Title: Factorization method in inverse scattering

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Factorization method in inverse scattering

Synonyms

Kirsch's factorization method; operator factorization method (rarely used); linear sampling method (ambiguous and to avoid—the name is very rarely used for the factorization method by now but has been employed more frequently after the first papers on the method appeared).

Definition

The factorization method for inverse scattering provides an explicit and theoretically sound characterization for the support of a scattering object using multi-static far field measurements at fixed frequency: A point z belongs to the scatterer if and only if a special test function belongs to the range of the square root of a certain operator that can be straightforwardly computed in terms of far field data. This characterization yields a fast and easy-to-implement numerical algorithm to image scattering objects. A crucial ingredient of the proof of this characterization is a factorization of the measurement operator, which explains the method's name. There are basically two variants of the method leading to different characterization criteria: If the far field operator F is normal, the above characterization applies for the square root $(F^*F)^{1/4}$ of F itself; otherwise, one considers the square root of $F_\sharp := |\mathrm{Re}F| + \mathrm{Im}F$ where $\mathrm{Re}F$ and $\mathrm{Im}F$ are the selfadjoint and non-selfadjoint part of F, respectively.

Overview

The factorization method was first introduced by Andreas Kirsch [15, 16] for time-harmonic inverse obstacle and inverse medium scattering problems where the task is to determine the support of the scatterer from multi-static far field measurements at fixed frequency (roughly speaking, from measurements of the far field pattern of scattered waves in several directions and for several incident plane waves). The method follows the spirit of the linear sampling method and can be seen as a refinement of the latter technique. Both methods try to determine the support of the scatterer by deciding whether a point z in space is inside or outside the scattering object. When the far field operator F is normal, the factorization method's criterion for this decision is whether or not special test functions ϕ_z, parametrized by z and explicitly known for homogeneous background media, are contained in the range of the linear operator $(F^*F)^{1/4}$. Indeed, when the point z is outside the scatterer then ϕ_z is not contained in the range of $(F^*F)^{1/4}$ whereas ϕ_z belongs to this range when z is inside the scatterer. The factorization method can be used for imaging by computing the norm of a possible solution g_z to $(F^*F)^{1/4}g_z = \phi_z$ using Picard's criterion for many sampling points z from a grid covering a region of interest. Plotting these norms then yields a picture of the scattering object. If F fails to be normal, a variant of the method based on $F_\sharp := |\mathrm{Re}F| + \mathrm{Im}F$ provides analogous analytical results and imaging algorithms.

These algorithms are very efficient compared to other techniques solving inverse scattering problems since their numerical implementation basically requires the computation of the singular value decomposition of a discretization of the far field operator F. A further attractive feature of the method is its independence of the nature of the scattering object; for instance, the factorization method yields the same object characterization and imaging algorithm for penetrable and impenetrable objects, such that a mathematical model describing the scatterer does not need to be known in advance.

The analysis of the factorization method is based on functional analytic results on range identities for operator factorizations of the form $F = H^*TH$. Under appropriate assumptions, these results state, roughly speaking, that the range of the square root of F equals the range of H^*. Moreover, via unique continuation results and fundamental solutions it is usually not difficult to show that the range of H^* characterizes the scattering object, since the far field ϕ_z of a point source at z belongs to this range if and only if z belongs to the scattering object. Combining these two results hence provides a direct characterization of the scattering object in terms of the range of the square root of F.

Differences to the Linear Sampling Method and Limitations

The fundamental difference between the factorization method and the linear sampling method is that the latter one considers an operator equation for the measurement operator itself, while the Factorization method considers the corresponding equation for the "square root" of this operator. Due to this difference, the factorization method is able to provide an mathematically rigorous and exact characterization of the scattering object that is fully explicit and merely based on the measurement operator. Note that the linear sampling method does not share this feature, since, for points z inside the scatterer, the theorem that is usually employed to justify that method claims that there exist approximate solutions to a certain operator equation. It remains however unclear how to actually determine or to compute these approximate solutions. Several variants of the standard version of the linear sampling method are able to cope with this problem, see, e.g., Arens and Lechleiter [5] or Audibert and Haddar [1].

To obtain a mathematically rigorous characterization of the scatterer's support, the Factorization method however requires the inverse scattering problem under investigation to satisfy several structural assumptions that are not required by the linear sampling method (or other sampling methods). The reason is a functional analytic result on range identities for operator factorizations that is the backbone of the method. First, the measurement operator F defined on a Hilbert space V (imagine the far field operator defined on L^2 of the unit sphere) needs to have a self-adjoint factorization of the form $F = H^*TH$ with a compact operator $H : V \to X$ and a bounded operator $T : X \to X^*$, where X is a reflexive Banach space. It is crucial that the outer operators of this factorization are adjoint to each other. Second, the middle operator T needs to be a compact perturbation of a coercive operator: $T = T_1 + T_2$ such that $\mathrm{Re}\,\langle T_1\phi,\, \phi\rangle_{X^*\times X} \geq c\|\phi\|_X^2$ for all $\phi \in X$ and some $c > 0$, and such that T_2 is compact. There are several inverse scattering problems where at least one of these two conditions is violated. The first one does, for instance, not hold for near-field measurements when the wave number is different from zero. The coercivity assumption for the middle operator is violated, e.g., for electromagnetic scattering from a perfect conductor, for acoustic scattering from a scatterer that is partly sound-soft and partly sound-hard, and for scattering from an inhomogeneous medium that is partly stronger scattering and partly weaker scattering than the background medium. Consequently, providing theory that does not require either of the two conditions would be highly desirable.

In the first years after the invention of the method in Kirsch [15], the factorization method could only be applied to far field inverse scattering problems where the far field operator is normal. When the scatterer is absorbing, the far field operator fails to be normal and it was an open problem whether the factorization method applies to such problems. This problem has been solved by decoupling real and imaginary parts of the measurement operator, yielding range identities for the square root of the auxiliary operator $F_\sharp = (\mathrm{Re}(F)^* \, \mathrm{Re}(F))^{1/2} + \mathrm{Im}\,(F)$ that is easily computed in terms of F (see Grinberg [12]; Kirsch and Grinberg [20]).

Applications of the Factorization Method in Inverse Scattering

Even if the factorization method cannot be applied to all inverse scattering problems, there are many situations where the method provides the above-mentioned characterization of the support of the scattering object. To list only a few of them, the method has been successfully applied to inverse acoustic obstacle scattering from sound-soft, sound-hard or impedance obstacles, see Kirsch [15]; Grinberg [12], to inverse acoustic medium scattering problems, see Kirsch [16], to electromagnetic medium scattering problems, see Kirsch [19]; Kirsch and Grinberg [20], to inverse electromagnetic scattering problems at low frequency, see Gebauer et al [11], to inverse scattering problems for penetrable and impenetrable periodic structures, see Arens and Kirsch [4]; Arens and Grinberg [3]; Lechleiter and Nguyen [24], to inverse problems in elasticity, see Charalambopoulus et al [8], to inverse scattering problems in acousto-elasticity, see Kirsch and Ruiz [21], to inverse problems for stationary Stokes flows, see Lechleiter and Rienmüller [25], and to inverse scattering problems problems for limited aperture, see Kirsch and Grinberg [20, Section 2.3].

Apart from inverse scattering, the factorization method has been applied to a variety of inverse problems for partial differential equations. The monograph of Kirsch and Grinberg [20] and the review of Hanke and Kirsch [14] indicate a variety of other inverse problems treated by this method, and also further references for the factorization method in inverse scattering. Finally, we mention that the factorization method is linked to other sampling methods as the linear sampling method, see Arens [2]; Arens and Lechleiter [5], and the MUSIC algorithm, see Kirsch [18]; Arens et al [6].

An Example – Factorization Method for Inverse Medium Scattering

In this section, we consider a time-harmonic inverse medium scattering problem and explain in some detail how the factorization method works. This material is mostly from Kirsch [16, 18]; Kirsch and Grinberg [20]. We also indicate why there exist several variants of the method.

Scattering from an Inhomogeneous Medium

Time harmonic scattering theory considers waves $U(x,t) = u(x)\exp(-\mathrm{i}\omega t)$ with angular frequency $\omega > 0$ and time dependence $\exp(-\mathrm{i}\omega t)$. If we denote by c the space-dependent wave speed in $\mathbb{R}^3$, and by c_0 the constant wave speed in the background medium, then the wave equation $c^2 \Delta U - \partial_{tt} U = 0$ reduces to the Helmholtz equation

$$\Delta u + k^2 n^2 u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1}$$

with (constant) wave number $k = \omega/c_0 > 0$ and space-dependent refractive index $n = c_0/c$. In the following, we suppose that the refractive index equals one in the complement of a bounded Lipschitz domain D with connected complement; the domain D hence plays the role of the scattering object.

A typical direct scattering problem is the following: For an incident plane wave $u^i(x) = \exp(\mathrm{i}k\, x \cdot \theta)$, $x \in \mathbb{R}^3$, of direction $\theta \in \mathbb{S}^2 := \{x \in \mathbb{R}^3, \, |x| = 1\}$ we seek a total field u^t that solves (1). Moreover, the scattered field $u^s = u^t - u^i$ needs to satisfy the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial |x|} - \mathrm{i}k \right) u^s = 0 \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \in \mathbb{S}^2. \tag{2}$$

Sommerfeld's radiation condition acts as a boundary condition "at infinity" for the scattered field and guarantees uniqueness of solution to scattering problems on unbounded domains. Physically, this condition means that the scattered wave is created locally in D and propagates away from D. The scattering problem to find u^s when given u^i and n^2 is well-posed in standard function spaces under reasonable assumptions on the refractive index, see Colton and Kress [10]. Solutions to the exterior Helmholtz equation that satisfy the Sommerfeld radiation condition behave at infinity like an outgoing spherical wave modulated by a certain angular behaviour,

$$u(x) = \Phi(x)\Big(u_\infty(\hat{x}) + O\left(|x|^{-2}\right) \Big) \quad \text{as } |x| \to \infty, \qquad \Phi(x) := \frac{e^{\mathrm{i}k|x|}}{4\pi|x|}.$$

The function $u_\infty \in L^2(\mathbb{S}^2)$ is called the far field pattern of u.

In the following, we denote by $u_\infty(\hat{x}, \theta)$ the far field pattern in the direction $\hat{x} \in \mathbb{S}^2$ of the scattered wave caused by an incident plane wave of direction $\theta \in \mathbb{S}^2$. The refractive index n^2 is allowed to be real and positive, or complex valued with positive real part and non-negative imaginary part (further assumptions on n^2 will be stated where they are required).

Inverse Problem and Factorization

In an inverse medium scattering problem with far field data one seeks to determine properties of the scatterer from the knowledge of the far field pattern $u_\infty(\hat{x}, \theta)$ for all directions $\hat{x}$ in a given set of measurement directions and all θ in a given set of directions of incidence. Particularly, the factorization method solves the following inverse scattering problem: Given $u_\infty(\hat{x},\theta)$ for all $\hat{x}\in\mathbb{S}^2$ and all $\theta\in\mathbb{S}^2$, find the support D of the scattering object! Recall that $\overline{D}$ was defined to be the support of n^2-1.

A central tool for the factorization method is the far field operator F,

$$F:\, L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2) \qquad g\mapsto \int_{\mathbb{S}^2} u_\infty(\cdot,\theta)\, g(\theta)\,\mathrm{ds}(\theta).$$

This is an integral operator with continuous kernel $u_\infty(\cdot,\cdot)$ and the theory on integral equations states that F is a compact operator. By linearity of the scattering problem, F maps a density g to the far field of the scattered field for the incident Herglotz wave function

$$v_g(x)=\int_{\mathbb{S}^2} g(\theta)e^{\mathrm{i}k\,\theta\cdot x}\,\mathrm{ds}(\theta), \qquad x\in\mathbb{R}^3.$$

The restriction of a Herglotz wave function v_g on the obstacle D yields a bounded linear operator $H:\, L^2(\mathbb{S}^2)\to L^2(D)$, $g\mapsto v_g|_D$. Obviously, if we know $\{u_\infty(\hat{x},\theta):\, \hat{x},\theta\in\mathbb{S}^2\}$ for all directions $\hat{x},\theta\in\mathbb{S}^2$, then we also know F. Therefore we reformulate our inverse scattering problem as follows: Given F, determine the support D of the scatterer!

Theorem 1 (Factorization). *The far field operator can be factored as*

$$F=H^*TH,$$

where $T:\, L^2(D)\to L^2(D)$ *is defined by* $Tf=k^2(n^2-1)\,(f+v|_D)$, *and* $v\in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ *solves* $\Delta v+k^2n^2\,v=k^2(1-n^2)f$ *in* $\mathbb{R}^3$, *subject to the Sommerfeld radiation condition* (2).

The adjoint H^* of the Herglotz operator can be used to characterize the scatterer's support D: It holds that the far field $\Phi_\infty(\hat{x},z)=\exp(\mathrm{i}k\,\hat{x}\cdot z)$ of a point source $\Phi(x-z)=\exp(\mathrm{i}k|x-z|)/(4\pi|x-z|)$ at $z\in\mathbb{R}^3$ belongs to the range of H^* if and only if $z\in D$. Due to the factorization from the last theorem one would now like to link the range of H^* with the range of (some power of) the measurement operator F to obtain characterization results for the scatterer D.

Two Characterization Results

If the refractive index n^2 is real then F is a normal operator and consequently possesses a complete system of eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$ with associated eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$. Under suitable assumptions, this basis of eigenvectors allows to prove that the test functions $\Phi_\infty(\cdot, z)$ belong to the range of $(F^*F)^{1/4}$—the square root of F—if and only if the point z belongs to D. One key idea of the proof is that the orthonormal basis $\{\phi_j\}_{j\in\mathbb{N}}$ of $L^2(\mathbb{S}^2)$ transforms into a Riesz basis $\{\lambda_j^{-1/2} H\phi_j\}$ of a suitable subspace of $L^2(D)$ due the factorization of F. (This is a simplified statement, see Section 4 in [16] for the precise formulation.) Picard's criterion yields the following characterization of the scatterer:

$$z \in \mathbb{R}^3 \text{ belongs to } D \text{ if and only if } \sum_{j=1}^{\infty} \frac{\left|\langle \Phi_\infty(\cdot, z),\, \phi_j\rangle_{L^2(\mathbb{S}^2)}\right|^2}{|\lambda_j|} < \infty. \tag{3}$$

The main assumptions on n^2 for this result are that n^2 is real-valued, that $n^2 - 1$ does not change sign, and that k^2 is not an interior transmission eigenvalue, see Cakoni et al [7] for a definition.

If the refractive index takes imaginary values inside D, which corresponds to an absorbing scattering object, then the far field operator fails to be normal and the $(F^*F)^{1/4}$-variant of the factorization method does not work. However Grinberg and Kirsch [12; 18] showed that, under suitable assumptions, the auxiliary operator $F_\sharp = (\mathrm{Re}(F)^* \mathrm{Re}(F))^{1/2} + \mathrm{Im}\,(F)$ allows to prove that the ranges of $F_\sharp^{1/2}$ and of H^* are equal. Since the test functions $\Phi_\infty(\cdot, z)$ belong to the range of H^* if and only if $z \in D$, one can then conclude that $\Phi_\infty(\cdot, z)$ belongs to the range of $F_\sharp^{1/2}$ if and only if $z \in D$. Denote by $\{\psi_j\}_{j\in\mathbb{N}}$ the eigenvalues of the compact and selfadjoint operator $F_\sharp$ and by $\{\mu_j\}_{j\in\mathbb{N}}$ the corresponding eigenvalues. Using Picard's criterion we reformulate the characterization of D as follows:

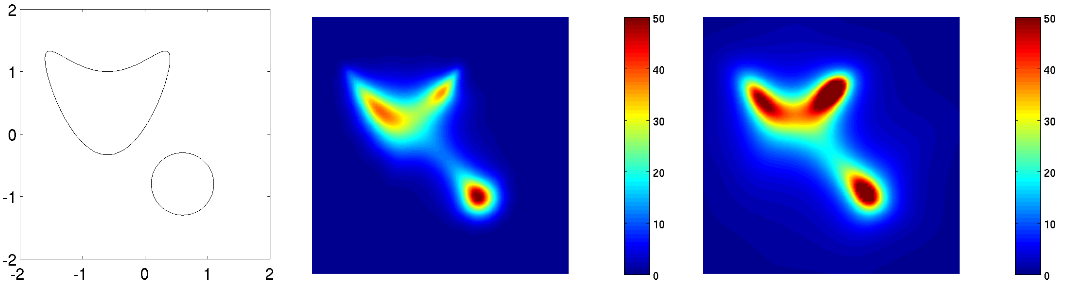
$$z \in \mathbb{R}^3 \text{ belongs to } D \text{ if and only if } \sum_{j=1}^{\infty} \frac{\left|\langle \Phi_\infty(\cdot, z),\, \psi_j\rangle_{L^2(\mathbb{S}^2)}\right|^2}{|\mu_j|} < \infty. \tag{4}$$

The main assumptions for this result are that $\mathrm{Re}\,(n^2-1)$ does not change sign The assumption that k^2 is not an interior transmission eigenvalue can be dropped for this variant of the method, but not for the $(F^*F)^{1/4}$-variant from (3), see Lechleiter [23].

Discretization

The criterion in (3) or (4) suggests the following algorithm to image the scattering object: Choose a discrete set of grid points in a certain test domain and plot the reciprocal of the series in (3) or (4) on this grid. Of course, in practice one can only plot a finite approximation to the infinite series afflicted with certain errors. Nevertheless, one might hope that plotting the reciprocal of the truncated series as a function of y leads to large and small values at points z inside and outside the scatterer D, respectively. However, the ill-posedness of the inverse scattering problem afflicts this imaging process, because we divide by small numbers λ_j or μ_j. For instance, if one only knows approximations λ_j^δ with $|\lambda_j^\delta - \lambda_j| \leq \delta$ and ϕ_j^δ with $\|\phi_j^\delta - \phi_j\| \leq \delta$ then the difference between $\left|\langle \Phi_\infty(\cdot, z), \phi_j^\delta \rangle\right|^2 / |\lambda_j^\delta|$ and the corresponding exact value is in general much larger than the noise level $\delta > 0$. Consequently, one needs to regularize the Picard series. Several methods are available: Tikhonov regularization, see Colton et al [9], regularization by truncation of the series, see Lechleiter [22], comparison techniques, see Hanke and Brühl [13], and noise subspace techniques, see Arens et al [6].

Figure 1 shows reconstructions for a two-dimensional inhomogeneous medium with piecewise constant index of refraction; n^2 equals 10 inside the inclusion, shown in Figure 1(a), and 1 outside the inclusion. The wave number is $k = 2$ and the reconstruction uses 32 incident and measurement directions uniformly distributed on the unit circle. These examples are reproduced from Arens et al [6] where further details can be found.



(a) (b) (c)

Fig. 1. Reconstructions of the support of an inhomogeneous medium using the factorization method, reproduced from Arens et al [6]. (a) The exact support of the scatterer (b) Reconstruction without artificial noise (c) Reconstruction with 5 percent artificial noise.

Key Results on Range Characterizations

The factorization method can be seen as a tool to pass the geometric information on the scattering object contained in the inaccessible operator H^* of the factorization $F = H^*TH$ to the measurement operator F. To this end, there are basically three functional analytic frameworks that can be used. As in the first section, we assume here that F is a compact operator on a Hilbert space V, that $H : V \to X$ is compact, and that $T : X \to X^*$ s bounded where X is a reflexive Banach space.

The first variant of the factorization method, the so-called $(F^*F)^{1/4}$-variant, requires F to be normal. In this case F possesses a complete basis of eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$ such that $F\phi_j = \lambda_j \phi_j$. The vectors $\psi_j = \lambda_j^{-1/2} H\phi_j$ satisfy

$$\langle T\psi_i,\, \psi_j\rangle_{X^*\times X} = \frac{\lambda_i}{|\lambda_i|}\delta_{i,j}, \qquad i,j \in \mathbb{N}.$$

If T is a compact perturbation of a coercive operator, and if the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfy certain geometric conditions, then Theorem 3.4 in Kirsch [15] proves that $\{\psi_j\}_{j\in\mathbb{N}}$ is a Riesz basis of X. This is the key step to prove that the range of $(F^*F)^{1/4}$ equals the range of H^*, see Kirsch [15, Theorem 3.6].

The second variant of the method, the so-called infimum criterion, was a first step towards the treatment of problems where F fails to be normal. In Kirsch [17, Theorem 2.3], it is shown that if there exist positive numbers $c_{1,2}$ such that

$$c_1\|T\psi\|_{X^*}^2 \leq |\langle T\psi,\, \psi\rangle_{X^*\times X}| \leq c_2\|T\psi\|_{X^*}^2$$

for all $\psi \in X$, then an element $g \in V$ belongs to the range of H^* is and only if

$$\inf\left\{|\langle F\phi,\, \phi\rangle_V|\,,\ \phi \in V,\ \langle g, \phi\rangle_V = 1\right\} > 0.$$

The drawback of this characterization is that the criterion whether or not a point belongs to the scatterer requires to solve an optimization problem. To get an image of the scattering object, one hence needs to solve an optimization problem for each sampling point in the grid.

The third variant of the method relies on the auxiliary operator $F_\sharp = (\mathrm{Re}(F)^* \,\mathrm{Re}(F))^{1/2} + \mathrm{Im}\,(F)$, see Grinberg [12]. For this operator, the equality of the ranges of $F_\sharp^{1/2}$ and H^* can be shown, e.g., under the conditions that T is injective, that the real part of T is a compact perturbation of a coercive operator, and that the imaginary part of T is non-negative, see Lechleiter [23].

Cross-references

Linear Sampling, Inverse Boundary Problems for Acoustic and Maxwell's Equations, Theoretical Aspects, Inverse Boundary Problems for Elasticity, Inverse medium problem, Inverse Scattering, Inverse Scattering, 1-D, Inversion Formulas in Inverse Scattering, Inhomogeneous Media Identification, Obstacle Identification, Crack Identification, Optical Tomography, Theory, Impedance Tomography, Adjoint Methods as Applied to Inverse Problems, Time Reversal, Theoretical Aspects

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