Inside-Outside Duality in Time-Harmonic Wave Scattering

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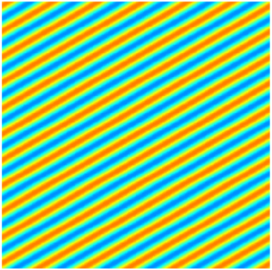
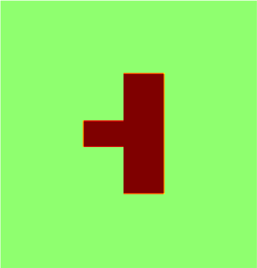
Joint work with Andreas Kirsch, Stefan Peters, Marcel Rennoch

Applied Inverse Problems Conference 2015

Wave Scattering

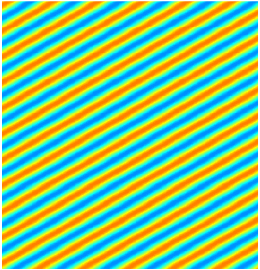


Incident field u^i



Scatterer D

Wave Scattering



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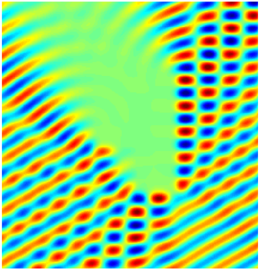
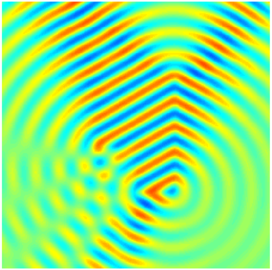
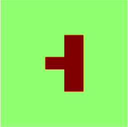


Total field u



Scattered Field u^s

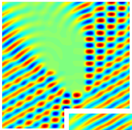
Wave Scattering

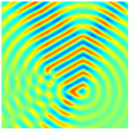


. . . are posed on exterior domains and link incident waves with outgoing scattered waves.

- Penetrable scatterer D, contrast $q > -1$, $\mathrm{supp}(q) = \overline{D}$
- Incident field $u^i(x, \theta) = \exp(\mathrm{i}k\, x \cdot \theta)$, $\theta \in \mathbb{S}$, wave number k,
- Total field $u(x, \theta) = u^i(x, \theta) + u^s(x, \theta)$ satisfies

$$\Delta u + k^2(1 + q)u = 0 \text{ in } \mathbb{R}^3$$

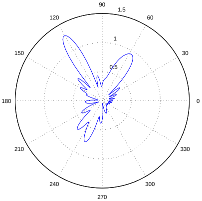




- Scattered field $u^s(\cdot, \theta) = u(\cdot, \theta) - u^i(\cdot, \theta)$ radiates:

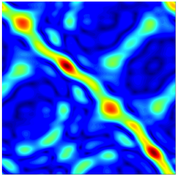
$$\frac{x}{|x|} \cdot \nabla u^s(x) - \mathrm{i}ku^s(x) = \mathcal{O}(|x|^{-2}) \text{ as } |x| \to \infty$$

Far Field Pattern and Far Field Operator



- Behaviour of scattered field at ∞:

$$u^s(r\hat{x},\theta) = \frac{e^{ikr}}{4\pi r}\left(u^\infty(\hat{x},\theta) + \mathcal{O}\left(\frac{1}{r}\right)\right) \text{ as } r \to \infty$$



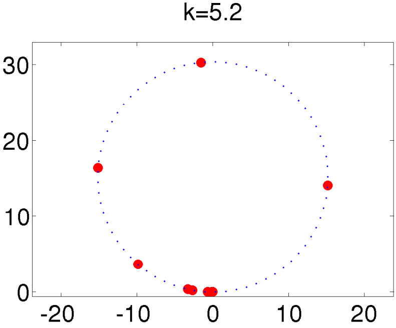
- Far field pattern: $u^\infty(\hat{x},\theta)$ for $\hat{x}$, $\theta \in \mathbb{S}$
- Far field operator $F:\ L^2(\mathbb{S}) \to L^2(\mathbb{S})$,

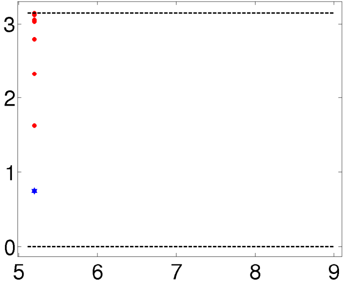
$$Fg = \int_{\mathbb{S}} u^\infty(\cdot,\theta)g(\theta)\,\mathrm{d}s$$

- Scatterer is non-absorbing $\Rightarrow F \in \mathcal{K}(L^2(\mathbb{S}))$ is normal
- Spectral representation: $Fg = \sum_{j=1}^\infty \lambda_j \langle g, g_j\rangle_{L^2(\mathbb{S})} g_j$

Inside-Outside Duality

Scattering from unit ball with refractive index 0.45, $k \mapsto F(k)$





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$\Leftrightarrow$ No scattered field in $\mathbb{R}^3 \setminus \overline{D}$ $\Leftrightarrow$ k^2 interior eigenvalue of D

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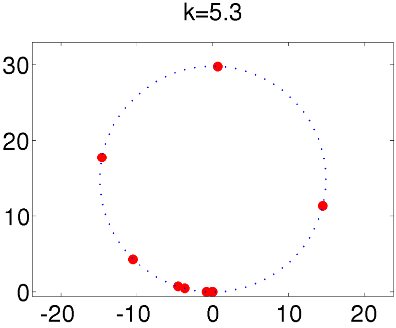

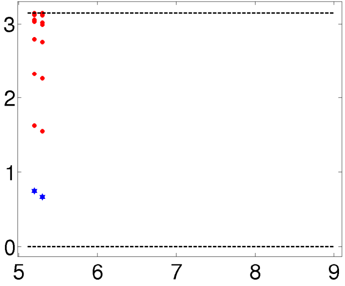

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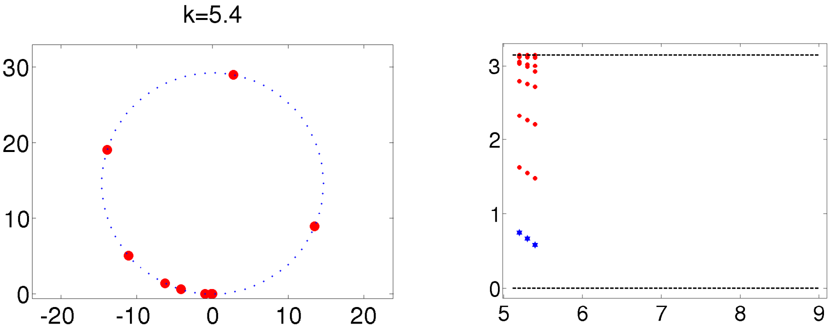

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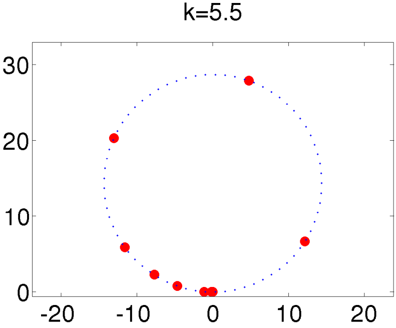

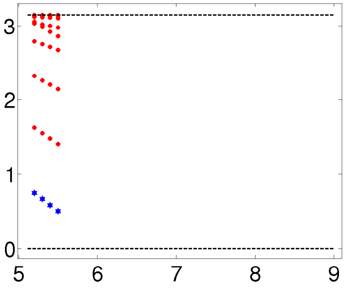

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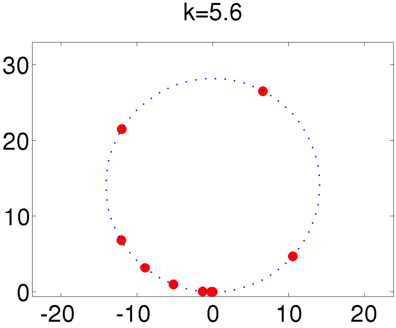

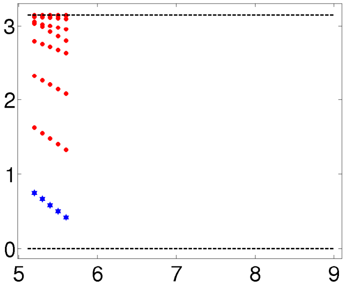

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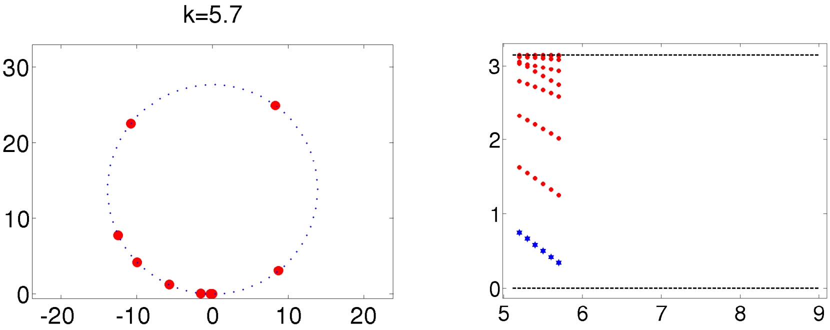

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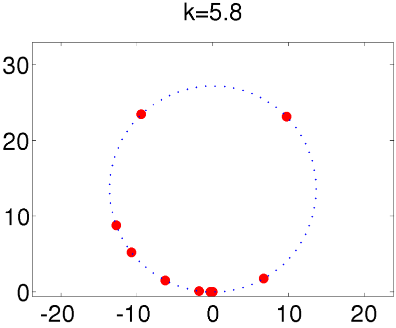

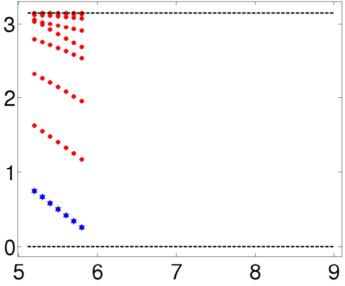

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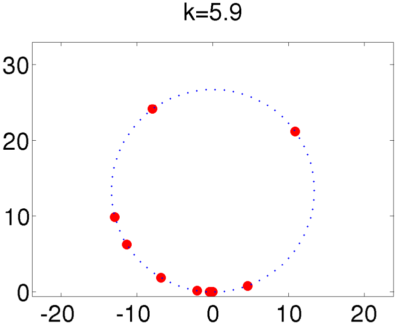

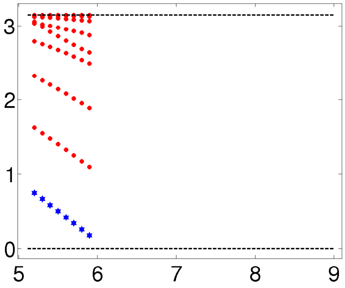

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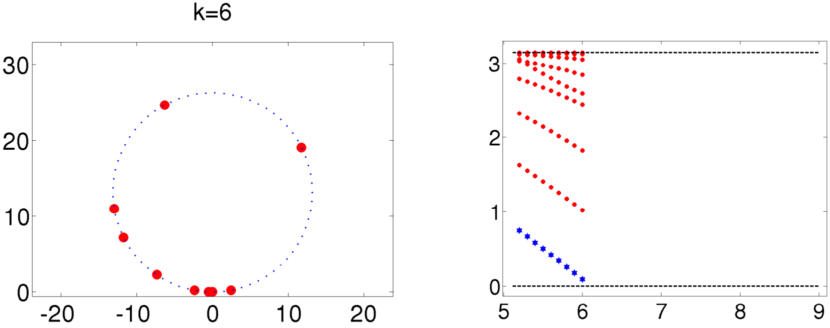

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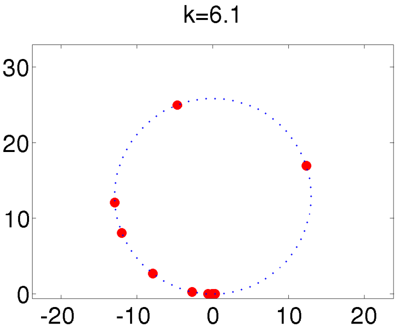

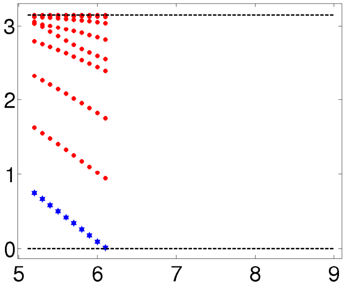

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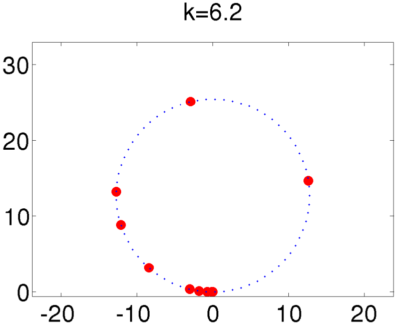

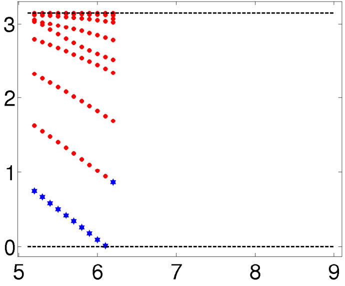

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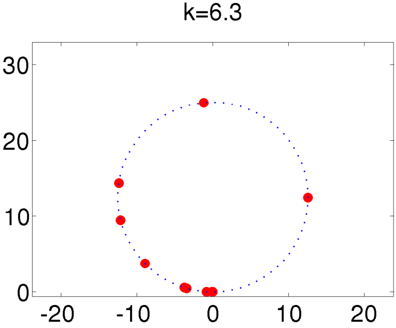

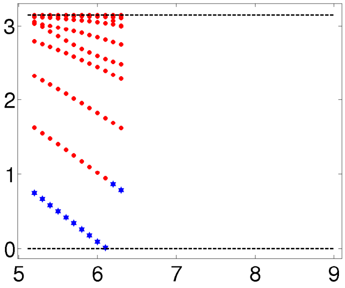

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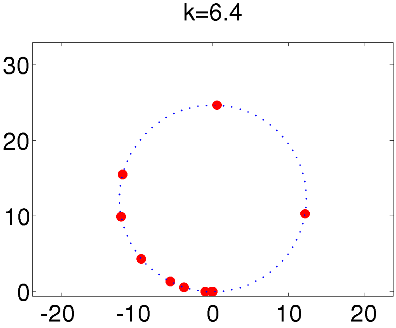

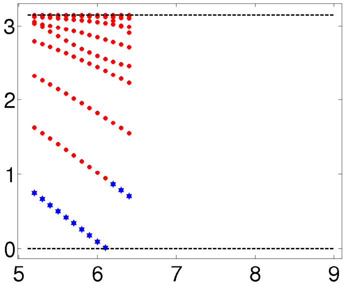

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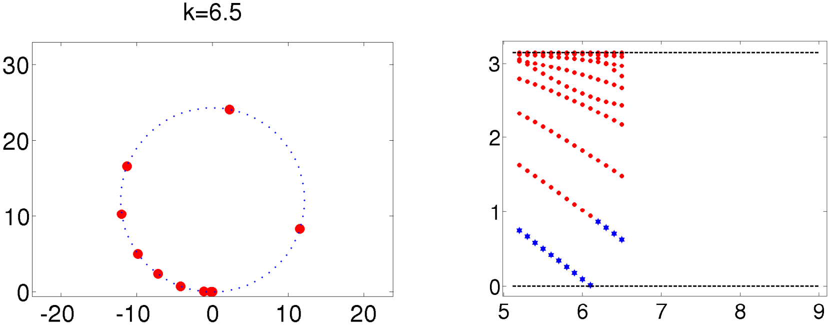

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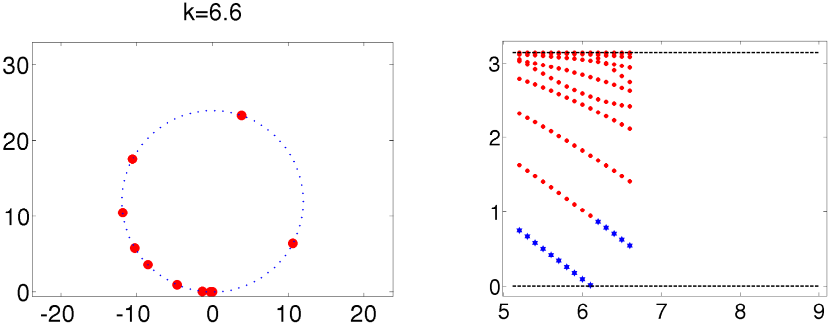

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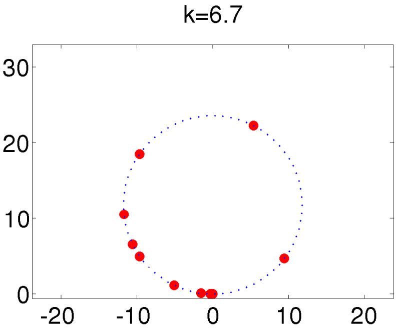

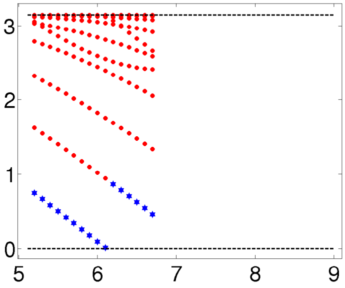

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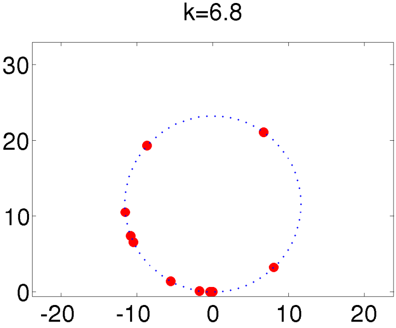

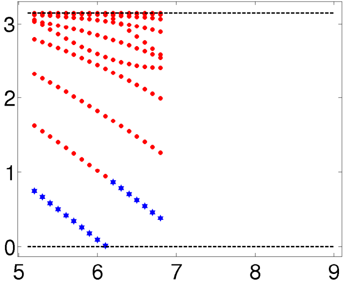

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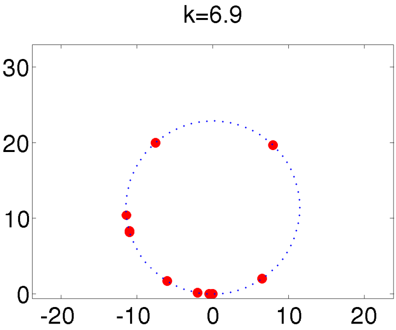

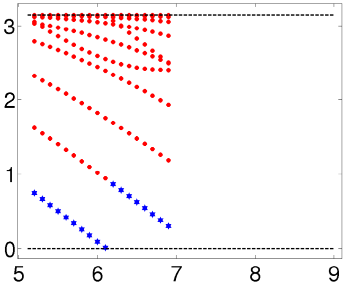

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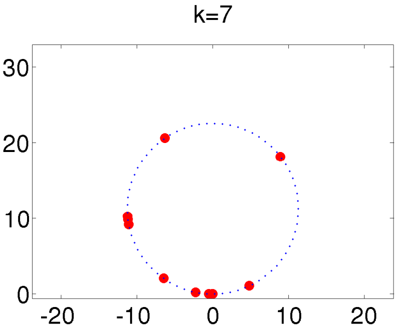

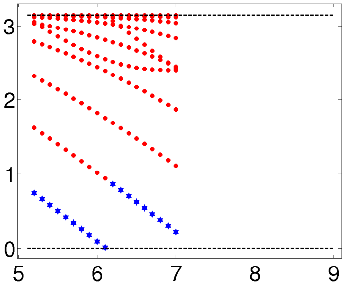

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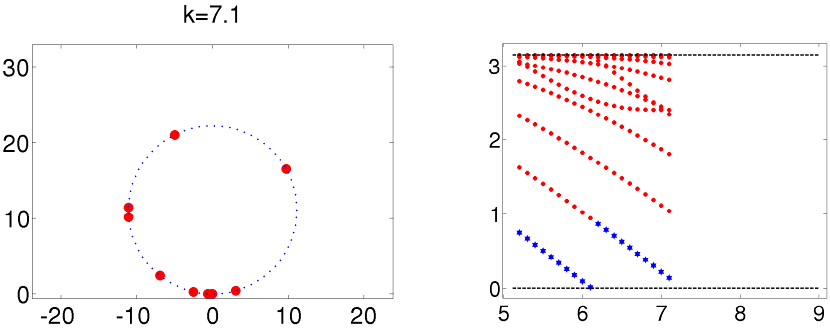

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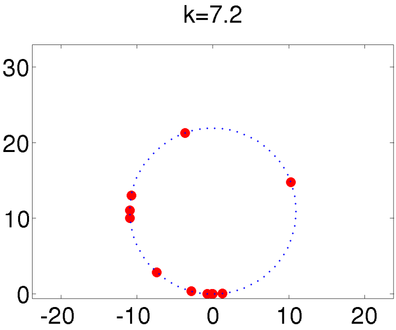

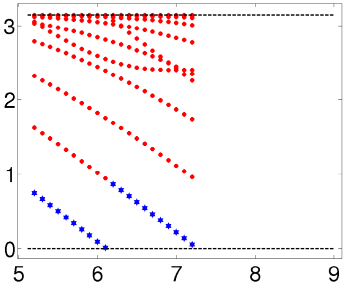

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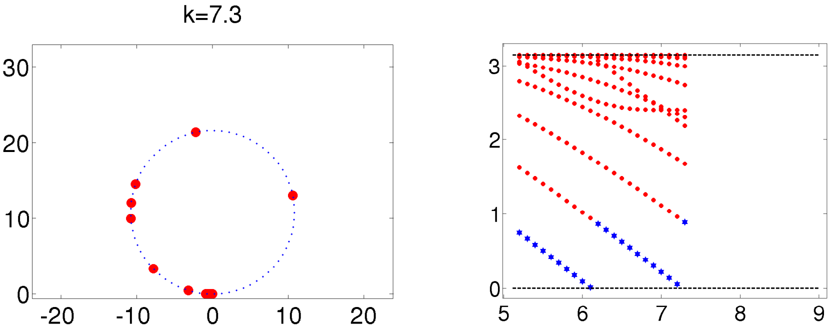

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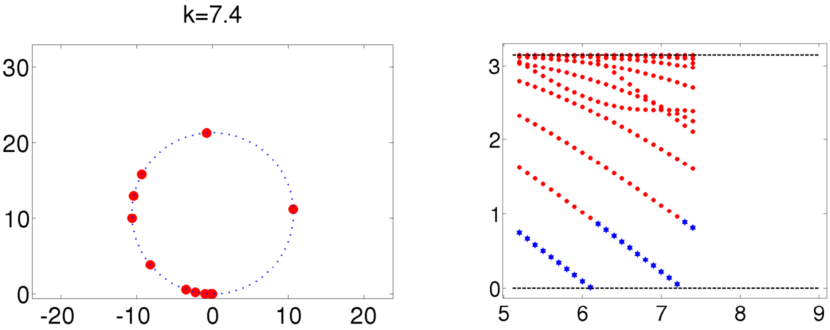

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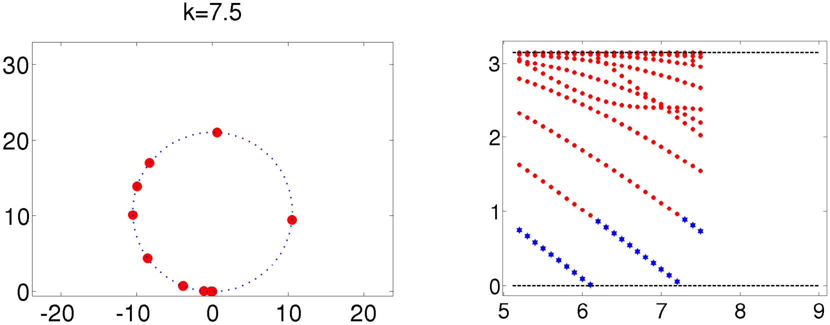

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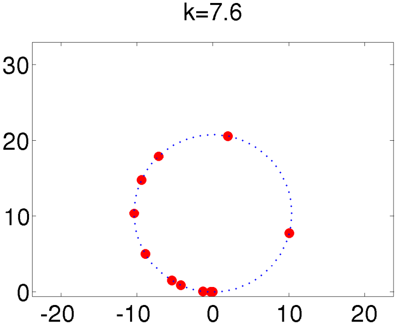

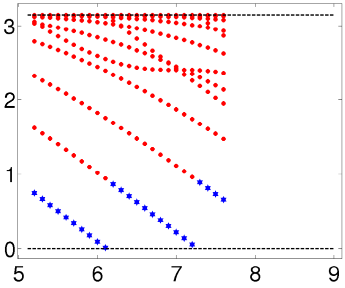

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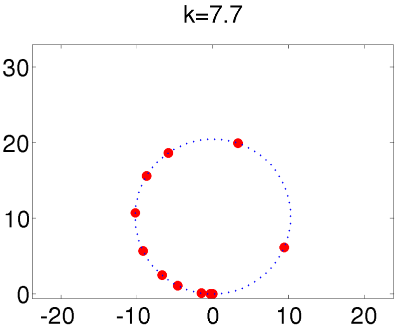

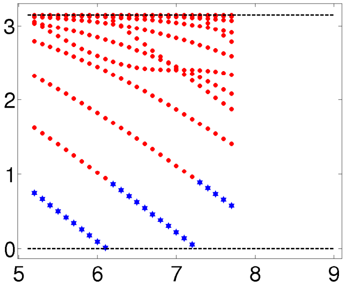

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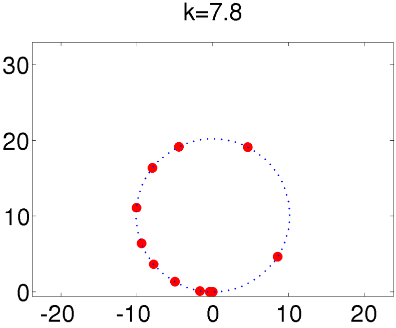

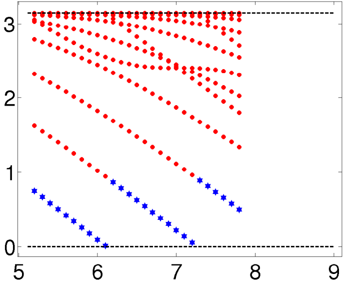

Roughly: Zero is eigenvalue of $F(k)$

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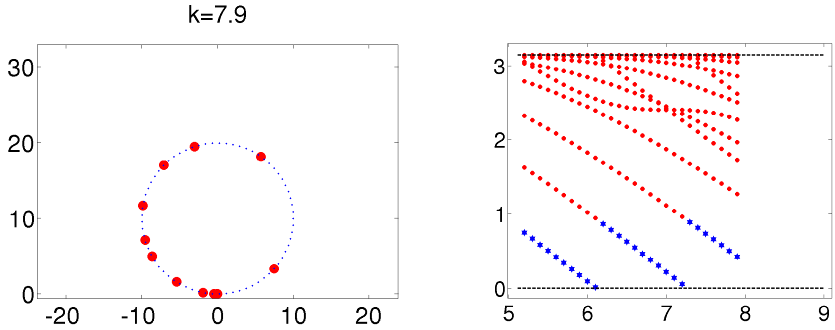

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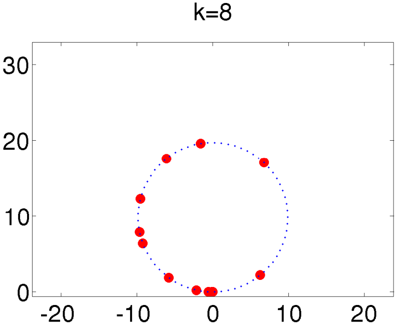

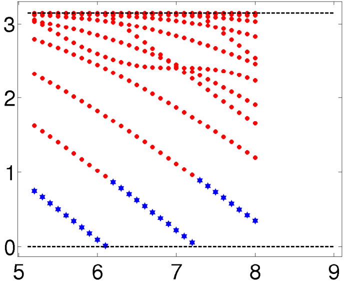

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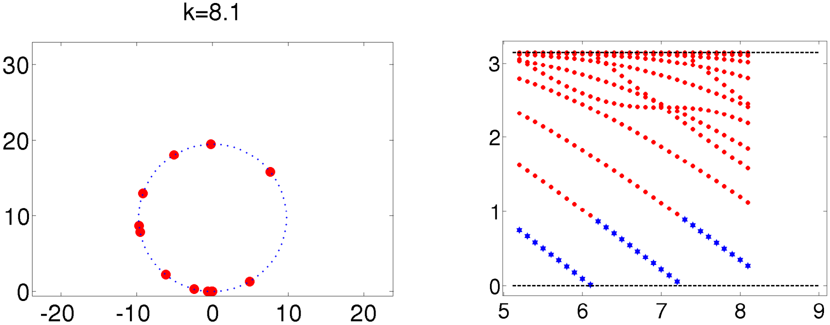

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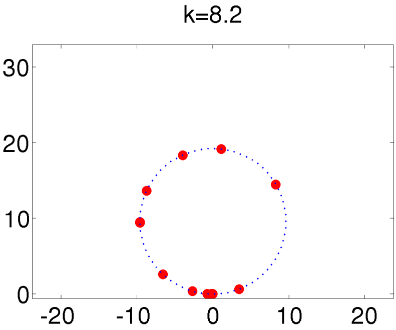

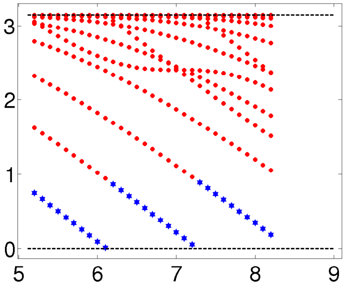

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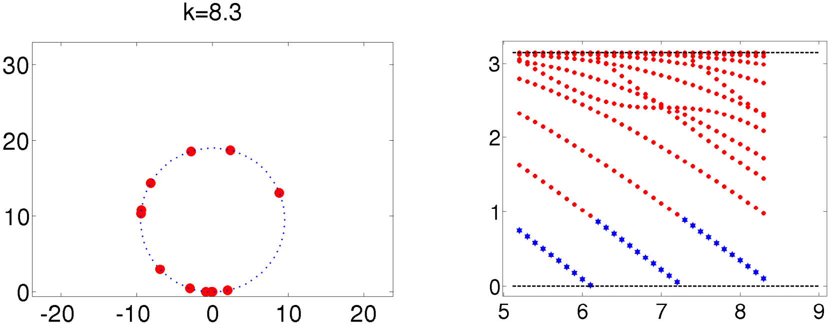

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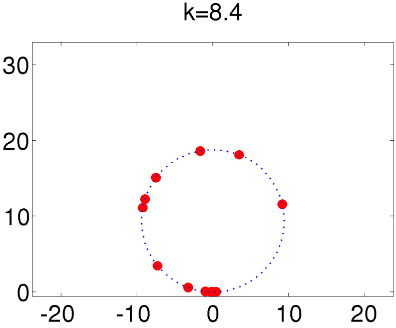

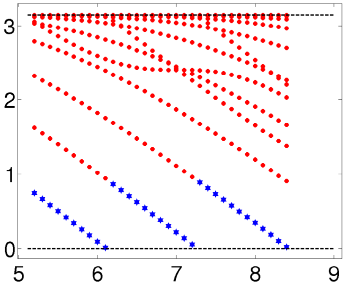

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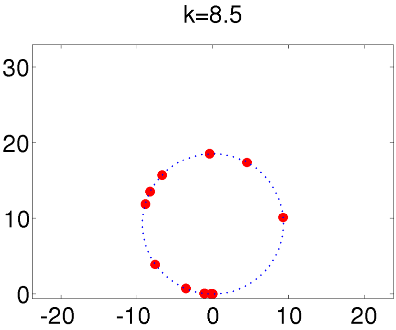

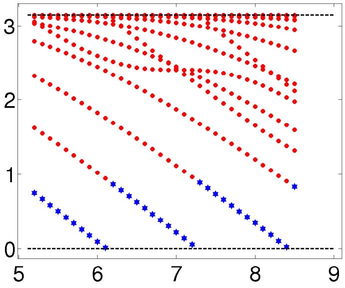

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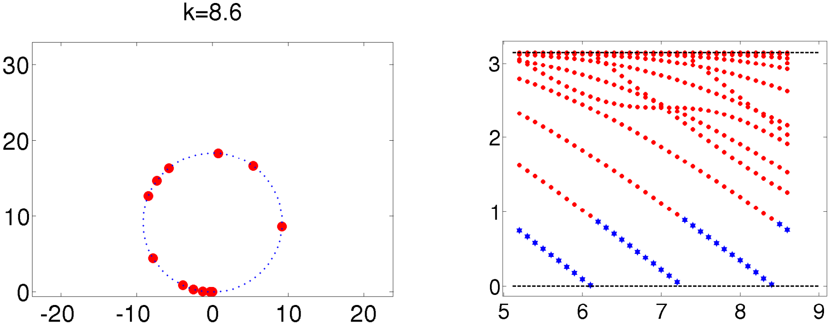

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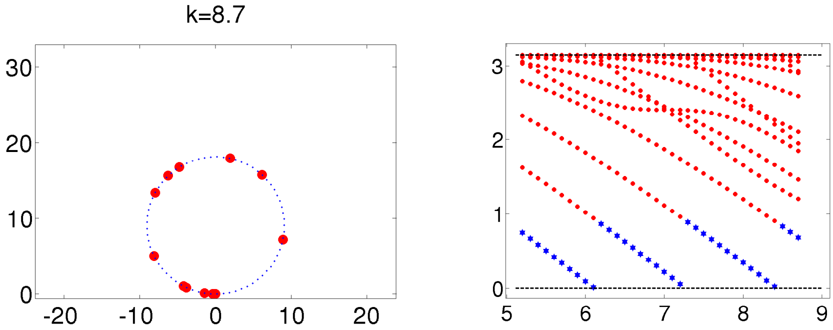

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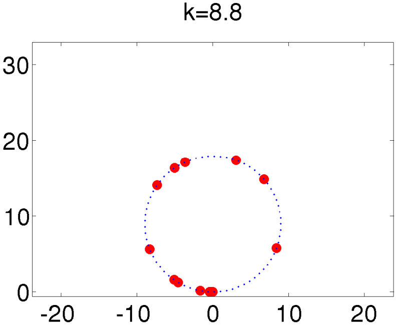

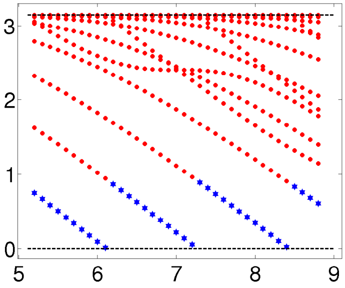

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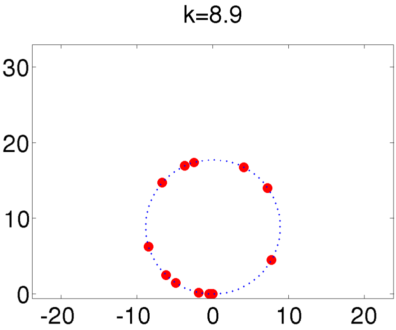

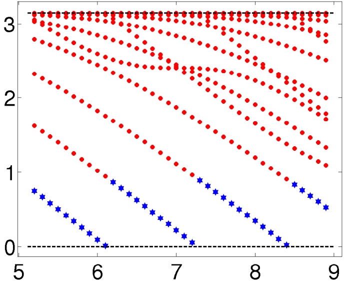

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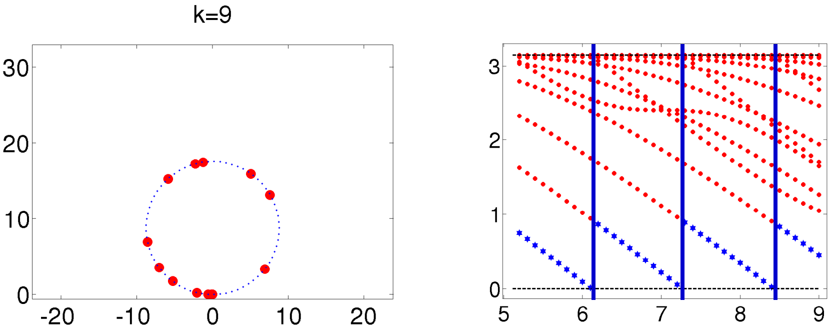

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Why Are Interior Eigenvalues Interesting?

- Well-known for Dirichlet eigenvalues:

$$\mathrm{vol}(\text{inscribed circle}(D)) \leq \frac{\pi}{\lambda_1^{\mathrm{D}}} j_{0,1}^2 \leq \mathrm{vol}(D)$$

- Interior transmission eigenvalues (ITEs) provide bounds for refractive index and $\mathrm{vol}(D)$: If k^2 is ITE,

$$1 + \sup_D(q) \geq \frac{\lambda_1^{\mathrm{D}}}{k^2} \qquad (\lambda_1^{\mathrm{D}} = \text{1st Dirichlet eig'value})$$

- Anisotropic model $\mathrm{div}((I + Q)\nabla u) + k^2 u = 0$ where unique determination from (multi-frequency) far field data fails:

$$1 + \sup_D |Q|_2 \geq \frac{\lambda_1^{\mathrm{D}}}{k^2}$$

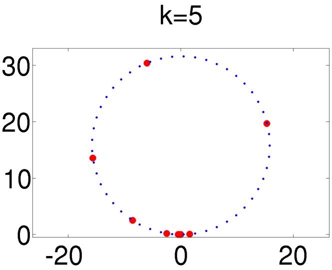
Cakoni, Colton & Monk '07; Colton, Cakoni & Haddar '09

Inside-Outside Duality

- It is possible to compute interior eig'vals from far field data
- Domain D and scattering model need not be known!

Theorem: If $\lambda_j(k) \stackrel{j\to\infty}{\longrightarrow} 0$ from fixed side:

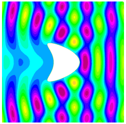
$$\left[\begin{array}{l} \lambda_j(k) \stackrel{k\to k_0}{\longrightarrow} 0 \text{ from "wrong" side} \\ \Longrightarrow k_0^2 > 0 \text{ is interior eig'val of } D \end{array} \right.$$



Reverse directions holds . . . ?

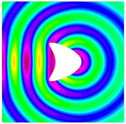
Overview

The Good Case: Impenetrable Scattering Problems



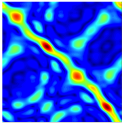
- Dirichlet scatterer $D \subset \mathbb{R}^3$ s.th. $\mathbb{R}^3 \setminus \overline{D}$ connected
- Radiating scattered field $u^s = u^s(\cdot, \theta)$ solves

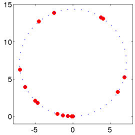
$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}, \;\; u^s = -u^i(\cdot, \theta) \text{ on } \partial D$$



- Far field pattern $u^\infty(\hat{x}, \theta)$, $\hat{x}, \theta \in \mathbb{S}$,

$$Fg = \int_{\mathbb{S}} u^\infty(\cdot, \theta) g(\theta) \, \mathrm{d}s \text{ is normal}$$





- Eig'vals λ_j accumulate at 0 from left:
  $\mathrm{Re}\, \lambda_j < 0$ for j large enough
- Eig'value problem: Find $u_0 \in H_0^1(D)$ s.th. $\Delta u_0 + k_0^2 u_0 = 0$ in D

Eckmann & Pillet '95, '97; L & Peters '14

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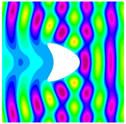

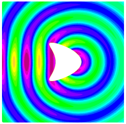

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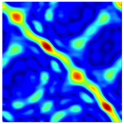

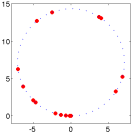

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Link Cotangent of Smallest Phase with F

- Polar representation: $\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j)$, $\vartheta_j \in [0, \pi)$
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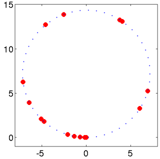

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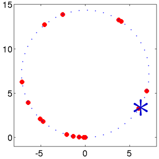

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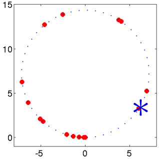

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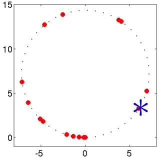

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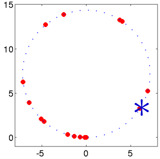

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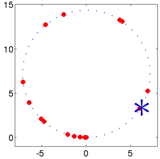

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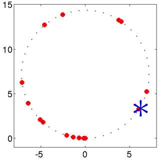

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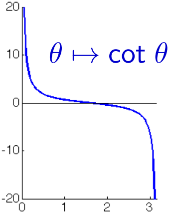
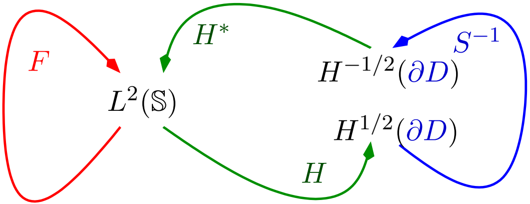


Factorization of F



- If k^2 no Dirichlet eigenvalue of $-\Delta$: $F = -H^* S^{-1} H$
- $H : L^2(\mathbb{S}) \to H^{1/2}(\partial D)$ Herglotz operator (dense range!)

$$g \mapsto v_g|_D\,, \quad v_g(x) := \int_{\mathbb{S}} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\,\mathrm{d}s(\theta)$$

- S^{-1}: $H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ inverse of single-layer operator

Plugging things together ...

Lemma

If k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D:

$$\cot \vartheta_* = \max_{g \in L^2(\mathbb{S})} \frac{\mathrm{Re}\,\langle F_k g, \, g\rangle}{\mathrm{Im}\,\langle F_k g, g\rangle}$$

- $k \mapsto \mathrm{Im}\,\langle S_k \varphi, \, \varphi\rangle \leq 0$ vanishes at $k = k_0$ iff k_0^2 is Dirichlet eigenvalue and $\varphi = (\partial w / \partial \nu)$ for eigenfunction $w \in H_0^1(D)$
- If $\vartheta_*(k_j) \to 0$ but k^2 is no Dirichlet eig'value: $\exists$ normalized φ_j s.th. $\mathrm{Im}\,\langle S_{k_j} \varphi_j, \varphi_j\rangle \to 0$, which yields a contradiction.

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- If $\vartheta_*(k_j) \to 0$ but k^2 is no Dirichlet eig'value: $\exists$ normalized φ_j s.th. $\mathrm{Im}\,\langle S_{k_j} \varphi_j, \varphi_j \rangle \to 0$, which yields a contradiction.

Plugging things together ...

Lemma

If k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D:

$$\cot \vartheta_* = \max_{\psi \in H^{1/2}(\partial D)} \frac{\mathrm{Re}\, \langle S^{-1}\psi, \psi \rangle}{\mathrm{Im}\, \langle S^{-1}\psi, \psi \rangle}$$

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Does the Reverse Direction Hold . . . ?

Lemma (Characterization via Smallest Phase)

If k_0^2 *is Dirichlet eig'value, if* $\operatorname{Im}\langle S_{k_0}\varphi_0, \varphi_0\rangle = 0$, *and if (!)*

$$\alpha := \left[\frac{\mathrm{d}}{\mathrm{d}k}\langle S_k\varphi_0, \varphi_0\rangle\right]\Big|_{k=k_0} > 0, \quad \textit{then} \quad \lim_{k \nearrow k_0} \vartheta_*(k) = 0$$

Proof by Taylor's formula: As $k \nearrow k_0$,

$$\cot \vartheta_*(k) = \max_{\varphi \in H^{1/2}} \frac{\operatorname{Re}\langle S_k\varphi, \varphi\rangle}{\operatorname{Im}\langle S_k\varphi, \varphi\rangle} \overset{\varphi=\varphi_0}{\geq} \frac{\alpha(k-k_0) + \operatorname{Re}(r(k))}{\operatorname{Im}(r(k))} \to \infty$$

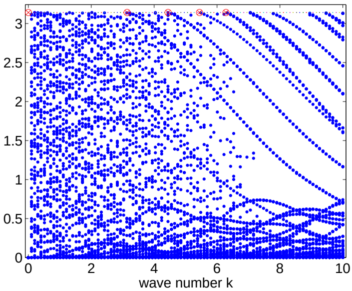
Finally: $\alpha = 2k_0 \int_D |\mathrm{SL}_{k_0}\varphi_0|^2 \, \mathrm{d}x > 0$ by Green's formula

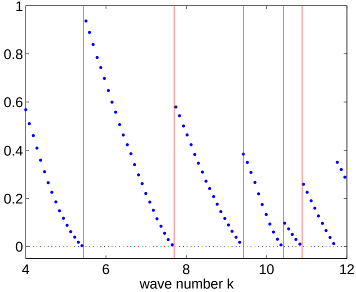
Inside-Outside Duality for Dirichlet Obstacles

Theorem

k_0^2 *is a Dirichlet eigenvalue of* D *if and only if* $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$

Similar characterization for interior Neumann or Robin eigenvalues





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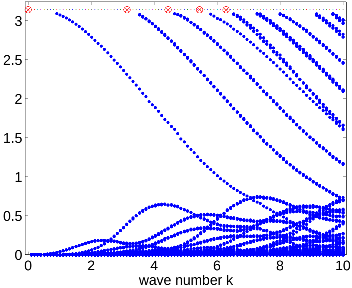
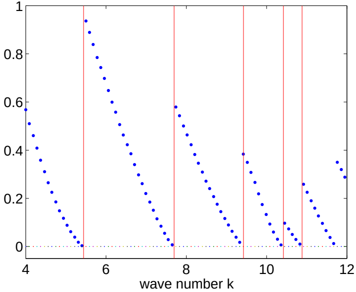
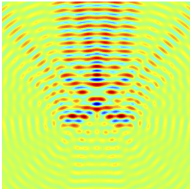




Overview

Scattering from Penetrable Medium



- Scattering from penetrable medium D
- Contrast $q : \mathbb{R}^3 \to \mathbb{R}$, supp$(q) = \overline{D}$
- Radiating scattered wave $u^s(\cdot, \theta)$ satisfies

$$\Delta u^s + k^2(1+q)u^s = -k^2 q u^i(\cdot,\theta) \text{ in } \mathbb{R}^3$$

- Far field operator F normal (as q real-valued) and injective iff there is no interior transmission eigenpair (v, w) solving

$$\Delta v + k^2 v = 0 \quad \text{and} \quad \Delta w + k^2(1+q)w = 0 \qquad \text{in } D,$$

$$v = w \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D,$$

s.th. $v = v_g$ additionally is a Herglotz wave function

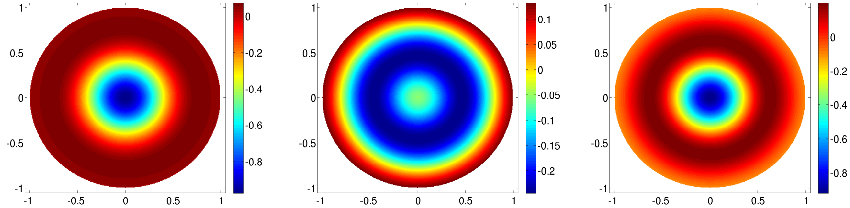
Interior Transmission Eigenvalues

- Find eigenpair $(u, w) \in L^2(D) \times L^2(D)$ with $u - w \in H^2_0(D)$ and transmission eigenvalue $k_0^2 > 0$ such that

$$\Delta u + k_0^2(1+q)u = 0 \quad \text{in } D, \qquad \Delta w + k_0^2 w = 0 \quad \text{in } D,$$

$$u = w \quad \text{on } \partial D, \qquad \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D$$

- Example: $D = B(0,1) \subset \mathbb{R}^3$, $q \equiv 2$



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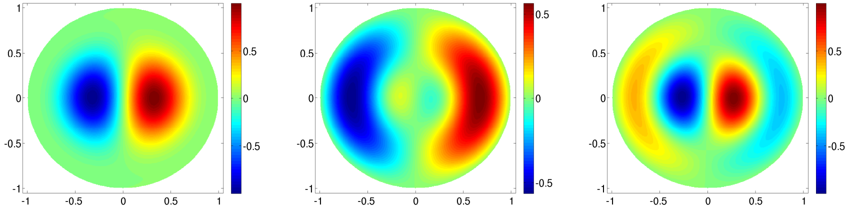

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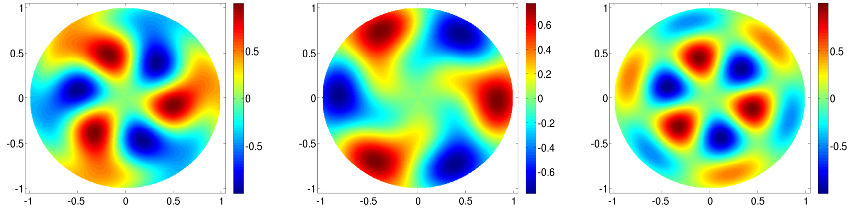

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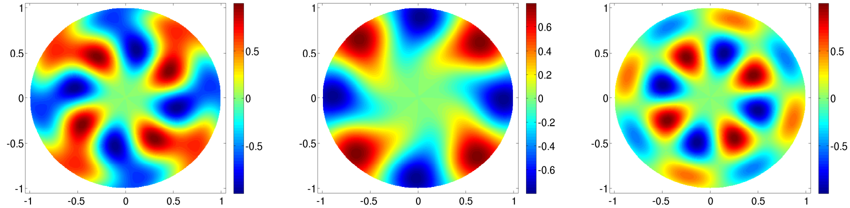
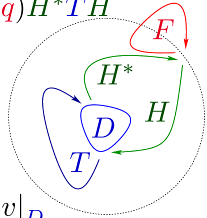


Factoriyation of Far Field Operator

- Far field operator F has factorization $F = k^2 \text{sign}(q) H^* T H$
- Herglotz operator $H : L^2(\mathbb{S}) \mapsto L^2(D)$,



$$g \mapsto v_g|_D \,, \quad v_g(x) = \int_{\mathbb{S}} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathrm{d}s(\theta)$$

- Solution operator $T : L^2(D) \to L^2(D)$, $f \mapsto f + v|_D$ where $v \in H^1_{\text{loc}}(\mathbb{R}^3)$ is radiating solution to

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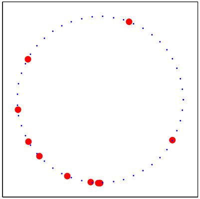

- If $q \gtrless 0$ in $D \Rightarrow$ Eigenvalues λ_j tend to zero in $j \in \mathbb{N}$ from right/left: Re $\lambda_j \gtrless 0$ for large j

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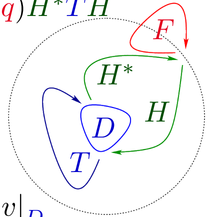

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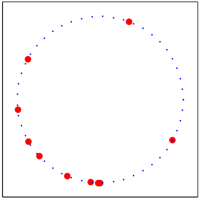

The Cotangent of the Smallest Phase and F

Assume from now on that $\text{sign}(q) = -1$ in D (i.e., $-1 < q(x) < 0$) and consider $\vartheta_* = \max_{j\in\mathbb{N}} \vartheta_j$

Theorem

If k^2 is no ITE, then

$$\cot \vartheta_*(k) = \max_{g\in L^2(\mathbb{S})} \frac{\text{Re}\,\langle F_k g,\, g\rangle}{\text{Im}\,\langle F_k g, g\rangle}$$

- $X_k = \text{closure}_{L^2(D)} \mathcal{R}(H) = \{v \in L^2(D) : (\Delta + k^2)v = 0 \text{ in } L^2(D)\}$
- Recall: $Hg = v_g|_D$ with $v_g(x) = \int_{\mathbb{S}} e^{\mathrm{i}k\,x\cdot\theta} g(\theta)\,\mathrm{d}s$

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Difference Compared to Impenetrable Obstacles: k-Dependent Spaces!

$$X_k = \overline{\mathcal{R}(H)}^{L^2(D)} = \left\{ v \in L^2(D) : (\Delta v + k^2 v) = 0 \text{ in } L^2(D) \right\}$$

Theorem (Characterization of ITEs via $\operatorname{Im} T_k$)

(a) If $k_0^2 > 0$ is ITE with eigenpair (u, w), then $w \in X_{k_0} \setminus \{0\}$ and $\operatorname{Im} \langle T_{k_0} w, w \rangle_{L^2(D)} = 0$

(b) If $\operatorname{Im} \langle T_k w, w \rangle_{L^2(D)} = 0$ for $k > 0$ and $w \in X_k \setminus \{0\}$, then there is $u \in L^2(D)$ such that (u, w) is eigenpair for ITE k^2

- Recall: $\cot \vartheta_*(k) = \max_{w \in X_k} \dfrac{\operatorname{Re} \langle T_k w, \, w \rangle_{L^2(D)}}{\operatorname{Im} \langle T_k w, \, w \rangle_{L^2(D)}}$
- As for Dirichlet case: If $\vartheta_*(k) \to 0$ as $k \to k_0$, then k_0^2 is ITE

... and the Reverse Direction?

- Assume that $P_k : L^2(D) \to X_k$ is any projection onto X_k that is differentiable in k, i.e., $k \mapsto P_k \in C^1(\mathbb{R}_{>0}; L^2(D))$
- Rewrite: $\cot \vartheta_*(k) = \max_{w \in X_k} \frac{\mathrm{Re}\, \langle T_k w,\, w\rangle_{L^2(D)}}{\mathrm{Im}\, \langle T_k w,\, w\rangle_{L^2(D)}}$
- Good news: $k \mapsto \langle T_k w, w\rangle$ is differentiable. At ITE k_0^2 with eig'pair (u, w), $\left[\frac{\mathrm{d}}{\mathrm{d}k}\langle T_k w, w\rangle\right]\Big|_{k=k_0} = 2k_0 \int_D |\nabla(u-w)|^2 \,\mathrm{d}x$

Lemma

If $k_0^2 > 0$ is a transmision eigenvalue with eigenpair (u, w) such that $\left[\frac{\mathrm{d}}{\mathrm{d}k}\langle T_k P_k w,\, P_k w\rangle\right]\Big|_{k=k_0} > 0$, then $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$

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Lemma

If $k_0^2 > 0$ is a transmision eigenvalue with eigenpair (u, w) such that $\left[\frac{\mathrm{d}}{\mathrm{d}k} \langle T_k P_k w,\ P_k w \rangle \right] \Big|_{k=k_0} > 0$, then $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$

The bad news ...

Denote the crucial derivative at ITE k_0^2 by

$$\alpha := \left[\frac{\mathsf{d}}{\mathsf{d}k}\langle T_k P_k w,\ P_k w\rangle\right]\Big|_{k=k_0}$$

If $k_0^2 > 0$ is a transmission eigenvalue with eigenpair (u, w), then for any projection P_k there holds

$$\begin{aligned}\alpha &= 2k_0\left[\int_D \left[|\nabla(u-w)|^2 + 2\mathrm{Re}\left((u-w)\overline{w}\right)\right] \mathsf{d}x\right] \\ &= 2k_0\left[\mathrm{Re}\,\langle(2-q)(u-w), w\rangle - k_0^2\|(1+q)^{1/2}(u-w)\|^2\right]\end{aligned}$$

Sketch of Results for Scalar Case

- If $q = q_0$ is constant in D and either positive and large enough or negative and small enough, then derivative α from last slide has a sign for the first few (smallest) ITE
- Perturbation argument (in $L^\infty(D)$): Same works also for variable contrasts, but bounds get more restrictive
- Possible to show that α cannot be negative for infinite number of positive ITEs ...

Other Scattering Models: Anisotropic Scatterers

- Scattering from penetrable medium D described by contrast function $Q : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$; support of Q is $\overline{D}$
- Acoustics: Wave field satisfies

$$\mathrm{div}((I_3 + Q)\nabla u) + k^2 u = 0 \quad \text{in } \mathbb{R}^3$$

- Electromagnetics: Magnetic field H satisfies

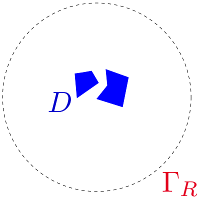
$$\nabla \times ((I_3 + Q)\nabla \times H) - k^2 H = 0 \quad \text{in } \mathbb{R}^3$$

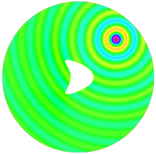
- Penetrable structures where support D possesses holes

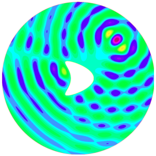
L & Peters '15; L & Rennoch '15

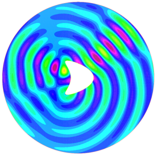
Overview

Near Field Data



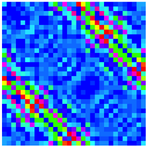






- Dirichlet scatterer $D \subset \mathbb{R}^3$ s.th. complement $\mathbb{R}^3 \setminus \overline{D}$ connected
- Incident point sources $u^i(\cdot, y) = \Phi(\cdot, y)$ for source points $y \in \Gamma_R := \partial B(0, R)$
- Radiating scattered field $u^s = u^s(\cdot, y)$ solves $(\Delta + k^2)u^s(\cdot, y) = 0$ in $\mathbb{R}^3 \setminus \overline{D}$, subject to $u^s(\cdot, y) = -u^i(\cdot, y)$ on ∂D
- Near field operator N on $L^2(\Gamma_R)$,

$$f \mapsto \int_{\Gamma_R} u^s(\cdot, y) f(y)\, \mathrm{d}s$$



The Duality Statement

- Phase conjugation operator Z_k on Γ_R (outgoing $\rightsquigarrow$ incoming)
- Factorization $Z_k N_k = -\mathcal{H}^* S_k^{-1} \mathcal{H}$ but $Z_k N_k$ is not normal
- Numerical range $W(Z_k N_k) = \{(Z_k N_k f, f)_{L^2(\Gamma_R)} : \|f\| = 1\}$
- If k_0^2 is no interior Dirichlet eigenvalue $\Rightarrow$ Define smallest phase $\vartheta_*(k) = \min\{\vartheta \in (0, 2\pi] : \exists z = r \exp(\mathrm{i}\vartheta) \in W(Z_k N_k)\}$

Theorem (Inside-Outside Duality for Near Field Data)

The smallest phase $\vartheta_*(k)$ of $W(Z_k N_k)$ tends to zero as $k \nearrow k_0$ if and only if k_0^2 is an interior Dirichlet eigenvalue of D.

Hu, Yang, Zhang & Zhang '14

Summary

- Interior eigenvalues can be rigorously characterized from spectrum of far-field operators for many wave numbers
- Full characterization for interior eigenvalues of impenetrable scatterers (Dirichlet, Neumann, Robin)
- Conditional characterization for ITEs of penetrable scatterers
- Convergent regularized numerical algorithm to compute ITEs from far field data

Thanks for your attention!

$k \mapsto \lambda_j(k)$