miXFEM- an XFEM toolbox to tackle multiphysics problems with FEniCS

Mischa Jahn
The Center for Industrial Mathematics (ZeTeM), University of Bremen

Motivation

- Our research group works on the modeling and simulation of various applications involving moving or evolving geometries described by multiphysics problems.
- Most applications are tackled by specifically developed approaches (and codes) based on conventional finite element methods.
- Reusing and extending the (in-house) codes requires a lot of time and implementation work.

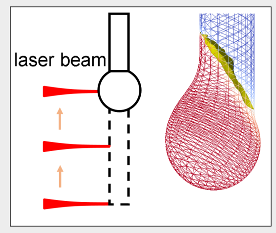

Fig. 1: Material accum. process

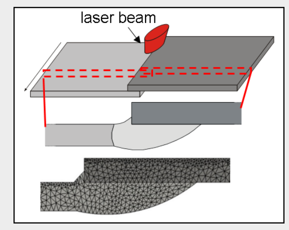

Fig. 2: Laser-welded joints

Approach

- Develop one flexible framework to simulate different processes (2D/3D) that is easy to maintain and extend.
- Use the eXtended finite element method and enhance the automated code generation approach of `FEniCS` to consider multiphysics problems with moving/evolving boundaries and discontinuities.

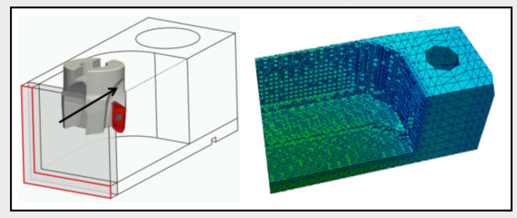
Fig. 3: Thermomechanical distortion in drilling

Mathematical background

Level set method

- A hold-all domain Ω is subdivided into N_{dom} subdomains $\Omega_i(t)$ by the zero level sets $\Gamma_i(t)$ of (continuous and scalar) functions $\varphi_i\colon \Omega \times [t_0, t_f] \to \mathbb{R}$, $i = 1, \ldots, N_{\text{dom}} - 1$.
- Maintaining algorithms like reinitialization and mass correction techniques are included.

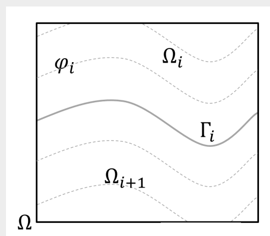

Fig. 4: $\Omega_i(t)$ and $\Omega_{i+1}(t)$ separated by $\Gamma_i(t)$.

eXtended finite element method

- For creating eXtended approximation spaces V_h^{XFEM}, all $\Gamma_i(t)$ are linearly approximated by $\Gamma_{i,h}(t) := \{\boldsymbol{x} \in \Omega : \mathbb{I}_{\text{lin}}\varphi_i(\boldsymbol{x}, t) = 0\}$.
- Use Heaviside functions to enrich conventional FEM spaces such as

$$V_{\text{cg},h}^m = \{v_h \in C^0(\Omega) : v_h|_S \in \mathcal{P}^m(S),\ S \in \mathcal{S}_h\}, \quad \text{or}$$
$$V_{\text{dg},h}^m = \{v_h \in L^2(\Omega) : v_h|_S \in \mathcal{P}^m(S),\ S \in \mathcal{S}_h\}.$$

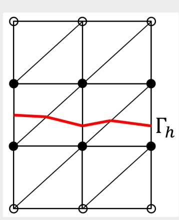

Fig. 5: $\Gamma_{i,h}(t)$ and enriched DOFs ($\mathcal{P}^1$ elements)

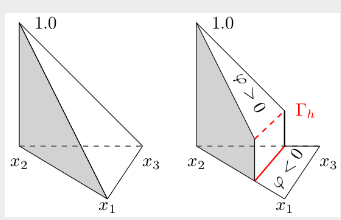

Fig. 6: Standard basis functions v_j and enriched basis functions Hv_j for $\mathcal{P}^1$ elements

- Enriched functions $u_h \in V_h^{\text{XFEM}}$ then have the representation

$$u_h = \sum_{j\in\mathcal{N}} u_j v_j + \sum_{i=1}^{N_{\text{dom}}-1}\left(\sum_{k\in\mathcal{N}_i} u_{i,k} v_{i,k}\right) = \boldsymbol{u}_h \cdot \boldsymbol{v}_h, \quad \text{with}$$

$$\boldsymbol{u}_h = [\underbrace{u_1, \ldots, u_{N_{\mathcal{B}}}}_{\text{std. coefficients}}, \underbrace{u_{1,1}, \ldots, u_{1,|\mathcal{N}_1|}}_{\text{coefficients for } \Gamma_{1,h}}, \ldots, \underbrace{u_{N_{\text{enr}},1}, \ldots, u_{N_{\text{enr}},|\mathcal{N}_{N_{\text{enr}}}|}}_{\text{coefficients for } \Gamma_{N_{\text{enr}},h}}]^T,$$

$$\boldsymbol{v}_h = [\underbrace{v_1, \ldots, v_{N_{\mathcal{B}}}}_{\text{std basis functions}}, \underbrace{v_{1,1}, \ldots, v_{1,|\mathcal{N}_1|}}_{\text{add. basis functions for } \Gamma_{1,h}}, \ldots, \underbrace{v_{N_{\text{enr}},1}, \ldots, v_{N_{\text{enr}},|\mathcal{N}_{N_{\text{enr}}}|}}_{\text{add. basis functions for } \Gamma_{N_{\text{enr}},h}}]^T.$$

- Boundary and interface conditions are imposed with Nitsche's method. Thereby, problems involving weak discontinuous features can also be considered by enforcing continuity across interfaces.

`miXFEM` (multiple interfaces XFEM) [1,2]

Fig. 7: Implementation design: Extended FFC and toolboxes for XFEM and level set.

- Implementation of a level set toolbox [3] (including maintaining and stabilization techniques) to describe discontinuities and their evolution by zero levels.
- Implementation of a flexible XFEM framework, partly based on the `PUM` library [4], to use automated code generation for solving multiphysics problems involving discontinuities.

Academic example

Given $\mathbb{R}^2 \supset \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Gamma_1 \cup \Gamma_2$ with polygonal $\partial\Omega = \Gamma_{\text{D}} \cup \Gamma_{\text{N}}$ and sufficiently smooth data, find $u \in V_h^{\text{XFEM}}$ s.t. it solves the problem

$$\begin{aligned}
\xi u - \nabla \cdot (\kappa \nabla u) &= f && \text{in } \Omega_1 \cup \Omega_2 \cup \Omega_3\\
u &= g_{\text{D}} && \text{on } \Gamma_{\text{D}}\\
\kappa \frac{\partial u}{\partial \vec{n}} &= g_{\text{N}} && \text{on } \Gamma_{\text{N}}\\
[\![\kappa \nabla u]\!] \cdot \vec{n}_1 &= g_{\Gamma_1} && \text{on } \Gamma_1\\
[\![u]\!] &= q_{\Gamma_1} && \text{on } \Gamma_1\\
u|_{\Omega_2} &= g_2 && \text{on } \Gamma_2\\
u|_{\Omega_3} &= g_3 && \text{on } \Gamma_2
\end{aligned}$$

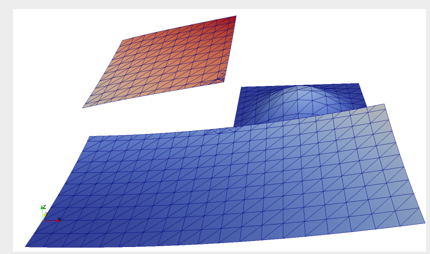
Fig. 8: Academic example.

Real world application: Laser-welded hybrid joints

Process is modeled by the heat equation with two discontinuities (*material* and *solid-liquid interface*), similar to the academical problem defined above.

- Ω is separated into different *materials* by the (stationary) zero level of φ_1 (illustrated in white).
- The evolution of the *interface*, defined the zero level of φ_2 (illustrated in yellow), is modeled by the Stefan problem.

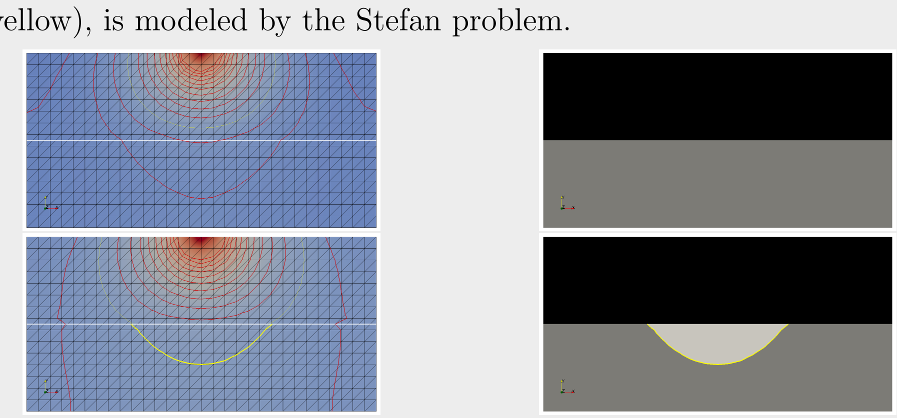
Fig. 9: Time levels of laser welding process, without and with molten domain.

Real world application: Keyhole-based laser welding

Process is modeled by the heat equation with two discontinuities (*keyhole* and *solid-liquid interface*).

- *Keyhole* geometry (zero level of φ_1) is fixed but moves with prescribed (welding) velocity.
- *Interface* (zero level of φ_2) evolution is described by the Stefan problem.

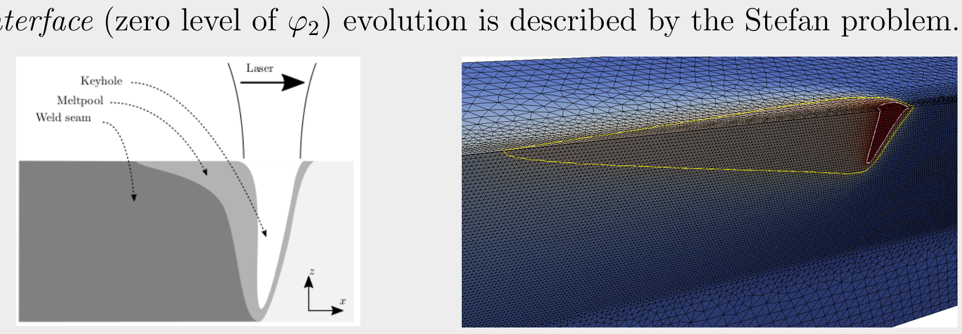

Fig. 10: Keyhole-based welding process.

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