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On Inverse Wave Problems – Part 2: Applications

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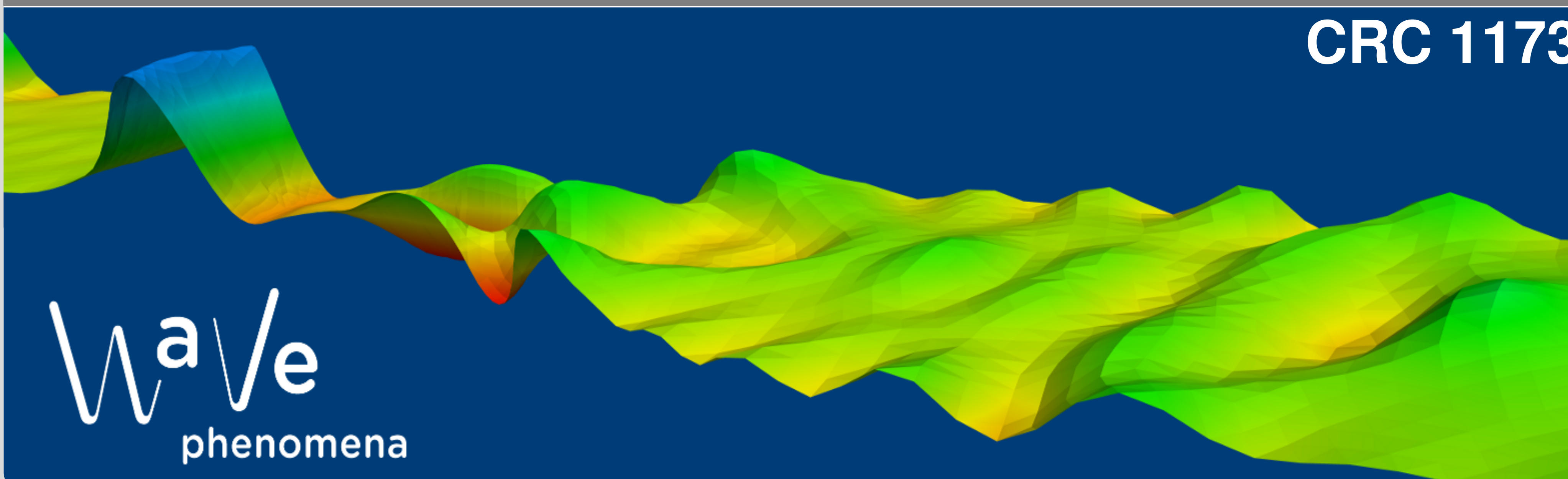


Outline

Seismic Tomography

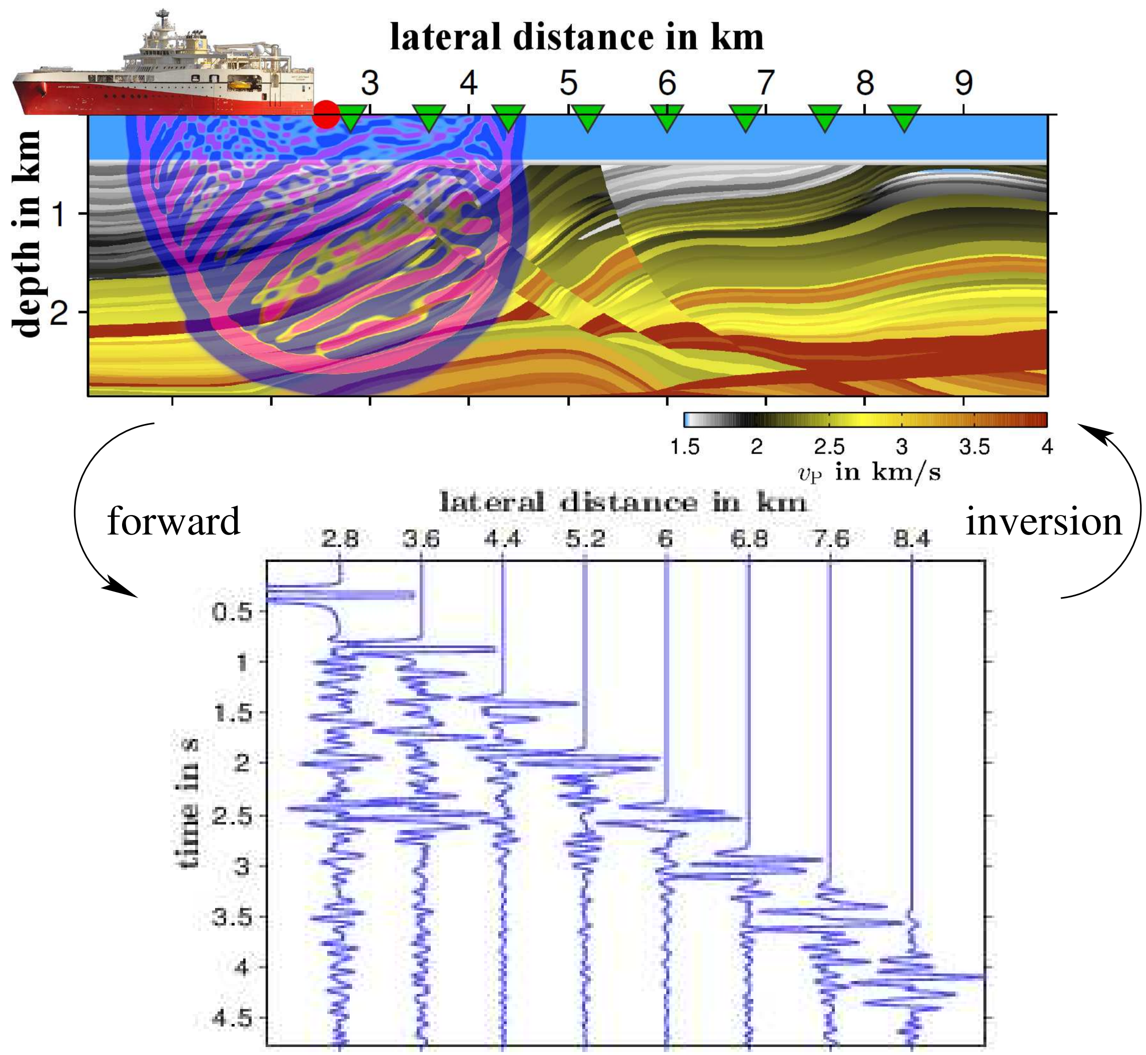
Inverse electromagnetic scattering

Summary

▷ Seismic Tomography
Inverse electromagnetic scattering
Summary

Seismic Tomography

The setting



Symes, *The seismic reflection inverse problem*, Inverse Problems 25, 123008 (2009)

The direct problem: elastic wave equation

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$D \subset \mathbb{R}^3$ bounded Lipschitz domain (or an exterior of such a domain).

$\boldsymbol{\sigma} : [0,\infty) \times D \to \mathbb{R}^{3\times 3}_{\text{sym}}$ stress tensor, $\mathbf{v} : [0,\infty) \times D \to \mathbb{R}^3$ velocity field

$$\partial_t \boldsymbol{\sigma}(t,x) = C\big(\mu(x), \lambda(x)\big)\boldsymbol{\varepsilon}(\mathbf{v}(t,x)) \qquad \text{in } [0,\infty) \times D,$$
$$\varrho(x)\partial_t \mathbf{v}(t,x) = \operatorname{div}_x \boldsymbol{\sigma}(t,x) + \mathbf{f}(t,x) \qquad \text{in } [0,\infty) \times D,$$

where $\varrho : D \to \mathbb{R}$ mass density, $\mathbf{f} : [0,\infty) \times D \to \mathbb{R}^3$ volume force,

$$C(m,\ell)\boldsymbol{\varepsilon} \;=\; 2m\,\boldsymbol{\varepsilon} + \ell\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I}, \quad \boldsymbol{\varepsilon} \in \mathbb{R}^{3\times 3}_{\text{sym}},\ m,\ell \in \mathbb{R} \quad \text{Hook's law}$$

$\mu(x)$, $\lambda(x)$ Lamé parameters

$\varepsilon(\mathbf{v}) := \frac{1}{2}\big[(\nabla_x \mathbf{v})^\top + \nabla_x \mathbf{v}\big]$ (linearized) strain

- Initial and boundary conditions will be specified below.

Elastic wave equation (continued)

$$C \colon \mathcal{D}(C) \subset \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{R}^{3\times 3}_{\mathsf{sym}})$$

$$\mathcal{D}(C) = \big\{(m,\ell)^\top \in \mathbb{R}^2 : \mathrm{c}^{-1} \le 2m + 3\ell \le \mathrm{c},\ \mathrm{c}^{-1} \le m \le \mathrm{c}\big\}$$

$$\widetilde{C}(m,\ell) := C(m,\ell)^{-1} = C\left(\frac{1}{4m}, -\frac{\ell}{2m(3\ell+2m)}\right)$$

$$\partial_t \boldsymbol{\sigma}(t,x) = C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}(\mathbf{v}(t,x))$$

$$\iff \quad \widetilde{C}\big(\mu(x),\lambda(x)\big)\partial_t \boldsymbol{\sigma}(t,x) = \boldsymbol{\varepsilon}(\mathbf{v}(t,x))$$

Wave equation as abstract evolution equation

- $\mathcal{P} := \left\{ (\mu, \lambda, \varrho)^\top \in L^\infty(D)^3 : \mathrm{c}^{-1} \le \varrho, \mu \le \mathrm{c},\ \mathrm{c}^{-1} \le 2\mu + 3\lambda \le \mathrm{c} \text{ a.e.} \right\}$
- $X = L^2(D, \mathbb{R}^{3\times 3}_{\mathsf{sym}}) \times L^2(D, \mathbb{R}^3)$ with inner product

$$\left((\boldsymbol{\sigma}, \mathbf{v})^\top, (\boldsymbol{\psi}, \mathbf{w})^\top \right)_X := \int_D \left(\boldsymbol{\sigma} : \boldsymbol{\psi} + \mathbf{v} \cdot \mathbf{w} \right) \mathrm{d}x$$

For fixed $(\mu, \lambda, \varrho)^\top \in \mathcal{P}$ we define $B \in \mathcal{L}(X)$ by

$$B \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} \widetilde{C}(\mu, \lambda) & \mathbf{0} \\ \mathbf{0} & \varrho\, I \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{v} \end{pmatrix}$$

which is self-adjoint and uniformly positive definite.

Wave equation as abstract evolution equation (cont'd)

Now, the elastic wave equation may be written as

$$B\,\partial_t\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix} + A\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{\sigma}(0,\cdot)\\ \mathbf{v}(0,\cdot)\end{pmatrix} = \begin{pmatrix}\boldsymbol{\sigma}_0\\ \mathbf{v}_0\end{pmatrix},$$

with $A\colon \mathcal{D}(A) \subset X \to X$,

$$A := -\begin{pmatrix} 0 & \boldsymbol{\varepsilon}\\ \operatorname{div}_x & 0\end{pmatrix},$$

where

$$\mathcal{D}(A) = \Big\{(\boldsymbol{\sigma},\mathbf{v})^\top \in H\big(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathsf{sym}}\big) \times H^1_{\partial D_D}(D,\mathbb{R}^3) : \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \text{ on } \partial D_N\Big\},$$

$$\partial D = \partial D_D \,\dot\cup\, \partial D_N, \quad \operatorname{vol}_2(\partial D_D) > 0.$$

Lemma A is maximal monotone ($-A$ dissipative and $\mathrm{R}(I+A) = X$).

Existence, uniqueness, regularity

- mild/weak solution exists in $\mathcal{C}\big([0,\infty), X\big)$ for

$$\mathbf{f} \in L^1\big((0,\infty), L^2(D,\mathbb{R}^3)\big) \quad \text{and} \quad (\boldsymbol{\sigma}_0, \mathbf{v}_0)^\top \in X.$$

- classical solution exists in $\mathcal{C}\big([0,\infty), \mathcal{D}(A)\big) \cap \mathcal{C}^1\big([0,\infty), X\big)$ for

$$\mathbf{f} \in W^{1,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big) \quad \text{and } (\boldsymbol{\sigma}_0, \mathbf{v}_0)^\top \in \mathcal{D}(A).$$

- classical solution exists in $\mathcal{C}^1\big([0,\infty), \mathcal{D}(A)\big) \cap \mathcal{C}^2\big([0,\infty), X\big)$ for

$$\mathbf{f} \in W^{2,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big), \quad (\boldsymbol{\sigma}_0, \mathbf{v}_0)^\top \in \mathcal{D}(A),$$

$$\varrho^{-1}\big[\operatorname{div}\boldsymbol{\sigma}_0 + \mathbf{f}(0)\big] \in H^1_{\partial D_D}(D,\mathbb{R}^3), \quad C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0) \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}),$$

$$C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0)\mathbf{n} = \mathbf{0} \text{ on } \partial D_N.$$

The inverse problem of seismic imaging

Let $(\boldsymbol{\sigma}_0, \mathbf{v}_0)^\top \in \mathcal{D}(A)$ and $\mathbf{f} \in W^{1,1}\big([0,T], L^2(D,\mathbb{R}^3)\big)$ for $T > 0$.

Then, the parameter-to-solution map

$$\Phi\colon \mathcal{P} \subset L^\infty(D)^3 \to \mathcal{C}\big([0,T], X\big), \quad (\mu, \lambda, \varrho)^\top \mapsto (\boldsymbol{\sigma}, \mathbf{v})^\top$$

is well defined.

Let $R\colon \mathcal{C}([0,T], X) \to \mathbb{R}^N$ be a (continuous) measurement operator.

Given $w \in \mathbb{R}^N$ find $(\mu, \lambda, \varrho)^\top \in \mathcal{P}$ such that

$$R\Phi(\mu, \lambda, \varrho) = w.$$

Solving above problem is called full waveform inversion in seismic imaging.

Full waveform inversion is locally ill-posed

Theorem The equation

$$\Phi(\mu, \lambda, \varrho) = (\boldsymbol{\sigma}, \mathbf{v})^\top$$

is locally ill-posed at any interior point of $\mathcal{P}$.

Proof: We factorize

$$\Phi = F \circ V$$

where

$$F \colon \mathcal{B} \subset \mathcal{L}(X) \to \mathcal{C}\big([0,T], X\big), \quad B \mapsto (\boldsymbol{\sigma}, \mathbf{v})^\top,$$

and

$$V \colon \mathcal{P} \subset L^\infty(D)^3 \to \mathcal{B}, \quad (\mu, \lambda, \varrho)^\top \mapsto \begin{pmatrix} \widetilde{C}(\mu, \lambda) & 0 \\ 0 & \varrho I \end{pmatrix}.$$

Note that F is the mapping considered in the abstract theory.

Full waveform inversion is locally ill-posed (cont'd)

Define sequences

$$\mu_k := \mu + r_1 e_k, \qquad \lambda_k := \lambda + r_2 e_k, \qquad \varrho_k := \varrho + r_3 e_k$$

where $r_i \in [0, r]$, $r > 0$ sufficiently small, $r_1 + r_2 + r_3 > 0$, and

$$e_k := \chi_{\mathrm{B}_{1/k}(\xi)} \quad \text{for one fixed } \xi \in D.$$

We have that

$$\{(\mu_k, \lambda_k, \varrho_k)\}_k \in \mathrm{B}_r(\mu, \lambda, \varrho) \quad \text{but} \quad (\mu_k, \lambda_k, \varrho_k) \not\to (\mu, \lambda, \varrho).$$

It remains to show

$$\Phi(\mu_k, \lambda_k, \varrho_k) \to \Phi(\mu, \lambda, \varrho).$$

Full waveform inversion is locally ill-posed (cont'd)

Define $E_k \in \mathcal{L}(X)$ by

$$E_k := V(\mu_k, \lambda_k, \varrho_k) - V(\mu, \lambda, \varrho) = \begin{pmatrix} \widetilde{C}(\mu_k, \lambda_k) - \widetilde{C}(\mu, \lambda) & 0 \\ 0 & r_3 e_k I \end{pmatrix}$$

and show that

- E_k is monotone, symmetric, bounded,
- $\lim_{k\to\infty} \|E_k\binom{\boldsymbol{\sigma}}{\mathbf{v}}\|_X = 0$ for all $(\boldsymbol{\sigma}, \mathbf{v})^\top \in X$, and
- $\widehat{r} \lesssim \|E_k\|_{\mathcal{L}(X)} \lesssim r$ for all k where $0 < \widehat{r} < r$.

As

$$\Phi(\mu_k, \lambda_k, \varrho_k) = F\big(V(\mu, \lambda, \varrho) + E_k\big) \to F\big(V(\mu, \lambda, \varrho)\big) = \Phi(\mu, \lambda, \varrho)$$

the claimed local ill-posedness follows. ✓

Fréchet derivative of the forward map

Theorem The parameter-to-solution map

$$\Phi\colon \mathcal{P} \subset L^\infty(D)^3 \to \mathcal{C}\big([0,T],X\big)$$

is Fréchet differentiable at $(\mu,\lambda,\varrho)^\top$. In fact,

$$\Phi'(\mu,\lambda,\varrho)\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{pmatrix} \overline{\boldsymbol{\sigma}} \\ \overline{\mathbf{v}} \end{pmatrix}$$

where $(\overline{\boldsymbol{\sigma}},\overline{\mathbf{v}})^\top \in \mathcal{C}\big([0,T],X\big)$ is the mild solution of

$$\partial_t \overline{\boldsymbol{\sigma}}(t,x) = C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}(\overline{\mathbf{v}}(t,x)) + C\big(h_1(x),h_2(x)\big)\boldsymbol{\varepsilon}(\mathbf{v}(t,x)),$$

$$\varrho(x)\partial_t \overline{\mathbf{v}}(t,x) = \operatorname{div}\overline{\boldsymbol{\sigma}}(t,x) - h_3(x)\partial_t \mathbf{v}(t,x),$$

in $[0,T]\times D$ with $\overline{\boldsymbol{\sigma}}(0)=\mathbf{0}$, $\overline{\mathbf{v}}(0)=\mathbf{0}$. Here, $\mathbf{v} = \Phi(\mu,\lambda,\varrho)_2$.

Fréchet derivative of the forward map (continued)

Proof: As $\Phi = F \circ V$ we have that

$$\Phi'(\mu, \lambda, \varrho) = F'\big(V(\mu, \lambda, \varrho)\big)V'(\mu, \lambda, \varrho).$$

The assertion follows from

$$V'(\mu, \lambda, \varrho)\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{pmatrix} \widetilde{C}'(\mu, \lambda) & 0 \\ 0 & I \end{pmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= \begin{pmatrix} -\widetilde{C}(\mu, \lambda)C(h_1, h_2)\widetilde{C}(\mu, \lambda) & 0 \\ 0 & h_3 I \end{pmatrix}$$

and the abstract result applied to F. ✓

Second order wave equation

$$\rho(x)\partial_{tt}\mathbf{v}(t,x) = \operatorname{div}\big[C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}\big(\mathbf{v}(t,x)\big)\big] + \mathbf{g}(t,x)$$

with $\mathbf{v}(0,\cdot) = \mathbf{v}_0$, $\partial_t\mathbf{v}(0,\cdot) = \mathbf{v}_1$ in D, and

$\mathbf{v} = \mathbf{0}$ on $[0,T]\times\partial D_D$, $C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{n} = \mathbf{0}$ on $[0,T]\times\partial D_N$.

Set

$$\boldsymbol{\sigma}(t,x) := C\big(\mu(x),\lambda(x)\big)\int_0^t \boldsymbol{\varepsilon}\big(\mathbf{v}(s,x)\big)\,\mathrm{d}s.$$

Then, $(\boldsymbol{\sigma},\mathbf{v})$ solves

$$\partial_t\boldsymbol{\sigma}(t,x) = C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}\big(\mathbf{v}(t,x)\big),$$

$$\rho(x)\,\partial_t\mathbf{v}(t,x) = \operatorname{div}\boldsymbol{\sigma}(t,x) + \int_0^t \mathbf{g}(s,x)\,\mathrm{d}s + \rho(x)\mathbf{v}_1(x)$$

with $\mathbf{v}(0,\cdot) = \mathbf{v}_0$, $\boldsymbol{\sigma}(0,\cdot) = \mathbf{0}$ in D, and

$\mathbf{v} = \mathbf{0}$ on $[0,T]\times\partial D_D$, $\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}$ on $[0,T]\times\partial D_N$.

Second order wave equation (continued)

Conclusion

- Our results carry over to the second order wave equation.
- Thus, we regain and extend results of

  LECHLEITER/SLASCHE 2015 and BOEHM/ULBRICH 2015

  on the Fréchet differentiability of the parameter-to-solution map.

Seismic Tomography

▷ Inverse electromagnetic scattering

Summary

Inverse electromagnetic scattering

The Maxwell system

$\mathbf{E} = \mathbf{E}(t, \mathbf{x})$ and $\mathbf{H} = \mathbf{H}(t, \mathbf{x})$ electric and magnetic fields, resp.

$$\underbrace{\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}}_{= B} \partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = -\underbrace{\begin{pmatrix} \sigma I & -\mathrm{curl}_x \\ \mathrm{curl}_x & 0 \end{pmatrix}}_{= A} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \underbrace{\begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}}_{= f} \quad \text{in } (0,T) \times D$$

with bc: $\mathbf{n} \times \mathbf{E} = 0$ on $(0,T) \times \partial D$

ic: $\mathbf{E}(0, \cdot) = \mathbf{e}_0$, $\mathbf{H}(0, \cdot) = \mathbf{h}_0$

where $\mathbf{J}_{e/m} = \mathbf{J}_{e/m}(t, x)$ current/magnetic density

$\varepsilon = \varepsilon(x)$ permittivity

$\mu = \mu(x)$ permeability

$\sigma = \sigma(x)$ conductivity

Abstract settings

- $\mathcal{P} := \left\{(\varepsilon,\mu)^\top \in L^\infty(D)^2 : \mathrm{c}^{-1} \le \varepsilon,\mu \le \mathrm{c} \text{ a.e.}\right\}$
- $X = L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^3)$

If $(\varepsilon,\mu)^\top \in \mathcal{P}$ then $B \in \mathcal{L}(X)$ is self-adjoint and uniformly pos. definite.

- $\sigma \in L^\infty(D),\ \sigma \ge 0$
- $\mathcal{D}(A) = H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D)$

Lemma $A\colon \mathcal{D}(A) \subset X \to X$ is maximal monotone.

Electromagnetic scattering operator

Let $(\mathbf{e}_0, \mathbf{h}_0)^\top \in \mathcal{D}(A)$ and $(\mathbf{J}_e, \mathbf{J}_m)^\top \in W^{1,1}\big([0,T], X\big)$ for $T > 0$.

Then, the parameter-to-solution map is well defined:

$$\Phi\colon \mathcal{P} \subset L^\infty(D)^2 \to \mathcal{C}\big([0,T], X\big), \quad (\varepsilon, \mu)^\top \mapsto (\mathbf{E}, \mathbf{H})^\top.$$

We factorize again

$$\Phi = F \circ V$$

where

$$F\colon \mathcal{B} \subset \mathcal{L}(X) \to \mathcal{C}\big([0,T], X\big), \quad B \mapsto (\mathbf{E}, \mathbf{H})^\top,$$

and

$$V\colon \mathcal{P} \subset L^\infty(D)^2 \to \mathcal{B}, \quad (\varepsilon, \mu)^\top \mapsto \begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}.$$

Conclusion: Fréchet-differentiability and local ill-posedness hold for inverse electromagnetic scattering as well.

Seismic Tomography
Inverse electromagnetic scattering
▷ Summary

Summary

Things to remember

$$Bu'(t) + Au(t) = f(t),\ \ t \geq 0, \quad u(0) = u_0$$

$B \in \mathcal{L}(X)$ pos. self-adjoint; $A\colon \mathcal{D}(A) \subset X \to X$ maximal monotone; X Hilbert space

$$F\colon B \mapsto u$$

We have investigated the

- Fréchet-differentiability of F,
- Local ill-posedness of $F(\cdot) = u$,

and applied our abstract findings to

- the elastic wave equation (seismic tomography),
- Maxwell's system (inverse electromagnetic scattering).

