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On Inverse Wave Problems Part 1: Theory

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Inverse Problems for PDEs, Bremen, March 2016

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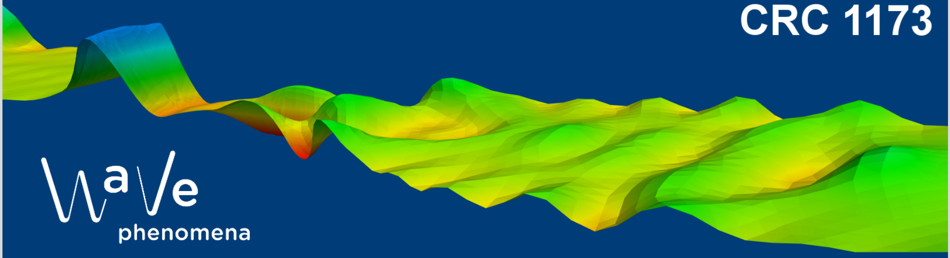

Plan of Talk

(A) Evolution Equations
(B) Smoothness With Respect to Parameters
(C) Local Ill-Posedness

Reference: A. Kirsch, A. Rieder: Inverse Problems for Abstract Evolution Equations With Applications in Electrodynamics and Elasticity. Submitted to *Inverse Problems*.

(A) Evolution Equations

Time – dependent problems are formulated as abstract evolution equations

$$u'(t) \;=\; -Au(t) \;+\; f(t)\,, \quad t \geq 0\,, \qquad u(0) = u_0\,.$$

Here, $A : X \supset \mathcal{D}(A) \to X$ (unbounded) operator in Hilbert space X, $f : \mathbb{R}_{\geq 0} \to X$, and $u_0 \in \mathcal{D}(A)$.

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Note: A is independent of t! Spaces:

$$\begin{aligned}
C\big(\mathbb{R}_{\geq 0}, \mathcal{D}(A)\big) \quad &= \quad \big\{u : \mathbb{R}_{\geq 0} \to \mathcal{D}(A) : \ u \text{ continuous}\big\}\,,\\
C^1\big(\mathbb{R}_{\geq 0}, X\big) \quad &= \quad \big\{u : \mathbb{R}_{\geq 0} \to X : u \text{ cont. F-differentiable}\big\}\,,\\
L^1\big(\mathbb{R}_{>0}, X\big) \quad &= \quad \big\{u : \mathbb{R}_{>0} \to X : u \text{ meas. and Bochner integrable}\big\}\,,\\
W^{1,1}\big(\mathbb{R}_{>0}, X\big) \quad &= \quad \big\{u \in C(\mathbb{R}_{\geq 0}, X) : u' \in L^1(\mathbb{R}_{>0}, X)\big\}\,.
\end{aligned}$$

Analogously, $C^k(\mathbb{R}_{>0}, Y)$ and $W^{k,1}_{loc}(\mathbb{R}_{>0}, X)$ are defined for $k \in \mathbb{N}$.

First homogeneous equation ($f = 0$):

Theorem (Hille - Yosida)
Let $u_0 \in \mathcal{D}(A)$ and A maximal monotone; that is, $(Av, v)_X \geq 0$ for all $v \in X$ and $A + I : \mathcal{D}(A) \to X$ is surjective. Then there exists a unique solution $u \in C\big(\mathbb{R}_{\geq 0}, \mathcal{D}(A)\big) \cap C^1(\mathbb{R}_{\geq 0}, X)$ of

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(iii) let ε tend to zero. $\blacksquare$

Remark: A maximal monotone $\Rightarrow$ $\mathcal{D}(A)$ dense and A closed

Reference: H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011.

Estimate $\|u(t)\|_X \leq \|u_0\|_X$ implies that operators $S(t) : u_0 \to u(t)$ have bounded extensions $S(t) : X \to X$ with $\|S(t)\|_{\mathcal{L}(X)} \leq 1$. Familly $S(t)$ forms continuous semigroup of contractions; that is,

- $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$,
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Now inhomogeneous equation:

Theorem Let A maximal monotone, $u_0 \in \mathcal{D}(A)$, and $f \in W^{1,1}_{loc}(\mathbb{R}_{>0}, X)$. There there exists a unique solution $u \in C\big(\mathbb{R}_{\geq 0}, \mathcal{D}(A)\big) \cap C^1(\mathbb{R}_{\geq 0}, X)$ of

$$u'(t) \;=\; -Au(t) \;+\; f(t)\,, \quad t \geq 0\,, \qquad u(0) = u_0\,,$$

which is given by the variation-of-constant formula

$$u(t) \;=\; S(t)u_0 \;+\; \int_0^t S(t-s)\, f(s)\, ds\,, \quad t \geq 0\,.$$

If only $u_0 \in X$ and $f \in L^1_{loc}(\mathbb{R}_{>0}, X)$ then this defines mild solution.

Mild solution

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\,f(s)\,ds\,, \quad t \geq 0\,,$$

is weak solution; that is,

$$\frac{d}{dt}\big(u(t),\psi\big)_X = -\big(u(t),A^*\psi\big)_X + \big(f(t),\psi\big)_X$$

for a.a. $t \geq 0$ and $\psi \in \mathcal{D}(A^*)$ where $A^* : X \supset \mathcal{D}(A^*) \to X$ denotes the adjoint of A.

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Stability:

$u_0 \in \mathcal{D}(A)$, and $f \in W^{1,1}_{loc}(\mathbb{R}_{>0}, X)$, and u classical solution:

$$\|u'(t)\|_X \;+\; \|u(t)\|_{\mathcal{D}(A)} \;\leq\; c\big[\|u_0\|_{\mathcal{D}(A)} + \|f\|_{W^{1,1}((0,t),X)}\big]\,, \quad t > 0\,,$$

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Regularity

Theorem Let A be maximal monotone, $f \in W^{k,1}_{loc}(\mathbb{R}_{>0}, X)$ for some $k \in \mathbb{N}$ and

$$u_{0,\ell} := (-A)^{\ell} u_0 + \sum_{j=0}^{\ell-1} (-A)^j f^{(\ell-1-j)}(0) \ \in \ \mathcal{D}(A) \text{ for } \ell = 0, \dots, k-1 \,.$$

Then $u \in C^k(\mathbb{R}_{\geq 0}, X) \cap C^{k-1}\big(\mathbb{R}_{\geq 0}, \mathcal{D}(A)\big)$ and

$$\|u^{(\ell)}(t)\|_X \ \leq \ \|u_{0,\ell}\|_X \ + \ \|f^{(\ell)}\|_{L^1((0,t),X)}\,, \quad t \geq 0\,,\ \ell = 0, \dots, k,$$

where $u_{0,k} := -Au_{0,k-1} + f^{(k-1)}(0) \in X$.

(B) Smoothness With Respect to Parameters

How to model parameters? Define, for $0 < \gamma_- < \gamma_+$,

$$\mathcal{B} := \left\{ B \in \mathcal{L}(X) : \begin{array}{l} B \text{ self adjoint}, \\ \gamma_- \|v\|_X^2 \leq (Bv, v)_X \leq \gamma_+ \|v\|_X^2 \ \forall v \in X \end{array} \right\}$$

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Thus: $B^{-1}A$ is maximal monotone in X with respect to inner product $(u, v)_B := (Bu, v)_X$.

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Corollary Results of previous section hold for $\|\cdot\|_X$ replaced by equivalent (uniformly in $\mathcal{B}$) norm $\|\cdot\|_B$.

Therefore, for fixed $u_0 \in \mathcal{D}(A)$ and $f \in W^{1,1}_{loc}(\mathbb{R}_{>0}, X)$ and $T > 0$ the nonlinear mapping (parameter-to-solution map) $\mathcal{F} : B \mapsto u|_{[0,T]}$ is well defined from $\mathcal{B}$ into $C^1([0,T], X) \cap C([0,T], \mathcal{D}(A))$.

Continuity

Lemma Let A be maximal monotone, $u_0 \in \mathcal{D}(A)$, $\hat{B}$, $\hat{B} + B \in \mathcal{B}$, and $f \in W^{2,1}_{loc}(\mathbb{R}_{>0}, X)$ and $\hat{v}_0 := \hat{B}^{-1}\big(Au_0 - f(0)\big) \in \mathcal{D}(A)$. Let $\hat{u}, \tilde{u} \in C^1(\mathbb{R}_{\geq 0}, X) \cap C\big(\mathbb{R}_{\geq 0}, \mathcal{D}(A)\big)$ be the solutions of

$$\hat{B}\hat{u}'(t) + A\hat{u}(t) = f(t) \quad \text{and} \quad (\hat{B} + B)\tilde{u}'(t) + A\tilde{u}(t) = f(t)$$

for $t > 0$ and $\hat{u}(0) = \tilde{u}(0) = u_0$.

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for $t > 0$ and $\hat{u}(0) = \tilde{u}(0) = u_0$. Then there exists c, depending only on A, γ_+, γ_-, $\hat{v}_0$, and f, such that for all $T > 0$:

$$\|\hat{u}(t) - \tilde{u}(t)\|_X + \|\hat{u}'(t) - \tilde{u}'(t)\|_X \ \leq \ c(1 + T)\,\|B\|_{\mathcal{L}(X)}$$

for $0 \leq t \leq T$.

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Theorem Let $T > 0$, $f \in W^{1,1}_{loc}(\mathbb{R}_{>0}, X)$, and $u_0 \in \mathcal{D}(A)$. Then $\mathcal{F} : B \mapsto u|_{[0,T]}$ is continuous from $\mathcal{B}$ into $C^1\big([0, T], X\big)$.

Differentiability

Theorem Let $T > 0$, $f \in W^{1,1}_{loc}(\mathbb{R}_{>0}, X)$, and $u_0 \in \mathcal{D}(A)$. Then $\mathcal{F} : \mathcal{B} \to C\big([0, T), X\big)$ is Fréchet differentiable at $\hat{B} \in \mathrm{int}(\mathcal{B})$ and $\mathcal{F}'(\hat{B})B = \overline{u}|_{[0,T]}$ where $\overline{u} \in C(\mathbb{R}_{\geq 0}, X)$ is the mild solution of

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Here, $\hat{u} \in C^1(\mathbb{R}_{\geq 0}, X) \cap C\big(\mathbb{R}_{\geq 0}, \mathcal{D}(A)\big)$ is the (classical) solution of $\hat{B}\hat{u}'(t) + A\hat{u}(t) = f(t)$, $t \geq 0$, $\hat{u}(0) = u_0$.

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Remarks: Here, $\mathcal{F} : \mathcal{B} \to C\big([0, T), X\big)$ rather than $\mathcal{F} : \mathcal{B} \to C^1\big([0, T], X\big) \cap C\big([0, T], \mathcal{D}(A)\big)$!

Differentiability into the latter space needes stronger regularity assumptions on u_0 and f (order $k = 2$ in above theorem).

In applications, only $\mathcal{F} : \mathcal{B} \to L^2\big([0, T), X\big)$ or $\mathcal{O} \circ \mathcal{F}$ with (linear) observation operator $\mathcal{O} : L^2\big([0, T), X\big) \to Z$ is considered.

(C) Local Ill-Posedness

Definition Equation $\mathcal{F}(x) = y$ is locally ill-posed at $\hat{x} \in \mathcal{D}(\mathcal{F})$ satisfying $\mathcal{F}(\hat{x}) = y$ if in any neighborhood of $\hat{x}$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathcal{F})$ such that

$$\lim_{k\to\infty} \|\mathcal{F}(x_k) - y\|_Y = 0\,, \text{ however } \|x_k - \hat{x}\|_X \not\to 0 \text{ for } k \to \infty.$$

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Theorem Let $u_0 \in \mathcal{D}(A)$ and $f \in W^{1,1}_{loc}(\mathbb{R}_{>0}, X)$. Then $\mathcal{F}(B) = u$ is locally ill-posed at any $\hat{B} \in \mathcal{D}(\mathcal{F})$ satisfying $\mathcal{F}(\hat{B}) = u$ if for any $r \in (0, 1]$ there exists $\hat{r} \in (0, r)$ and a sequence of bounded, symmetric and monotone operators $E_k : X \to X$ with $\hat{B} + E_k \in \mathcal{D}(\mathcal{F})$ and $\hat{r} \le \|E_k\| \le r$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} E_k v = 0$ for all $v \in X$.

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In the applications (see next talk by the other Andreas) $X = L^2(\Omega)$ and $B \in \mathcal{D}(\mathcal{F}) \Leftrightarrow (Bv)(x) = \rho(x)v(x)$ a.e. for $\rho \in L^\infty(\Omega)$. Then one can take $(E_k v)(x) = r\,\chi_{K_k}(x)v(x)$ a.e. with $K_k = \{x \in \Omega : |x - z| \leq 1/k\}$ for some $z \in \Omega$.

Thank you for your attention!