An XFEM toolbox for FEniCS

Mischa Jahn, Andreas Luttmann, Timo Klock
The Center for Industrial Mathematics (ZeTeM), University of Bremen

Zentrum für Technomathematik
Universität Bremen
SFB 747
DFG

Introduction to `miXFEM` (multiple interfaces XFEM)

Idea: Automated code generation for problems with arbitrary discontinuities by using an extended `FEniCS` form compiler and a C++ library.

Design principle: Modular structure consisting of (independent) toolboxes.

- Implementation of a level set toolbox (including reinitialization and mass correction techniques) which can be used for describing a discontinuity.
- Implementation of a flexible XFEM framework, partly based on the `PUM` library [3], to solve stationary and instationary problems with strong and weak discontinuities.

miXFFC	**miXFEM**	**Level set toolbox**
Extension of the FFC for automated code generation for problems with discontinuities	C++ implementation providing methods for solving problems with discontinuities	Toolbox for the solution of the level set problem including reinit. and mass corr. techniques

Fig. 1: Implementation design: Extended FFC and toolboxes for XFEM and level set.

Mathematical background

Level set toolbox [1]

- A discontinuity $\Gamma(t)$ is described by the zero level set of a (continuous and scalar) function $\varphi\colon \Omega \times [t_0, t_f] \to \mathbb{R}$, separating Ω into $\Omega(t) = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$, cf. Fig. 2.
- Maintaining algorithms like reinitialization and mass correction techniques are included.

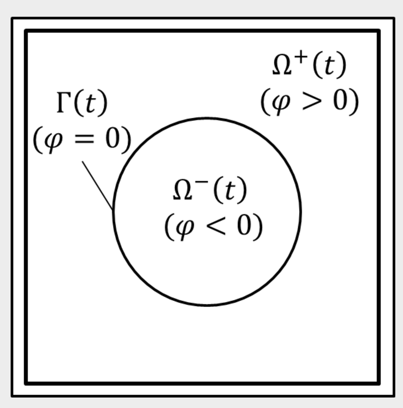


Fig. 2: $\Omega^+(t)$ and $\Omega^-(t)$ separated by $\Gamma(t)$.

eXtended finite elements [2]

A standard function space V_h^{FEM} with basis functions v_i, $i \in \mathcal{N}$, is (only) enriched strongly, weak discontinuities are handled by enforcing continuity with Nitsche's approach.

- The discrete approximation of $\Gamma(t)$, which is considered for enriching a function space V_h^{FEM}, is given by $\Gamma_h(t) := \{\boldsymbol{x} \in \Omega_h \,|\, I_{\text{lin}}\varphi_h(\boldsymbol{x}, t) = 0\}$.
- Enriched DOFs are $\tilde{\mathcal{N}} := \{i \in \mathcal{N} \,|\, \text{meas}_{d-1}(\Gamma_h \cap \text{supp}(v_i)) > 0,\ v_i \in V_h^{\text{FEM}}\}$ so that a function $u_h^{\text{XFEM}} \in V_h^{\text{XFEM}}$ reads
$$u_h^{\text{XFEM}} = \sum_{i \in \mathcal{N}} u_i v_i + \sum_{j \in \tilde{\mathcal{N}}} \tilde{u}_j H v_j,$$
where u_i are the coefficients and H denotes the Heaviside function.

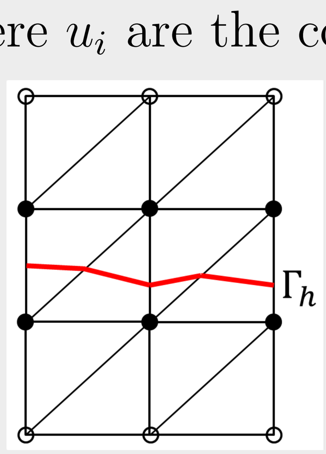


Fig. 3: Enriched DOFs $\tilde{\mathcal{N}}$ for $\mathcal{P}_1$ elements

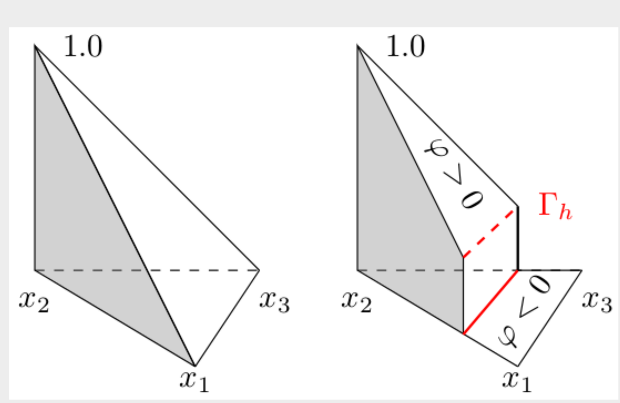


Fig. 4: Standard basis functions v_i and enriched basis functions Hv_i for $\mathcal{P}_1$ elements

- For intersected elements, a local mesh consisting of sub-elements is created and the quadrature rule is modified to consider each sub-element's contribution.

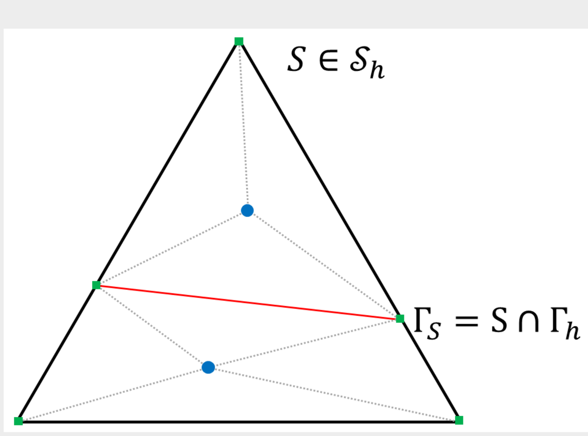


Fig. 5: Creating sub-elements of a cell in 2D.

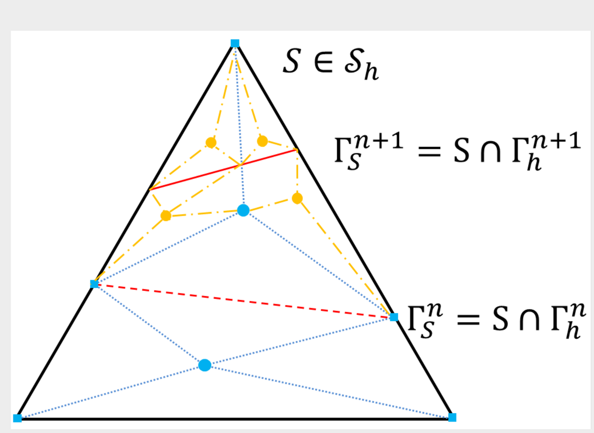


Fig. 6: Creating sub-elements of a cell for time-dependent problems in 2D.

- The L_2-projection is used to interpolate an arbitrary function f onto V_h^{XFEM}
$$\int_\Omega (P_h^{\text{XFEM}} f) v dx = \int_\Omega f v dx, \quad \forall v \in V_h^{\text{XFEM}}.$$

Stefan problem (model problem)

On $\Omega \subset \mathbb{R}^2$, define $u(\cdot, t_0)$ and $\varphi(\cdot, t_0)$ and consider the problem
$$\partial_t u - \nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega^+(t) \cup \Omega^-(t),\ t \in (t_0, t_f),$$
$$\partial_t \varphi + \vec{V} \cdot \nabla \varphi = 0 \quad \text{in } \Omega \times [t_0, t_f],$$
$$\text{with } u(\cdot, t) = u_\Gamma \text{ on } \Gamma(t) \quad \text{and} \quad [\![\kappa \nabla u \cdot \vec{n}]\!] = L\vec{V} \cdot \vec{n} \text{ on } \Gamma(t)$$
and suitable (Dirichlet/Neumann) boundary conditions.

```
[...]
# Bilinear form and linear form for the heat equation with disc. coeff. k:
a0 = u * v * dx +  deltat * k * dot(nabla_grad(v), nabla_grad(u)) * dx
a1 = - deltat('+') * k('+') * dot(grad(u('+')), n(phi('+'))) * v('+') * dc(0) \
     - deltat('+') * k('+') * dot(grad(v('+')), n(phi('+'))) * u('+') * dc(0) \
     + deltat('+') * lambda('+') * u('+') * v('+')             * dc(0)
a2 =   deltat('+') * k('-') * dot(grad(u('-')), n(phi('+'))) * v('-') * dc(0) \
     + deltat('+') * k('-') * dot(grad(v('-')), n(phi('+'))) * u('-') * dc(0) \
     + deltat('+') * lambda('+') * u('-') * v('-')             * dc(0)
[...]
L0 = v * u_old * dx + deltat * f * v * dx
L1 = - deltat('+') * k('+') * dot(grad(v('+')), n(phi('+'))) * u_Gamma('+') * dc(0) \
     + deltat('+') * lambda('+') * u_Gamma('+') * v('+')             * dc(0)
L2 =   deltat('+') * k('-') * dot(grad(v('-')), n(phi('+'))) * u_Gamma('-') * dc(0) \
     + deltat('+') * lambda('+') * u_Gamma('-') * v('-')             * dc(0)
[...]
a = a0 + a1 + a2
L = L0 + L1 + L2
```

Fig. 7: Parts of the corresponding UFL file using Nitsche's approach

Exemplary convergence plots for fixed Δt resp. h using $\mathcal{P}_1$ elements

- Scenario 1, the interface is a-priori known and prescribed, see Fig. 8.
- Scenario 2, the interface is unknown and part of the solution, see Fig. 9.

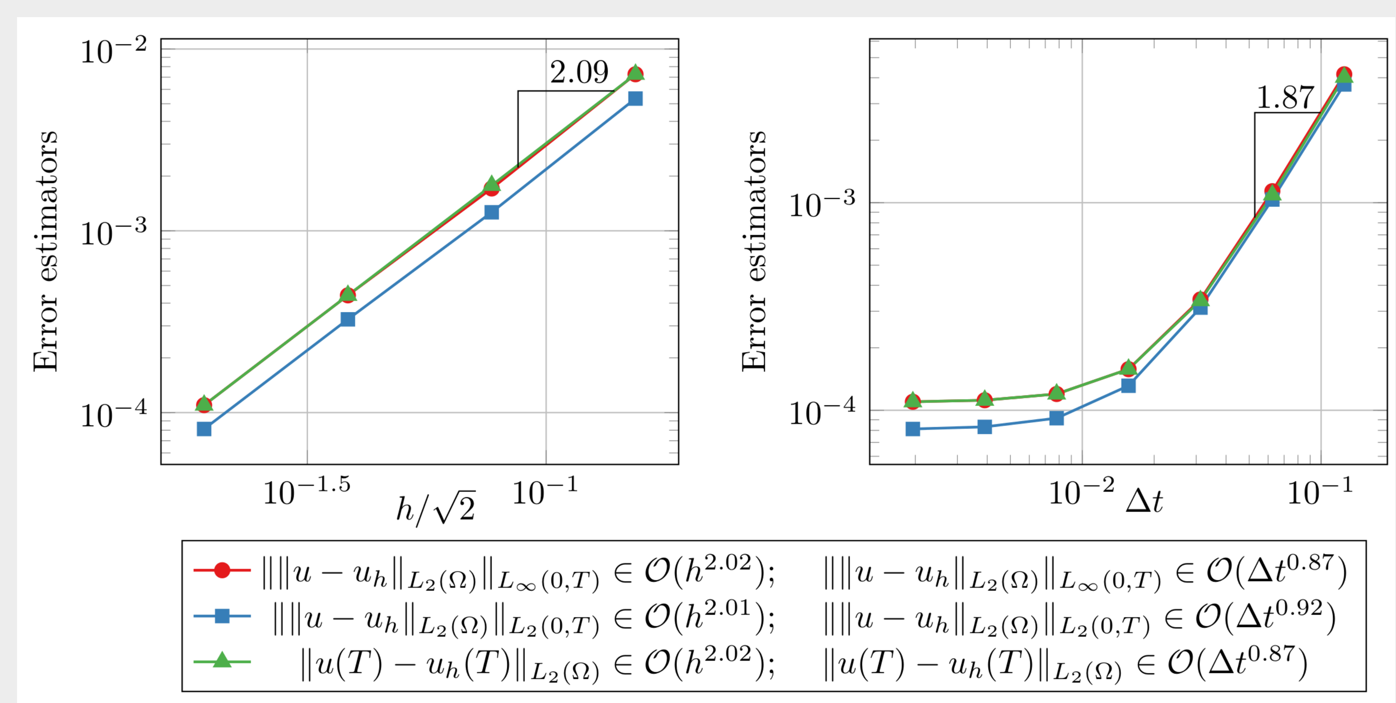


Fig. 8: Convergence plots scenario 1, interface is a-priori known

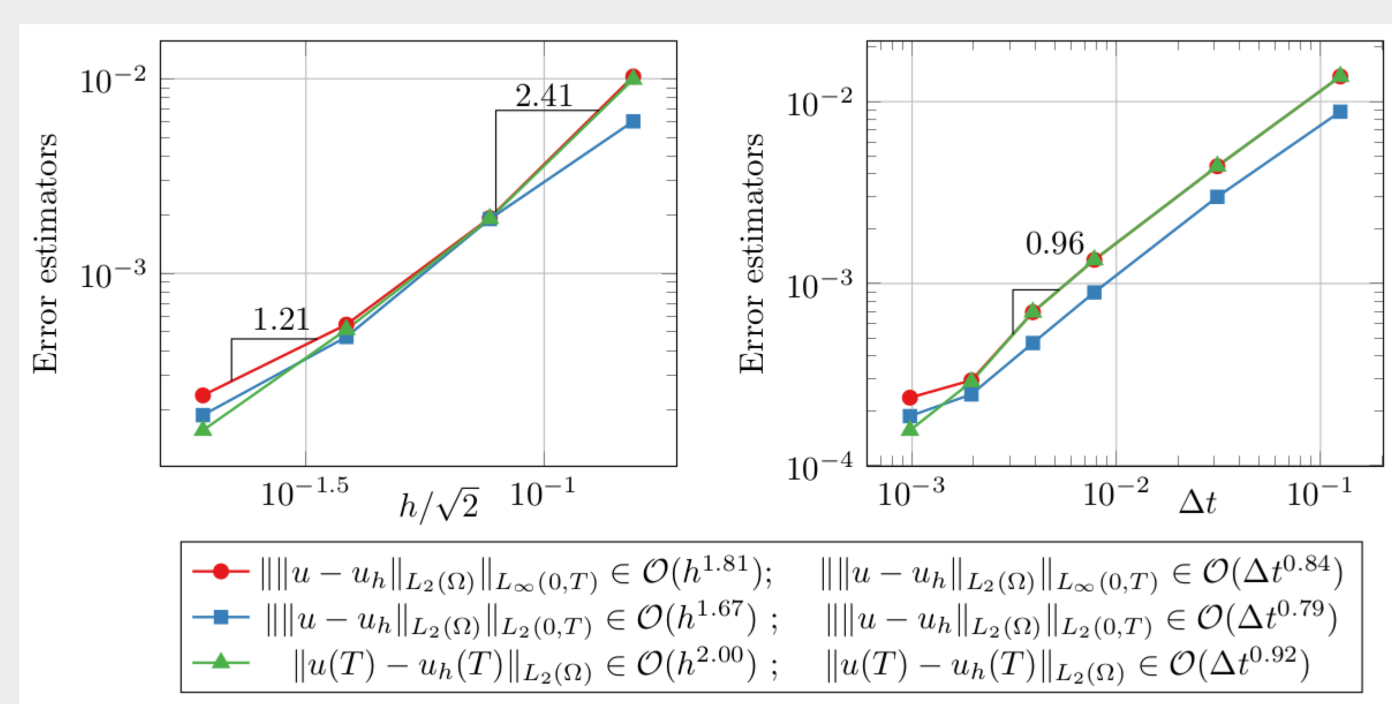
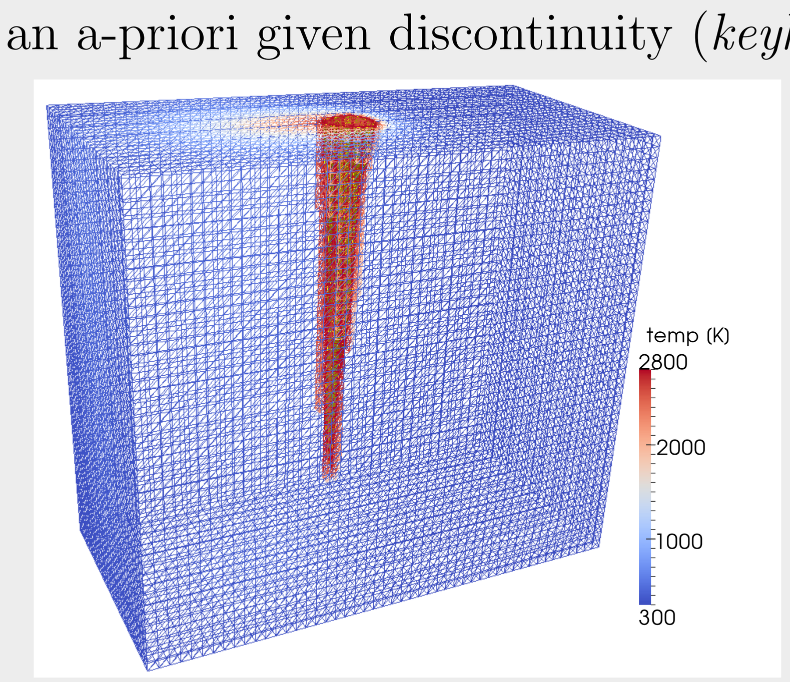
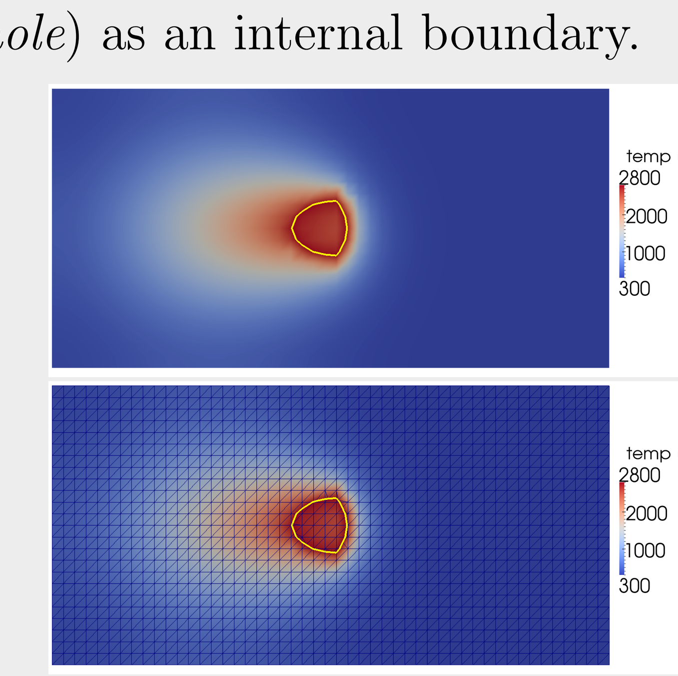


Fig. 9: Convergence plots scenario 2, interface is part of the solution

Outlook: Keyhole-based laser welding (real world application)

On $\Omega \subset \mathbb{R}^3$ consider a welding process described by the heat equation with an a-priori given discontinuity (*keyhole*) as an internal boundary.





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