Imaging an acoustic waveguide from surface data in the time domain

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Workshop Inverse Problems for PDEs

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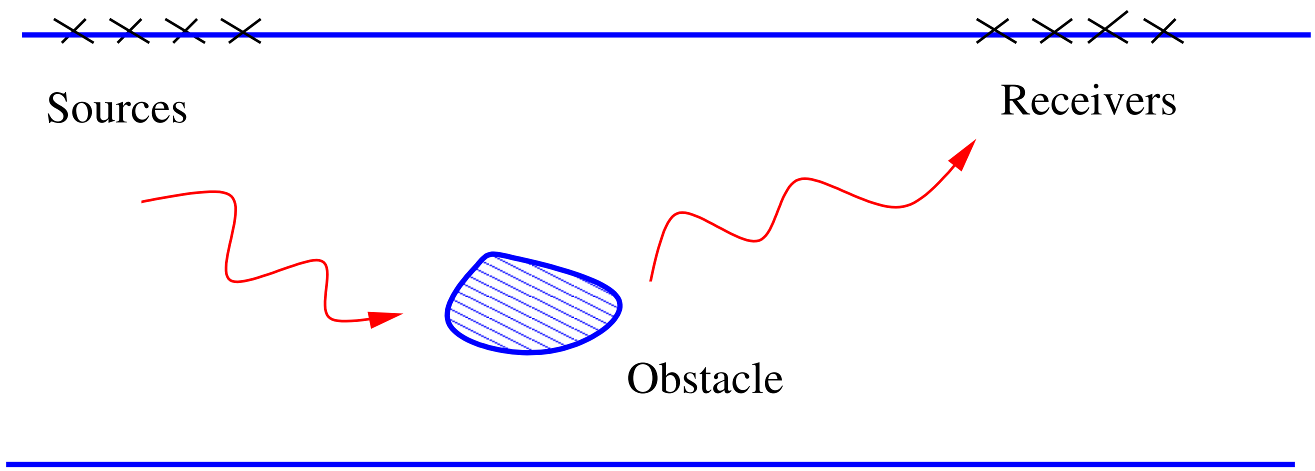
Statement of the inverse problem

Geometry : a 2D acoustic waveguide in the time domain



Objective : find a sound soft obstacle from measurements at surface receivers of scattered waves due to surface sources

A brief state of the art

Sampling Methods in waveguides:

- **Frequency domain**

  Acoustics : Xu, Mawata & Lin (2000) : LSM, Charalambopoulos, Gintides, Kiriaki & Kirsch (2006), Arens, Gintides & Lechleiter (2011) : Factorization Method, Bourgeois & Lunéville (2008) : Modal approach, Monk & Selgas (2012) : RG-LSM

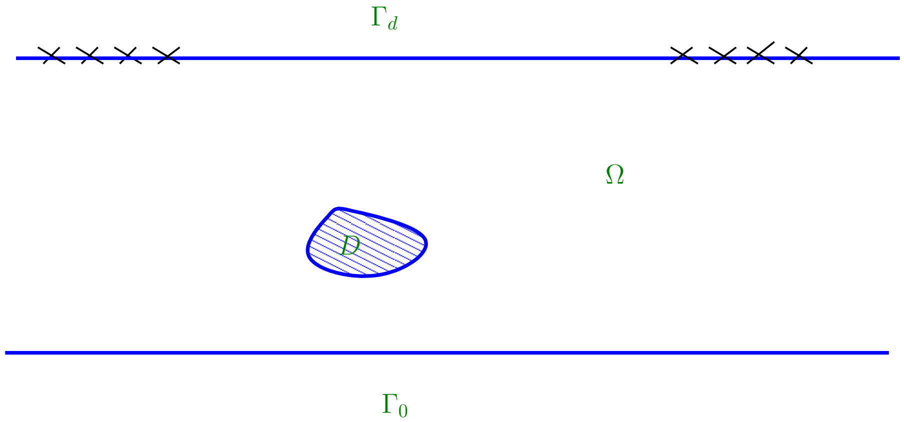
  Elasticity : Bourgeois, Le Louër & Lunéville (2011)

  Electromagnetism : Borcea & Nguyen (2016)

- **Time domain**

  Monk & Selgas (2016) : first temptative

The forward problem in the time domain



$$\left\{\begin{array}{cclll} \frac{1}{c^2}\partial_t^2 v - \Delta v & = & 0 & \text{in} & \Omega \times (0,+\infty) \\ \partial_\nu v & = & f\,\chi & \text{on} & \Gamma_d \times (0,+\infty) \\ \partial_\nu v & = & 0 & \text{on} & \Gamma_0 \times (0,+\infty) \\ v & = & 0 & \text{on} & \partial D \times (0,+\infty) \\ v, \partial_t v & = & 0 & \text{on} & \Omega \times \{0\} \end{array}\right.$$

Data : space function $f = \delta(\cdot - x_1^{m\pm})$ with $x_1^{m\pm} = \pm(R + m\delta)$ and $m = 0, \cdots, M-1$, time function χ compactly supported in $[0,+\infty)$

$\longrightarrow$ correctly choose M, δ and χ !

The forward problem in the frequency domain

Apply the **Fourier transform** :

$$u := \widehat{v}(x,\omega) = \int_{\mathbb{R}} v(x,t)e^{i\omega t}\,dt$$

$$\left\{\begin{array}{ccc} (\Delta + k^2)u = 0 & \text{in} & \Omega \\ \partial_\nu u = \phi & \text{on} & \Gamma_d \\ \partial_\nu u = 0 & \text{on} & \Gamma_0 \\ u = 0 & \text{on} & \partial D \\ + \text{ Radiation condition} & & \end{array}\right.$$

with

$$k = \omega/c \quad \phi = \widehat{\chi}(\omega) f$$

A problem in the time domain $\longrightarrow$ Harmonic problems for multiple frequencies

The guided modes

- The guided modes : find u such that

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad W \\ \partial_\nu u = 0 & \text{on} \quad \Gamma \end{cases}$$

- θ_n and λ_n $(n \in \mathbb{N})$: Neumann eigenfunctions and eigenvalues of the 1D operator $-\Delta$ in transverse section Σ

- The θ_n form a complete basis of $L^2(\Sigma)$ while $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \longrightarrow +\infty$

- Guided modes : $u_n^\pm(x_1, x_2) = \theta_n(x_2)e^{\pm i\beta_n x_1}$, $\beta_n = \sqrt{k^2 - \lambda_n}$ for $n = 0, \cdots, N-1$ (propagating modes) and $\beta_n = i\sqrt{\lambda_n - k^2}$ for $n \geq N$

- Assumption on k : $\beta_n \neq 0$ for all $n \in \mathbb{N}$

The fundamental solution

For $y \in W$, the solution to

$$\begin{cases} -(\Delta + k^2)G(\cdot, y) = \delta_y & \text{in} \quad W \\ \partial_\nu G(\cdot, y) = 0 & \text{on} \quad \Gamma_0 \cup \Gamma_d \\ + \text{ Radiation condition} & \end{cases}$$

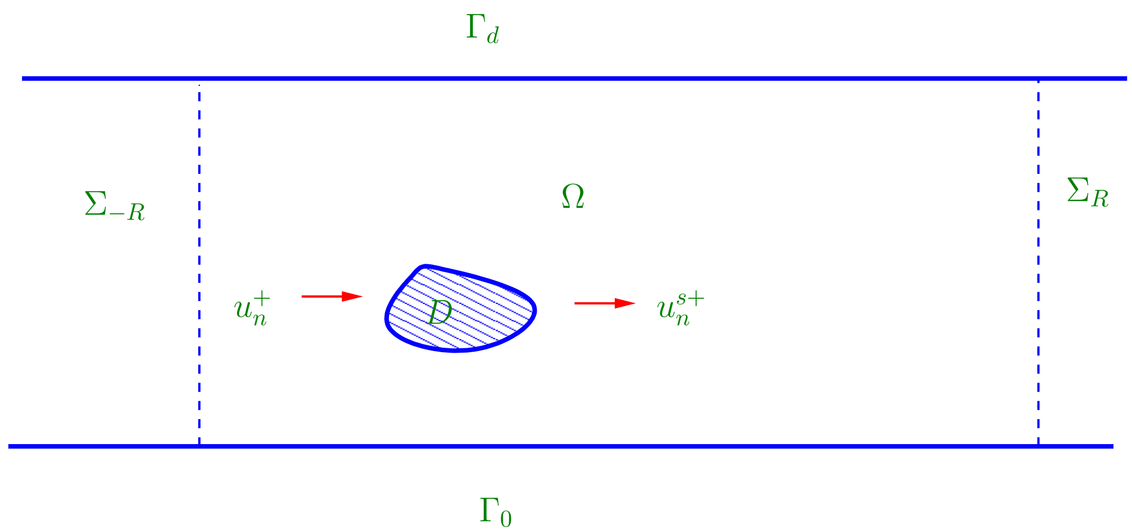
and for $y \in \Gamma_d$, the solution to

$$\begin{cases} -(\Delta + k^2)G(\cdot, y) = 0 & \text{in} \quad W \\ \partial_\nu G(\cdot, y) = \delta_y & \text{on} \quad \Gamma_d \\ \partial_\nu G(\cdot, y) = 0 & \text{on} \quad \Gamma_0 \\ + \text{ Radiation condition} & \end{cases}$$

are given by

$$G(x, y) = -\sum_{n \in \mathbb{N}} \frac{e^{i\beta_n |x_1 - y_1|}}{2i\beta_n} \theta_n(x_2)\theta_n(y_2)$$

An ideal inverse problem



Inverse problem with **guided modes** : we measure on $\hat{\Sigma} = \Sigma_{-R} \cup \Sigma_{+R}$ the scattered fields $u_n^{s\pm}$ associated to the incident fields $u_n^{\pm}$ for all $n \in \mathbb{N}$: find D

Projection of scattered fields on $\hat{\Sigma}$:

$$u_n^{s+}|_{\Sigma_{-R}} = \sum_{m\in\mathbb{N}} S_{mn}^{+-}\,\theta_m, \quad u_n^{s+}|_{\Sigma_{+R}} = \sum_{m\in\mathbb{N}} S_{mn}^{++}\,\theta_m,$$

$$u_n^{s-}|_{\Sigma_{-R}} = \sum_{m\in\mathbb{N}} S_{mn}^{--}\,\theta_m, \quad u_n^{s-}|_{\Sigma_{+R}} = \sum_{m\in\mathbb{N}} S_{mn}^{-+}\,\theta_m$$

The Linear Sampling Method

Theorem : for $z \in W$, if $G(\cdot, z)|_{\hat{\Sigma}} \in R(\mathcal{N})$ then $z \in D$

Equation $\mathcal{N}h = G(\cdot, z)|_{\hat{\Sigma}}$ in $L^2(\hat{\Sigma})$ is equivalent to

$$\forall m \in \mathbb{N}, \quad \left\{ \begin{array}{l} \displaystyle\sum_{n \in \mathbb{N}} \frac{e^{i\beta_n R}}{2i\beta_n} \left(S^{+-}_{mn}\, h^-_n + S^{--}_{mn}\, h^+_n \right) = \frac{e^{i\beta_m (R+z_1)}}{2i\beta_m} \theta_m(z_2) \\ \displaystyle\sum_{n \in \mathbb{N}} \frac{e^{i\beta_n R}}{2i\beta_n} \left(S^{++}_{mn}\, h^-_n + S^{-+}_{mn}\, h^+_n \right) = \frac{e^{i\beta_m (R-z_1)}}{2i\beta_m} \theta_m(z_2), \end{array} \right.$$

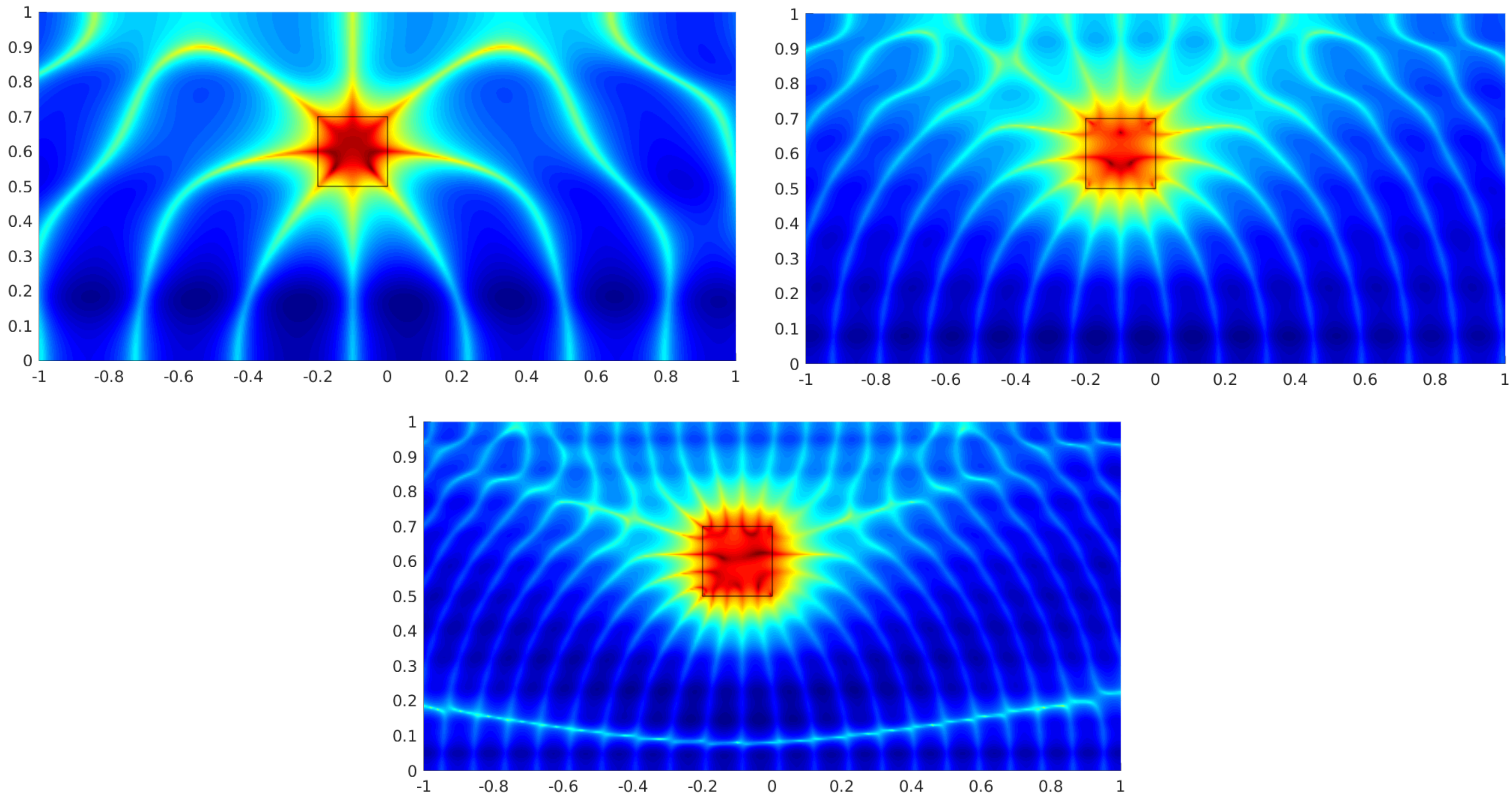
with

$$z = (z_1, z_2), \quad h = (h^-, h^+), \; h^- = \sum_{n \in \mathbb{N}} h^-_n\, \theta_n, \; h^+ = \sum_{n \in \mathbb{N}} h^+_n\, \theta_n$$

Indicator function : $\psi(z) = 1/||h(z)||_{L^2(\hat{\Sigma})}$ characterizes the defect

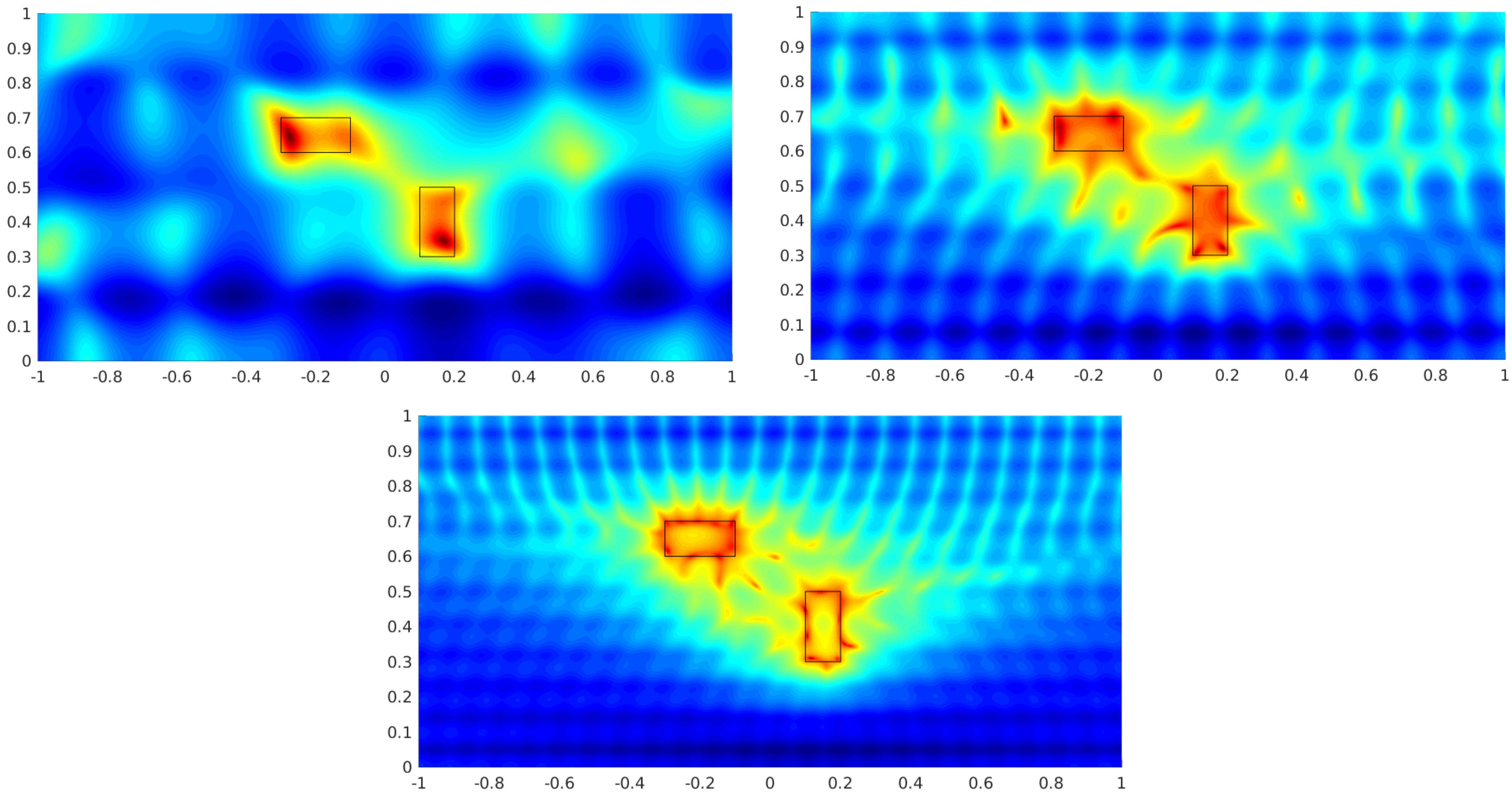
Discretization : $m, n = 0, \cdots, N-1$ (N : number of propagating modes) $\longrightarrow$ scattering matrix S, LSM matrix $U_{mn} = (e^{i\beta_n R}/2i\beta_n) S_{mn}$

Some numerical results : frequency domain



Number of propagating modes : $N = 4$, $N = 8$ and $N = 12$

Some numerical results : frequency domain



Number of propagating modes : $N = 4$, $N = 8$ and $N = 12$

The case of surface data

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad \Omega \\ \partial_\nu u = \phi & \text{on} \quad \Gamma_d \\ \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \\ u = 0 & \text{on} \quad \partial D \\ + \text{ Radiation condition} & \end{cases}$$

Incident field for $\phi = \delta(\cdot - y_1)$, **left source** and **right receiver** :

$$u^i(x_1, x_2) = -\sum_{n \in \mathbb{N}} \frac{e^{i\beta_n |x_1 - y_1|}}{2i\beta_n} \theta_n(x_2)\theta_n(d)$$

$$= -\sum_{n \in \mathbb{N}} \frac{e^{-i\beta_n y_1}}{2i\beta_n} \theta_n(d) \underbrace{e^{i\beta_n x_1} \theta_n(x_2)}$$

$$= -\sum_{n \in \mathbb{N}} \frac{e^{-i\beta_n y_1}}{2i\beta_n} \theta_n(d)\, u_n^+(x_1, x_2)$$

The case of surface data (cont.)

Scattered field $u^s = u - u^i$:

$$u^s(x_1, x_2) = -\sum_{n\in\mathbb{N}} \frac{e^{-i\beta_n y_1}}{2i\beta_n}\theta_n(d)\, u_n^{s+}(x_1, x_2)$$

For $y_1 = -R - s\delta$,

$$u^s(x_1, x_2) = -\sum_{n\in\mathbb{N}} \frac{e^{i\beta_n R}}{2i\beta_n}\theta_n(d)e^{is\beta_n\delta}\, u_n^{s+}(x_1, x_2)$$

For $x_1 = R + r\delta$,

$$u_n^{s+}(x_1, d) = \sum_{m\in\mathbb{N}} (u_n^{s+}(R, \cdot), \theta_m)_R\, e^{i\beta_m(x_1-R)}\theta_m(d) = \sum_{m\in\mathbb{N}} S_{mn}^{++}\, e^{ir\beta_m\delta}\theta_m(d)$$

Then

$$u^s(x_1, d) = -\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} e^{ir\beta_m\delta}\,\theta_m(d)\, \underbrace{\frac{e^{i\beta_n R}}{2i\beta_n}\, S_{mn}^{++}}_{U_{mn}^{++}}\,\theta_n(d)\, e^{is\beta_n\delta}$$

The measurement matrix

The measurement matrix : for $r, s = 0, \cdots, M-1$,

$$M^{++}_{rs} = -\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{ir\beta_m\delta}\,\theta_m(d)\,U^{++}_{mn}\,\theta_n(d)\,e^{is\beta_n\delta}$$

Factorization

$$M = -RUE^t$$

- M : measurement matrix
- R, E: reception and emission matrices
- U : LSM matrix

For $\phi = \delta(\cdot - y_1)$, $R = E = VT$ with $T = \mathrm{Diag}(\theta_n(d))$ and V is the Vandermonde Matrix

$$V_{mn} = e^{im\beta_n\delta}, \quad m = 0, \cdots, M-1, \quad n = 0, \cdots, N-1$$

Optimization of sources/receivers

We have to invert V^*V for $M \geq N$: how to choose M and δ ?

Vandermonde Matrix $V = (V_{mn})$, $m = 0, \cdots, M-1$, $n = 0, \cdots, N-1$:

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i\, f_0} & e^{2\pi i\, f_1} & \cdots & e^{2\pi i\, f_{N-1}} \\ \cdots & \cdots & \cdots & \cdots \\ e^{2\pi i\,(M-1) f_0} & e^{2\pi i\,(M-1) f_1} & \cdots & e^{2\pi i\,(M-1) f_{N-1}} \end{bmatrix}$$

with

$$f_n = \frac{1}{2\pi}\beta_n\delta = \sqrt{1 - n^2\frac{\lambda^2}{4d^2}}\,\frac{\delta}{\lambda}$$

λ : wavelength, d : waveguide thickness

Invertibility of V^*V : true if for all $n' \neq n$, $f_{n'} - f_n \notin \mathbb{Z}$

$\longrightarrow$ achieved for $\delta \leq \lambda$

Conditioning of V^*V

A classical question in **signal processing**...

Condition number : $\kappa(V) = \sqrt{\sigma_{\max}(V^*V)/\sigma_{\min}(V^*V)}$

Remark : for $n, n' = 0, \cdots, N-1$,

$$(V^*V)_{nn'} = \sum_{m=0}^{M-1} e^{-2\pi i(f_n - f_{n'})m}$$

$$= \begin{cases} M & \text{if} \quad f_n - f_{n'} \in \mathbb{Z} \\ \dfrac{1 - e^{-2\pi i(f_n - f_{n'})M}}{1 - e^{-2\pi i(f_n - f_{n'})}} & \text{if} \quad f_n - f_{n'} \notin \mathbb{Z}. \end{cases}$$

Uniform case : if $M/N \in \mathbb{N}$ and $f_n = n/N$, $n = 0, \cdots, N-1$, then

$V^*V = M\, I_N$

$$\longrightarrow \kappa(V) = 1 \text{ : optimal conditioning !}$$

Conditioning of V^*V

What about the general case ?

Wrap-around distance on $[0,1]$:

$$d_w(f,g) = \inf_{q\in\mathbb{Z}} |f-g+q|$$

Example : $d_w(0.1, 0.9) = 0.2$

Theorem (A. Moitra, 2015) : If the minimal separation

$$\Delta = \min_{n,n'=0,\cdots,N-1,\, n\neq n'} d_w(f_n, f_{n'})$$

is such that $M > 1/\Delta + 1$, then

$$\kappa(V) \leq \sqrt{\frac{M+1/\Delta-1}{M-1/\Delta-1}}$$

$\longrightarrow$ when $M \to +\infty$, optimal conditioning is recovered !

The proof

The Selberg's majorant and minorant : for $[a, b]$ and $L > 0$, there exist two functions S_+ and S_- such that :

1. S_+ and S_- are integrable
2. $S_-(t) \leq \chi_{[a,b]}(t) \leq S_+(t)$ for all $t \in \mathbb{R}$
3. $\widehat{S}_\pm(x) = 0$ for $|x| \geq L$

and

$$\int_{-\infty}^{+\infty} S_\pm(t)\, dt = b - a \pm \frac{1}{L}$$

Moreover

$$|S_\pm(t)| \leq \frac{C}{1+t^2}, \quad \forall t \in \mathbb{R}$$

$\longrightarrow$ The functions $S_\pm$ can serve as test fonctions for the Dirac comb

The proof (cont.)

We choose $a = 0$, $b = M - 1$ and $L = \Delta$ in $S_\pm$:

Consider $v(x) = \sum_{n=0}^{N-1} u_n e^{-2\pi i\, f_n x}$ for $U = (u_0, u_1, \cdots, u_{N-1}) \in \mathbb{C}^N$

$$\sum_{m=0}^{M-1} |v(m)|^2 = \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} u_n \overline{u}_{n'} \underbrace{\sum_{m=0}^{M-1} e^{-2\pi i (f_n - f_{n'})m}}$$

$$= \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} (V^* V)_{nn'} u_n \overline{u}_{n'} = ||VU||^2$$

Interpretation with **Dirac comb** $D = \sum_{l \in \mathbb{Z}} \delta_l$: since $\chi_{[0,M-1]}(t) \leq S_+(t)$ for all $t \in \mathbb{R}$

$$\sum_{m=0}^{M-1} |v(m)|^2 = \left\langle \chi_{[0,M-1]} D, |v|^2 \right\rangle \leq \left\langle D, S_+ |v|^2 \right\rangle$$

The proof (cont.)

$$D = \sum_{l \in \mathbb{Z}} e^{-2\pi i\, lx} \;\Rightarrow\; \sum_{m=0}^{M-1} |v(m)|^2 \leq \sum_{l \in \mathbb{Z}} \left\langle e^{-2\pi i\, lx}, S_+ |v|^2 \right\rangle$$

$$= \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} \sum_{l \in \mathbb{Z}} u_n \overline{u}_{n'} \underbrace{\int_{\mathbb{R}} e^{-2\pi i\, lx} S_+(x) e^{-2\pi i\,(f_n - f_{n'})x}\, dx}_{\widehat{S}_+(f_n - f_{n'} + l)}$$

Minimal separation : $\Delta = \inf_{n \neq n'} \inf_{l \in \mathbb{Z}} |f_n - f_{n'} + l|$ and the support of $\widehat{S}_+$ is contained in $[-\Delta, \Delta]$

$$\sum_{m=0}^{M-1} |v(m)|^2 \leq \widehat{S}_+(0) \sum_{n=0}^{N-1} |u_n|^2, \; \widehat{S}_+(0) = \int_{-\infty}^{+\infty} S_+(t)\, dt = M - 1 + 1/\Delta$$

Final estimate : $||VU||^2 \leq (M - 1 + 1/\Delta)||U||^2$

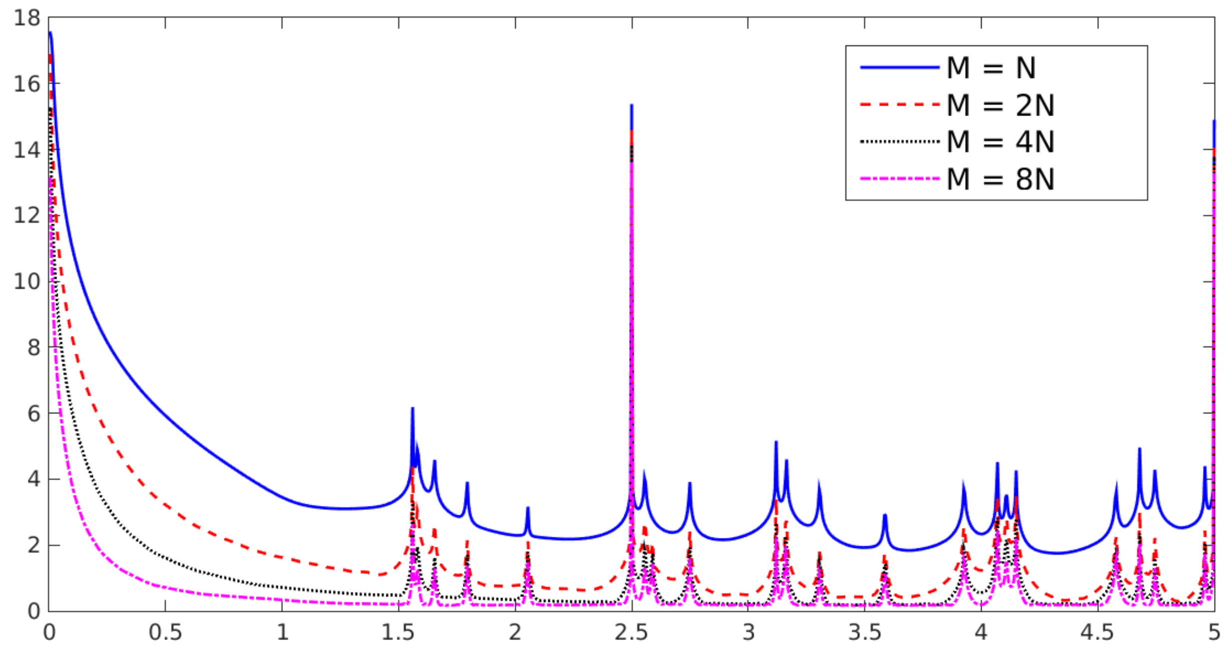
Optimization of sources/receivers

Conclusion : M and Δ have to be large !

In our particular case :

$$\Delta = \left(1 - \sqrt{1 - \frac{\lambda^2}{4d^2}}\right)\frac{\delta}{\lambda} \quad \longrightarrow \quad \text{Optimal } \delta \text{ is } \lambda$$

Log of the condition number $\kappa(V)$ versus δ/λ :



Back to the time domain problem

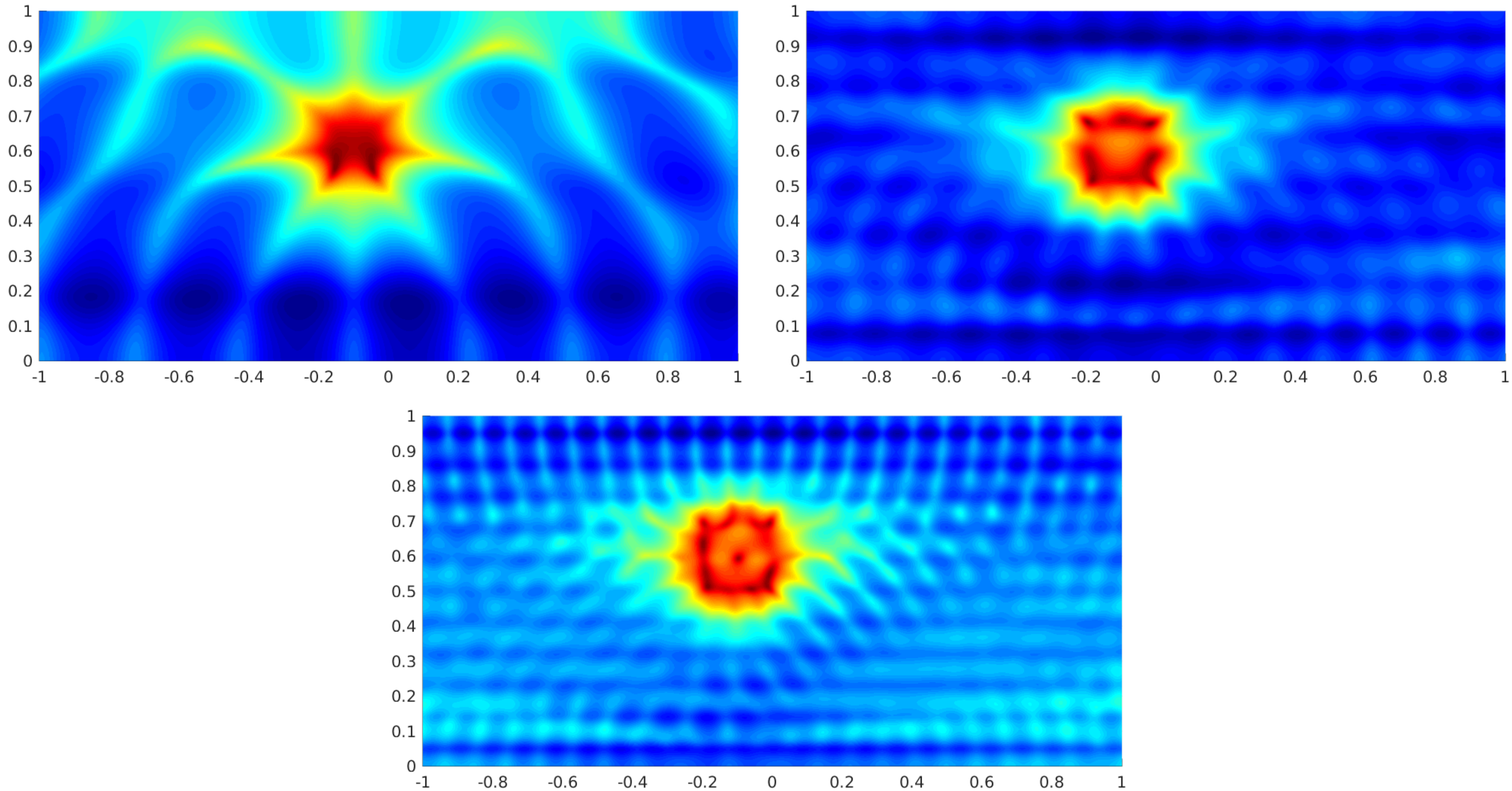
Methodology:

- Take the Fourier transform of the data
  $\longrightarrow$ Measurement matrix $M(\omega)$
- Invert the matrices E and R for each frequency ω
  $\longrightarrow$ LSM matrix $U(\omega)$
- Solve the LSM modal formulation for each frequency ω
  $\longrightarrow$ Indicator function $\psi(\omega)$
- Compute a global indicator function Ψ :

$$\Psi = \left(\int_{\omega_-}^{\omega_+} \frac{\max_{z \in G} |\psi(z,\omega)|^2}{|\psi(\cdot,\omega)|^2} \, d\omega \right)^{-1/2}$$

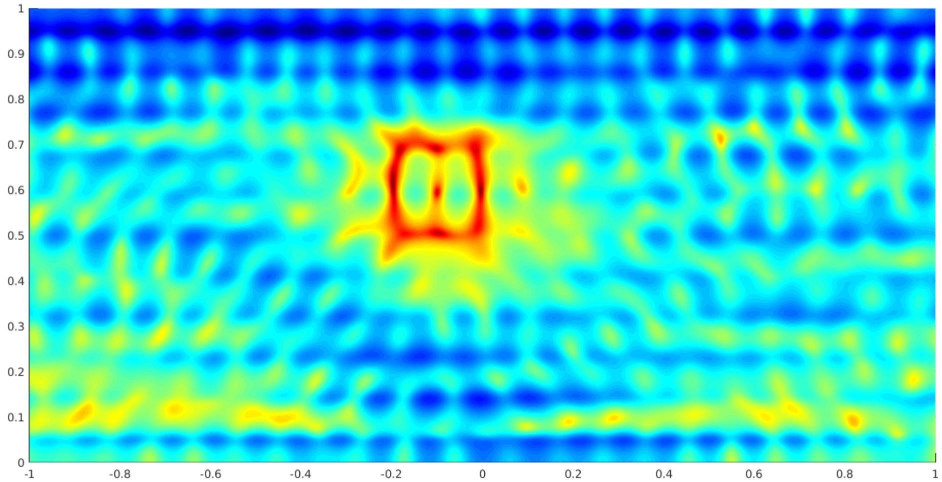
G : sampling grid

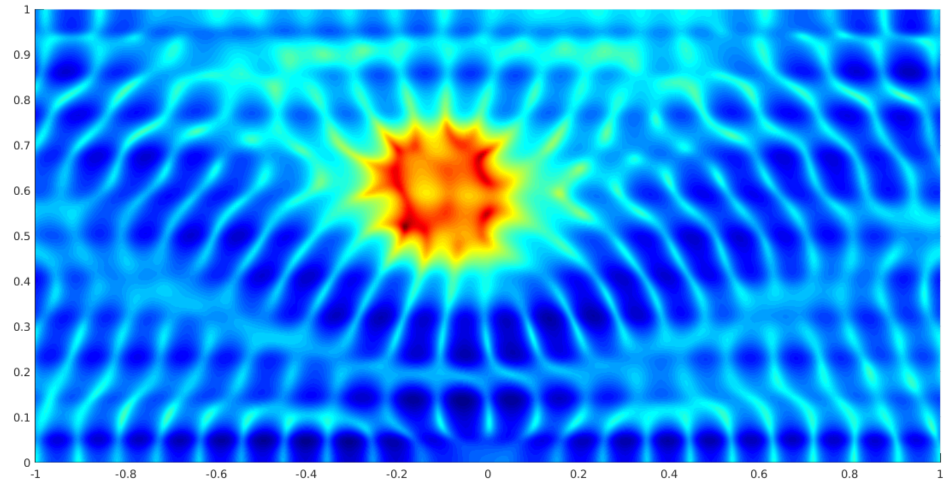
Some numerical results : time domain

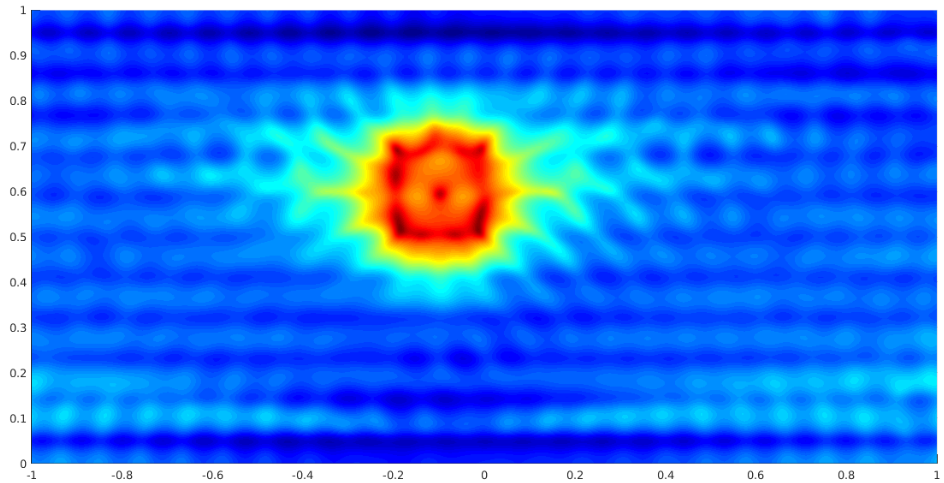


Inversion using one single frequency corresponding to $N = 4$, $N = 8$ and $N = 12$

Impact of the number of sources/receivers

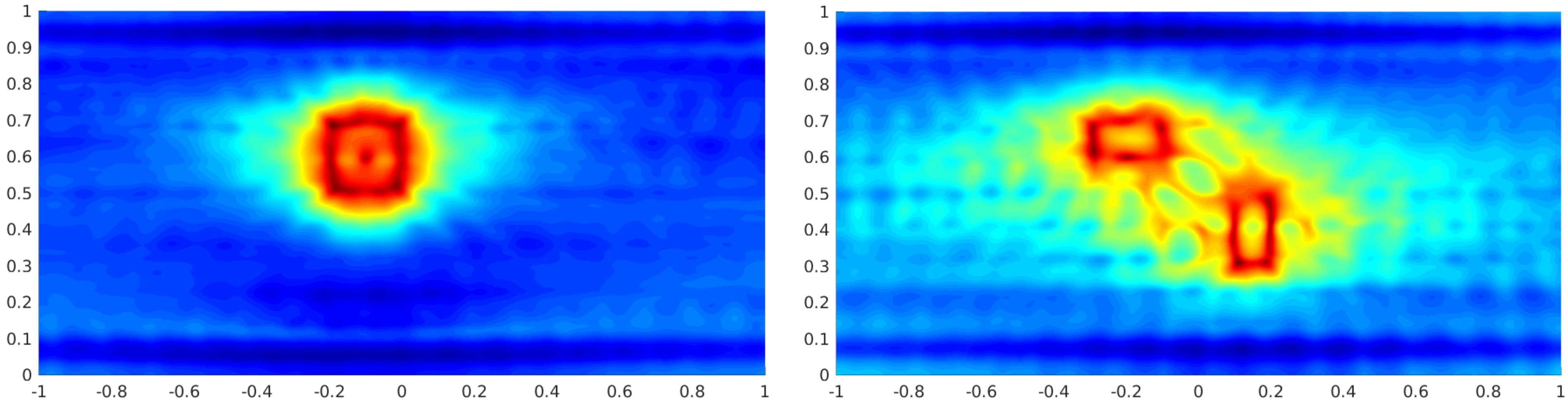






Inversion using a single frequency corresponding to $N = 12$:
$M = N$, $M = 2N$ and $M = 3N$

Some numerical results : time domain



Inversion using multiple frequencies

Some perspectives

- Extension to elasticity (NDT by ultrasonics) : the modal approach in more difficult
- Extension to more realistic defects : corrosion, cracks
- Experimental validation with true data

Thank you for your attention !

About the synthetic data

Some difficulties related to the time domain:

- From a numerical point of view, the forward computation is not that easy, many choices $\longrightarrow$ Our choice : FE in space, FD in time, PMLs,...

- In a waveguide, the scattering field is slowly decaying with respect to time $\longrightarrow$ the forward computation has to be long !

Possible remedy : $\hat{\chi}(\omega)$ should avoid the cut-off frequencies λ_n