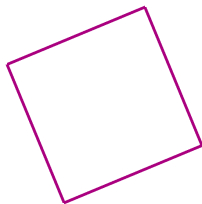
$$T_j(t) = (t \quad 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_j \\ X_{j+1} \end{pmatrix}$$

with $X_{N+1} = X_1$, $t \in [0, 1]$

$$\Gamma = \{T_j(t), \quad t \in [0, 1], j = 1, \dots, N\}$$

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%



	Target	In. Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

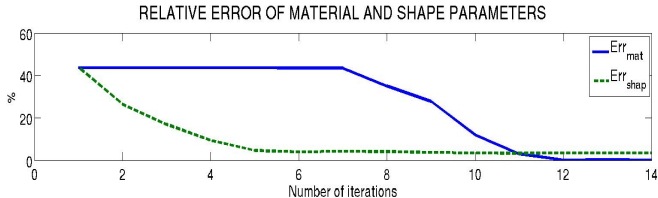
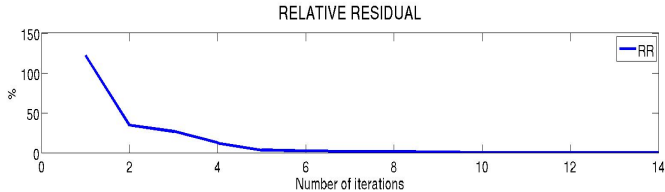
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	$\text{Err}_{mat}(\%)$	$\text{Err}_{shap}(\%)$	α
1	66.67	121.35	43.60	39.91	2
2	100	33.13	43.61	22.76	2
3	133.33	23.91	43.61	12.36	2
4	166.67	13.19	43.60	5.92	2
5	200	3.00	43.57	2.21	2
6	233.33	1.34	43.52	1.20	2
7	266.67	1.88	43.43	1.31	$2 \cdot 10^{-6}$
8	300	1.02	15.44	0.74	$2 \cdot 10^{-6}$
9	333.33	0.27	10.82	0.22	$2 \cdot 10^{-6}$
10	366.67	0.10	3.84	0.09	$2 \cdot 10^{-6}$
11	400	0.02	0.13	0.01	$2 \cdot 10^{-6}$

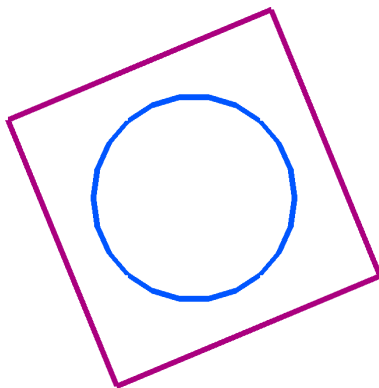
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%



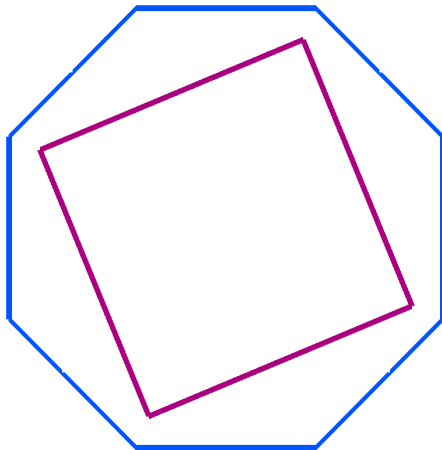
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



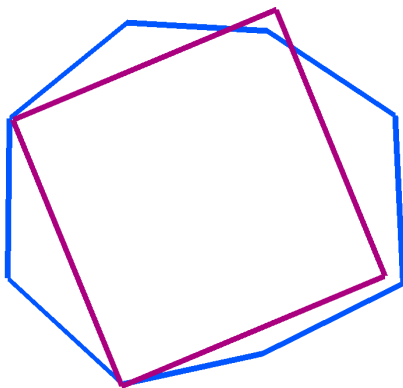
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=1



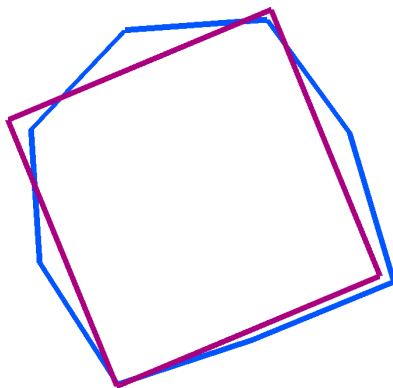
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=2



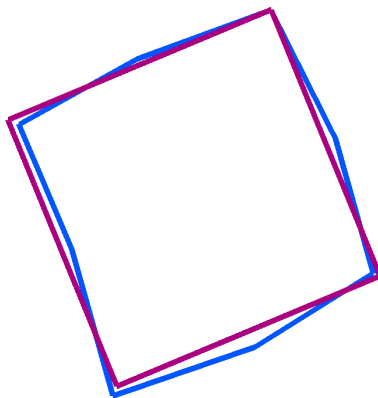
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=3



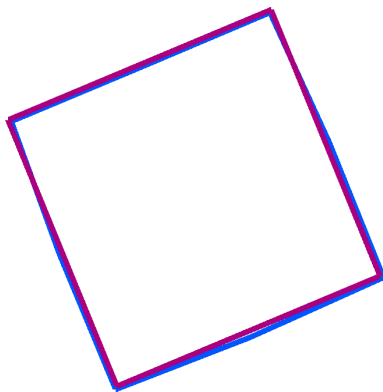
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=4



Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=5



B-spline representation

s_1, s_2, \dots, s_N

B-spline representation

$$s_1, s_2, \dots, s_N \rightarrow \mathbf{X}_j = s_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}, j = 1, \dots, N,$$

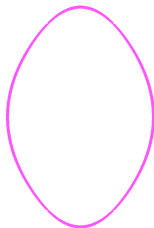
$$T_j(t) = \frac{1}{2} \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X}_{j-1} \\ \mathbf{X}_j \\ \mathbf{X}_{j+1} \end{pmatrix}$$

with $\mathbf{X}_{N+1} = \mathbf{X}_1$ and $\mathbf{X}_0 = \mathbf{X}_N$.

$$\Gamma = \{ T_j(t), \quad t \in [0, 1], \quad j = 1, \dots, N \}$$

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%



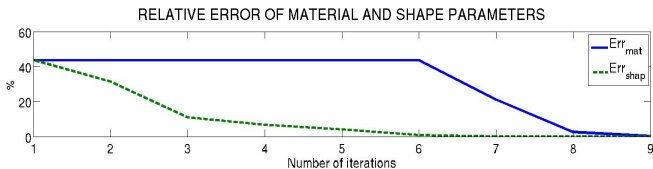
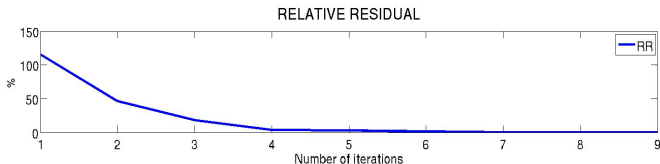
	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

Rugby ball, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
1	66.67	115.47	43.60	43.85	1
2	100	46.13	43.61	31.37	1
3	133.337	18.19	43.60	10.90	1
4	166.67	3.54	43.60	6.71	1
5	200	2.79	43.60	4.05	1
6	233.33	1.21	43.59	0.89	10^{-9}
7	266.67	0.29	21.04	0.06	10^{-9}
8	300	0.04	2.61	0.01	10^{-9}

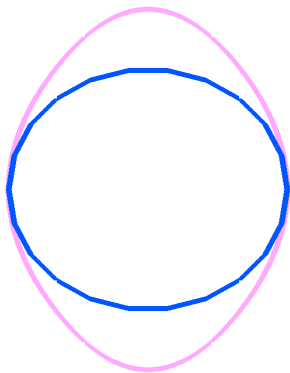
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%



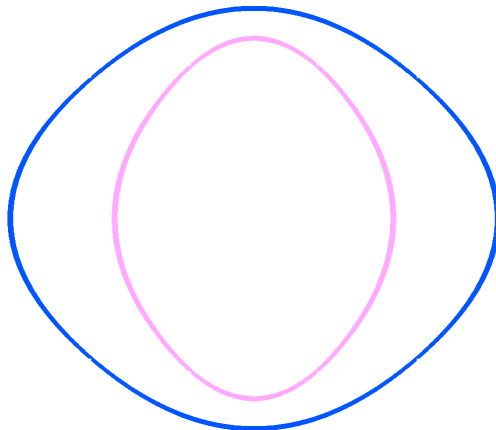
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



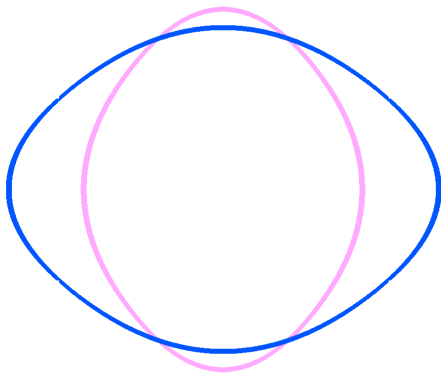
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=1



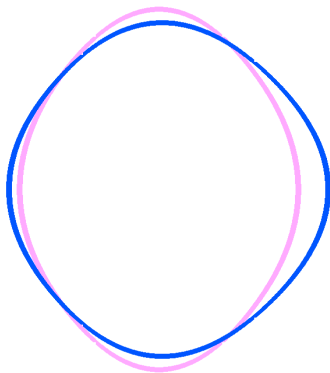
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=2



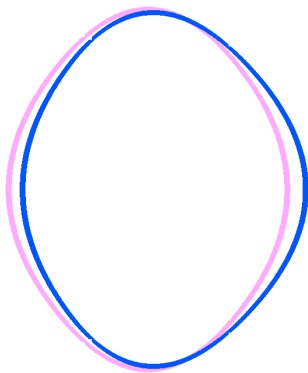
B-Spline representation

**Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=3**



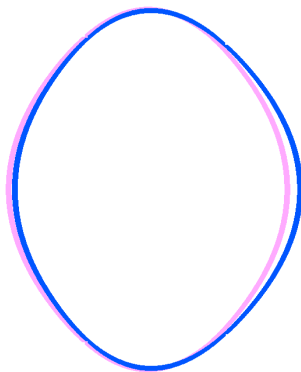
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=4



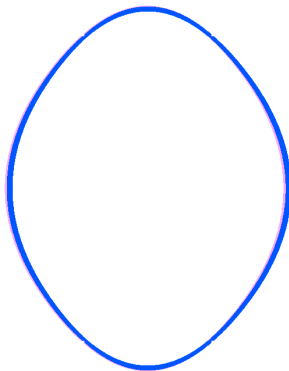
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=5



B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%
#Iter=6



Fourier Parametrization

$$\Gamma = \{r(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)\}$$

$$\text{where } r^M(\theta) = s_1 + \sum_{k=1}^M s_{2k} \cos(k\theta) + s_{2k+1} \sin(k\theta)$$

$$\Gamma = \left\{ \sum_{k=1}^M s_j \phi_j(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi) \right\}$$

where $\phi_1 = 1$, $\phi_{2k} = \cos(k\theta)$, $\phi_{2k+1} = \sin(k\theta)$ for $k = 1, \dots, M$

Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%



	Target	In.Guess
s_1	0.01	0.0125
s_2	0.007	0
s_3	0.0025	0

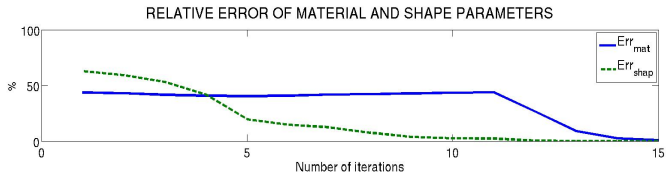
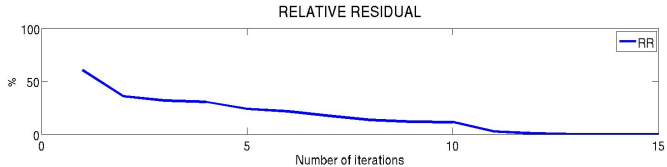
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
1	800	60.51	43.60	62.94	12
2	800	35.74	42.94	59.18	12
3	800	31.77	41.68	53.23	12
5	800	23.97	40.30	19.76	12
7	800	17.39	41.82	12.40	12
9	800	11.75	42.86	3.68	12
11	200	2.60	43.78	2.31	10^{-6}
12	200	0.81	26.55	0.28	10^{-6}
13	200	0.08	8.99	0.02	10^{-6}
14	200	0.01	2.46	0.02	10^{-6}
15	200	0.01	0.64	0.01	10^{-6}

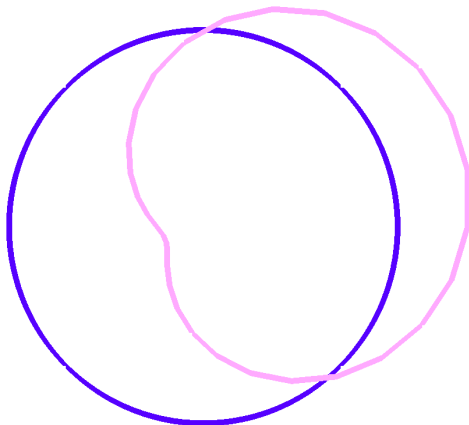
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%



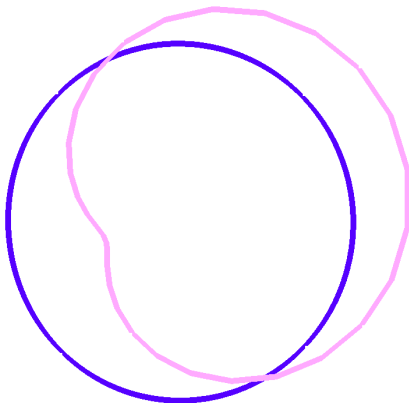
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



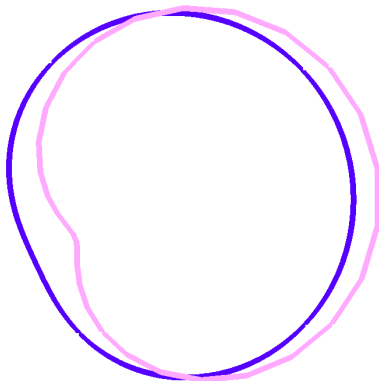
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=3



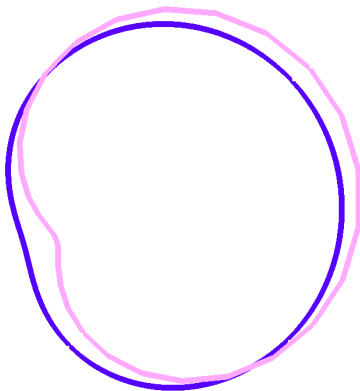
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=4



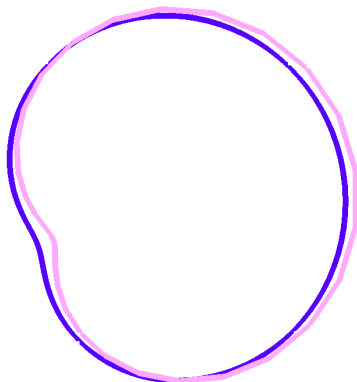
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=5



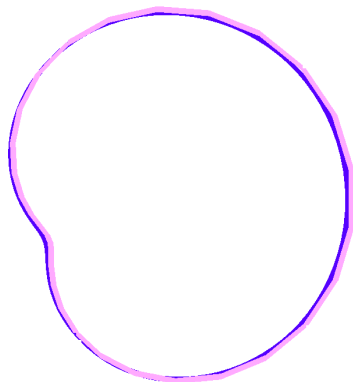
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=7



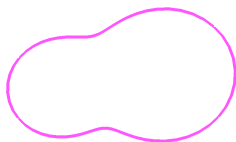
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=9



Fourier Parametrization

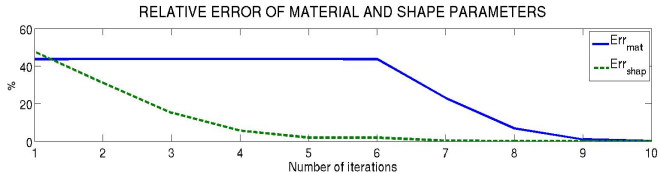
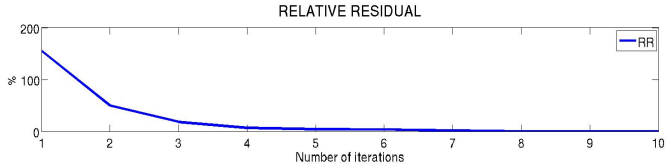
Peanut, Steel media, Algorithm1, Noise Level= 0%



	Target	In. Guess
s_1	0.01	0.0125
s_2	0.002	0
s_3	0.0005	0
s_4	0.004	0
s_5	0.001	0

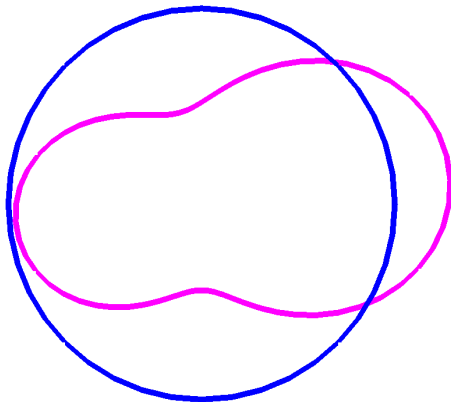
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%



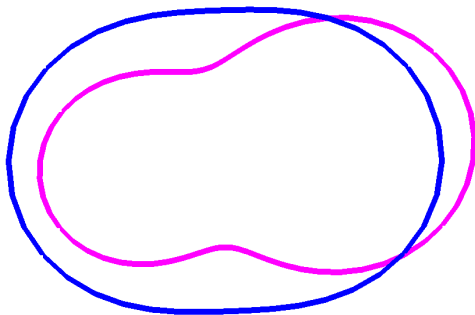
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



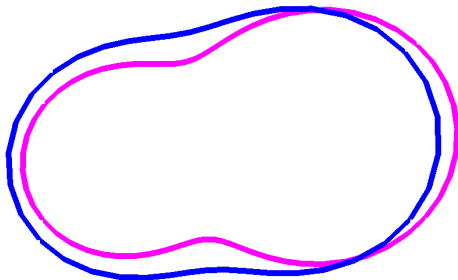
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=1



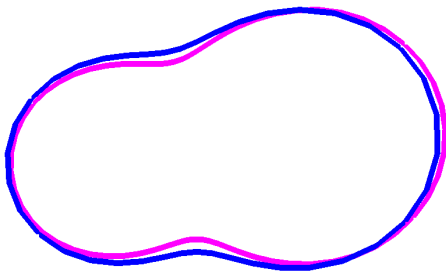
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=2



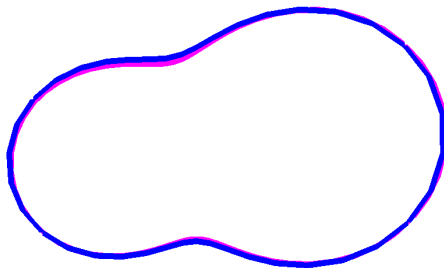
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=3



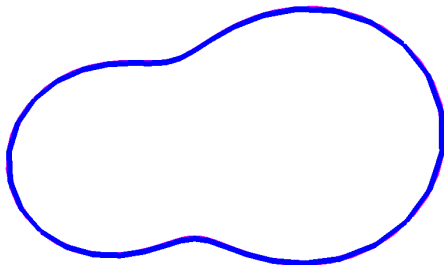
Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=4**



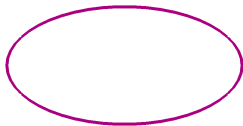
Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=6**



Elliptic coordinate system

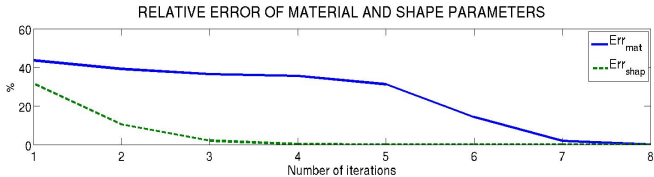
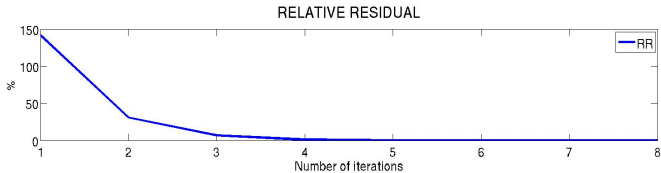
Ellipse, Steel media, Algorithm1, Noise Level= 0%
Various regularization parameters



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%



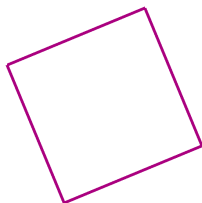
Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	267	141.64	43.60	31.62	0.05	0.5
2	267	30.94	39.14	10.41	0.005	0.5
3	267	6.88	36.52	2.08	$5 \cdot 10^{-4}$	0.5
4	267	0.96	35.58	0.28	$5 \cdot 10^{-5}$	0.005
5	267	0.29	31.28	0.09	$5 \cdot 10^{-6}$	0.005
6	267	0.12	14.23	0.03	$5 \cdot 10^{-7}$	0.005
7	267	0.02	1.97	0.00	$5 \cdot 10^{-8}$	0.005
8	267	0.00	0.08	0.00	$5 \cdot 10^{-9}$	0.005

Polygonal parametrization

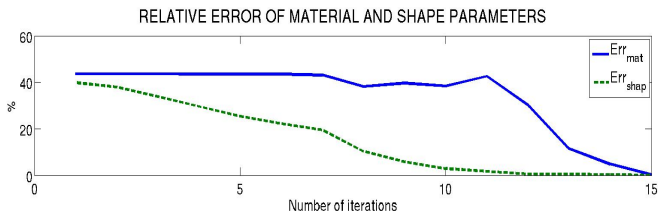
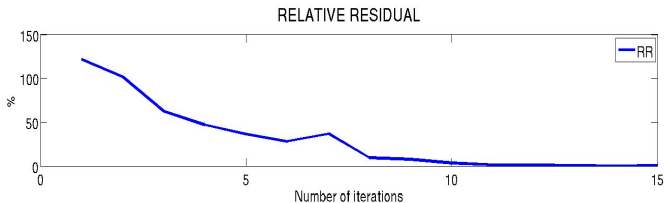
Square, Steel media, Algorithm1, Noise Level= 0%
Various regularization parameters



	Target	In. Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%



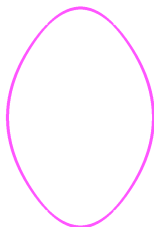
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	67	121.92	43.60	39.91	200	200
3	133	62.34	43.59	33.79	200	200
5	200	36.36	43.55	25.33	20	200
7	267	36.78	43.04	19.46	0.2	200
9	333	7.58	39.64	5.77	0.002	200
11	400	1.36	42.62	1.70	$2 \cdot 10^{-5}$	200
12	433	1.34	30.20	0.56	$2 \cdot 10^{-6}$	200
13	467	0.55	11.40	0.45	$2 \cdot 10^{-7}$	200
14	500	0.24	4.76	0.22	$2 \cdot 10^{-8}$	200
15	533	0.40	0.15	0.10	$2 \cdot 10^{-9}$	200

B-spline representation

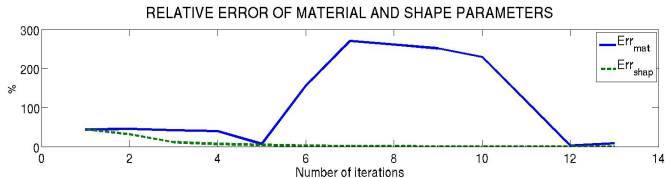
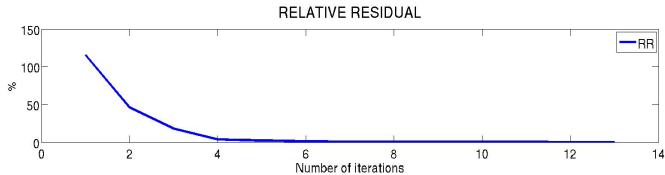
Rugby ball, Steel media, Algorithm1, Noise Level= 0%
Various regularization parameters



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%



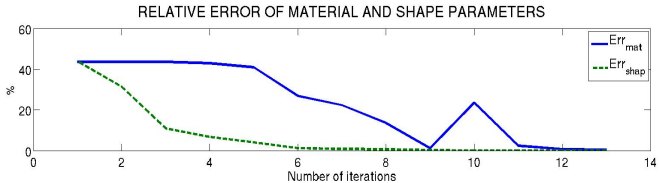
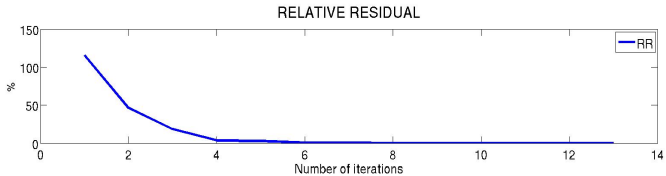
B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	67	115.47	43.60	43.85	0.01	1
2	100	46.15	45.09	31.38	0.001	1
3	133	18.20	41.67	10.97	$2 \cdot 10^{-4}$	1
4	167	3.67	39.45	6.75	$2 \cdot 10^{-5}$	1
5	200	2.36	6.83	4.18	$2 \cdot 10^{-6}$	1
6	233	1.29	155.33	1.77	$2 \cdot 10^{-6}$	1
7	267	0.51	270.43	1.12	$2 \cdot 10^{-6}$	1
9	333	0.42	251.06	0.37	$2 \cdot 10^{-6}$	1
11	400	0.30	116.71	0.21	$2 \cdot 10^{-6}$	1
12	433	0.29	2.50	0.23	$2 \cdot 10^{-6}$	1

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%



B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	67	115.47	43.60	43.85	1	1
2	100	46.13	43.61	31.37	0.1	1
3	133	18.19	43.56	10.90	0.01	1
4	167	3.55	42.91	6.71	10^{-3}	1
5	200	2.70	40.91	4.05	10^{-4}	1
6	233	0.74	26.80	1.19	10^{-5}	1
7	267	0.38	22.34	0.78	10^{-6}	1
8	300	0.26	13.54	0.58	10^{-7}	1
9	333	0.19	1.10	0.29	10^{-8}	1
10	367	0.08	23.52	0.05	10^{-9}	1
11	400	0.03	2.40	0.02	10^{-10}	1
12	433	0.01	0.73	0.01	10^{-11}	1

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation=100
 - ▶ **WHILE** stagnation > 0.75

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation=100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ Update shape parameters
 - ▶ Update material parameters

Computational Methodology

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ $\alpha_s \rightarrow \text{FFP } p_{\infty}(\lambda^n, \mu^n, \Gamma^n)$
 - ▶ Fréchet Derivative, $\frac{\partial p_{\infty}}{\partial s_i} \quad i = 1, \dots, r$
 - ▶ Recovery $s_1^{n+1}, \dots, s_N^{n+1}$, $n = n + 1$
 - ▶ stagnation = $\frac{p_{\infty}^n - p_{\infty}^{n+1}}{p_{\infty}^n} \cdot 100$
 - ▶ Update material parameters

Computational Methodology

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation=100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ Update shape parameters
 - ▶ stagnation=100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ $\alpha_m \rightarrow \text{FFP } p_{\infty}(\lambda^n, \mu^n, \Gamma^{n+1})$
 - ▶ Fréchet Derivative $\frac{\partial p_{\infty}}{\partial \lambda}, \frac{\partial p_{\infty}}{\partial \mu}$
 - ▶ Recovery, λ^{n+1}, μ^{n+1} , $n=n+1$
 - ▶ stagnation = $\frac{p_{\infty}^n - p_{\infty}^{n+1}}{p_{\infty}^n} \cdot 100$

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ Update shape parameters
 - ▶ Update material parameters

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ Update shape parameters
 - ▶ Update material parameters
- ▶ Increase the frequency (*) $\rightarrow p_{\infty, meas}$

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level= 0%

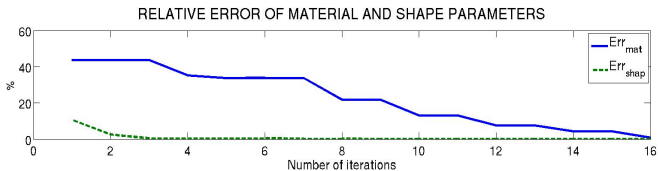
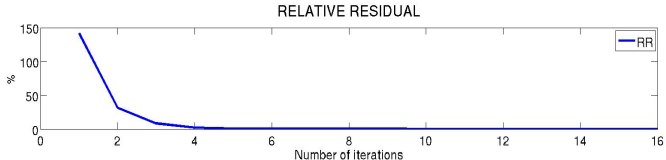
$$s_1, s_2 \rightarrow \Gamma = \{(s_1 \cos \theta, s_2 \sin \theta), \quad \theta \in [0, 2\pi)\}$$



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level= 0%

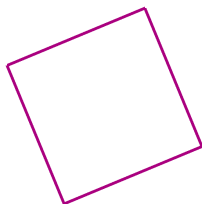


Elliptic coordinate system

	# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
S_i	1	267	141.63	43.60	10.68	2
S_i	2	267	31.79	43.60	2.58	2
S_i	3	267	8.36	43.60	0.49	2
λ, μ	4	267	2.13	35.10	0.49	$2 \cdot 10^{-30}$
λ, μ	5	267	0.96	33.71	0.49	$2 \cdot 10^{-30}$
λ, μ	6	267	0.84	33.74	0.49	$2 \cdot 10^{-30}$
S_i	7	267	0.84	33.74	0.19	2
λ, μ	8	267	0.49	21.71	0.19	$2 \cdot 10^{-30}$
S_i	9	267	0.34	21.71	0.10	2
λ, μ	10	267	0.27	13.03	0.10	$2 \cdot 10^{-30}$
S_i	11	267	0.18	13.03	0.05	2
λ, μ	12	267	0.14	7.57	0.05	$2 \cdot 10^{-30}$
S_i	13	267	0.09	7.57	0.03	2
λ, μ	14	267	0.07	4.32	0.03	$2 \cdot 10^{-30}$
λ, μ	16	267	0.04	0.76	0.01	$2 \cdot 10^{-30}$

Polygonal parametrization

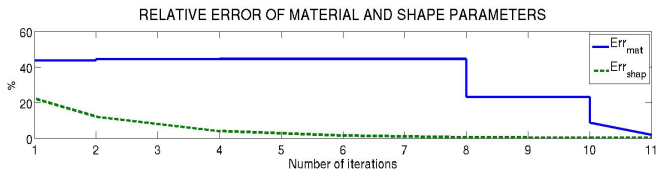
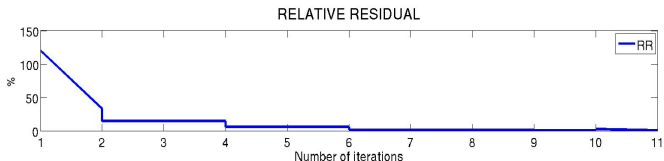
Square, Steel media, Algorithm2, Noise Level= 0%



	Target	In. Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level= 0%



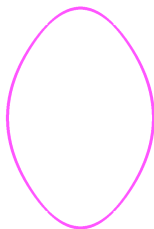
Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level= 0%

	# Iter	k_f	TRR(%)	$Err_{mat}(\%)$	$Err_{shap}(\%)$	α
s_j	1	67	120.28	43.60	22.22	1
s_j	2	83	32.79	43.60	12.13	1
λ, μ	2	100	14.79	44.51	12.13	$3, 5 \cdot 10^{-28}$
s_j	4	100	14.78	44.51	3.95	1
λ, μ	4	117	5.86	44.55	3.95	$3, 5 \cdot 10^{-28}$
s_j	6	117	5.79	44.55	1.50	1
λ, μ	6	133	1.45	44.54	1.50	$3, 5 \cdot 10^{-28}$
s_j	8	133	1.45	44.54	0.48	1
λ, μ	8	150	1.40	23.27	0.48	$3, 5 \cdot 10^{-28}$
s_j	10	150	0.89	23.27	0.35	1
λ, μ	10	167	2.39	8.68	0.35	$3, 5 \cdot 10^{-28}$
λ, μ	11	167	0.91	1.83	0.35	$3, 5 \cdot 10^{-28}$

B-spline representation

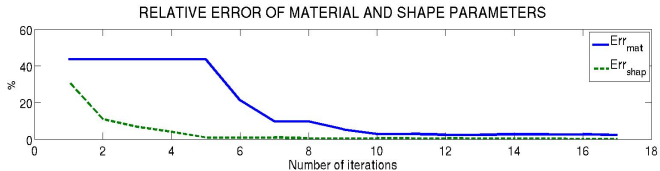
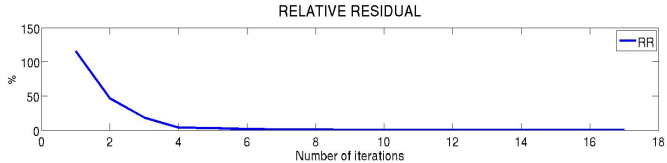
Rugby ball, Steel media, Algorithm2, Noise Level= 0%



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm2, Noise Level= 0%



B-spline representation

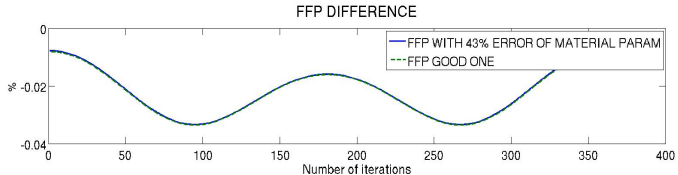
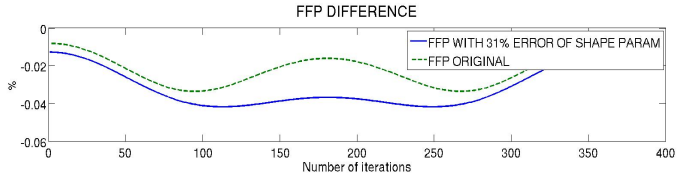
Rugby ball, Steel media, Algorithm2, Noise Level= 0%

	# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
s_j	1	66.67	115.47	43.60	31.37	1
s_j	2	100	46.13	43.60	10.90	1
s_j	3	133.33	18.19	43.60	6.71	1
s_j	4	150	3.54	43.60	4.05	1
s_j	5	183.33	2.79	43.60	0.89	1
λ, μ	6	200	1.21	21.19	0.89	$2 \cdot 10^{-28}$
λ, μ	7	200	0.93	9.69	0.89	$2 \cdot 10^{-28}$
s_j	8	200	0.89	9.69	0.39	1
λ, μ	9	233.33	0.21	5.35	0.39	$2 \cdot 10^{-28}$
λ, μ	10	233.33	0.13	2.90	0.39	$2 \cdot 10^{-28}$
s_j	11	233.33	0.12	2.90	0.39	1
λ, μ	12	250	0.11	2.43	0.39	$2 \cdot 10^{-28}$

FFP DIFFERENCE

- ▶ $\text{Err}_{\text{mat}} = 43\% \rightarrow \text{TRR} = 0.95\%$
- ▶ $\text{Err}_{\text{shape}} = 31\% \rightarrow \text{TRR} = 104\%$

FFP DIFFERENCE



Elliptic coordinate system

WORK IN PROGRESS

Ellipse, Steel media, Algorithm1

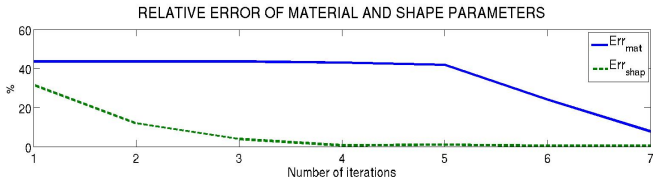
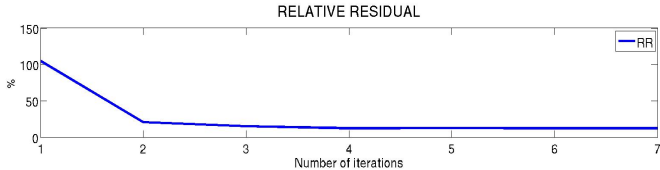
Various regularization parameters, Noise Level= 2%



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 2%



Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 2%

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	133	104.76	43.60	31.62	0.5	0.5
2	167	20.76	43.59	11.95	0.05	0.5
3	200	15.17	43.53	3.84	0.005	0.5
4	233	12.25	42.98	0.65	$5 \cdot 10^{-4}$	0.5
5	267	12.46	41.81	0.99	$5 \cdot 10^{-5}$	0.5
6	300	12.32	24.13	0.39	$5 \cdot 10^{-6}$	0.5
7	333	12.32	7.81	0.45	$5 \cdot 10^{-7}$	0.5

Table: Development of Total and Relative Errors

Conclusions

- ▶ The selection of regularization parameters and frequencies seems to be very sensitive with respect to the problem

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- ▶ The selection of regularization parameters and frequencies seems to be very sensitive with respect to the problem
- ▶ It is easier to deal with the Algorithm 1
- ▶ The first recovery of shape parameters seems to be efficient

Conclusions

- ▶ Recompute the numerical computations with bigger noise level

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- ▶ Recovery of shape and material parameters of an elastic object immersed in an inhomogeneous media

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- ▶ Recompute the numerical computations with bigger noise level
- ▶ Recovery of shape and material parameters of an elastic object immersed in an inhomogeneous media
- ▶ Numerical computations to retrieve shape and material parameters of an elastic anisotropic object

Conclusions

- ▶ Recompute the numerical computations with bigger noise level
- ▶ Recovery of shape and material parameters of an elastic object immersed in an inhomogeneous media
- ▶ Numerical computations to retrieve shape and material parameters of an elastic anisotropic object
- ▶ Employ multiple angles of incidence

DANKESCHÖN

Cases depending on Lamé coefficients

Steel

$$\lambda = 9.695 \cdot 10^9 \text{ N/m}^2 \quad \mu = 7.617 \cdot 10^9 \text{ N/m}^2$$

$$\text{Err}_{mat}^0 = 43.59\%$$

$$\text{Err}\lambda^0 = 48.42\%$$

$$\text{Err}\mu^0 = 34.35\%$$

Aluminium

$$\lambda = 5.10878772 \cdot 10^9 \text{ N/m}^2 \quad \mu = 2.63165868 \cdot 10^9 \text{ N/m}^2$$

$$\text{Err}_{mat}^0 = 41.26\%$$

$$\text{Err}\lambda^0 = 2.13\%$$

$$\text{Err}\mu^0 = 89.99\%$$

Elliptic coordinate system

Ellipse, Aluminium media, Algorithm1, Noise Level= 0%

$$s_1, s_2 \rightarrow \Gamma = \{(s_1 \cos \theta, s_2 \sin \theta), \quad \theta \in [0, 2\pi)\}$$

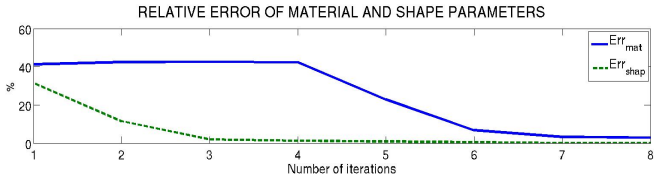
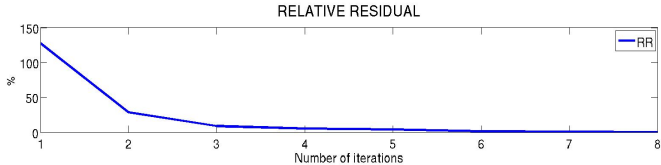


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

$$k_f = 267, \alpha = 0.5, \alpha^4 = 0.005$$

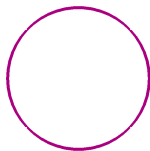
Elliptic coordinate system

Ellipse, Aluminium media, Algorithm1, Noise Level= 0%



Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 0%

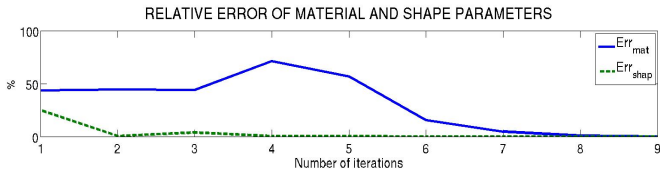
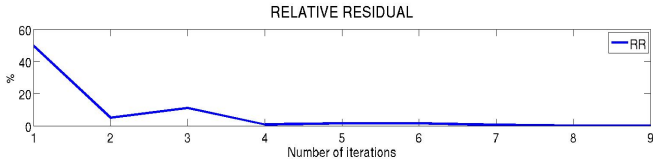


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.01	0.0075

$$k_f = 267, \alpha = 1, \alpha^3 = 5 \cdot 10^{-6}$$

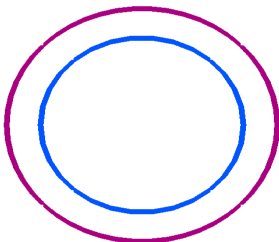
Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 0%



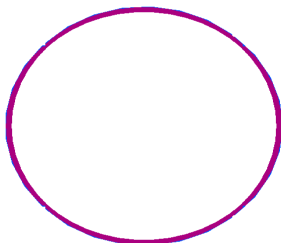
Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 0$



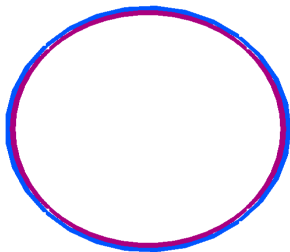
Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 0%
#Iter = 1



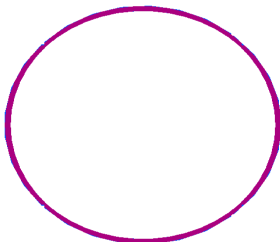
Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 2$



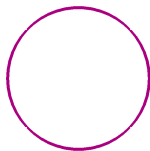
Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 0%
 $\#Iter = 3$



Elliptic coordinate system

Circle, Aluminium media, Algorithm1, Noise Level= 0%

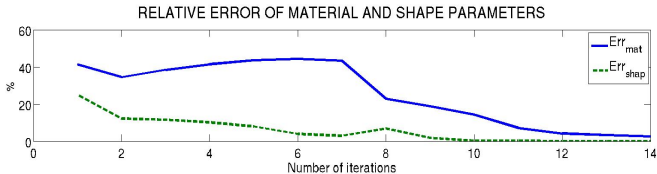
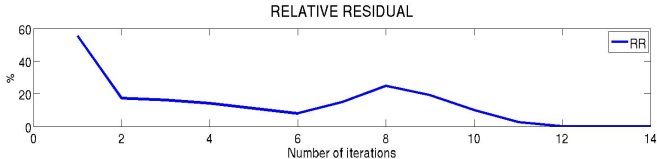


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.01	0.0075

$$k_f = 267, \alpha = 1, \alpha^7 = 0.05$$

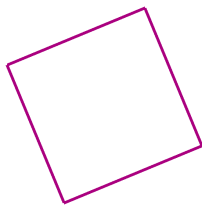
Elliptic coordinate system

Circle, Aluminium media, Algorithm1, Noise Level= 0%



Polygonal parametrization

Square, Aluminium media, Algorithm1, Noise
Level= 0%

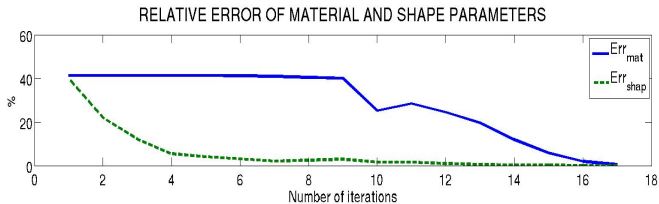
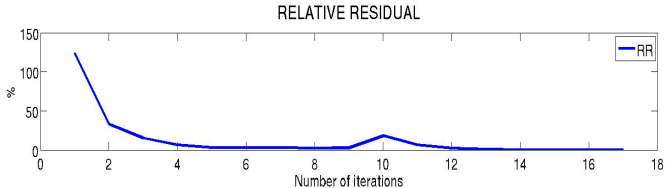


	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

$$k_f \in [67, 333], \alpha = 1, \alpha^9 = 0.05$$

Polygonal parametrization

Square, Aluminium media, Algorithm1, Noise Level= 0%



Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%



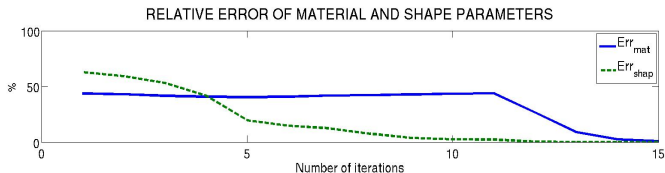
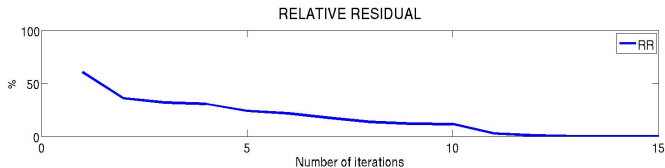
	Target	In.Guess
s_1	0.01	0.0125
s_2	0.007	0
s_3	0.0025	0

Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%

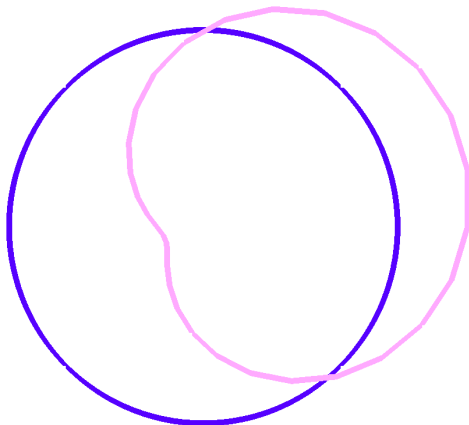
# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
1	800	60.51	43.60	62.94	12
2	800	35.74	42.94	59.18	12
3	800	31.77	41.68	53.23	12
5	800	23.97	40.30	19.76	12
7	800	17.39	41.82	12.40	12
9	800	11.75	42.86	3.68	12
11	200	2.60	43.78	2.31	10^{-6}
12	200	0.81	26.55	0.28	10^{-6}
13	200	0.08	8.99	0.02	10^{-6}
14	200	0.01	2.46	0.02	10^{-6}
15	200	0.01	0.64	0.01	10^{-6}

Fourier Parametrization



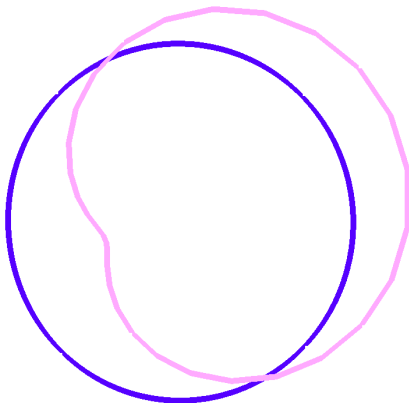
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



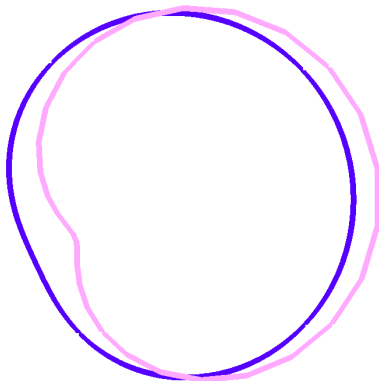
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=3



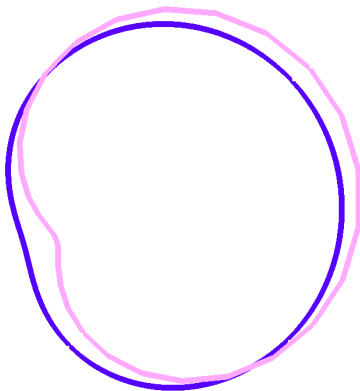
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=4



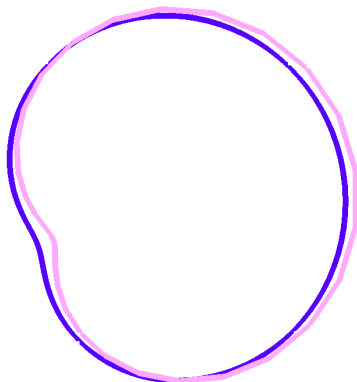
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=5



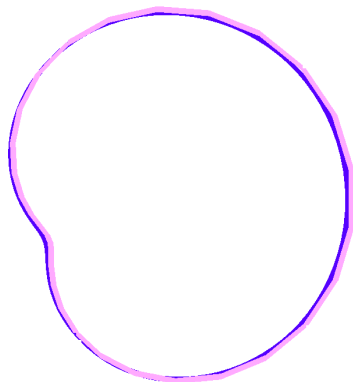
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=7



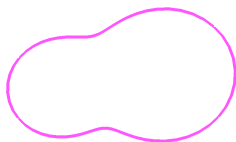
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=9



Fourier Parametrization

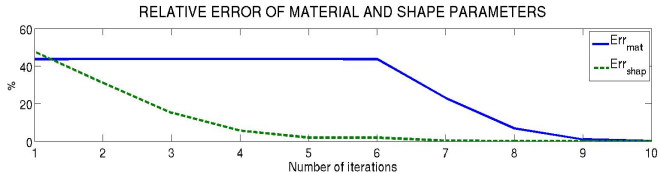
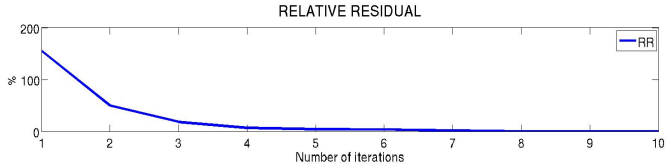
Peanut, Steel media, Algorithm1, Noise Level= 0%



	Target	In. Guess
s_1	0.01	0.0125
s_2	0.002	0
s_3	0.0005	0
s_4	0.004	0
s_5	0.001	0

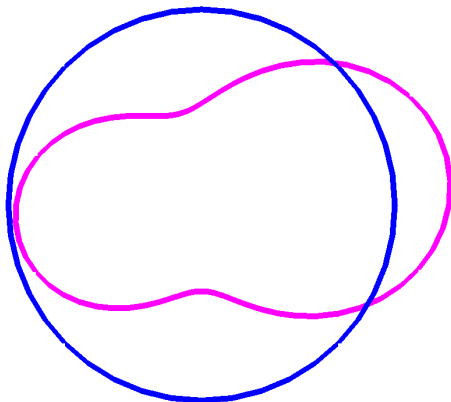
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%



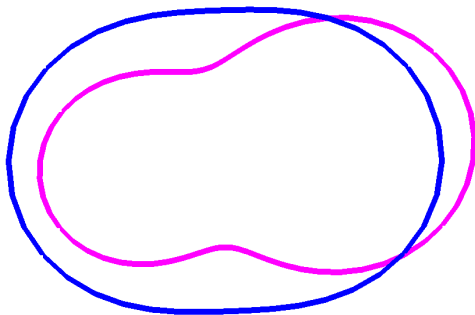
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



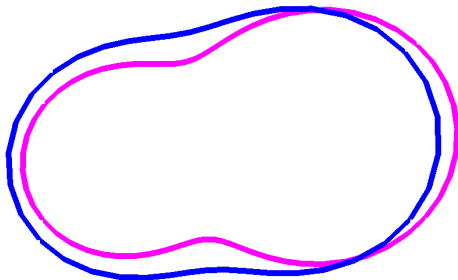
Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=1**



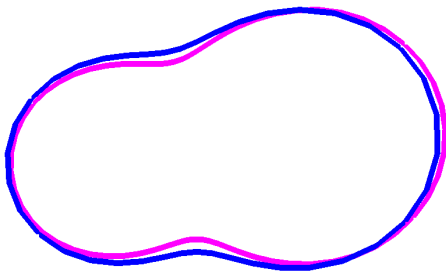
Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=2**



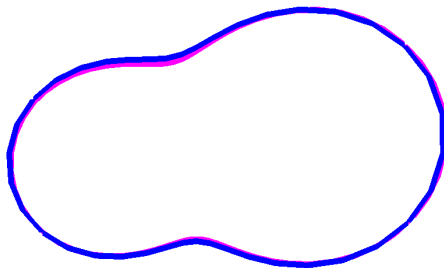
Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=3**



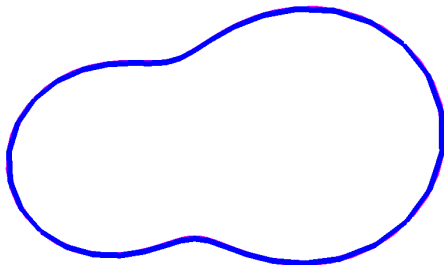
Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=4**



Fourier Parametrization

**Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=6**



Computational Methodology

Algorithm 2

- ▶ Initialization $p_{\infty, meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma(s_1^0, \dots, s_r^0)$
- ▶ **WHILE** Relative Residual (RR) > 0.01 && $n < \text{MaxNumberIter}$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ $\alpha_s \rightarrow \text{FFP } p_{\infty}(\lambda^n, \mu^n, \Gamma^n)$
 - ▶ Fréchet Derivative, $\frac{\partial p_{\infty}}{\partial s_j}$ $i = 1, \dots, r$
 - ▶ Recovery $s_1^{n+1}, \dots, s_r^{n+1}$, $n = n + 1$
 - ▶ stagnation = $\frac{p_{\infty}^n - p_{\infty}^{n+1}}{p_{\infty}^n} \cdot 100$
 - ▶ stagnation = 100
 - ▶ **WHILE** stagnation > 0.75
 - ▶ $\alpha_m \rightarrow \text{FFP } p_{\infty}(\lambda^n, \mu^n, \Gamma^{n+1})$
 - ▶ Fréchet Derivative $\frac{\partial p_{\infty}}{\partial \lambda}, \frac{\partial p_{\infty}}{\partial \mu}$
 - ▶ Recovery, λ^{n+1}, μ^{n+1} , $n = n + 1$
 - ▶ stagnation = $\frac{p_{\infty}^n - p_{\infty}^{n+1}}{p_{\infty}^n} \cdot 100$
- ▶ Increase the frequency (*) $\rightarrow p_{\infty, meas}$

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level= 0%

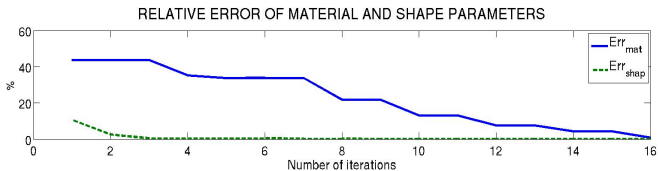
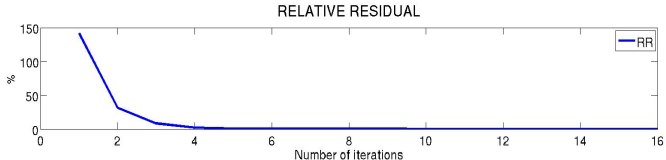
$$s_1, s_2 \rightarrow \Gamma = \{(s_1 \cos \theta, s_2 \sin \theta), \quad \theta \in [0, 2\pi)\}$$



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level= 0%

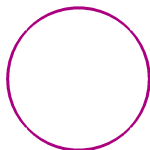


Elliptic coordinate system

	# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
S_i	1	267	141.63	43.60	10.68	2
S_i	2	267	31.79	43.60	2.58	2
S_i	3	267	8.36	43.60	0.49	2
λ, μ	4	267	2.13	35.10	0.49	$2 \cdot 10^{-30}$
λ, μ	5	267	0.96	33.71	0.49	$2 \cdot 10^{-30}$
λ, μ	6	267	0.84	33.74	0.49	$2 \cdot 10^{-30}$
S_i	7	267	0.84	33.74	0.19	2
λ, μ	8	267	0.49	21.71	0.19	$2 \cdot 10^{-30}$
S_i	9	267	0.34	21.71	0.10	2
λ, μ	10	267	0.27	13.03	0.10	$2 \cdot 10^{-30}$
S_i	11	267	0.18	13.03	0.05	2
λ, μ	12	267	0.14	7.57	0.05	$2 \cdot 10^{-30}$
S_i	13	267	0.09	7.57	0.03	2
λ, μ	14	267	0.07	4.32	0.03	$2 \cdot 10^{-30}$
λ, μ	16	267	0.04	0.76	0.01	$2 \cdot 10^{-30}$

Elliptic coordinate system

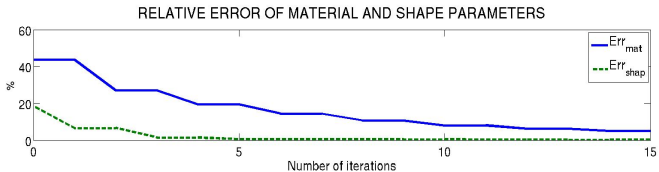
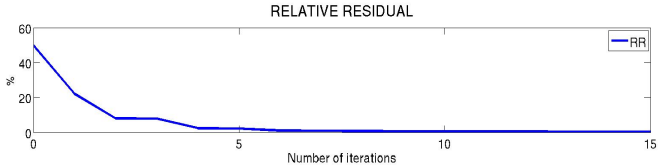
Circle, Aluminium media, Algorithm1, Noise Level= 0%



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.01	0.0075

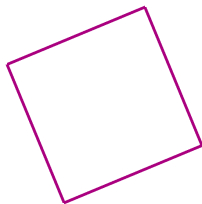
Elliptic coordinate system

Circle, Steel media, Algorithm2, Noise Level= 0%



Polygonal parametrization

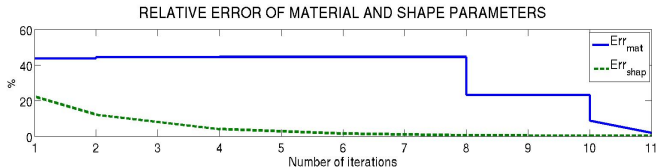
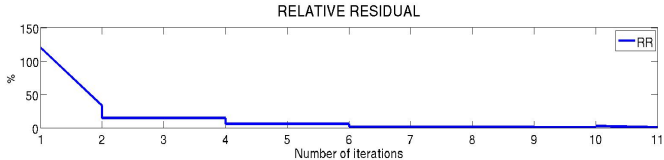
Square, Steel media, Algorithm2, Noise Level= 0%



	Target	In. Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level= 0%



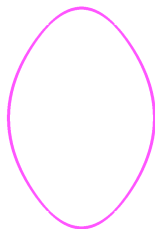
Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level= 0%

	# Iter	k_f	TRR(%)	$Err_{mat}(\%)$	$Err_{shap}(\%)$	α
s_j	1	67	120.28	43.60	22.22	1
s_j	2	83	32.79	43.60	12.13	1
λ, μ	2	100	14.79	44.51	12.13	$3, 5 \cdot 10^{-28}$
s_j	4	100	14.78	44.51	3.95	1
λ, μ	4	117	5.86	44.55	3.95	$3, 5 \cdot 10^{-28}$
s_j	6	117	5.79	44.55	1.50	1
λ, μ	6	133	1.45	44.54	1.50	$3, 5 \cdot 10^{-28}$
s_j	8	133	1.45	44.54	0.48	1
λ, μ	8	150	1.40	23.27	0.48	$3, 5 \cdot 10^{-28}$
s_j	10	150	0.89	23.27	0.35	1
λ, μ	10	167	2.39	8.68	0.35	$3, 5 \cdot 10^{-28}$
λ, μ	11	167	0.91	1.83	0.35	$3, 5 \cdot 10^{-28}$

B-spline representation

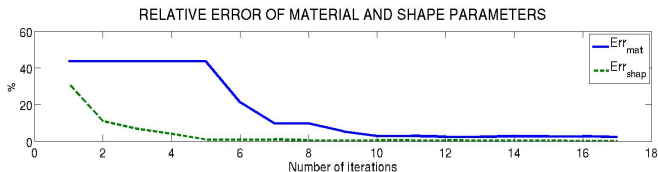
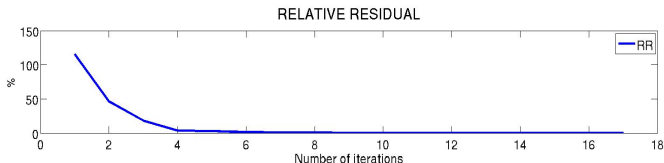
Rugby ball, Steel media, Algorithm2, Noise Level= 0%



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm2, Noise Level= 0%



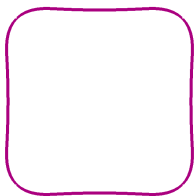
B-spline representation

Rugby ball, Steel media, Algorithm2, Noise Level= 0%

	# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
s_j	1	66.67	115.47	43.60	31.37	1
s_j	2	100	46.13	43.60	10.90	1
s_j	3	133.33	18.19	43.60	6.71	1
s_j	4	150	3.54	43.60	4.05	1
s_j	5	183.33	2.79	43.60	0.89	1
λ, μ	6	200	1.21	21.19	0.89	$2 \cdot 10^{-28}$
λ, μ	7	200	0.93	9.69	0.89	$2 \cdot 10^{-28}$
s_j	8	200	0.89	9.69	0.39	1
λ, μ	9	233.33	0.21	5.35	0.39	$2 \cdot 10^{-28}$
λ, μ	10	233.33	0.13	2.90	0.39	$2 \cdot 10^{-28}$
s_j	11	233.33	0.12	2.90	0.39	1
λ, μ	12	250	0.11	2.43	0.39	$2 \cdot 10^{-28}$

B-spline representation

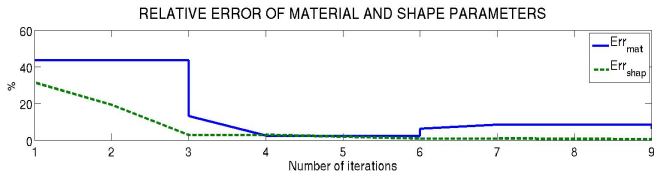
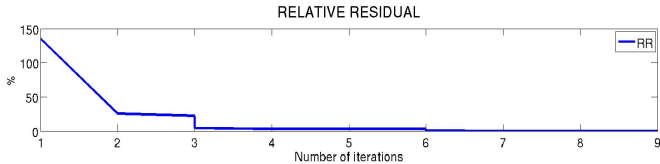
Rounded Square, Steel media, Algorithm2, Noise Level= 0%



	Target	In. Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175
s_5	0.01	0.0175
s_6	0.015	0.0175
s_7	0.01	0.0175
s_8	0.015	0.0175

B-spline representation

**Rounded square, Steel media, Algorithm2, Noise
Level= 0%**



B-spline representation

**Rounded square, Steel media, Algorithm2, Noise
Level= 0%**

	# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
s_j	1	66.67	135.42	43.60	31.38	1
s_j	2	100	25.52	43.60	19.22	1
s_j	3	133.33	22.42	43.60	2.89	1
λ, μ	4	166.67	4.03	13.09	2.89	$3.5 \cdot 10^{-28}$
λ, μ	5	166.67	3.26	2.46	2.89	$3.5 \cdot 10^{-28}$
s_j	6	166.67	3.19	2.46	0.93	1

Elliptic coordinate system

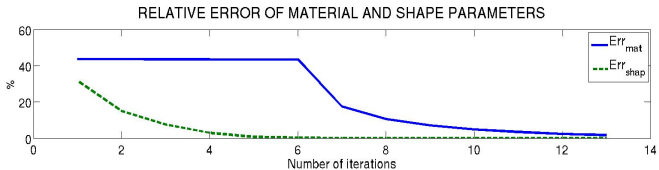
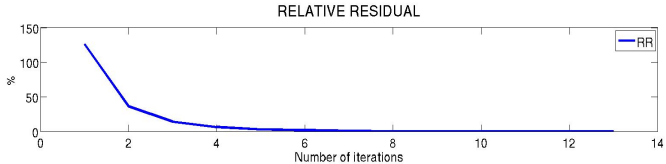
Ellipse, Steel media, Algorithm1, Noise Level= 2%



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

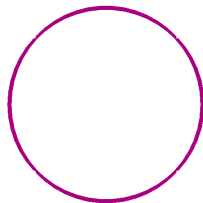
Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level= 2%



Elliptic coordinate system

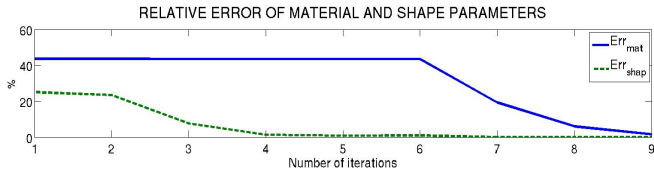
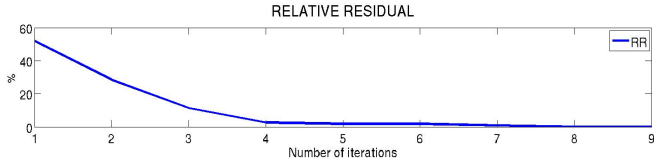
Circle, Steel media, Algorithm1, Noise Level= 2%



Shape par.	
s_1	0.01
s_2	0.01

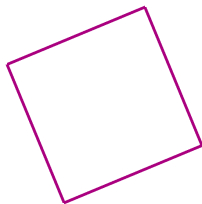
Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level= 2%



Polygonal parametrization

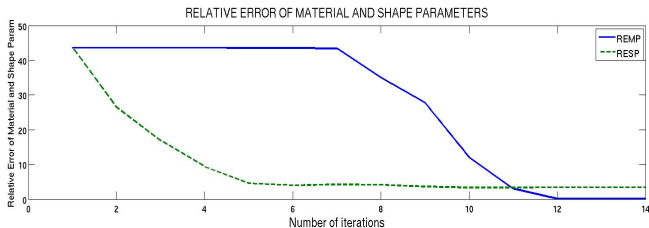
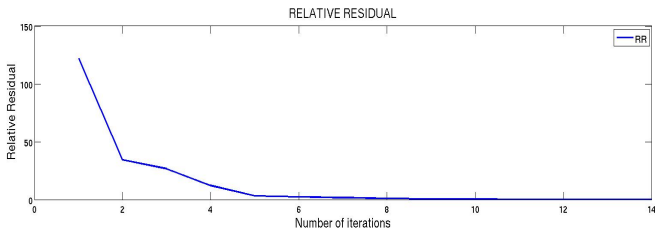
Square, Steel media, Algorithm1, Noise Level= 2%



	Target	In. Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 2%



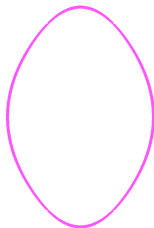
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 2%

# Iter	k_f	TRR(%)	$\text{Err}_{mat}(\%)$	$\text{Err}_{shap}(\%)$	α
1	67	121.74	43.60	43.85	2
2	100	34.31	43.61	26.47	2
3	133	26.70	43.61	16.92	2
4	167	12.09	43.60	9.35	2
5	200	3.18	43.55	4.61	2
6	233	2.34	43.50	3.97	2
7	267	1.61	43.42	4.23	$2 \cdot 10^{-6}$
8	300	1.09	34.92	4.09	$2 \cdot 10^{-6}$
9	333	0.47	27.75	3.64	$2 \cdot 10^{-6}$
10	367	0.25	12.03	3.35	$2 \cdot 10^{-6}$
11	400	0.09	3.02	3.33	$2 \cdot 10^{-6}$
12	433	0.02	0.12	3.37	$2 \cdot 10^{-6}$

B-spline representation

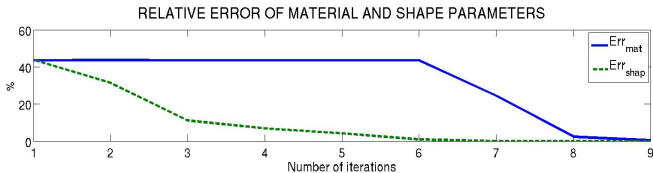
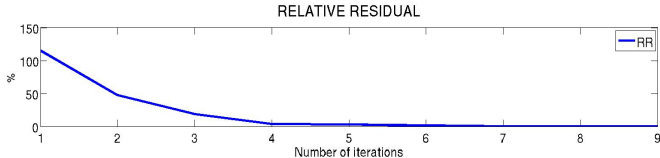
Rugby ball, Steel media, Algorithm1, Noise Level= 2%



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 2%

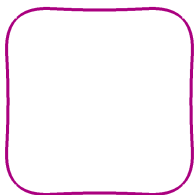


Rugby ball, Steel media, Algorithm1, Noise Level= 2%

# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
1	67	114.95	43.60	43.85	1
2	100	47.13	43.61	31.42	1
3	133	18.55	43.60	11.07	1
4	167	3.61	43.60	6.85	1
5	200	2.83	43.60	4.14	1
6	233	1.22	43.59	0.90	10^{-7}
7	266	0.32	24.43	0.10	10^{-7}
8	300	0.04	2.42	0.01	10^{-7}
9	333	0.01	0.46	0.00	10^{-7}

B-spline representation

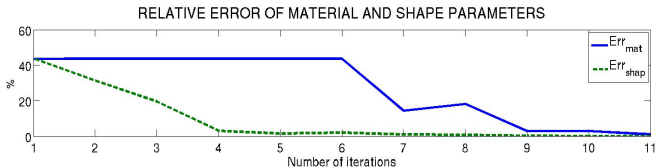
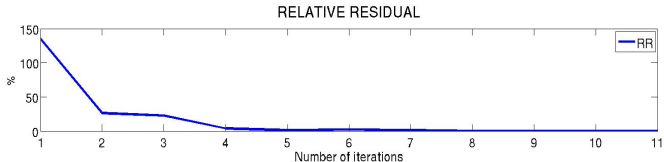
**Rounded Square, Steel media, Algorithm1, Noise
Level= 2%**



	Target	In. Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175
s_5	0.01	0.0175
s_6	0.015	0.0175
s_7	0.01	0.0175
s_8	0.015	0.0175

B-spline representation

Rounded Square, Steel media, Algorithm1, Noise Level= 2%



Rounded Square, Steel media, Algorithm1, Noise Level= 2%

# Iter	k_f	TRR(%)	Err _{mat} (%)	Err _{shap} (%)	α
1	67	135.73	43.60	43.85	1
2	100	26.34	43.61	31.34	1
3	133	22.63	43.64	19.51	1
4	167	3.94	43.72	3.00	1
5	200	1.31	43.71	1.51	1
6	233	2.21	43.70	1.96	10^{-7}
7	267	1.23	14.29	0.92	10^{-7}
8	300	0.28	18.13	0.64	10^{-7}
9	333	0.10	2.82	0.18	10^{-7}