A data assimilation procedure in blood flow simulations

J. Tiago, T. Guerra, A. Sequeira
CEMAT (IST, ULisboa)

Workshop on Inverse Problems for PDEs
Bremen, March 31, 2016

INSTITUTO SUPERIOR TÉCNICO

cemat IST CENTRO de MATEMÁTICA E APLICAÇÕES

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR Portugal

Motivation

- The progress of medical imaging techniques, blood flow modelling, and computational capacity allow us to consider the possibility of obtaining patient specific simulations.
- Numerical simulations must be reliable
- Important flow indicators as the Wall Shear Stress are highly sensitive to the geometry and the parameters in the model. But those are hard to "guess"!
    - Missing link: use the available data to make simulations reliable...

Motivation

- The progress of medical imaging techniques, blood flow modelling, and computational capacity allow us to consider the possibility of obtaining patient specific simulations.
- Numerical simulations must be reliable
- Important flow indicators as the Wall Shear Stress are highly sensitive to the geometry and the parameters in the model. But those are hard to "guess"!
    - Missing link: use the available data to make simulations reliable...

The Cardiovascular System - too big to model in detail

Fig : *medicalnoises.com*

- Aorta with characteristic diameter - 2.5 cm
- 50 arteries with diameter - $1 - 10$ mm
- 10^3 arteries with diameter - $0.5 - 1$ mm
- 10^4 arterioles with diameter $0.01 - 0.5$ mm
- 10^6 capillaries with diameter - $0.006 - 0.01$ mm

[Thiriet, Parker, *Physiology and pathology of the cardiovascular system*,Cardiovascular Mathematics, 2009]

Optimal Control in Cardiovascular Mathematics

Optimal Control techniques can be useful for modeling the cardiovascular system:

- Shape optimization: Rozza, Quarteroni,...
- Inverse problems (parameter estimation): Gerbeau, Moireau, Figueroa, Veneziani, Vergara, ...
- Flow reconstruction: D'Elia, Perego, Veneziani,...
- Boundary reconstruction: Gambaruto, Tiago, Sequeira

Optimal Control in Cardiovascular Mathematics

Optimal Control techniques can be useful for modeling the cardiovascular system:

- Shape optimization: Rozza, Quarteroni,...
- Inverse problems (parameter estimation): Veneziani, Vergara,...
- Flow reconstruction: D'Elia, Perego, Veneziani,...
- Boundary reconstruction: Gambaruto, T., Sequeira

Data Assimilation

- Flow reconstruction can be seen as Data Assimilation (DA) problem: use known information to obtain/improve the accuracy of the simulations
- Different approaches for DA - we consider the Variational Approach: solve a certain optimal control problem!

First suggested in:

[D'Elia, A. Veneziani, *Methods for assimilating blood velocity measures in hemodynamics simulations: preliminary results, Procedia Comput. Sci., 2010.*]

Data Assimilation

- Flow reconstruction can be seen as Data Assimilation (DA) problem: use known information to obtain/improve the accuracy of the simulations
- Different approaches for DA - we consider the Variational Approach: solve a certain optimal control problem!

First suggested in:

[D'Elia, A. Veneziani, *Methods for assimilating blood velocity measures in hemodynamics simulations: preliminary results, Procedia Comput. Sci., 2010.*]

Blood Composition

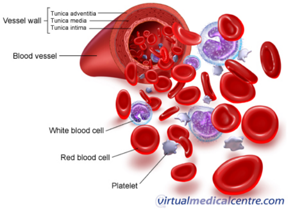


Fig : *myvmc.com*

Blood is a suspension of particles in the plasma (92% of water)

- Red Blood Cells (RBC) - $6-8\ \mu m/\ 4-6\times10^6$ per mm^3
- White Blood Cells - $8-18$ μm / $4-10\times10^6$ per mm^3
- RBC - 45% of blood volume.

[Quarteroni, Formaggia, *Modelling the Cardivascular System*, 2006]

non-Newtonian rheology

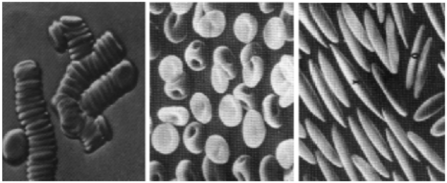


Fig : RBC form three-dimensional structures at low shear rates mainly under pathological conditions.

- *Fahraeus* - 1929.
- *Chien* - 1970.
- *Robertson, Sequeira, Kameneva* - 2008.

Viscosity laws for shear-thinning behavior

- $\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1+(\lambda\dot{\gamma})^2)^{\frac{1-n}{2}}}$ $\rightarrow$ Carreau
  - $\mu_0 = 4.56 \cdot 10^{-2}\ Pa.s, \quad \mu_\infty = 3.2 \cdot 10^{-3}\ Pa.s$
  - $n = 0.344, \quad \lambda = 10.3\ s$
- $\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1+(\lambda\dot{\gamma})^b)^{\frac{1-n}{b}}}$ $\rightarrow$ Carreau-Yasuda
  - $\mu_0 = 6.57 \cdot 10^{-2}\ Pa.s, \quad \mu_\infty = 4.47 \cdot 10^{-3}\ Pa.s$
  - $n = 0.34, \quad b = 1.76, \quad \lambda = 10.4\ s$
- $\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1+(\lambda\dot{\gamma})^b)^a}$ $\rightarrow$ Generalized Cross
  - $\mu_0 = 1.6 \cdot 10^{-1}\ Pa.s, \quad \mu_\infty = 3.6 \cdot 10^{-3}\ Pa.s$
  - $a = 1.23, \quad b = 0.64, \quad \lambda = 8.2\ s$

- *Robertson et al.*, Hem. Fl. Mod. An. Sim., 2008.
- *Gambaruto et al.*, Math. BioSc. Eng., 2011.
- *Bodnar, Sequeira, Prosi*, App. Math. Comp., 2011.

Model for the stationary case: medium size arteries with rigid walls - The Navier-Stokes (Generalized) Eq.

$$
\begin{cases}
-div\ (\tau(D\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\
div\,\mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u} = 0 & \text{on } \Gamma_{wall} \\
\mathbf{u} = \mathbf{g} & \text{on } \Gamma_{in} \\
(-p\mathbf{I} + 2\mu D\mathbf{u})\cdot\mathbf{n} = 0 & \text{on } \Gamma_{out}.
\end{cases}
$$

where $\tau(D\mathbf{u}) = 2\mu(\sqrt{D\mathbf{u} : D\mathbf{u}})D\mathbf{u} = 2\mu(\dot{\gamma})D\mathbf{u}$

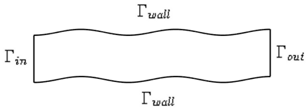


Fig : Domain representation

The optimal control problem: velocity tracking

$$
\begin{cases}
-div\ (\tau(D\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\
div\,\mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u} = 0 & \text{on } \Gamma_{wall} \\
\mathbf{u} = \mathbf{g} & \text{on } \Gamma_{in} \\
(-p\mathbf{I} + 2\mu D\mathbf{u})\cdot\mathbf{n} = 0 & \text{on } \Gamma_{out}.
\end{cases}
$$

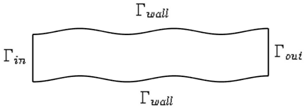


Fig : Domain representation

The optimal control problem: velocity tracking

$$\begin{cases} -div\ (\tau(D\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ div\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_{wall} \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_{in} \\ (-p\mathbf{I} + 2\mu D\mathbf{u})\cdot\mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases}$$

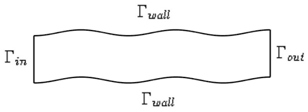


Fig : Domain representation

The optimal control problem: velocity tracking

$$\mathbf{J}(\mathbf{u}, \mathbf{g}) = \int_{\Omega_{part}} |\mathbf{u} - \mathbf{u_d}|^2 dx$$

The optimal control problem: velocity tracking

$$\mathbf{J}(\mathbf{u}, \mathbf{g}) = w_1 \int_{\Omega_{part}} |\mathbf{u} - \mathbf{u}_d|^2 dx + w_2 \int_{\Gamma_{in}} |\mathbf{g}|^2 ds$$

The optimal control problem: velocity tracking

$$\mathbf{J}(\mathbf{u}, \mathbf{g}) = w_1 \int_{\Omega_{part}} |\mathbf{u} - \mathbf{u}_d|^2 dx + w_2 \int_{\Gamma_{in}} |\nabla_s \mathbf{g}|^2 ds$$

The optimal control problem: velocity tracking

$$\mathbf{J}(\mathbf{u}, \mathbf{g}) = w_1 \int_{\Omega_{part}} |\mathbf{u} - \mathbf{u}_d|^2 dx + w_2 \int_{\Gamma_{in}} |\mathbf{g}|^2 ds + w_3 \int_{\Gamma_{in}} |\nabla_s \mathbf{g}|^2 ds$$

The optimal control problem: velocity tracking

Minimize

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = w_1 \int_{\Omega_{part}} |\mathbf{u}-\mathbf{u}_d|^2 dx + w_2 \int_{\Gamma_{in}} |\mathbf{g}|^2 ds + w_3 \int_{\Gamma_{in}} |\nabla_s \mathbf{g}|^2 ds$$

subject to

$$\begin{cases} -div\ (\tau(D\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ div\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_{wall} \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_{in} \\ (-p\mathbf{I} + 2\mu D\mathbf{u})\cdot\mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases}$$

Remark:

In *"D'Elia, A. Veneziani, Methods for assimilating blood velocity measures in hemodynamics simulations: preliminary results, Procedia Comput. Sci., 2010."* *the control was taken as the pressure - Neumann control.*

Theoretical frame: Non-Newtonian case

Existence results and optimality conditions for distributed and boundary control, Neumann type.

$$\mathbf{J}(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_{\Omega} |\mathbf{u} - \mathbf{u}_d|^2 dx + \frac{w2}{2} \int_{\Omega} |v|^2 ds$$

for generalized Newtonian models

Eduardo Casas, Fernandez, Nadir Arada, Telma Guerra.

Theoretical frame: Newtonian case

Existence results and optimality conditions for full Dirichlet boundary control

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_{\Omega} |\mathbf{u}-\mathbf{u}_d|^2 dx + \frac{w2}{2}\int_{\partial\Omega} |g|^2 ds$$

Gunzburger, Casas, Manservisi, Fursikov, De los Reyes, Kunish,....

- Gunzburger, Hou, Svobodny, RAIRO- Mod. Math. An. Num., 1991
- De los Reyes, Kunisch, Num. An., 2005
- Manservisi, I.J. Num. An. Mod., 2007
- De los Reys, Yousept, J. Math. An. App., 2009

Except in

- Fursikov, Rannacher, Ad. Math. F. Mech. 2009

Theoretical frame: Newtonian case

Existence results and optimality conditions for full Dirichlet boundary control

$$\mathbf{J}(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_{\Omega} |\mathbf{u} - \mathbf{u}_d|^2 dx + \frac{w2}{2} \int_{\partial\Omega} |g|^2 ds$$

Gunzburger, Casas, Manservisi, Fursikov, De los Reyes, Kunish,....

- Gunzburger, Hou, Svobodny, RAIRO- Mod. Math. An. Num., 1991
- De los Reyes, Kunisch, Num. An., 2005
- Manservisi, I.J. Num. An. Mod., 2007
- De los Reys, Yousept, J. Math. An. App., 2009

Except in

- Fursikov, Rannacher, Ad. Math. F. Mech. 2009

Theoretical Frame: Direct Problem

Theorem

Let $g \in \mathbf{H}_0^1(\Gamma_{in})$ such that $\|g\|_{\mathbf{H}_0^1(\Gamma_{in})} \leq \rho$, for $\rho > 0$ sufficiently small, and $f \in \mathbf{L}^{\frac{3}{2}}(\Omega)$. Then, there exists a unique weak solution $u \in \mathbf{V}_{\Gamma_{wall}}$ of the Navier-Stokes eq which verifies

$$\|u\|^2_{\mathbf{H}^1(\Omega)} \leq \alpha\left(\|g\|^2_{\mathbf{H}_0^1(\Gamma_{in})}\right) + \|f\|^2_{\mathbf{L}^{\frac{3}{2}}(\Omega)}, \qquad (1)$$

where $\alpha(s) = c(s^2 + s)$.

$$\mathbf{V}_{\Gamma_{wall}} = \left\{\mathbf{v} \in H^1(\Omega) : \gamma_{\Gamma_{wall}}\mathbf{v} = 0, \, div\, \mathbf{v} = 0\right\}$$

$$H_0^1(\Gamma_{in}) = \left\{v \in L^2(\Gamma_{in}) \mid \nabla_s v \in L^2(\Gamma_{in}), \, \gamma_{\partial\Gamma_{in}} v = 0\right\}$$

[Guerra, Sequeira, Tiago, *Existence of optimal boundary control for the Navier-Stokes equations with mixed boundary conditions*, Port. Math, 2015].

Theoretical Frame: Direct Problem - Some tools

- $H^{\frac{1}{2}}_{00}(\Gamma) = \left\{ g \in L^2(\Gamma) \mid \exists v \in H^1(\Omega),\ v_{|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega),\ \gamma_\Gamma v = g,\ \gamma_{\partial\Omega\setminus\Gamma} v = 0 \right\}$
  closed subspace of $H^{\frac{1}{2}}(\Gamma)$.
- The continuous embeddings $H^1_0(\Gamma) \subset H^{\frac{1}{2}}_{00}(\Gamma)$ and $H^{\frac{1}{2}}_{00}(\Gamma) \subset L^2(\Gamma)$

  [R. Dautray, J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, 2000]
- Extension operator from $\hat{H}^{\frac{1}{2}}(\Gamma_{in} \cup \Gamma_{out})$ to $\mathbf{V}_{\Gamma_{wall}}$
  $\hat{H}^{\frac{1}{2}}(\Gamma_1 \cup \Gamma_2) = \left\{ (g_1, g_2) \in H^{\frac{1}{2}}_{00}(\Gamma_1) \times H^{\frac{1}{2}}_{00}(\Gamma_2) \mid \int_{\Gamma_1} g_1 \cdot n\, ds + \int_{\Gamma_2} g_2 \cdot n\, ds = 0 \right\}$
- Fixed point argument + Estimates for Stokes solution

Theoretical Frame: Control Problem - Existence Result

Minimize

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = w_1 \int_{\Omega_{part}} |\mathbf{u}-\mathbf{u}_d|^2 dx + w_2 \int_{\Gamma_{in}} |\mathbf{g}|^2 ds + w_3 \int_{\Gamma_{in}} |\nabla_s \mathbf{g}|^2 ds$$

[$\Omega_{part} = \cup_{i=1}^{m} s_i$ where S_i are cross sections of the "vessel" domain]

subject to

$$\begin{cases} -\mu \triangle \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ div\, \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_{wall} \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_{in} \\ \mu \dfrac{d\mathbf{u}}{d\mathbf{n}} - \mathbf{n}p = 0 & \text{on } \Gamma_{out}. \end{cases}$$

$$\mathbf{g} \in \mathcal{U}, \quad \mathbf{u} \in \mathbf{V}_{\Gamma_{wall}} = \left\{\mathbf{v} \in H^1(\Omega) : \gamma_{\Gamma_{wall}} \mathbf{v} = 0,\ div\, \mathbf{v} = 0\right\}$$

$$\mathcal{U} = \left\{ g \in H_0^1(\Gamma_{in}) \mid \|g\|_{H_0^1(\Gamma)} \le \rho \right\} \subset \mathcal{U}_0$$

$$\mathcal{U}_0 = \left\{ g \in H_0^1(\Gamma_{in}) : \text{s.t. NSEq have a unique weak solution} \right\}.$$

Theoretical Frame: Control Problem - Tools

Direct Method of the Calculus of Variations

- Non-emptiness of admissible set
- Compactness of minimizing sequences
- J is weakly l.s.c (we need to check continuity and convexity properties)

Numerical Approach - Discretization

- Discretize then Optimize approach -
    - Discretize using Finite Element Methods + SUPG stabilization:

Minimize

$$\mathbf{J}(\mathbf{u}, \mathbf{g}) = w_1 \int_{\Omega_{part}} |\mathbf{u} - \mathbf{u}_d|^2 dx + w_2 \int_{\Gamma_{in}} |\nabla_s \mathbf{g}|^2 ds$$

subject to

$$\begin{cases} -div\ (\tau(D\mathbf{u})) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ div\, \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_{wall} \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_{in} \\ (-p\mathbf{I} + 2\mu D\mathbf{u}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases}$$

Numerical Approach - Discretization

- Discretize then Optimize approach -
  - Discretize using Finite Element Methods + SUPG stabilization:

Minimize

$$J(U,G) = w_1 \langle U - U_d, M(U - U_d) \rangle_{N_u} + w_2 \langle G, NG \rangle_{N_g}$$

subject to

$$\begin{cases} \mathbf{Q}(U) + \mathbf{N}(U)U + B^T P = F \\ BU = 0\,. \end{cases}$$

$$U = U(G)$$

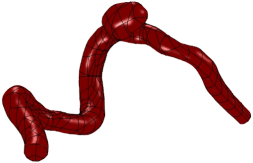
Numerical Approach- Optimization

- Solve the Nonlinear Mathematical Programming (NMP) problem
  - Use Sequential Quadratic Programming (in SNOPT - Sparse Nonlinear Optimization + SQOPT)

$$\begin{cases} \min_{g} Q(u(g), g) \\ L(u(g), g) = 0 \\ l1 \leq G(u(g), g) \leq l2 \end{cases}$$

[Gill, Murray, Saunders, *Siam Review, 2005*]

3D simulations: the case of a brain aneurysm



$\longleftarrow$

- Computational domain Segmented from Medical Images
- Mesh Tetrahedral and Hexahedral (boundary layers) elements
- Stabilized P1-P1 FEM. 213k dofs
- Finite Element Meth. solved with Comsol Multiphysics

Direct Solution - Data to be used

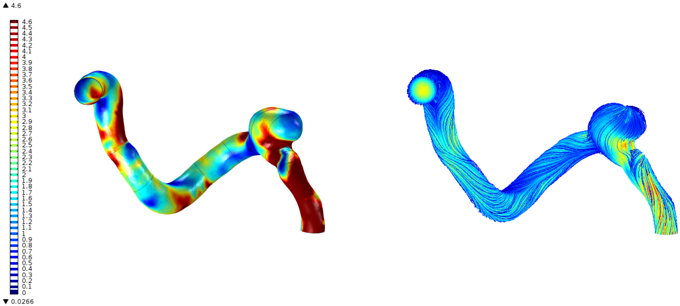
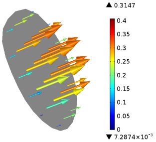


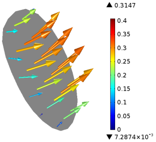
Fig : Left: WSS (N/m^2). Right: Streamlines.

Naive Solution: laminar inflow boundary, same flow rate Q



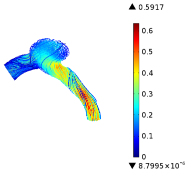
(a) Laminar Inlet for the naive solution (u_Q).

(b) Domain of interest

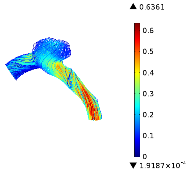


(c) True velocity (u_d) at the inlet .

Naive Solution: laminar inflow boundary, same flow rate Q



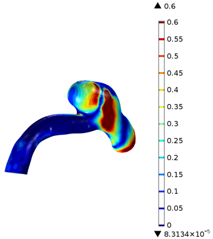
(a) Streamlines for the *naive* u_Q.



(b) Wanted Streamlines (u_d).

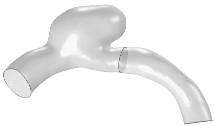
	Relative Error
$\mathbf{u}_Q$ vs $\mathbf{u}_d$	0.15199

Table : $\mathbf{u}_Q$ error relative to $\mathbf{u}_d$



Control Problem

$$\min \mathbf{J}(\mathbf{u},\mathbf{v}) = 1.e^4 \int_{\Omega_{part}} |\mathbf{u}-\mathbf{u}_d|^2 dx + 1.e^{-3} \int_{\Gamma_{in}} |\nabla_s \mathbf{v}|^2 dx$$



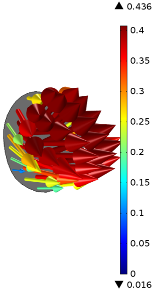
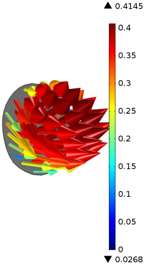
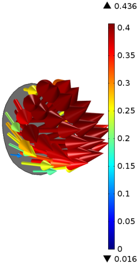


Fig : Left: Observations $\mathbf{u}_d$ in Ω_{part}. Right: Zoom view

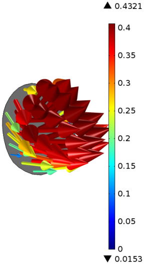
Numerical Results: 3D control



(a) *Naive* u_Q at observed section.



(b) Data u_d.



(c) Controlled u.

Numerical Results: 3D control

Weights	Rel. Error Ω_{part}	Rel. error Ω
$\mathbf{u}$ vs $\mathbf{u}_d$, $w_1 = 1e^4$	0.06778	0.11732
$\mathbf{u}$ vs $\mathbf{u}_d$, $w_1 = 1e^5$	$\boxed{0.01381}$	0.0883
$\mathbf{u}_Q$ vs $\mathbf{u}_d$	0.10584	0.15199

Table : Errors relative to $\mathbf{u}_d$ for $\mathbf{u}_Q$ and the controlled solutions $\mathbf{u}$

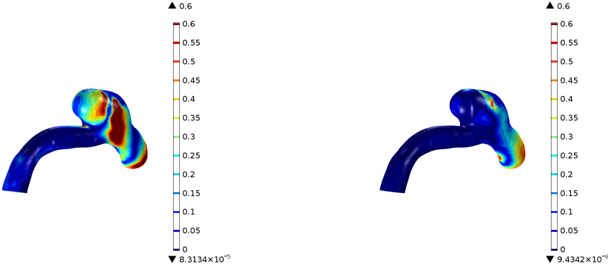
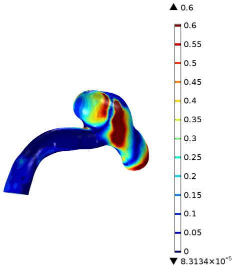


Fig : Left: Wall Shear Stress Relative Error of u_Q with respect to u_d. Right: Wall Shear Stress Relative Error of u with respect to u_d.

Numerical Results: 3D control - enriched data

Weights	Rel. Error Ω_{part}	Rel. error Ω
$\mathbf{u}$ vs $\mathbf{u}_d$, $w_1 = 1e^4$	$\boxed{0.0279}$	0.07548
$\mathbf{u}$ vs $\mathbf{u}_d$, $w_1 = 1e^5$	$\boxed{0.00524}$	$\boxed{0.03478}$
$\mathbf{u}_Q$ vs $\mathbf{u}_d$	0.10584	0.15199

Table : Errors relative to $\mathbf{u}_d$ for $\mathbf{u}_Q$ and the controlled solutions $\mathbf{u}$



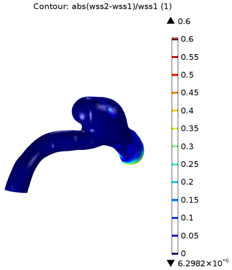


Fig : Left: Wall Shear Stress Relative Error of u_Q with respect to u_d. Right: Wall Shear Stress Relative Error of u with respect to u_d.

Future Work

- Theoretical frame:
  - Regularity;
  - Time dependent problems.
- Sensitivity analysis to understand best locations for velocity measurements and control boundaries.

Future Work

- Theoretical frame:
  - Regularity;
  - Time dependent problems.
- Sensitivity analysis to understand best locations for velocity measurements and control boundaries.
- For the velocity reconstruction, use real data and time dependent model;
  - Deal with the presence of errors (from data or/and from the model) - Kalman filter.
  - Order reduction.

Future Work

- Theoretical frame:
  - Regularity;
  - Time dependent problems.
- Sensitivity analysis to understand best locations for velocity measurements and control boundaries.
- For the velocity reconstruction, use real data and time dependent model;
  - Deal with the presence of errors (from data or/and from the model) - Kalman filter.
  - Order reduction.

References

- D'Elia, Perego, Veneziani, *A variational Data Assimilation procedure for the incompressible Navier-Stokes equations in hemodynamics*, J. Sci. Comput., 2011.

- Guerra, Tiago, Sequeira, *Optimal control in blood flow simulations*, Int. J. Nonlin. Mech., Vol. 64, pp. 57-59, 2014.
- Tiago, Gambaratuto, Sequeira, *Patient-specific blood flow simulations: setting Dirichlet boundary conditions for minimal error with respect to measured data*, Math. Model. Nat. Phenom., 2014.
- Guerra, Sequeira, Tiago, *Existence of optimal boundary control for the Navier-Stokes equations with mixed boundary conditions*, Portugal. Math., 2015.
- Tiago, Guerra, Sequeira, *A velocity tracking approach for the Data Assimilation problem in blood flow simulations*, subm.

Thank you for your attention!

References

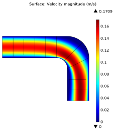
- D'Elia, Perego, Veneziani, *A variational Data Assimilation procedure for the incompressible Navier-Stokes equations in hemodynamics*, J. Sci. Comput., 2011.

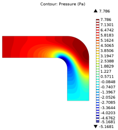
- Guerra, Tiago, Sequeira, *Optimal control in blood flow simulations*, Int. J. Nonlin. Mech., Vol. 64, pp. 57-59, 2014.
- Tiago, Gambaratuto, Sequeira, *Patient-specific blood flow simulations: setting Dirichlet boundary conditions for minimal error with respect to measured data*, Math. Model. Nat. Phenom., 2014.
- Guerra, Sequeira, Tiago, *Existence of optimal boundary control for the Navier-Stokes equations with mixed boundary conditions*, Portugal. Math., 2015.
- Tiago, Guerra, Sequeira, *A velocity tracking approach for the Data Assimilation problem in blood flow simulations*, subm.

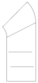
Thank you for your attention!

Comparison pressure vs velocity control

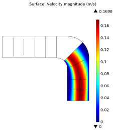
Data and computational domain with observed sections

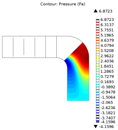




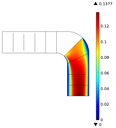


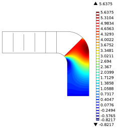
Velocity control





Pressure control





Numerical Results: 3D control - Avoiding Crimes...

Errors when data is generated with different mesh, and interpolated

Solutions	Relative Error on Ω_{part}
$\mathbf{u}$ vs $\mathbf{u}_d$, $w_1 = 1e^3$, $w_3 = 1e^{-3}$	0.08349
$\mathbf{u}_Q$ vs $\mathbf{u}_d$	0.17919

Table : Errors relative to $\mathbf{u}_d$ for $\mathbf{u}_Q$ and the controlled solutions $\mathbf{u}$

Fig : Left: Observations corresponding to downstream section

Table : Comparison between different degrees of freedom, Non-Newtonian, $U_0 = 0.0662 m/s$;

Cost function	**Derived values**	35934 **dofs**	87230 **dofs**	216871 **dofs**
$(1e^6, 1e^6, 1e^{-3})$	$\frac{\|\|\mathbf{u}-\mathbf{u}_d\|\|_2}{\|\|\mathbf{u}_d\|\|_2}$(all)	$3.118494e^{-5}$	$3.115543e^{-5}$	$3.111570e^{-5}$
	$\|\|\mathbf{u}-\mathbf{u}_d\|\|_2$(all)	$2.309949e^{-8}$	$2.307909e^{-8}$	$2.304955e^{-8}$

Wall Reconstruction

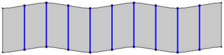
We can now choose the uncertain velocity profiles as the solution of the optimal control problem

$$\min_{\mathbf{v}\in\mathcal{A}} \mathbf{J}(\mathbf{v}) = w_1 \int_{\Omega_{Obs}} |\mathbf{u}-\mathbf{u}_d|^2\,dx + w_2 \int_{\Gamma_c} |\nabla \mathbf{v}|^2\,dx \tag{2}$$

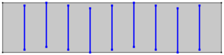
subject to

$$\begin{cases} -div\,\tau + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 & \text{in } \Omega \\ div\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_{wall} \setminus \Gamma_c \\ \mathbf{u} = \mathbf{v} & \text{on } \Gamma_c \\ (-p\mathbf{I} + 2\mu D\mathbf{u})\cdot\mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases} \tag{3}$$

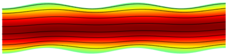
Example: Wavy Channel



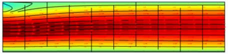
True geometry and observation sections



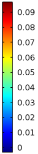
Approximated geometry

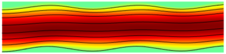


True solution

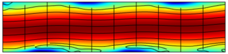


Initial guess

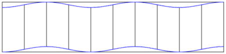




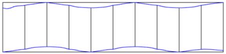
True solution extrapolated



Controlled solution



Zero contour for true solution extrapolated



Zero contour of controlled solution

Computational time: desktop ws 4 cores x 2GB RAM

Optimal Control Tolerance	Initial Cost	Final Cost	Relative Error	Computational Time [/min]	Major Iterations	Accumulated Min Iterations
10^{-2}	4.69259	0.02958	0.00456	30	926	1724
10^{-3}	4.69259	0.02792	0.00378	49	1353	2151
10^{-4}	4.69259	0.02751	0.00374	61	1506	2303

Noisy observations - 1D Control

We assume that

$$\mathbf{v} = v_c \times (1-(y/R)^2-(z/R)^2) \text{ and } w_3 = 0$$

1-Dimensional control problem!

Table : Comparison between controlled problem (CP) and direct problem (DP) with different noise (ε).

Noise	Problem type	$\frac{\|\mathbf{u}-\mathbf{u}_d\|_2}{\|\mathbf{u}_d\|_2}$(all)	$\frac{\|\mathbf{u}-\mathbf{u}_d\|_2}{\|\mathbf{u}_d\|_2}$(obs)	$\|\mathbf{u}-\mathbf{u}_d\|_2$(all)	$\|\mathbf{u}-\mathbf{u}_d\|_2$(obs)
$\varepsilon = 0.05\frac{U_0}{3}$	CP	$3.110867e^{-7}$	$4.779611e^{-7}$	4.982782^{-11}	$8.082039e^{-10}$
	DP	0.252264	0.259325	$4.040642e^{-5}$	$4.385203e^{-4}$
$\varepsilon = 0.1\frac{U_0}{3}$	CP	$7.486919e^{-6}$	$7.665531e^{-6}$	$1.199198e^{-9}$	$1.296196e^{-8}$
	DP	0.243455	0.250345	$3.899548e^{-5}$	$4.233367e^{-4}$
$\varepsilon = 0.2\frac{U_0}{3}$	CP	$9.905021e^{-6}$	$1.002028e^{-5}$	$1.586511e^{-9}$	$1.694369e^{-8}$
	DP	0.250710	0.258988	$4.015753e^{-5}$	$4.379504e^{-4}$