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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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On the Past Ten Years and the Future Development of Nonlinear Dynamics and Systems Theory (ND&ST)

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On the analysis of journal, conference papers and books with titles containing the words related to "nonlinear mechanics", "nonlinear dynamics", and "nonlinear analysis", we saw a large number of such papers, where theory and applications were being investigated. See, for example, (70Kxx MSC 2010) for nonlinear dynamics and (93–XX MSC 2010) for systems theory. With this observation, we set up a new scientific journal, entitled "Nonlinear Dynamics and Systems Theory" (ND&ST), in 2001. The scopes of the journal also include topics on stability theory and its applications.

Over the last 10 years, the members of the Editorial Board, especially the past and current Regional Editors C. Corduneanu, C. Cruz-Hernandez, H.I. Freedman, A.D.C. Jesus (former), M. Ikeda (former), J. Mildowney (former), S. Omatru (former), Peng Shi, S. Sivasundaram (former), K.L. Teo, and J. Wu (former), have made significant contributions to the improved quality of the published papers. They also helped shape the directions and focuses of the Journal. All papers received are subject to a rigorous reviewing process. Approximately 30% of the submitted papers were rejected in 2010.

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In addition to regular papers, ND&ST also allocates a section called "Personage in Science" for some issues, publishing short biographical sketches, reviews of results, and

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lists of main works by A.M. Lyapunov, N.N. Bogoluybov, Yu.A. Mitropolsky, V.I. Zubov and other scholars who have made fundamental contributions to the development of nonlinear dynamics and systems theory. ND&ST has so far published 5 Special Issues listed below:

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Complex Network Synchronization of Coupled Time-Delay Chua Oscillators in Different Topologies

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Abstract: In this paper, complex network synchronization of coupled hyperchaotic nodes (described by time-delay Chua oscillators) in different topologies is reported. In particular, networks synchronization in nearest-neighbor, small-world, open ring, tree, star, and global topologies are achieved. For each topology, the number of hyperchaotic nodes is evaluated that can be connected in the dynamical networks for synchronization purpose, which is based on a particular coupling strength. In addition, complex network synchronization for the mentioned topologies with unidirectional and bidirectional coupling of hyperchaotic nodes is considerated.

Keywords: *complex networks; nearest-neighbor topology; small-world topology; open ring topology; tree topology; star topology; global topology; network synchronization; hyperchaotic time-delay Chua oscillator.*

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1 Introduction

Many systems in nature, applied sciences, and technology are constituted by a large number of highly interconnected dynamical units, the so-called complex dynamical networks. Some typical examples are coupled biological and chemical systems, neural networks, social interacting species, the Internet or the World Wide Web.

Complex networks of dynamical systems have been recently proposed as models in many diverse fields of applications (see e.g. [3, 17] and references therein). Recently, particular attention has been focused on the problem of making a network of dynamical systems synchronize to a common behavior. Typically, the complex network consists of N identical nonlinear dynamical systems coupled through the edges of the network itself [3, 17].

Synchronization is an important property of dynamical systems and even more when the dynamical systems have chaotic behavior, since achieving synchronization of chaotic systems provides superior alternatives to be explored, in complex network synchronization with chaotic nodes. The most works on network synchronization is about network configurations with regular coupling, see for example [8, 34], while the study in random network synchronization has been smaller, see e.g. [7, 13].

In particular, there is an increased interest in complex network synchronization of dynamical chaotic systems, which has led many scientists to consider the phenomenon of synchronization in large-scale networks with coupled chaotic oscillators like nodes, see e.g. [1, 2, 5, 8, 9, 18–27, 30, 31, 34, 35]. This type of network synchronization has been with topologies completely regular and global networks, see for example [26]. The main benefit of these simple architectures is that we can focus on the complexity caused by the nonlinear dynamics of the nodes, without taking into account the additional complexity, characteristic of the network topology.

In this paper, we synchronize complex dynamical networks of coupled hyperchaotic nodes in different topologies. In particular, each uncoupled dynamical system is described by a nonlinear set of time-delay Chua oscillators, which generate very complex behavior including hyperchaotic motion. This study presents network synchronization in nearest-neihbor, small-world, open ring, tree, star, and global coupling topologies; which are the most widely used in network communication systems. The complex network synchronization is achieved in two different way: with unidirectional and bidirectional coupling. In addition, network synchronization is evaluated according to a particular coupling strength for each topology.

The rest of the paper is outlined as follows. Section 2 describes the mathematical preliminaries, some important definitions, description of the networks, topologies, characteristics, network synchronization conditions, probability conditions, etc. Section 3 shows network synchronization with time-delay Chua oscillator like nodes, then it's performed network synchronization with each of the topologies with unidirectional and bidirectional coupling. Section 4 gives the conclusions of the results.

2 Preliminaries

We consider a complex dynamical network of N identical nodes, linearly coupled through the first state variable of each node, each node being a n-dimensional dynamical system.

The state equations of the network are given by:

$$\begin{aligned}
\dot{x}_{i1} &= f_1(\mathbf{x}_i) + s\sum_{j=1}^{N} a_{ij}x_{j1}, \qquad i = 1, 2, ..., N.\\
\dot{x}_{i2} &= f_2(\mathbf{x}_i),\\
&\vdots\\
\dot{x}_{in} &= f_n(\mathbf{x}_i),
\end{aligned} \tag{1}$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, ..., x_{in})^{\mathrm{T}} \in \mathbb{R}^n$ are the state variables of node i, $f_i(\mathbf{0}) = \mathbf{0}$, $s > 0$ represents the coupling strength of the network, and the coupling matrix $\mathbf{A} = (a_{ij})_{(N\times N)} \in \mathbb{R}^{N\times N}$ represents the coupling configuration of the complex dynamical network. If there is a connection between node i and node j, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$ $(i \neq j)$.

In this paper, we only consider symmetric and diffusive coupling. In particular, we assume that:
(i) $\mathbf{A}$ is a symmetric and irreducible matrix.
(ii) The off-diagonal elements a_{ij} $(i \neq j)$ of coupling matrix $\mathbf{A}$, are either 1 (when a connection between node i and node j) or 0 (when a connection between node i and node j is absent).
(iii) The elements of the principal diagonal of $\mathbf{A}$ satisfy

$$a_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij} = -\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ji}, \qquad i = 1, 2, ..., N. \tag{2}$$

The above conditions imply that one eigenvalue of the coupling matrix $\mathbf{A}$ is zero, with multiplicity 1, and all the other eigenvalues of $\mathbf{A}$ are strictly negative.

Given the dynamics of an isolated node and the coupling strength, stability of the synchronization state of the complex dynamical network (1) can be characterized by those nonzero eigenvalues of the coupling matrix $\mathbf{A}$. A typical result states that the complex dynamical network (1) will synchronize if these eigenvalues are negative enough [34].

Lemma 2.1 [31] *Consider the dynamical network (1). Let λ_1 be the largest nonzero eigenvalue of the coupling matrix $\mathbf{A}$ of the network. The synchronization state of network (1) defined by $\mathbf{x}_1 = \mathbf{x}_2 = ... = \mathbf{x}_n$ is asymptotically stable, if*

$$\lambda_1 \quad \leq \quad -\frac{T}{s}, \tag{3}$$

where $s > 0$ is the coupling strength of the network and $T > 0$ is a positive constant such that zero is an exponentially stable point of the following n-dimensional system:

$$\dot{z}_1 = f_1(z) - Tz_1, \quad \dot{z}_2 = f_2(z), \quad \dot{z}_n = f_n(z). \tag{4}$$

System (4) corresponds an *isolated node* with self-feedback $-Tz_1$. Condition (3) means that the complex dynamical network (1) will synchronize provided that λ_1 is negative enough, e.g. it is sufficient to be less than $-T/s$, so that the self-feedback term $-Tz_1$ could stabilize the isolated node (4).

3 Complex Network Topologies

3.1 Nearest-neighbor coupled network topology

The coupling configuration in nearest-neighbor consists of N arranged nodes in ring where each node i coupled to its nearest-neighbors nodes. The corresponding coupling matrix is given by

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}. \quad (5)$$

The eigenvalues of the coupling matrix $\mathbf{A}_{nc}$ are given by [31]:

$$\Big\{ -4sin^2\left(\frac{k\pi}{N}\right),\ k = 0, 1, \ldots, N-1 \Big\}. \quad (6)$$

Therefore, according to Lemma 2.1, the nearest-neighbor coupled dynamical network will asymptotically synchronize if [31]:

$$4sin^2\left(\frac{\pi}{N}\right) \geq \frac{T}{s}. \quad (7)$$

3.2 Small-world coupled network topology

Aiming to describe a transition from a regular network to a random network, Watts and Strogatz [32] introduced an interesting model, called the *small-world* (SW) network. The original SW model can be described as follows. Take an one-dimensional network of N arranged nodes in a ring with connections between only nearest neighbors. We "rewire" each connection with some *probability p*. Rewiring in this context means shifting one end of the connection to a new node chosen at random from the whole network, with the constraint that no two different nodes can have more than one connection among them, and no node can have a connection with itself.

However, there is a possibility for the SW model to be broken into unconnected clusters. This problem can be circumvented by a slight modification of the SW model, suggested by Newman and Watts [15], which is called the *NW model*. In the NW model, we do not break any connection between any two nearest neighbors. We add with *probability* p a connection between each other pair of nodes. Likewise, we do not allow a node to be coupled to another node more than once, or coupling of a node with itself. For $p = 0$, it reduces to the originally nearest-neighbor coupled network; for $p = 1$, it becomes a globally coupled network. In this paper, we are interested in probabilities with $0 < p < 1$.

From a coupling matrix point of view, a complex dynamical network (1) with new connections in small-world is determined as follows: if $a_{ij} = 0$, this element can change to $a_{ij} = a_{ji} = 1$ according to the *probability p*. Then, we recompute the diagonal elements according to Eq. (2). We denote the new *small-world coupling matrix* as $\mathbf{A}_{swc}(p; N)$ and let $\lambda_{1\,swc}(p; N)$ be its largest nonzero eigenvalue. According to Lemma 2.1, if

$$\lambda_{1\,swc}(p; N) \leq -\frac{T}{s}, \quad (8)$$

then the corresponding complex dynamical network (1) with small-world connections will synchronize [31].

3.3 Open ring coupled network topology

Open ring configuration consists of N arranged nodes in a ring, but in this case, the last node is not connected to the first node. The corresponding coupling matrix is given by

$$\mathbf{A}_{rc} = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}. \tag{9}$$

This matrix have an eigenvalue at 0 and others $N-1$ are at -1 [30].

3.4 Star coupled network topology

A complex dynamical network with star coupling consists of a single node (called the common node or central node) of the complex dynamical network connected with the remaining $N-1$ nodes. The coupling matrix is given by

$$\mathbf{A}_{sc} = \begin{bmatrix} 1-N & 1 & ... & ... & 1 \\ 1 & -1 & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 1 & & & & -1 \end{bmatrix}. \tag{10}$$

The eigenvalues of the coupling matrix $\mathbf{A}_{sc}$ are $\{0, -N, -1, \dots, -1\}$ [30].

3.5 Globally coupled network topology

In this coupling configuration, the N nodes are connected with others; that is any two nodes are connected directly. All nodes are connected to the same number $(N-1)$ of nodes. Thus, the coupling matrix is given by

$$\mathbf{A}_{gc} = \begin{bmatrix} 1-N & 1 & 1 & \dots & 1 \\ 1 & 1-N & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1-N \end{bmatrix}. \tag{11}$$

This matrix has a single eigenvalue at 0 and others $N-1$ at $-N$ [30]. Lemma 2.1 implies that the global coupled network will asymptotically synchronize, if

$$N \geq -\frac{T}{s}.$$

3.6 Tree coupled network topology

The network topology in tree can be viewed as a collection of star networks arranged in a hierarchy way, the network begins with a master node and this in turn is connected to other slave nodes which are connected with the rest of the nodes. The coupling matrix for this network is given by

$$\mathbf{A}_{tc} = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 1 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \quad (12)$$

This matrix has an eigenvalue at 0 and the others $N-1$ are at -1 [30].

4 Network Synchronization with Time-Delay Chua Oscillator Like Nodes

4.1 Time-delay Chua oscillator

The time-delay Chua oscillator is a physical system, which presents well-defined hyperchaotic dynamics confirmed experimentally and numerically. The state equations describing the time-delay Chua oscillator in dimensionless form are given by [4,6,29]:

$$\begin{aligned} \dot{x}_1 &= \alpha(-x_1 + x_2 - f(x_1)), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\beta x_2 - \gamma x_3 - \beta\varepsilon \sin(\sigma x_1 (t-\tau)), \end{aligned} \quad (13)$$

with nonlinear function defined by

$$f(x_1) = bx_1 + \frac{1}{2}(a-b)(|x_1+1| - |x_1-1|).$$

The parameters which are obtained hyperchaotic dynamics are: $\alpha = 10$, $\beta = 19.53$, $\gamma = 0.1636$, $a = -1.4325$, $b = -0.7831$, $\sigma = 0.5$, $\varepsilon = 0.2$, and by using the time-delay $\tau = 0.001$. The initial conditions of the oscillator are $\mathbf{x}(0) = (1.1, 0.1, 0.5)$. The generated hyperchaotic attractors by the time-delay Chua oscillator (13) are shown in Figure 1.

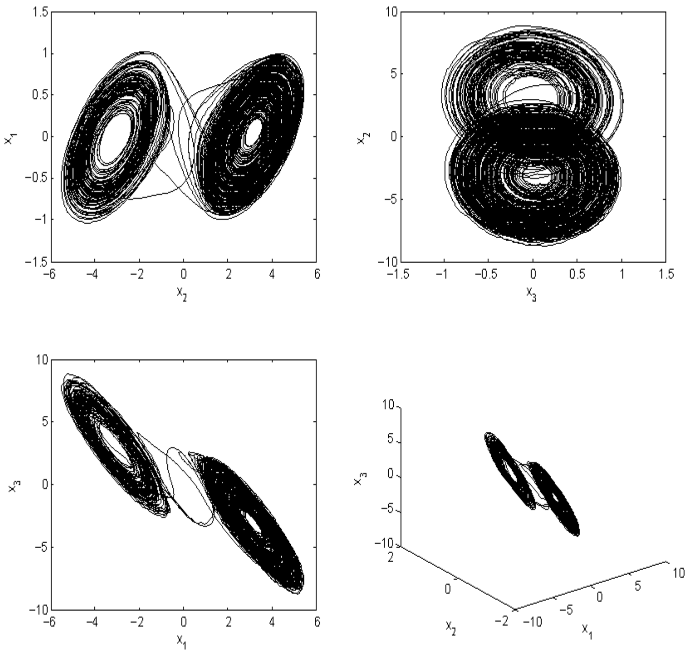

Figure 1: Hyperchaotic attractors generated by the time-delay Chua oscillator.

4.2 Synchronization in nearest-neighbor coupled networks

4.2.1 Bidirectional network synchronization

The state equations for N hyperchaotic nodes of the complex dynamical network (1) are given by

$$\begin{aligned} \dot{x}_{i1} &= \alpha(-x_{i1} + x_{i2} - f(x_{i1})) + s\sum_{j=1}^{N}(a_{ij}x_{j1}), \qquad i = 1, 2, ..., N, \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3}, \\ \dot{x}_{i3} &= -\beta x_{i2} - \gamma x_{i3} - \beta\varepsilon \sin(\sigma x_{i1}\,(t-\tau)), \end{aligned} \tag{14}$$

with nonlinear function defined by

$$f\,(x_1) = bx_1 + \frac{1}{2}\,(a-b)\,(|x_1+1| - |x_1-1|)\,.$$

For $T = 30$, the isolated node time-delay Chua oscillator (13) stabilizes at a point as is shown in Figure 2. The coupling strength chosen is $s = 25$ and the connection grade of the dynamical network is $K = 2$.

For example, with $N = 5$ hyperchaotic nodes (time-delay Chua oscillators), the dynamical network is shown in Figure 3(a). The bidirectional coupling matrix is given

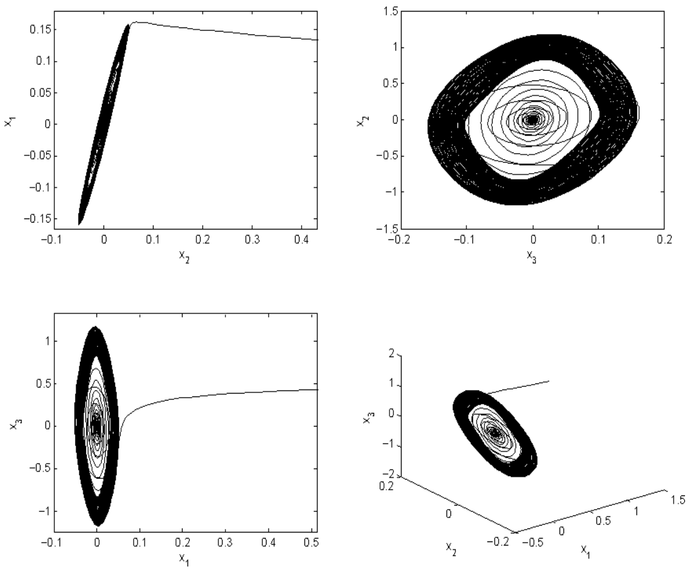

Figure 2: Attractors generated by an isolated time-delay Chua oscillator with feedback $-Tx_1$.

by

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{15}$$

The largest nonzero eigenvalue of $\mathbf{A}_{nc}$ is defined by (6) as follows

$$-4sin^2\left(\frac{\pi}{5}\right) = -1.382. \tag{16}$$

The network synchronization condition (3) is as follows

$$-1.382 \leq -\frac{30}{25} = -1.2. \tag{17}$$

With these chosen values, the condition (3) is fulfilled and therefore the dynamical network with $N = 5$ hyperchaotic nodes in nearest-neighbor will synchronize.

Figure 4 shows the first attractor (x_{i1} vs x_{i2}) of each node. While, Figure 5 illustrates synchronization among nodes, showing the first state of each hyperchaotic node.

With $N = 6$ hyperchaotic nodes, the network is shown in Figure 3(b). The coupling matrix is defined by

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{18}$$

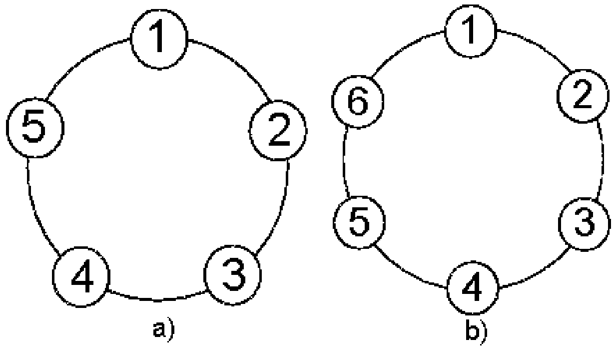

Figure 3: Network topologies in bidirectional nearest-neighbor: a) With $N = 5$ nodes. b) With $N = 6$ nodes.

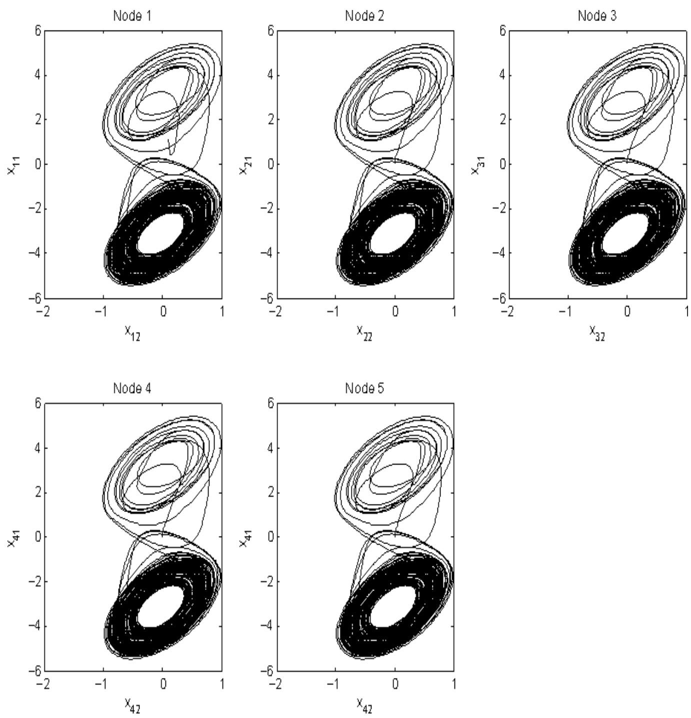

Figure 4: First attractor of each node with bidirectional synchronization for $N = 5$.

The largest nonzero eigenvalue is defined by (6),

$$-4sin^2\left(\frac{\pi}{6}\right) = -1. \tag{19}$$

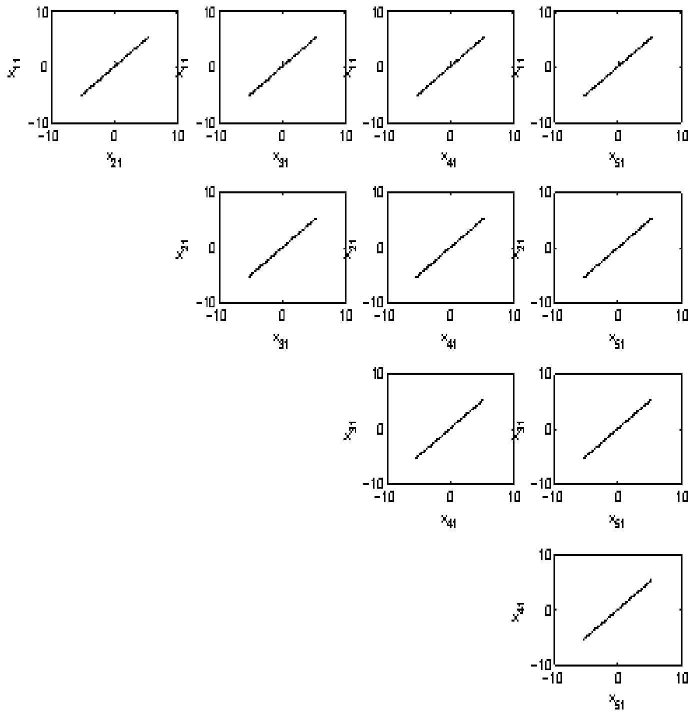

Figure 5: Synchronization among 5 hyperchaotic nodes in the bidirectional nearest-neighbor coupled network.

The network synchronization condition (3) is as follows

$$-1 \quad \leq \quad -\frac{30}{25} = -1.2.$$

Now, the network synchronization condition (3) is not fulfilled and therefore the dynamical network with 6 hyperchaotic nodes will not synchronize. Figure 6 illustrates the phase portrait among nodes, showing the first state of each node. It can be seen that there is no synchronization among nodes.

Therefore, we can say that for $s = 25$ and $N = 6$, the synchronization of the network in bidirectional nearest-neighbor configuration will synchronize up to with 5 hyperchaotic nodes.

4.2.2 Unidirectional network synchronization

With $N = 5$ hyperchaotic nodes (time-delay Chua oscillators), the dynamical network is shown in Figure 7(a). The unidirectional coupling matrix is given by

$$\mathbf{A}_{nc} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{20}$$

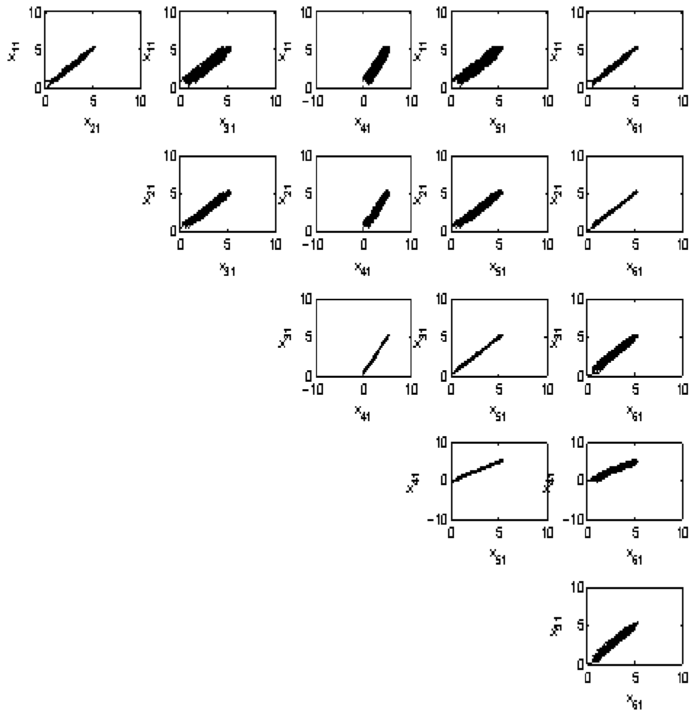

Figure 6: Phase portrait for no synchronization process among 6 hyperchaotic nodes of the dynamical network.

With a coupling strength $s = 10$ and previous values, the dynamical network with 5 nodes synchronizes. Figure 8 shows the first hyperchaotic attractors (x_{i1} vs x_{i2}) of each node. While, Figure 9 illustrates the synchronization among 5 hyperchaotic nodes, showing the first state of each node.

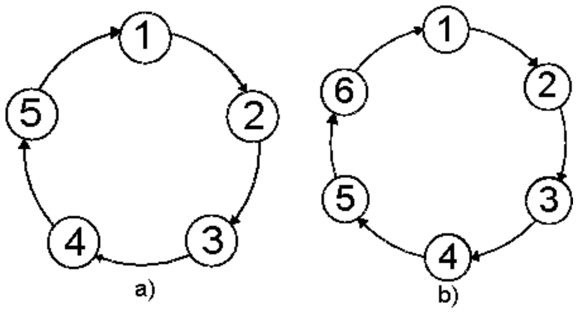

Figure 7: Network configuration in unidirectional nearest-neighbor: a) With $N = 5$ hyperchaotic nodes. b) With $N = 6$ hyperchaotic nodes.

With $N = 6$ hyperchaotic nodes, the dynamical network is shown in Figure 7(b). For

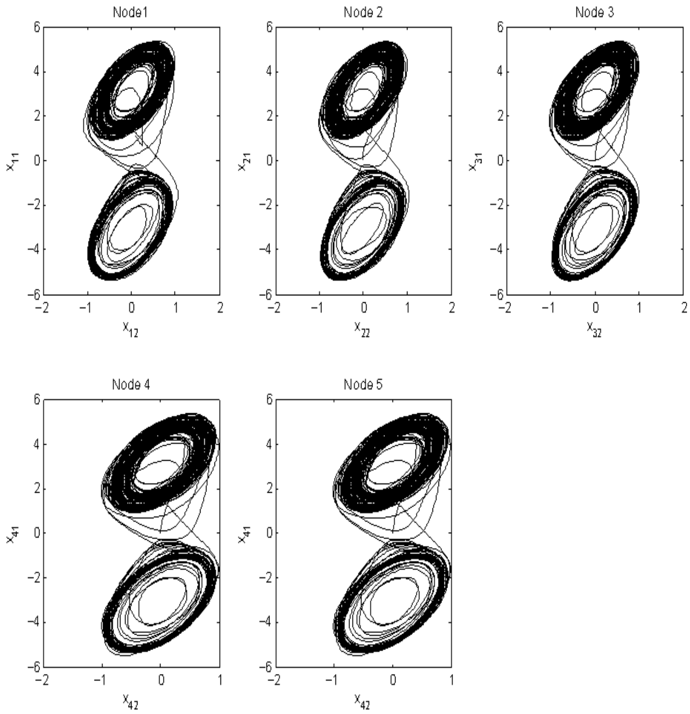

Figure 8: First attractor of each node with unidirectional synchronization for $N = 5$ hyperchaotic nodes.

this case, the coupling matrix is defined by

$$\mathbf{A}_{nc} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{21}$$

The largest nonzero eigenvalue is $-0.5 - 0.866i$. With a coupling strength $s = 10$ and previous values, the network with 6 hyperchaotic nodes will not synchronize. Figure 10 illustrates the synchronization errors among 6 hyperchaotic nodes, showing the first state of each node. It can be seen that there is no synchronization among nodes.

Therefore, we have that for coupling strength $s = 10$ and $N = 6$ hyperchaotic nodes, the dynamical network in unidirectional nearest-neighbor configuration will synchronize up to with $N = 5$ hyperchaotic nodes. In the sequel, we show how synchronize the mentioned complex dynamical network with few extra connections for $N \geq 6$ hyperchaotic nodes.

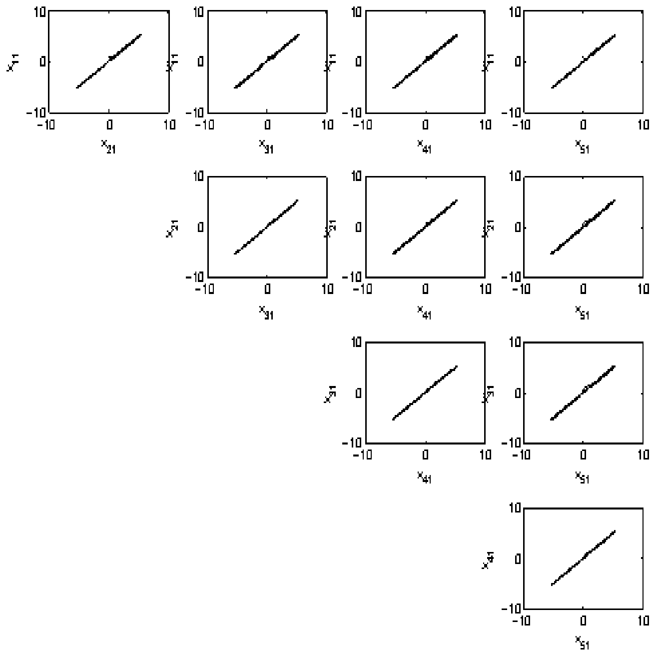

Figure 9: Synchronization among 5 hyperchaotic nodes in unidirectional nearest-neighbor coupled network.

4.3 Synchronization in small-world coupled networks

4.3.1 Bidirectional network synchronization

The nearest-neighbor coupled network with the values of previous parameters will not synchronize for $N \geq 6$, therefore, in this subsection we use the small-world configuration for synchronization of a number of hyperchaotic time-delay Chua oscillator nodes $N \geq 6$, of course without reaching the global coupling configuration, where network synchronization can be achieved without "any problem", besides having unnecessary connections that would increase construction costs and higher energy consumption, while with some new connections we can achieve complete synchronization of the complex dynamical network by using the small-world configuration.

Based on the coupling matrix (12) with $N = 6$ hyperchaotic nodes, the bidirectional nearest-neighbor configuration can build the new coupling matrix in small-world as follows [31]. In the nearest-neighbor coupling matrix $\mathbf{A}_{nc}$, the elements $a_{ij} = a_{ji} = 0$ can change to $a_{ij} = a_{ji} = 1$ according to a chosen probability p. In this case, we used a probability of connection $p = 0.15$, we choose a dynamical network with two new connections. With these new connections the dynamical network is shown in Figure 11(a). Then, we recompute the diagonal elements according to Eq. (2), the coupling matrix

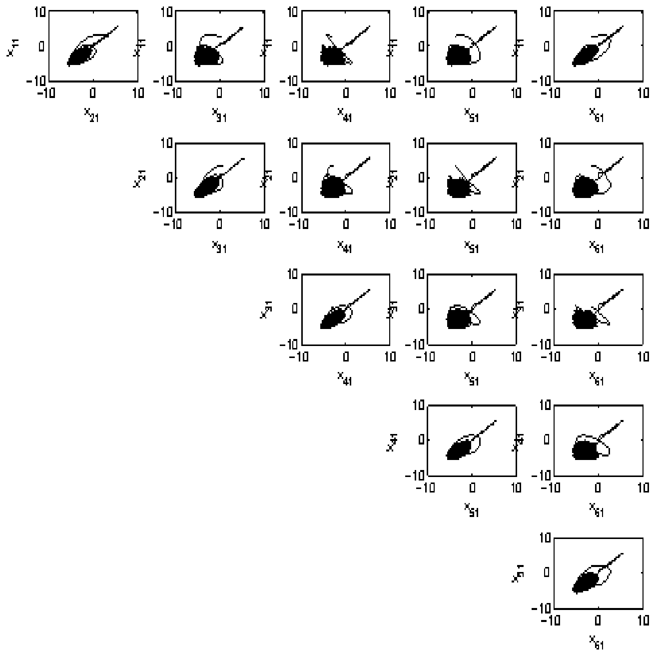

Figure 10: Phase portrait for no synchronization process among 6 hyperchaotic nodes of the unidirectional network.

with two new connections is defined by

$$\mathbf{A}_{swc} = \begin{bmatrix} -3 & 1 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}.$$

The largest nonzero eigenvalue is -1.3088. The network synchronization condition (3) is given by

$$-1.3088 \leq -\frac{30}{25} = -1.2. \tag{22}$$

With these chosen values, the condition (3) is fulfilled and therefore the complex dynamical network with 6 hyperchaotic nodes will synchronize. Figure 12 shows the first attractor of each node.

Figure 13 illustrates the network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

Figure 14 shows the numerical values of λ_{1swc} as a function of the number of N hyperchaotic nodes. In this figure each pair of values N and λ_{1swc} is obtained by averaging the results of 20 runs, implemented in the Matlab programming language. The above results imply that, for any given coupling strength s, we have: For any given

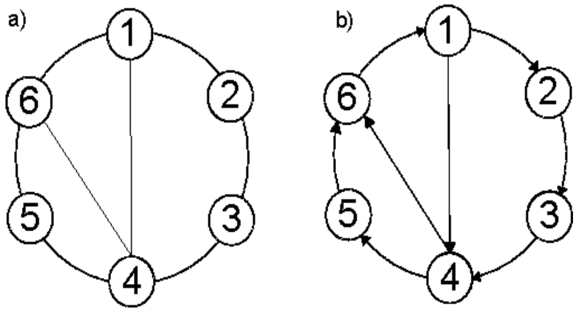

Figure 11: Small-world network configuration with $N = 6$ hyperchaotic nodes. a) Bidirectional coupling. b) Unidirectional coupling.

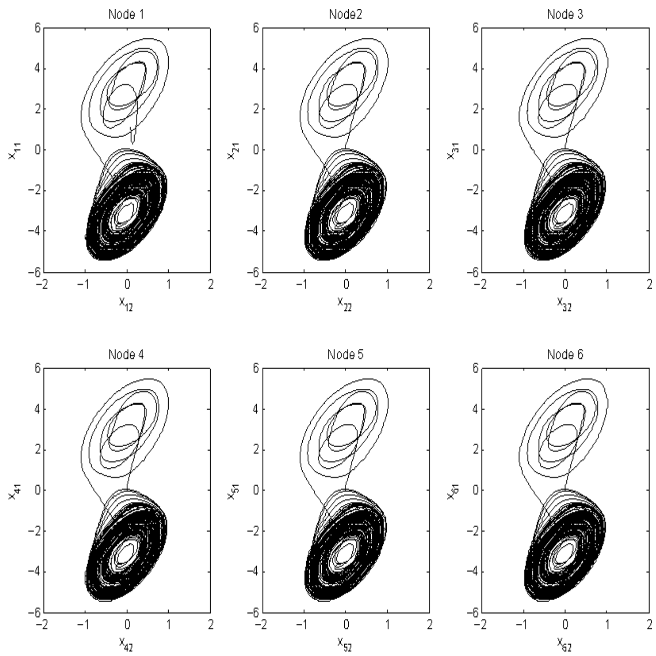

Figure 12: First hyperchaotic attractor of each node in bidirectional small-world configuration with $N = 6$ nodes.

N hyperchaotic nodes, there exists a critical value $\overline{\lambda}_{1swc}$, such that if $\overline{\lambda}_{1swc} \geq \lambda_{1swc}$, then the small-world connected network will synchronize.

4.3.2 Unidirectional network synchronization

Based on the coupling matrix (12) with $N = 6$ hyperchaotic nodes, the unidirectional nearest-neighbor configuration can build the new coupling matrix in small-world as follows [31]. In the nearest-neighbor coupling matrix $\mathbf{A}_{nc}$, the elements $a_{ij} = a_{ji} = 0$ can change to $a_{ij} = a_{ji} = 1$ according to a chosen probability p. In this case, we used a

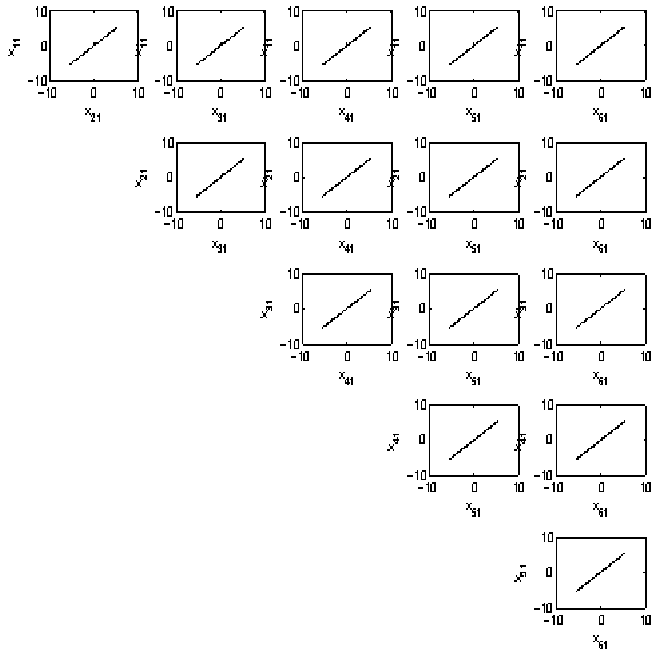

Figure 13: Synchronization in the first states among 6 hyperchaotic nodes of the network in bidirectional small-world configuration.

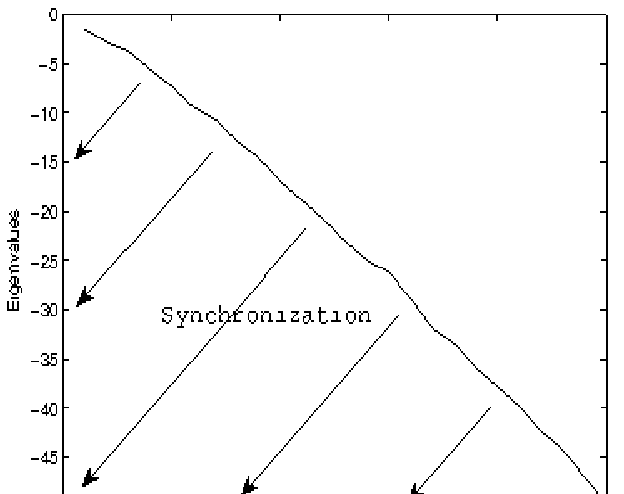

Figure 14: Number of nodes in function of the eigenvalues to achieve complex network synchronization.

probability of connection $p = 0.15$, we choose a dynamical network with two new connections. With these new connections the dynamical network is shown in Figure 11(b). Then, we recompute the diagonal elements according to Eq. (2), the coupling matrix with two new connections is defined by

$$\mathbf{A}_{swc} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}.$$

The largest nonzero eigenvalue is $-0.9293 + 0.7587i$.

With these two new connections and previous values, the unidirectional small-world complex dynamical network with 6 nodes will synchronize. Figure 15 shows the first hyperchaotic attractor of each node. Figure 16 illustrates the complex network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

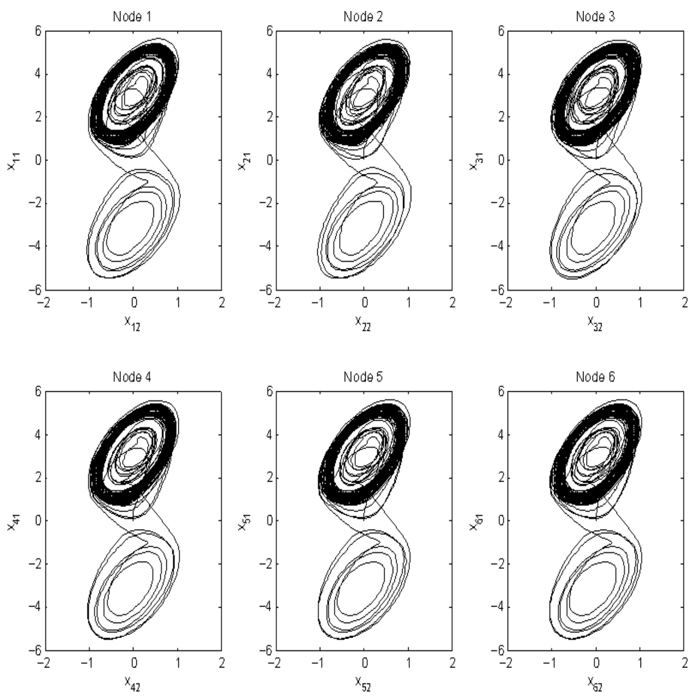

Figure 15: First hyperchaotic attractor of each node in unidirectional small-world configuration with $N = 6$.

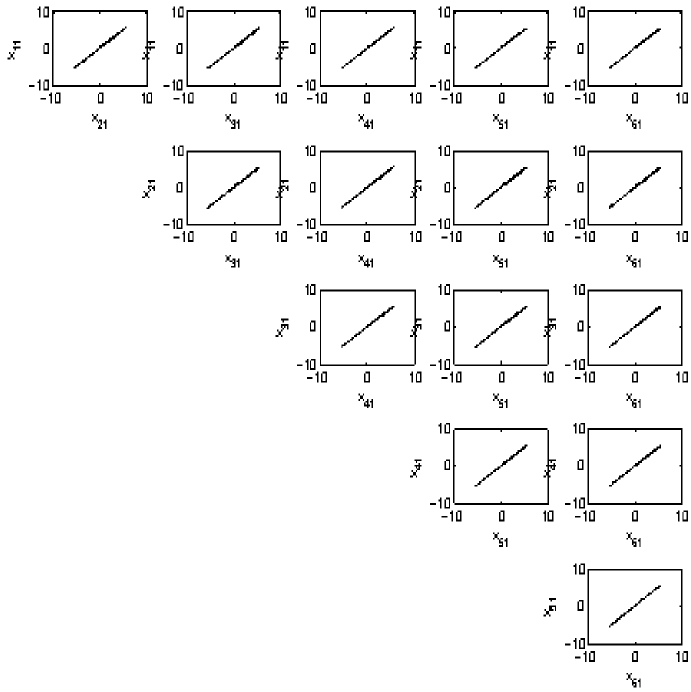

Figure 16: Synchronization in the first states among 6 hyperchaotic nodes of the network in unidirectional small-world configuration.

4.4 Synchronization in open ring coupled networks

4.4.1 Bidirectional network synchronization

With $N = 6$ hyperchaotic nodes, the dynamical network is shown in Figure 17(a). The coupling matrix is given by

$$\mathbf{A}_{rc} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \qquad (23)$$

The largest nonzero eigenvalue is -0.2679. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 50$.

Figure 18 shows the first attractor (x_{i1} vs x_{i2}) of each node. Figure 19 illustrates the network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

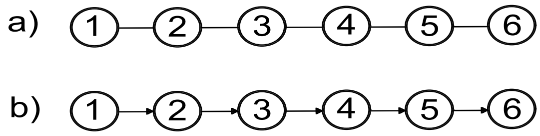

Figure 17: Network configuration in open ring with $N = 6$ hyperchaotic nodes: a) Bidirectional coupling. b) Unidirectional coupling.

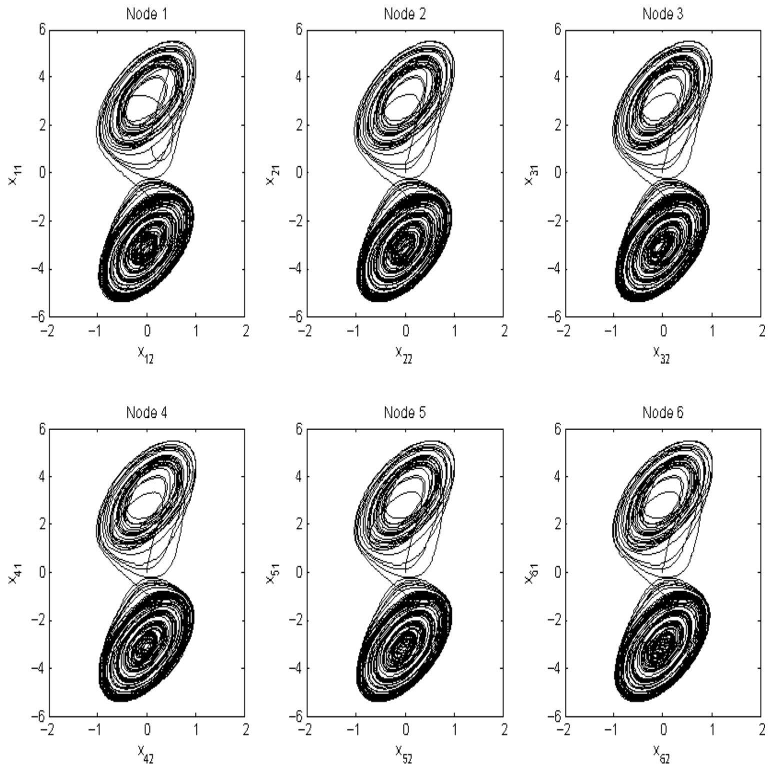

Figure 18: First attractor of each node in bidirectional open ring configuration with $N = 6$ hyperchaotic nodes.

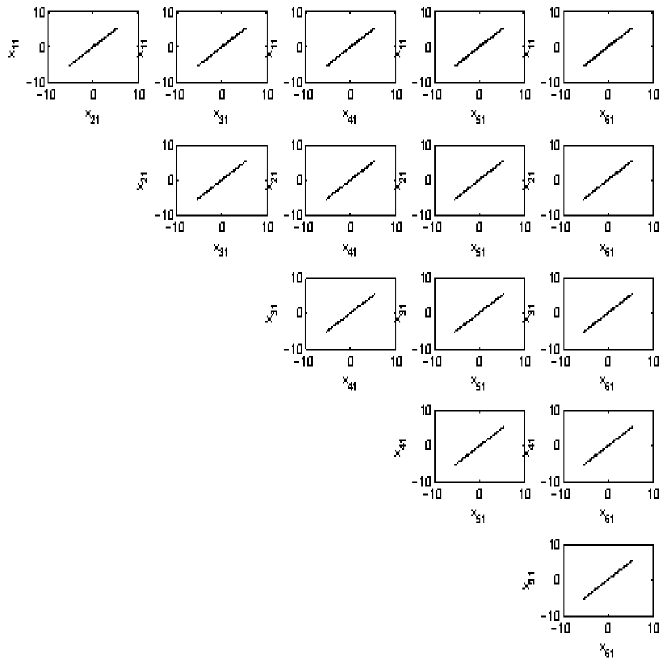

Figure 19: Synchronization among 6 nodes of the network in bidirectional open ring configuration.

4.4.2 Unidirectional synchronization

With a number $N = 6$ hyperchaotic nodes, the dynamical network is shown in Figure 17(b). The corresponding coupling matrix is defined by

$$\mathbf{A}_{rc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{24}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 50$.

Figure 20 illustrates the first attractor (x_{i1} vs x_{i2}) of each node. Figure 21 shows the network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

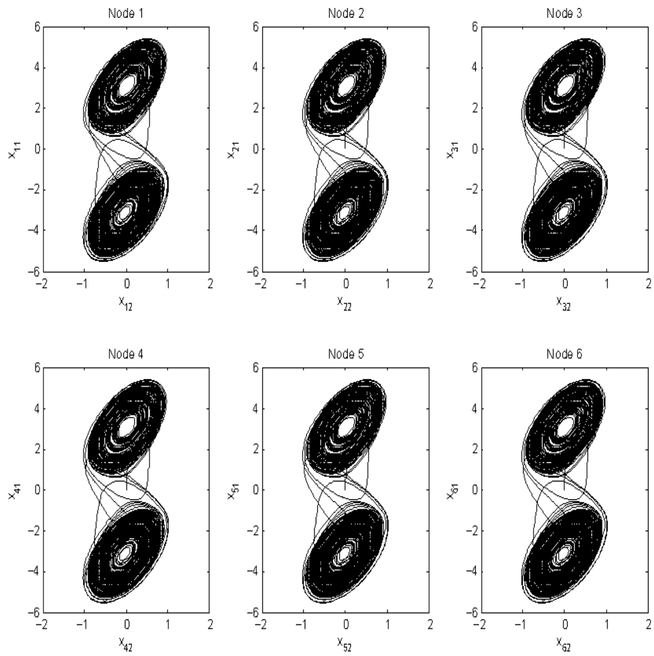

Figure 20: First attractor of each hyperchaotic node in unidirectional open ring configuration with $N = 6$ hyperchaotic nodes.

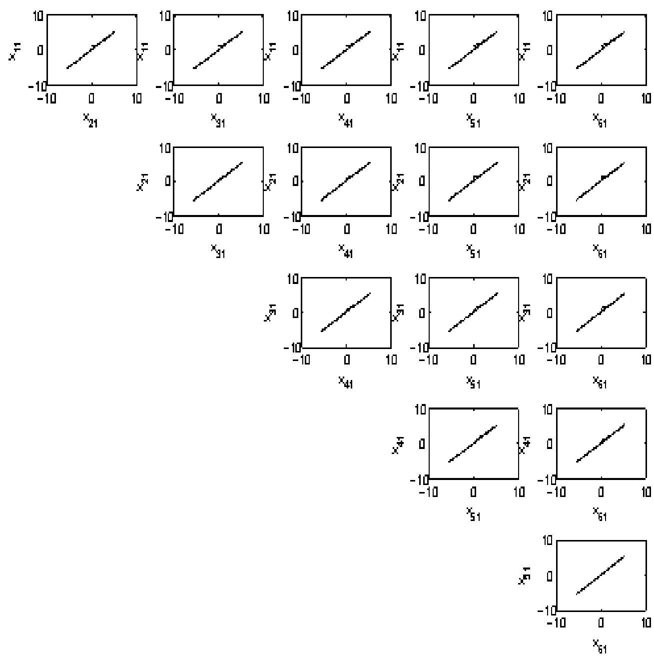

Figure 21: Synchronization among 6 hyperchaotic nodes of the network in unidirectional open ring configuration.

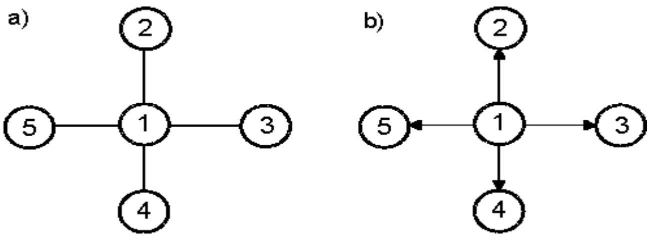

Figure 22: Network configuration in star with $N = 5$ hyperchaotic nodes: a) Bidirectional coupling. b) Unidirectional coupling.

4.5 Synchronization in star coupled networks

4.5.1 Bidirectional network synchronization

With $N = 5$ hyperchaotic nodes, the dynamical network is shown in Figure 22(a). The coupling matrix is given by

$$\mathbf{A}_{sc} = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{25}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this network synchronizes with a coupling strength $s = 20$.

Figure 23 shows the first attractor (x_{i1} vs x_{i2}) of each node. Figure 24 illustrates the synchronization among nodes, showing the first state of each node.

4.5.2 Unidirectional network synchronization

With $N = 5$ hyperchaotic nodes, the dynamical network is shown in Figure 22(b). The coupling matrix is given by

$$\mathbf{A}_{sc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{26}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 20$.

Figure 25 shows the first attractor (x_{i1} vs x_{i2}) of each hyperchaotic node. Figure 26 illustrates the synchronization among nodes, showing the first state of each node.

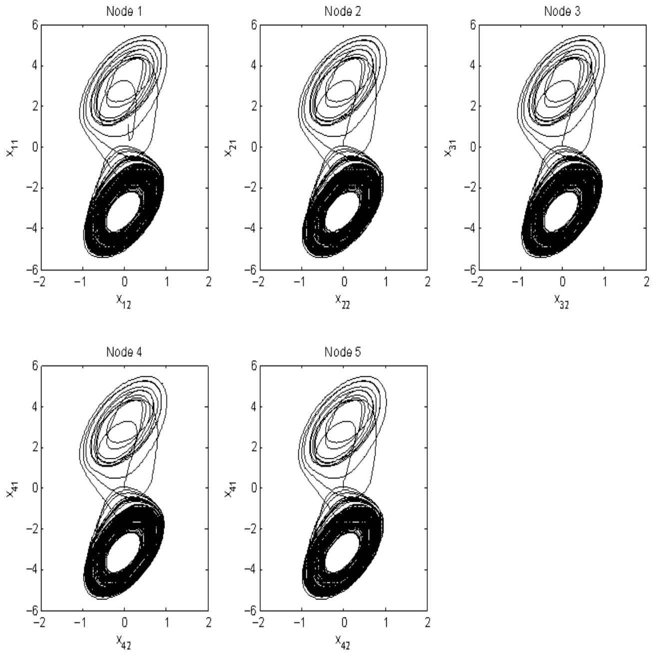

Figure 23: First hyperchaotic attractor of each node in bidirectional star configuration with $N = 5$.

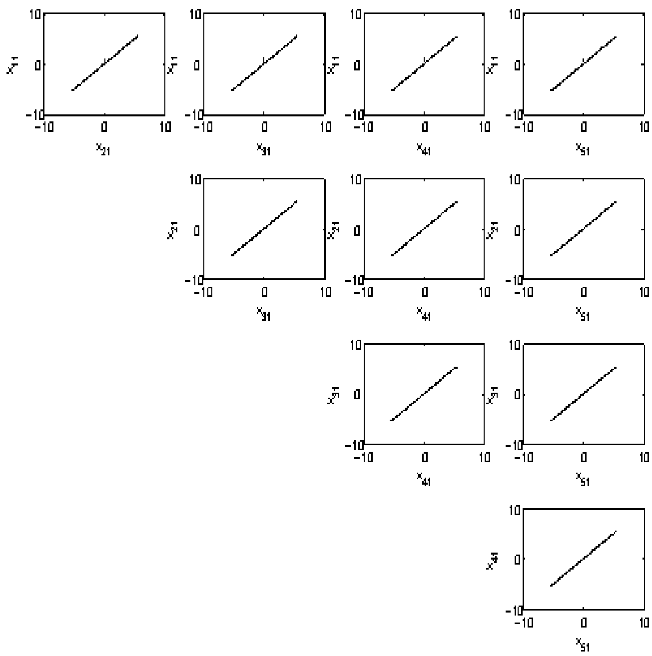

Figure 24: Synchronization among 5 hyperchaotic nodes of the network in bidirectional star configuration.

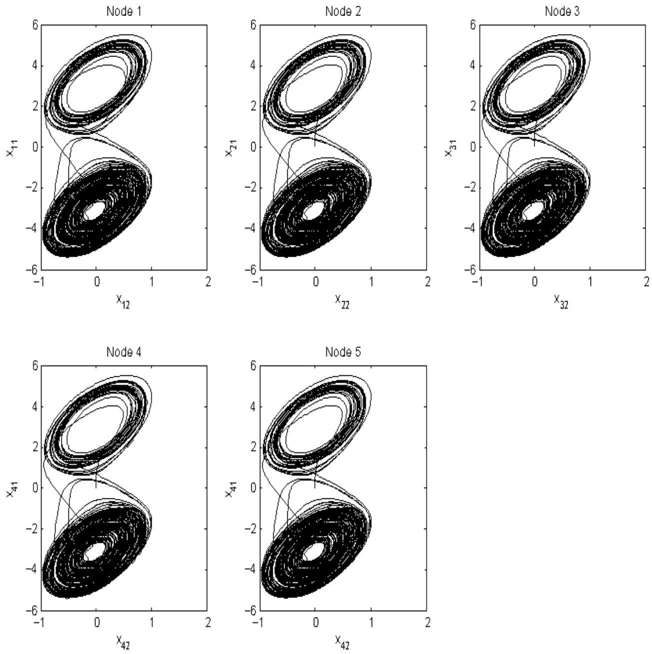

Figure 25: First hyperchaotic attractor of each node in unidirectional star configuration with $N = 5$ nodes.

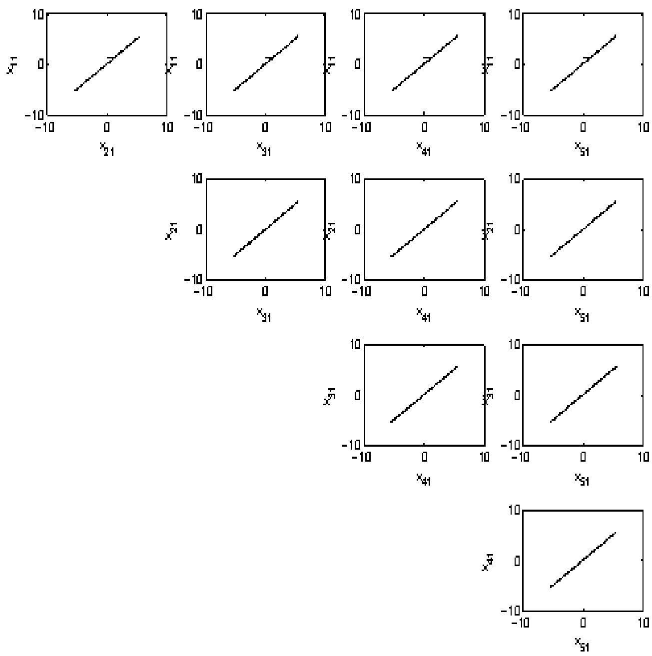

Figure 26: Synchronization among 5 hyperchaotic nodes of the network in unidirectional star configuration.

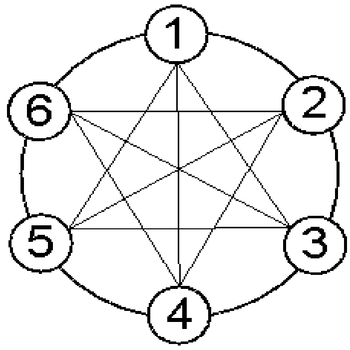

Figure 27: Network in global coupling with $N = 6$ hyperchaotic nodes.

4.6 Synchronization in global coupled networks

With $N = 6$ hyperchaotic nodes, the network is shown in Figure 27. The coupling matrix is defined by

$$\mathbf{A}_{gc} = \begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -5 & 1 & 1 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & 1 & 1 & -5 \end{bmatrix}. \tag{27}$$

The largest nonzero eigenvalue is -6. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 10$. Figure 28 shows the first attractor (x_{i1} vs x_{i2}) of each 6 hyperchaotic nodes. Figure 29 illustrates the synchronization among 6 hyperchaotic nodes, showing the first state of each hyperchaotic node.

4.7 Synchronization in tree coupled networks

4.7.1 Bidirectional network synchronization

With $N = 7$ hyperchaotic nodes, the dynamical network is shown in Figure 30(a). The coupling matrix is given by

$$\mathbf{A}_{tc} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{28}$$

The largest nonzero eigenvalue is -0.2679. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 10$.

Figure 31 shows the first attractor (x_{i1} vs x_{i2}) of each node. Figure 32 illustrates the network synchronization among 7 hyperchaotic nodes, showing the first state of each node.

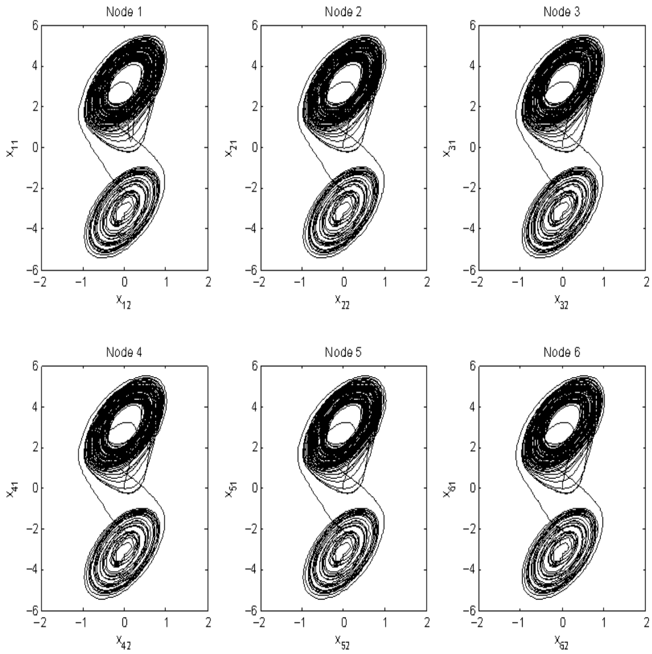

Figure 28: First hyperchaotic attractor of each node in global coupling with $N = 6$ nodes.

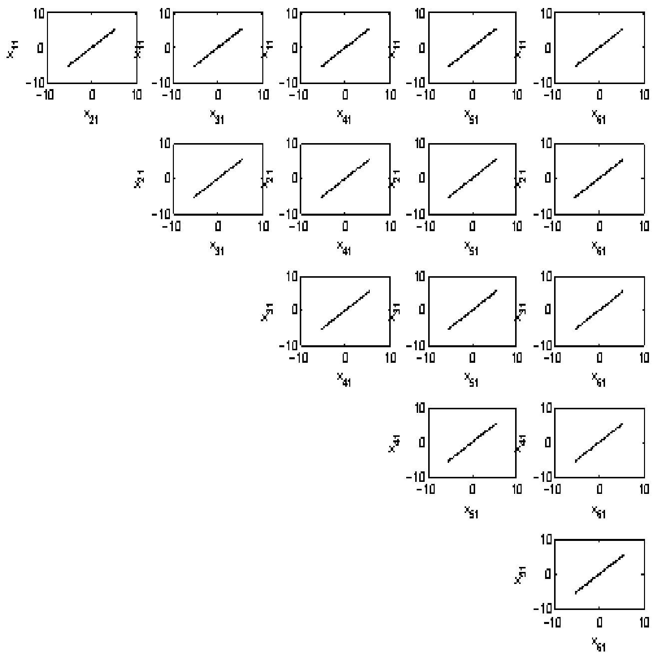

Figure 29: Synchronization among 6 hyperchaotic nodes of the dynamical network in global coupling.

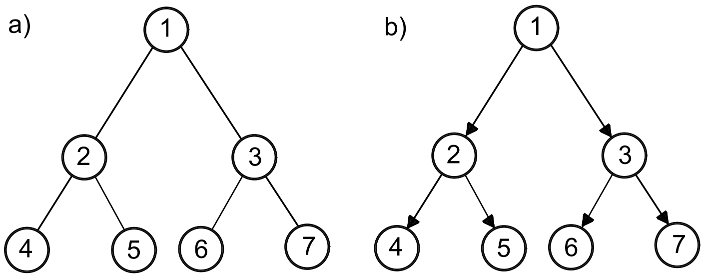

Figure 30: Network configuration in tree with $N = 7$ hyperchaotic nodes: a) Bidirectional coupling. b) Unidirectional coupling.

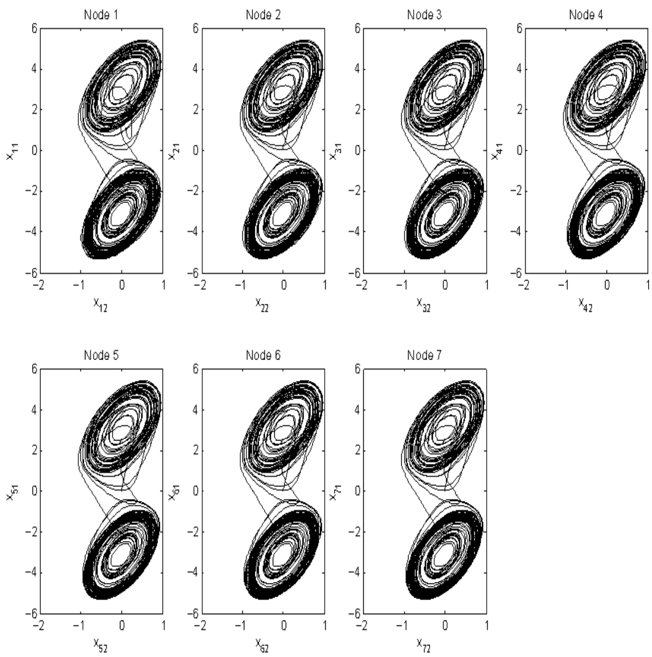

Figure 31: First hyperchaotic attractor of each node in bidirectional tree configuration with $N = 7$ nodes.

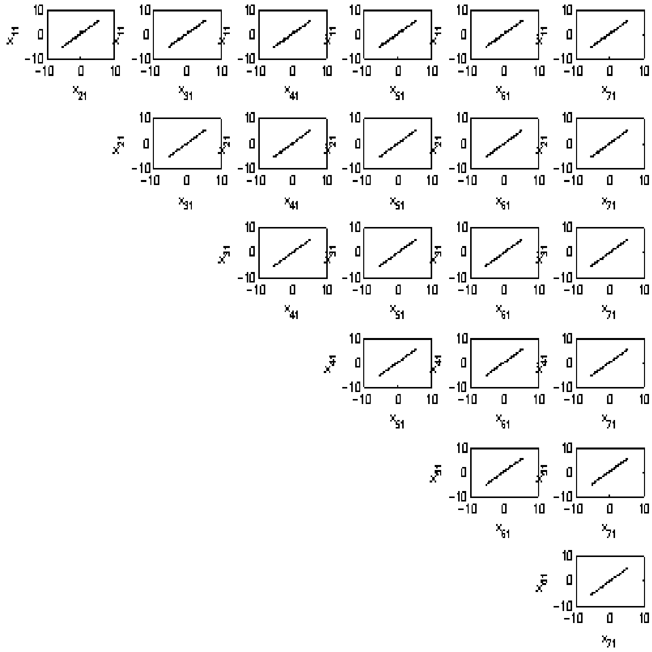

Figure 32: Synchronization among 7 hyperchaotic nodes of the network in bidirectional tree configuration.

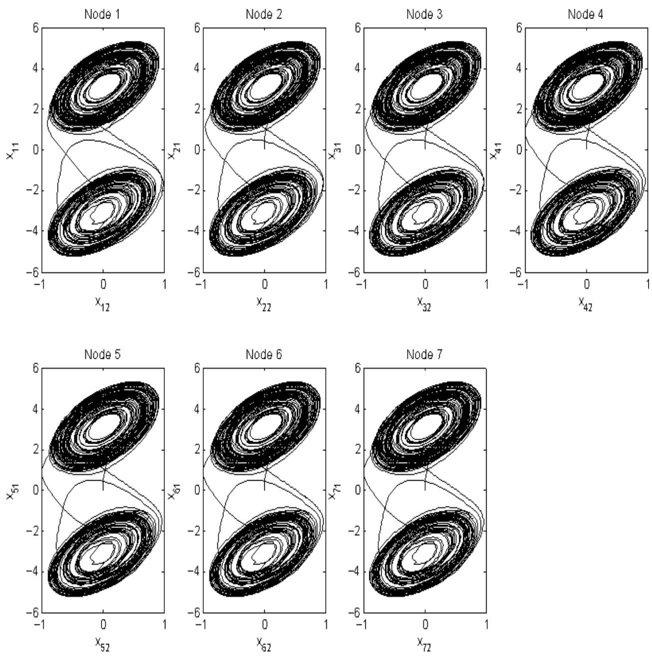

Figure 33: First hyperchaotic attractor of each node in unidirectional tree configuration with $N = 7$ nodes.

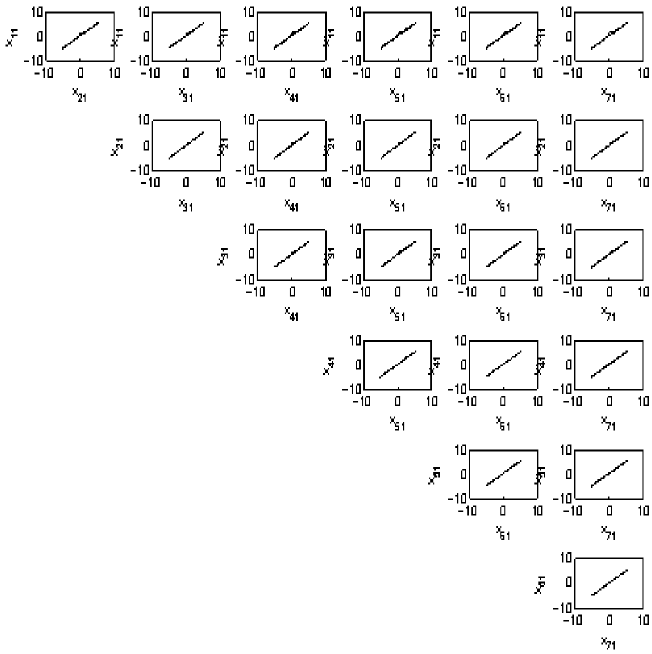

Figure 34: Synchronization among 7 hyperchaotic nodes of the network in unidirectional tree configuration.

4.7.2 Unidirectional network synchronization

With $N = 7$ hyperchaotic nodes, the dynamical network is shown in Figure 30(b). The coupling matrix is given by

$$A_{tc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{29}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this network synchronizes with a coupling strength $s = 20$. Figure 33 shows the first attractor (x_{i1} vs x_{i2}) of each hyperchaotic node. Figure 34 illustrates the synchronization among 7 hyperchaotic nodes, showing the first state of each node.

5 Conclusions

In this paper, synchronization of complex dynamical networks in various topologies was performed. Numerical results were obtained for network synchronization in nearest-neighbor configuration, small-world configuration, open ring configuration, tree configuration, star configuration, and global configuration topologies, by using the hyperchaotic time-delay Chua oscillator like nodes. One can see that each coupling configuration requires a different coupling strength, each topology also has its own characteristics with implication that this coupling strength is different for each case. As an example, we mention the first topology, the nearest-neighbor, where with a number of $N = 5$ hyperchaotic nodes, it was required a coupling strength $s = 25$ for bidirectional synchronization; instead, for unidirectional network synchronization and the same number of nodes a coupling strength $s = 10$ is enough to achieve network synchronization. For tree topology, the bidirectional network synchronization requires a coupling strength $s = 10$ to synchronize, however, the unidirectional synchronization requires a coupling strength $s = 20$ for synchronization, which is twice as large as that of bidirectional. However, for all topologies the synchronization of the network was achieved unidirectionally or bidirectionally.

Acknowledgment

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Application of Passivity Based Control for Partial Stabilization

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Abstract: In this paper, the problem of partial stabilization is considered for nonlinear control systems and a general approach for partial stabilization is proposed. In this approach, by introducing the notion of *partially passive systems*, some theorems for partial stabilization are developed. For this purpose, the nonlinear system is divided into two subsystems based on stability properties of system's states. The reduced control input vector (the vector that includes components of input vector appearing in the first subsystem), is designed based on the new passivity based control theorems, in such a way to guarantee asymptotic stability of the nonlinear system with respect to the first part of states vector.

Keywords: *nonlinear systems; partial stability; partial passivity; partial control.*

Mathematics Subject Classification (2000): 34D20, 37N35, 70K99, 74H55, 93C10, 93D15.

1 Introduction

For many of engineering problems, application of Lyapunov stability is required [1]– [3]. However, there are other physical systems like inertial navigation systems, spacecraft stabilization, electromagnetic, adaptive stabilization, guidance, etc. [4]– [12], where partial stability is necessary. In the mentioned applications, while the plant may be unstable in the standard sense, it is partially and not totally asymptotically stable. It means that naturally the plant is stable with respect to just some -and not all- of the state variables. For example, consider the equation of motion for the slider-crank mechanism depicted in Figure 1 [8]:

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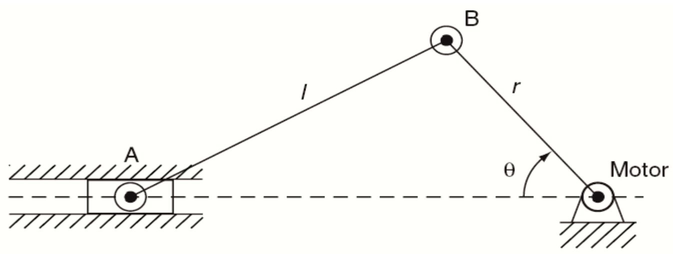

Figure 1: Slider-crank mechanism [8].

$$m(\theta(t))\ddot{\theta}(t) + c(\theta(t))\dot{\theta}^2(t) = u(t),$$
$$\theta(0) = \theta_0, \quad dot\theta(0) = \dot{\theta}_0, \quad t \geq 0,$$

where

$$m(\theta) = m_B r^2 + m_A r^2 \left(\sin\theta + \frac{r\cos\theta\sin\theta}{\sqrt{l^2 - r^2\sin^2\theta}}\right)^2,$$

$$c(\theta) = m_A r^2 \left(\sin\theta + \frac{r\cos\theta\sin\theta}{\sqrt{l^2 - r^2\sin^2\theta}}\right)\left(\cos\theta + r\frac{l^2(1 - 2\sin^2\theta) + r^2\sin^4\theta}{(l^2 - r^2\sin^2\theta)^{3/2}}\right)$$

and m_A and m_B are point masses, r and l are the lengths of the rods, and $u(.)$ is the control torque applied by the motor. Suppose that a feedback control law in the form of $u(.) = k(\theta, \dot{\theta})$ should be designed in a way that the angular velocity becomes constant; that is, $\dot{\theta}(t) \to \Omega$ as $t \to \infty$ where $\Omega > 0$. This implies that $\theta(t) = \Omega t \to \infty$ as $t \to \infty$. In addition, the angular position θ may not be disregarded. It is because $m(\theta)$ and $c(\theta)$ are functions of θ, and $\sin\theta$ does not converge to a limit. Consequently, it is obvious that the slider-crank mechanism is unstable in the standard sense; however, it is partially asymptotically stabilizable with respect to $\dot{\theta}$ [8].

In spite of variety of research papers in the ground of partial stability applications, there are only few papers in partial control design and advantages of partial control are not fully recognized. Furthermore, most of papers do not propose a general framework to design a partially stabilizing controller for nonlinear systems. In [6], the design of a partial controller is done for an Euler dynamical system. The references [5, 7] deal with several types of partial stabilization and control problems, such as permanent rotations of a rigid body, relative equilibrium of a satellite, stationary motions of a gimbaled gyroscope. Application of partial stabilization to achieve chaos synchronization is investigated in [10, 11].

In this article, a general approach for partial control design is proposed. This approach provides the possibility to transform the control problem into a simpler one by reducing the control input variables. For this purpose, the state vector of the system is separated into two parts and accordingly the nonlinear dynamical system is divided into two subsystems. The subsystems, hereafter, are referred to as the *first* and the *second* subsystems. The reduced control input vector (the vector that includes components of input vector which appear in the *first* subsystem) is designed based on new concept of passivity, i.e., *partial passivity* in such a way to guarantee asymptotic stability of the nonlinear system with respect to the first part of state vector.

The concept of passivity and its application in stability have been widely studied in many books and papers [13]– [18]. In this paper, introducing the notion of *partially passive systems*, a new approach for partial stabilization is developed.

The remainder of this paper is arranged as follows. First, the preliminaries on partial stability/control are given in Section 2. In Section 3, the theorems for partial control design are presented and explained in detail. Finally, conclusions are made in Section 4.

2 Preliminaries

In this section, the definitions and notations of partial stability are introduced. Consider a nonlinear system in the form:

$$\dot{x} = f(x), \qquad x(t_0) = x_0, \tag{1}$$

where $x \in R^n$ is the state vector. Let vectors x_1 and x_2 denote the partitions of the state vector, respectively. Therefore, $x = (x_1^T, x_2^T)^T$ where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$ and $n_1 + n_2 = n$. As a result, the nonlinear system (1) can be divided into two subsystems (the *first* and the *second* subsystems) as follows:

$$\begin{aligned} \dot{x}_1(t) &= F_1(x_1(t), x_2(t)), \qquad x_1(t_0) = x_{10}, \\ \dot{x}_2(t) &= F_2(x_1(t), x_2(t)), \qquad x_2(t_0) = x_{20}, \end{aligned} \tag{2}$$

where $x_1 \in D \subseteq R^{n_1}$, D is an open set including the origin, $x_2 \in R^{n_2}$ and $F_1 : D \times R^{n_2} \to R^{n_1}$ is such that for every $x_2 \in R^{n_2}$, $F_1(0, x_2) = 0$ and $F_1(., x_2)$ is locally Lipschitz in x_1. Also, $F_2 : D \times R^{n_2} \to R^{n_2}$ is such that for every $x_1 \in D$, $F_2(x_1, .)$ is locally Lipschitz in x_2, and $I_{x_0} = [0, \tau_{x_0}), 0 < \tau_{x_0} \leq \infty$ is the maximal interval of existence of solution $(x_1(t), x_2(t))$ of (2) $\forall t \in I_{x_0}$. Under these structures, the existence and uniqueness of solution is ensured. Stability of the dynamical system (2) with respect to x_1 can be defined as follows [8]:

Definition 2.1 The nonlinear system (2) is Lyapunov stable with respect to x_1 if for every $\epsilon > 0$ and $x_{20} \in R^{n_2}$, there exists $\delta(\epsilon, x_{20}) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(t)\| < \epsilon$ for all $t \geq 0$. This system is asymptotically stable with respect to x_1, if it is Lyapunov stable with respect to x_1 and for every $x_{20} \in R^{n_2}$, there exists $\delta = \delta(x_{20}) > 0$ such that $\|x_{10}\| < \delta$ implies $\lim_{t\to\infty} x_1(t) = 0$.

Now, in order to analyze partial stability, the following results are taken from [8].

Theorem 2.1 *Nonlinear dynamical system* (2) *is asymptotically stable with respect to* x_1 *if there exist a continuously differentiable function* $V : D \times R^{n_2} \to R$ *and class* K *functions,* $\alpha(.)$ *and* $\gamma(.)$*, such that:*

$$V(0, x_2) = 0, \qquad x_2 \in R^{n_2}, \tag{3}$$

$$\alpha\left(\|x_1\|\right) \leq V(x_1, x_2), \qquad (x_1, x_2) \in D \times R^{n_2}, \tag{4}$$

$$\frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1, x_2) + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1, x_2) \leq -\gamma(\|x\|), \qquad (x_1, x_2) \in D \times R^{n_2}. \tag{5}$$

Proof See [8]. □

Corollary 2.1 [8] *Consider the nonlinear dynamical system* (2)*. If there exist a positive definite, continuously differentiable function $V : D \to R$ and a class K function $\gamma(.)$, such that:*

$$\frac{\partial V(x_1)}{\partial x_1} F_1(x_1, x_2) \leq -\gamma(\|x\|), \qquad (x_1, x_2) \in D \times R^{n_2}, \tag{6}$$

then the nonlinear system (2) *is asymptotically stable with respect to x_1.*

Now, consider the following autonomous nonlinear control system:

$$\begin{aligned} \dot{x}_1(t) &= F_1(x_1, x_2, u(x_1, x_2)), \qquad x_1(t_0) = x_{10}, \\ \dot{x}_2(t) &= F_2(x_1, x_2, u(x_1, x_2)), \qquad x_2(t_0) = x_{20}, \end{aligned} \tag{7}$$

where $u \in R^m$ and $F_1 : D \times R^{n_2} \times R^m \to R^{n_1}$ is such that for every $x_2 \in R^{n_2}$, $F_1(0, x_2, 0) = 0$ and also $F_1(., x_2, .)$ is locally Lipschitz in x_1 and u. Also $F_2 : D \times R^{n_2} \times R^m \to R^{n_2}$ is such that for every $x_1 \in D$, $F_2(x_1, ., .)$ is locally Lipschitz in x_2 and u. These assumptions guarantee the local existence and uniqueness of the solution of the differential equations (7).

Definition 2.2 The nonlinear control system (7) is said to be asymptotically stabilizable with respect to x_1 if there exists some admissible feedback control law $u = k(x_1, x_2)$, which makes system (7) asymptotically stable with respect to x_1.

3 An Approach for Partial Control Design

Suppose that $\dot{x}_1$-*equation* in (7) is affine with respect to control input (the second subsystem may have the general dynamical form):

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1, x_2) + \sum_{i=1}^{m} g_{1i}(x_1, x_2) u_i, \\ \dot{x}_2(t) &= F_2(x_1, x_2, u), \end{aligned} \tag{8}$$

where u_i is the i^{th} component of input vector u. Also, g_{1i}, for $i = 1, 2, ..., m$ are the vectors which belong to R^{n_1}. Let us define:

$$r = \text{number of } (g_{1i} \neq 0)_{i=1,\dots,m},$$

where r indicates the number of control components of input vector which appear in $\dot{x}_1$-*equation.* Thus $0 \leq r \leq m$. Now, with respect to the value of r, two cases may be considered.

3.1 Case 1: $r \neq 0$.

By augmenting the r nonzero vectors g_{1i} in a matrix, i.e., G_1, the nonlinear control system (8) can be rewritten as follows:

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1, x_2) + G_1(x_1, x_2) u_r, \\ \dot{x}_2(t) &= F_2(x_1, x_2, u), \end{aligned} \tag{9}$$

where $u_r \in R^r$ is the reduced version of control input vector u, that contains r control variables appearing in $\dot{x}_1$-*equation*, $G_1(x_1, x_2)$ is a $n_1 \times r$ matrix where its columns are the r nonzero vectors g_{1i}. In this case, the task is to find an appropriate u_r, which guarantees partial stabilization of nonlinear system (9) with respect to x_1. Indeed, instead of design u, we design u_r to achieve partial stability and this approach lead to simplifying the controller design. Before this, some definitions about the new concept of passivity, i.e., partial passivity are introduced.

Definition 3.1 Consider the system (9) with output function (10):

$$y_r = h(x_1, x_2), \tag{10}$$

where $y_r \in R^r$ and h is a continuous function. The system (9)-(10) is partially passive (with respect to input u_r and output y_r) if there exists a continuously differentiable positive semi definite function $V : D \to R$ (called partially storage function) such that

$$u_r^T y_r \geq \dot{V}(x_1), \qquad (x_1, x_2, u_r) \in D \times R^{n_2} \times R^r. \tag{11}$$

Remark 3.1 It is important to note the difference between passive systems which have been proposed in literature and partially passive systems which is introduced in this paper. For this purpose, the definition of passive systems is taken from [13]. Consider the following nonlinear system

$$\dot{x} = f(x, u),$$

$$y = H(x),$$

where $x \in R^n$, $y, u \in R^n$, f is locally Lipshitz in (x, u) and H is continuous. The above system is passive with respect to input u and output y if there exists a continuously differentiable positive semidefinite function $V(x)$ (storage function) such that

$$u^T y \geq \dot{V}(x), \qquad (x, u) \in R^n \times R^m.$$

In Definition 3.1, by dividing the state vector x into two parts x_1 and x_2, the passivity concept only with respect to the first subsystem, i.e., $\dot{x}_1$-*equation* is considered (partial passivity). Also, the partial storage function (in Definition 3.1) is only function of a part of states, i.e., x_1, while the storage function in definition of passive systems is function of all states, i.e., x. In what follows some new lemma and theorems are proposed for partially passive systems.

Lemma 3.1 *Consider the nonlinear system* (9)*. Suppose there exists a positive definite, continuously differentiable function $V : D \to R$ such that:*

$$\frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1, x_2) \leq 0, \qquad (x_1, x_2) \in D \times R^{n_2}. \tag{12}$$

Take virtual output y_r as

$$y_r = h(x_1, x_2) = \frac{\partial V(x_1)^T}{\partial x_1} G_1(x_1, x_2). \tag{13}$$

Then the system (9)-(13) *is partially passive with respect to input u_r and output y_r.*

Proof Consider the following statement

$$u_r^T y_r - \frac{\partial V(x_1)^T}{\partial x_1}(f_1(x_1,x_2) + G_1(x_1,x_2)u_r) = u_r^T h - \frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1,x_2) - h^T u_r. \quad (14)$$

Since $u_r, y_r \in R^r$, thus $u_r^T h = h^T u_r$ are scalar terms. Therefore,

$$u_r^T y_r - \frac{\partial V(x_1)^T}{\partial x_1}(f_1(x_1,x_2) + G_1(x_1,x_2)u_r) = -\frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1,x_2), \quad (15)$$

where according to assumption (12), $\frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1,x_2) \leq 0$, therefore,

$$u_r^T y_r - \frac{\partial V(x_1)^T}{\partial x_1}(f_1(x_1,x_2) + G_1(x_1,x_2)u_r) \geq 0. \quad (16)$$

Consequently,

$$u_r^T y_r \geq \frac{\partial V(x_1)^T}{\partial x_1}(f_1(x_1,x_2) + G_1(x_1,x_2)u_r) = \dot{V}(x_1). \quad (17)$$

Hence, $u_r^T y_r \geq \dot{V}(x_1)$. Thus, by using the function $V(x_1)$ as the partial storage function candidate, the system is partially passive with respect to the input u_r and the output y_r (according to Definition 3.1). □

Theorem 3.1 *Consider the nonlinear dynamical system* (9)*. Suppose there exist a positive definite, continuously differentiable function* $V(x_1) : D \to R$ *and a class* K *function* $\gamma(.)$ *such that:*

$$\frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1,x_2) \leq -\gamma(\|x_1\|), \qquad (x_1,x_2) \in D \times R^{n_2}. \quad (18)$$

Then the state feedback control law (19)*, makes the system* (9) *asymptotically stable with respect to* x_1*.*

$$u_r = -\varphi(h(x_1,x_2)), \quad (19)$$

where $h(x_1,x_2) = \frac{\partial V(x_1)^T}{\partial x_1} G_1(x_1,x_2)$ *and* φ *is any smooth mapping such that* $\varphi(0) = 0$ *and* $h^T\varphi(h) > 0$ *for all* $h \neq 0$ *(It reads a function belonging to the first-third quadrant sector).*

Proof Let us define the virtual output function as follow

$$y_r = h(x_1,x_2) = \frac{\partial V(x_1)^T}{\partial x_1} G_1(x_1,x_2). \quad (20)$$

The derivative of $V(x_1)$ satisfies:

$$\begin{aligned} \dot{V}(x_1) &= \frac{\partial V(x_1)^T}{\partial x_1}\dot{x}_1 \\ &= \frac{\partial V(x_1)^T}{\partial x_1}(f_1(x_1,x_2) + G_1(x_1,x_2)u_r). \end{aligned} \quad (21)$$

Using (18)

$$\begin{aligned}\dot{V}(x_1) &\leq -\gamma\left(\|x_1\|\right) + \frac{\partial V(x_1)^T}{\partial x_1} G_1(x_1, x_2) u_r \\ &= -\gamma\left(\|x_1\|\right) + y_r^T u_r. \end{aligned} \tag{22}$$

Take,

$$u_r = -\varphi(y_r) \tag{23}$$

Therefore, $y_r^T u_r = -y_r^T \varphi(y_r) \leq 0$. As a result,

$$\dot{V}(x_1) \leq -\gamma\left(\|x_1\|\right). \tag{24}$$

Thus, according to Corollary 2.1, the control law (19) makes the nonlinear system (9) asymptotically stable with respect to x_1. □

Remark 3.2 There is great freedom in choosing φ which makes the possibility for u_r to satisfy some constraints. For instance, if u_r is constrained to $|u_{ri}| \leq k_i$ for $1 \leq i \leq r$, then $\varphi_i(y_r)$ can be chosen as $\varphi_i(y_r) = k_i sat(y_{ri})$ or $\varphi_i(y_r) = (2k_i/\pi)\tan^{-1}(y_{ri})$ (where u_{ri}, φ_i and y_{ri} are the i^{th} component of u_r, φ and y_r, respectively).

Remark 3.3 Consider the system (9). If condition (18) was not satisfied, by taking $u_r = \alpha(x_1, x_2) + \beta(x_1, x_2)v_r$, the appropriate functions α and β may be found such that condition (18) be satisfied for $f_{1new} = f_1 + G_1\alpha$. Then, the control law $v_r = -\varphi(h_1)$ may be designed for partial stabilization (where $h_1 = \frac{\partial V(x_1)}{\partial x_1}^T G_{1new} = \frac{\partial V(x_1)}{\partial x_1}^T G_1\beta$)

3.2 Case 2: $r = 0$.

It means that there is no component of control input vector in $\dot{x}_1$-*equation.* Therefore, the nonlinear system (8) can be rewritten as follows:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= F_2(x_1, x_2, u). \end{aligned} \tag{25}$$

In this case, the task is to find an appropriate u; which guarantees partial stabilization of the closed-loop system. Suppose that system (25) has the following structure,

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + G_1(x_1)x_2, \\ \dot{x}_2 &= f_2(x_1, x_2) + G_2(x_1, x_2)u. \end{aligned} \tag{26}$$

This system may be viewed as a cascade connection of two subsystems where x_2 is to be viewed as an input for the *first* subsystem. The system (26) is in the regular form. Assume that x_2 and u both belong to R^m (in other words, $n_2 = m$) and $G_2(x_1, x_2) \in R^{m\times m}$ is a nonsingular matrix. This assumption is not so restrictive and many design methods, which are based on regular forms, e.g., backstepping or sliding mode techniques use such an assumption [13, 14]. In this case, the task is to find an appropriate u; which guarantees partial stabilization of the closed-loop system.

Theorem 3.2 *Consider the nonlinear dynamical system* (26)*. Suppose there exist a positive definite, continuously differentiable function* $V : D \to R$ *and a class* K *function* $\gamma(.)$ *such that*

$$\frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1) \leq -\gamma\left(\|x_1\|\right). \tag{27}$$

Then the state feedback control law (28)*, makes the closed-loop nonlinear system* (26) *asymptotically stable with respect to* x_1 *and*

$$u = G_2^{-1}[-\frac{\partial \varphi(y)}{\partial y}\dot{y} - f_2(x_1, x_2)], \tag{28}$$

where $y = \frac{V(x_1)^T}{\partial x_1}G_1(x_1)$ *and* φ *is any locally Lipschitz function such that* $\varphi(0) = 0$ *and* $y^T\varphi(y) > 0$ *for all* $y \neq 0$*.*

Proof The derivative of $V(x_1)$ is given by

$$\begin{aligned} \dot{V}(x_1) &= \frac{\partial V(x_1)^T}{\partial x_1}\dot{x}_1 \\ &= \frac{\partial V(x_1)^T}{\partial x_1}(f_1(x_1) + G_1(x_1)x_2). \end{aligned} \tag{29}$$

Using (27), we have

$$\dot{V}(x_1) \leq -\gamma(\|x_1\|) + \frac{\partial V(x_1)^T}{\partial x_1}G_1(x_1)x_2. \tag{30}$$

Take,

$$y = \frac{\partial V(x_1)^T}{\partial x_1}G_1(x_1) \tag{31}$$

and

$$x_2 = -\varphi(y). \tag{32}$$

Then

$$\dot{V}(x_1) \leq -\gamma(\|x_1\|) + y^T x_2. \tag{33}$$

Since $y^T x_2 = -y^T\varphi(y) \leq 0$, thus $\dot{V}(x_1) \leq -\gamma(\|x_1\|)$ and according to Corollary 2.1, partial stabilization with respect to x_1 is achieved. Also,

$$\dot{x}_2 = -\frac{\partial \varphi(y)}{\partial y}\dot{y}. \tag{34}$$

In addition, from $\dot{x}_2$-*equation*, one has

$$\dot{x}_2 = f_2(x_1, x_2) + G_2(x_1, x_2)u. \tag{35}$$

Therefore, combination of (34) and (35) results in:

$$u = G_2^{-1}[-\frac{\partial \varphi}{\partial y}\dot{y} - f_2(x_1, x_2)]. \tag{36}$$

This feedback law guarantees partial stabilization of the closed-loop system. □

3.3 Design example

Consider the following system

$$\begin{aligned}\dot{z}_1 &= \frac{z_1^2 z_2}{z_3^2} + z_1 u_1,\\ \dot{z}_2 &= -z_2(1+z_1^2) + z_3 u_2,\\ \dot{z}_3 &= -z_2^2 \sin(z_3) - z_3,\end{aligned} \tag{37}$$

where $z_1, z_2 \in R$ and $z_3 \in [-\pi\ \pi]$. By separating the states into $x_1 = [z_2\ z_3]^T$ and $x_2 = z_1$, one has: $r = 1$ and $u_r = u_2$. The task is to design u_r according to Theorem 3.1 to achieve asymptotic stability with respect to x_1. For this purpose, first the condition (18) should be checked. By choosing $V(x_1) = \frac{1}{2}x_1^T x_1 = \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2$, one has:

$$\begin{aligned}\frac{\partial V(x_1)^T}{\partial x_1} f_1(x_1, x_2) &= [z_2\ z_3]\begin{bmatrix} -z_2(1+z_1^2) \\ -z_2^2\sin(z_3) - z_3\end{bmatrix}\\ &= -z_2^2(1+z_1^2) - z_2^2 z_3 \sin(z_3) - z_3^2\\ &= -z_2^2 - z_2^2 z_1^2 - z_2^2 z_3 \sin(z_3) - z_3^2\\ &\le -z_2^2 - z_3^2.\end{aligned} \tag{38}$$

Therefore, condition (18) is satisfied for $\gamma(\|x_1\|) = x_1^T x_1 = z_2^2 + z_3^2$. Now, by choosing h as,

$$\begin{aligned}h(x_1, x_2) &= \frac{\partial V(x_1)^T}{\partial x_1} G_1(x_1, x_2)\\ &= [z_2\ z_3]\begin{bmatrix} z_3 \\ 0 \end{bmatrix}\\ &= z_2 z_3.\end{aligned} \tag{39}$$

Then, the reduced input vector may be designed as

$$u_r = -\varphi(z_2 z_3), \tag{40}$$

where φ is any locally Lipschitz function such that $\varphi(0) = 0$ and $h^T\varphi(h) > 0$ for all $h \neq 0$. For example, by choosing $\varphi(h) = h$, then $u_r = -z_2 z_3$ which guarantees partial stabilization of system (37) with respect to x_1.

4 Conclusion

In this paper, a new approach for partial stabilization of nonlinear systems was proposed and it was shown that in this approach the controller synthesis can be simplified by reducing its variables. The reduced input vector was designed based on new introduced partial passivity concept. In the proposed design method, a virtual output with the same dimension as the reduced input vector was designed such that the nonlinear system was partially passive with respect to the reduced input vector and the virtual output vector. Then, the feedback law was designed as a first-third quadrant sector function of virtual output vector and it was shown that this law guarantees partial stabilization of the nonlinear system.

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Optical Soliton in Nonlinear Dynamics and Its Graphical Representation

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Abstract: The soliton arising from a robust balance between dispersion and nonlinearity is the solitary wave that maintains its shape while it travels at constant speed. The fiber Optical soliton in media and communication with quadratic nonlinearity and frequency dispersion are theoretically analyzed. The behavior of soliton solutions in the form of KdV partial differential equation have been investigated in the fiber optics solitons theory in communication engineering. In this study optical soliton is studied with illustrated graphical representation.

Keywords: *soliton solution, Korteweg-de Vries equation, Gaussian white noise, stochastic KdV equation, Fourier transform, nonlinear dynamics.*

Mathematics Subject Classification (2000): 35C08, 37K40, 35Q51.

1 Introduction

In recent years there have been important and tremendous developments in the study of nonlinear waves and a class of nonlinear wave equations which arise frequently in many engineering applications. The wide interest in this field comes from the understanding of special waves called *solitons* and the associated development of a method of solution to a class of nonlinear wave equations termed as the nonlinear Korteweg and de Vries (KdV) equation. A soliton phenomenon is an attractive field of present day research not only in nonlinear physics and mathematics but also in nonlinear dynamics and system engineering, specially in fiber optics and communication engineering. The soliton phenomenon was first pioneered by John Scott Russel in 1884, while he was conducting experiments on the Union Canal (near Edinburgh) to measure the relationship between the speed of a boat and its propelling force. Russel demonstrated the following findings as an independent dynamic entity moving with constant shape and speed:

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- (i) Solitary waves have the shape $h \sec h^2[k(x - vt)]$;
- (ii) A sufficiently large initial mass of water produces two or more independent solitary waves;
- (iii) Solitary waves cross each other without change of any kind;
- (iv) A wave of height h and travelling in a channel of depth d has a velocity given by the expression $v = \sqrt{g(d + h)}$ (where g is the acceleration of gravity) implying that a large amplitude solitary wave travels faster than one of low amplitude.

In 1895, Korteweg and de Vries published a theory of shallow water waves that reduced Russell's problem to its essential features (see [10] for details). However, the paper by Korteweg and de Vries was one of the first theoretical treatment in the soliton solution and thus a very important milestone in the history of the development of soliton theory. Another development of the 1960s was Toda's discovery of exact two-soliton interactions on a nonlinear spring-mass system (see for example [23]). The brief discussion of mathematical representation of soliton begins with the Wadati's paper published in 1983 [24]. Russel L. Herman a famous Mathematician improved soliton theory and found some improved results which represent a final solution of soliton [8]. Basically Wadati and Herman both used a non-linear third order partial differential equation known as Korteweg-de Vries equation, they started from this equation and finally gave mathematical assumption of soliton with graphical representation. We refer readers to [5, 7, 9, 10, 13, 17, 20, 23, 25] and the references therein for the detail studies about the history of soliton theory and KdV partial differential equation which is the basic foundation of solitons.

One of the active area of applications of solitons is fiber optics. Much experimentation has been done using solitons in fiber optics applications. In 1973, Robin Bullough [4] showed that solitons could exist in optical fibers while he was presenting the first mathematical report of the existence of optical solitons. He also proposed the idea of a soliton-based transmission system to increase performance of optical telecommunications. Now soliton is an essential tool in communication engineering. Recently the fiber optical soliton is dominating to the global telecommunication research by super performance data transmission in a long distances. See for examples [14,18,19] for more studies on solitons in communication systems. There are varieties of nonlinear equations representing the solitons in the nonlinear domain such as, general equal width wave equation (GEWE), general regularized long wave equation (GRLW), general Kortewegde Vries equation (GKdV), general improved Korteweg-de Vries equation (GIKdV), and Coupled equal width wave equations (CEWE), which are the important soliton equations. See for examples [1, 2] for more details about different varieties. Our aim is to investigate some aspects of Kortewegde Vries equation in soliton physics specially for the case of optical soliton in nonlinear dynamical systems in mathematical physics. We also provide an illustration with some graphical representations.

2 Analysis and Formulation of Soliton Solution

We consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where $u = u(x,t)$ which describes the elongation of the wave at the place x at time t. This equation is known as the KdV equation first derived in 1885 by Korteweg and de Vries to describe long-wave propagation on shallow water. But until recently its properties were not well understood [13]. However, the nonlinear shallow water wave equation can be written in the form

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\eta \frac{\partial \eta}{\partial x} + \frac{2}{3}\frac{\partial \eta}{\partial x} + \frac{1}{3}\sigma \frac{\partial^3 \eta}{\partial x^3}}, \tag{2}$$

where $\sigma = \frac{h^3}{3} - \frac{Th}{g\rho}$, h is the channel height, T is the surface tension, g is the gravitational acceleration and ρ is the density.The solutions to (1) are called *Solitons* or *Solitary waves.*

The nondispersive nature of the soliton solutions to the KdV equation arises not because the effects of dispersion are absent but because they are balanced by nonlinearities in the system. The presence of both phenomena can be appreciated by considering simplified versions of the KdV equation which can be calculated by eliminating the *nonlinear term* $u\frac{\partial u}{\partial x}$ as

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{3}$$

The equation (3) is now a linear version and the most elementary wave solution of this equation, called the harmonic wave is given by

$$u(x,t) = A \exp|i(kx + \omega t)|, \tag{4}$$

where k is the wave number and ω is the angular frequency. In order for the displacement $u(x,t)$ presented in equation (4) to be a solution of equation (3), ω and k must satisfy the relation

$$\omega = k^3. \tag{5}$$

The relation (5) is known as dispersion relation and it contains all the characteristics of the original differential equation. Two important concepts connected with the dispersion relation are called the *phase velocity* $v_p = \frac{\omega}{k}$ and the *group velocity* $v_g = \frac{\partial \omega}{\partial k}$. The phase velocity measures how fast a point of constant phase is moving, while the group velocity measures how fast the energy of the wave moves. The waves described by equation (3) are said to be dispersive because a wave with large k will have larger phase and group velocities than a wave with small k.

Now eliminating the *dispersive term* $\frac{\partial^3 u}{\partial x^3}$, we obtain the simple nonlinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{6}$$

admitting the wave solution in the form $u(x,t) = f(x - ut)$, where the function f is arbitrary. For such kind of waves, the important thing to note is that the velocity of a point of constant displacement u is equal to that displacement. As a result, the wave breaks; that is, portions of the wave undergoing greater displacements move faster than, and therefore overtake, those undergoing smaller displacements. This multivaluedness is a result of the nonlinearity and, like dispersion, leads to a change in form as the wave propagates. We refer readers to [6, 11, 15, 16, 21, 22] for further readings as well as recent developments on nonlinear dynamics and stability analysis in the system theory.

2.1 Mathematical derivation of KdV

We recall that KdV equation is the basic foundation of soliton solution. So we present here the brief sketch of calculation for the derivation of KdV equation in the form (1). In order to derive (1), we consider another nonlinear partial differential equation, called *Kadomtsev-Petviashvili equation* (or simply the KP equation) in two spatial and one temporal coordinate which describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate (see for details [3]).The normalized form of the equation is as follows:

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) \pm u_{yy} = 0, \tag{7}$$

where u_t, u_x, u_{xxx} and u_{yy} stand for the partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^3 u}{\partial x^3}$ and $\frac{\partial^2 u}{\partial y^2}$ respectively. The equation (7) can be calculated as

$$u_{xt} = u_{xxx} + 3u_{yy} - 6u_y u_{xx} - 6{u_x}^2 u_{xx}. \tag{8}$$

Thus the equation (8), after making a detailed calculation can be simplified as

$$u_t + 6uu_x + u_{xxx} = \xi(t). \tag{9}$$

Here $\xi(t)$ represents a time dependent *Gaussian white noise* . The stochastic process is called Gaussian white noise if its statistical average is zero i.e.; $\langle\xi(t)\rangle = 0$. See [7] for more details about Gaussian white noise.

Now a relation between two covariance functions in terms of Gaussian white noise is given by

$$\xi(t)\xi(T+t) = \sigma^2\delta(t). \tag{10}$$

For the Fourier transformation of stationary two times covariance function we obtain

$$\begin{aligned} F(\omega) &= \int\ dt\langle\xi(t)\xi(T+t)\rangle e^{i\omega T} \\ &\Longrightarrow F(\omega) = \sigma^2 \int\ dt\delta(t)e^{i\omega T} \\ &\Longrightarrow F(\omega) = \sigma^2. \end{aligned} \tag{11}$$

In other words, it is clear from the above that it does not depend upon ω because there is no co-relation in time. This is why it is called white noise.

Now for simplicity, let us assume a one-dimensional stochastic differential equation with additive noise,

$$\frac{dx(t)}{dt} = a(x(t),t) + \eta(t). \tag{12}$$

Here $a(x(t),t)$ is a Langiven (see for example [12]) equation which can be interpreted as a deterministic or average drift term perturbed by a noisy diffusion term $\xi(t)$. For the increase dx during a time step dt, we get

$$\begin{aligned} dx(t) &= a(x(t),t)dt + d\omega(t), \\ d\omega(t) &= \int_t^{t+dt} \eta(t')dt' \end{aligned} \tag{13}$$

and also we assume that

$$\begin{aligned} d\omega(t^2) &= \int_t^{t+dt} dt_1 \int_t^{t+dt} dt_2 \langle \eta(t_1)\eta(t_2)\rangle \\ &= \int_t^{t+dt} dt_1 \int_t^{t+dt} dt_2 \sigma^2 \delta(t-t') \\ &= \sigma^2 dt. \end{aligned} \tag{14}$$

Thus in the intervals $[t, t+dt]$ and $[t', t'+dt']$ which is a true successive step, we get

$$\langle d\omega\omega(t) d\omega' \rangle = 2\sigma\sigma(t-t'). \tag{15}$$

If $\delta = \varepsilon$ then in general we can write

$$\langle \xi(t)\xi(t')\rangle = 2\sigma\sigma(t-t'). \tag{16}$$

For such time dependent noise the stochastic equation can be transformed into unperturbed KdV equation [10] in the form

$$U_T + 6UU_X + U_{XXX} = 0. \tag{17}$$

Let us now introduce the Galilean transformation

$$\begin{cases} u(x,t) = U(X,T) + \omega(T), \\ X = x + m(t), \\ T = t, \\ m(t) = -\sigma \displaystyle\int_0^t \omega(t')dt'. \end{cases} \tag{18}$$

Under the above transformation we have from calculus that the derivatives transform as

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T} = \frac{\partial}{\partial X}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T} = -\sigma\omega(T)\frac{\partial}{\partial X} + \frac{\partial}{\partial T}. \tag{19}$$

Now using this transformation we have

$$\begin{aligned} \varepsilon(t) &= u_t + 6uu_x + u_{xxx} \\ &= (U+\omega)_T - 6\omega U_X + 6(U+\omega)U_X + U_{XXX} \\ &= U_T + 6UU_X + U_{XXX} + \omega_T. \end{aligned} \tag{20}$$

Let us now define

$$\xi = \omega_T \quad \text{or} \quad \omega(t) = \int_0^t \xi(t')dt'$$

which leads to the KdV equation.

A remarkable property of the KdV equation is that dispersion and nonlinearity balance each other and allow wave solutions that propagate without changing its form. An example of such a solution is one-soliton solution.

We next consider one-soliton solution. In such case, let us consider

$$U(X,T) = 2\eta\eta \sec^2(\eta(X - 4\eta^2 T - X_0)). \tag{21}$$

Then the above mentioned transformation leads directly to an exact solution of stochastic KdV equation

$$\begin{aligned} &u(x,t) = 2\eta^2 \sec h^2(\eta(x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt')) + \omega(t) \\ &\Longrightarrow \langle u(x,t)\rangle = 2\eta^2 \langle \sec h^2(\eta\eta(x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'))\rangle. \end{aligned} \tag{22}$$

Formally we can write

$$\sec h^2 z = \frac{4}{(e^z + e^{-z})^2} = \frac{4e^{-2|z|}}{(1 + e^{-2|z|})^2}. \tag{23}$$

Then according to [24], it can be presented by computing

$$\begin{aligned} \langle u(x,t)\rangle &= 8\eta^2 \textstyle\sum_{n=1}^{\infty} (-1)^{n+1} \langle \exp[2n\eta(x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'))\rangle \\ &= -2\frac{d}{dz}\frac{1}{1+e^{2z}} \\ &= -2\frac{d}{dz}\Big(\sum_{n=0}^{\infty}(e^{2z})^n\Big) = 2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{2nz}. \end{aligned} \tag{24}$$

In order to complete this composition some following useful relations (we are omitting the details) are needed. We have

- $\langle \omega(t)\rangle = 0$,
- $\langle \omega(t_1)\omega(t_2)\rangle = 2\varepsilon \min(t_1, t_2)$,
- $\langle \exp(c\omega(t))\rangle = \exp\Big(\frac{1}{2}c^2\langle \omega^2(t)\rangle\Big)$.

Applying this result, we get

$$\begin{aligned} \langle \exp\Big(\pm 12n\eta \int_0^t \omega(t')dt'\Big)\rangle &= \exp\Big(72n^2\eta^2 \int_0^t \int_0^t \langle \omega(t_1)\omega(t_2)dt_1 dt_2\rangle\Big) \\ &= \exp(48n^2\eta^2\varepsilon t^3). \end{aligned} \tag{25}$$

This leads to the following form

$$\begin{aligned} &\langle u(x,t)\rangle = 8\eta^2 \sum_{n=1}^{\infty}(-1)^{n+1} n e^{na+nb^2}, \\ &a = 2\eta(x - x_0 - 2\eta^2 t), \quad b = 48\eta^2\varepsilon t^3. \end{aligned} \tag{26}$$

In principle this result should be sufficient but we will go further to find analytically that gives an expression to this result.

Now differentiating the series with respect to a and b we obtain the partial differential equation

$$w_b = w_{aa}, \quad \text{where} \quad w(a,b) = \langle u(x,t)\rangle.$$

Furthermore we have $w(a,0) = 2\eta^2 \sec h^2 \frac{a}{2}$.

This is an initial value problem for the heat or diffusion equation on the real line. To solve this problem we use Fourier transformation. The Fourier transform is defined by

$$\tilde{w}(k,b) = \int_{-\infty}^{\infty} w(a,b) e^{-iak} da,$$

and the inverse transform is

$$w(a,b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{w}(k,b) e^{iak} dk.$$

The heat equation leads to the simple initial value problem

$$\tilde{w}_b = -k^2 \tilde{w} \quad \text{where} \quad \tilde{w}(k,0) = 2\eta^2 \int_{-\infty}^{\infty} \sec h^2 \frac{a}{2} e^{-iak} da = 8\eta^2 \frac{\pi k}{\sinh \pi k}.$$

Therefore, we obtain

$$\tilde{w}(k,b) = 8\eta^2 \frac{\pi k}{\sinh \pi k} e^{-bk^2}. \tag{27}$$

Thus the solution is found out from inverse Fourier transform as

$$u(x,t) = \frac{4\eta^2}{\pi} \int_{-\infty}^{\infty} \frac{\pi k}{\sinh \pi k} e^{iak - bk^2} dk. \tag{28}$$

However, the technique followed in [24] asserts that this simply can be calculated using the convolution theorem. Namely we note that

$$\tilde{w}(k,b) = \tilde{f}(k)\tilde{g}(k,b) \quad \text{for} \quad \tilde{f}(k) = 8\eta^2 \frac{\pi k}{\sinh \pi k} \quad \text{and} \quad \tilde{g}(k,b) = e^{-bk^2}.$$

The inverse transforms for these expressions are given by

$$f(a) = 2\eta^2 \sec h^2 \frac{a}{2} \quad \text{and} \quad g(a,b) = \frac{1}{\sqrt{4\pi b}} e^{\frac{a^2}{4b}}.$$

The last expression is just the statement for Fourier transformation of a Gaussian. Now from the Gaussian Convolution of the functions, we have

$$\begin{aligned} \langle u(x,t) \rangle = w(a,b) = (f * g)(a) &= \int_{-\infty}^{\infty} f(s) g(a-s) ds \\ &= \int_{-\infty}^{\infty} \Big(2\eta^2 \sec h^2 \frac{s}{2}\Big)\Big(\frac{1}{\sqrt{4\pi b}} e^{\frac{-(a-s)^2}{4b}}\Big) ds \\ &= \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{\frac{-(a-s)^2}{4b}} \sec h^2 \frac{s}{2} ds. \end{aligned} \tag{29}$$

This is the exact solution of stochastic KdV equation which we will now compare to any simulation results. The result of simulation solution of stochastic KdV equation is given in Figure 1.

Most of the focuses of any simulations are with respect to the asymptotic results that Wadati derived from the above solution. We will discuss the graphical representations of the behavior of soliton solution in two cases, namely, for small times and for large times.

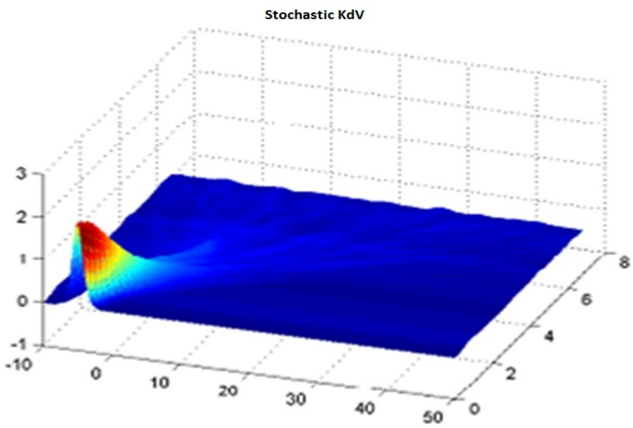

Figure 1: This graph represents the solution generated by doing a simulation of the stochastic KdV equation.

Case I: *For small times* $(e.g., b = 48\eta^2\varepsilon t^2 < 1)$.
In this case, it is a simple matter to show that

$$\langle u(x,t)\rangle = 2\eta^2 \sum_{0}^{\infty} \frac{b^n}{n!} \frac{\partial^{2n}}{\partial a^{2n}} \sec h^2 \frac{a}{2}. \tag{30}$$

Now using the equation(30) the numerically illustrated exact solution is given below in Figure 2.

We are now in a position to present here a comparison result between the simulation result of the stochastic KdV equation shown in Figure 1 and the result of exact solution shown in Figure 2. The comparison result is shown in the following Figure 3.

Case II: *For large times* $(e.g., b = 48\eta^2\varepsilon t^2 > 1)$.
In this case, the solution can be calculated as

$$\langle u(x,t)\rangle = \frac{4\eta^2}{\sqrt{\pi}}\Big(1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2)bB_n\pi^{2n}}{(2n)!} \frac{\partial^n}{\partial b^n}\Big) \frac{e^{\frac{-a^2}{4b}}}{\sqrt{b}} \tag{31}$$

and also the numerical simulation for this case is presented in Figure 4.

Now we will focus on the case when $t \longrightarrow \infty$ and the result can be approximated as

$$\langle u(x,t)\rangle \approx \frac{\eta}{\sqrt{3\pi\varepsilon}} \frac{1}{\sqrt[2]{t^3}} \exp\Big(-\frac{(x-x_0-4\eta^2 t)^2}{48\varepsilon t^3}\Big) \tag{32}$$

and the simulation result of equation (32) is presented in Figure 5.

Once again we present in Figure 6, the amplitude of the two solutions shown in the Figure 4 and in Figure 5.

Thus we end this section providing an extensive graphical illustrations about the shape and behavior of soliton solutions derived from the KdV equation. We also present

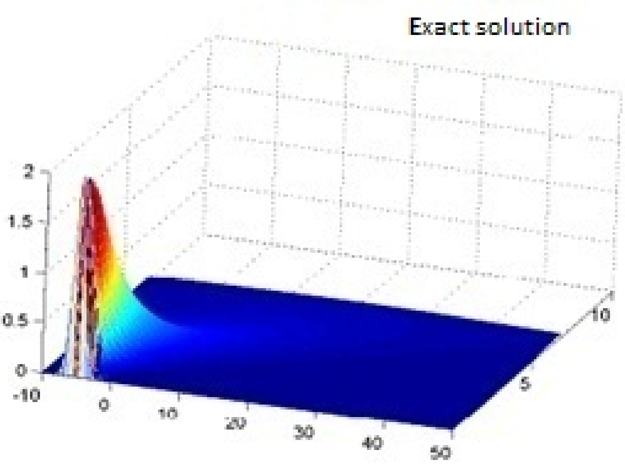

Figure 2: This graph represents the exact solution of the equation (30).

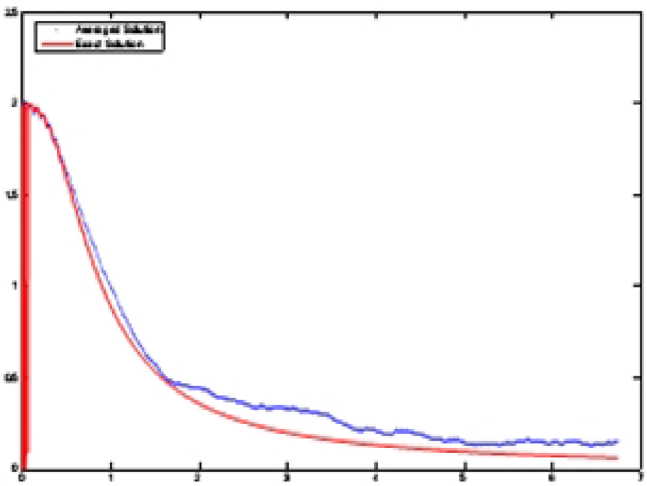

Figure 3: The physical graph of comparison of the amplitudes from the exact solution and a simulation solution.

the comparison result derived from the exact solution of stochastic KdV equation as well as the simulation result.

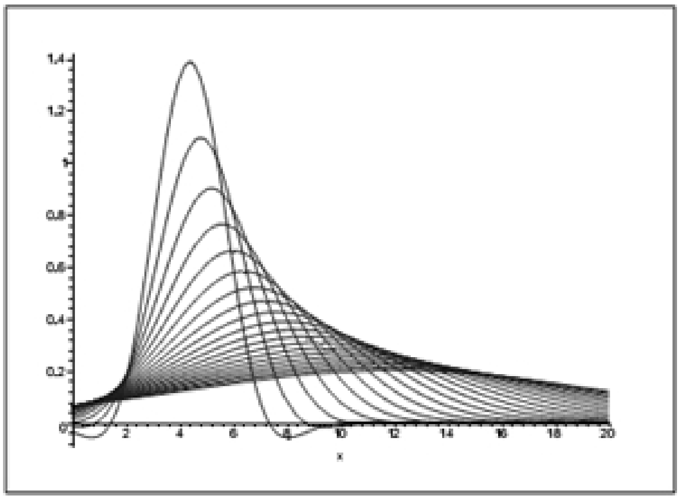

Figure 4: The graph for large times based on the solution using equation (31).

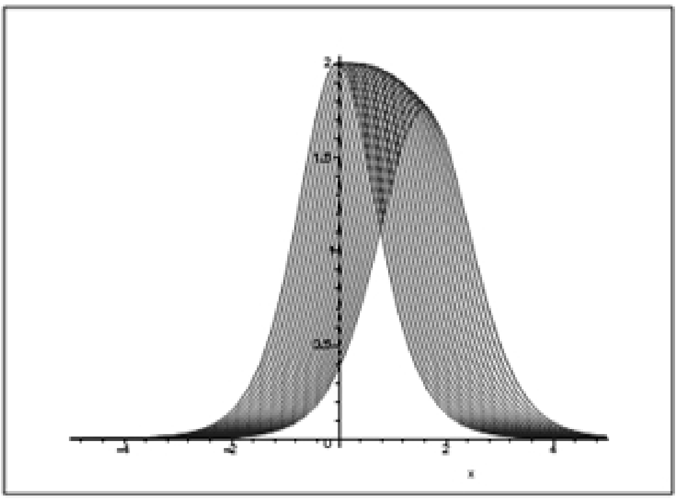

Figure 5: The solution for large times based upon equation (32).

3 A Particular Problem and Solution

In this section we will discuss the solution of a particular problem.There are several problems with Wadati's derivation. These also appear elsewhere in the literature references to Wadati's paper [24]. Here we discuss the alternatives of several shortcomings to Wadati's derivation by this particular problem.

First, we note that the series expansion for the $\sec h^2 z$ is not quite right. We should

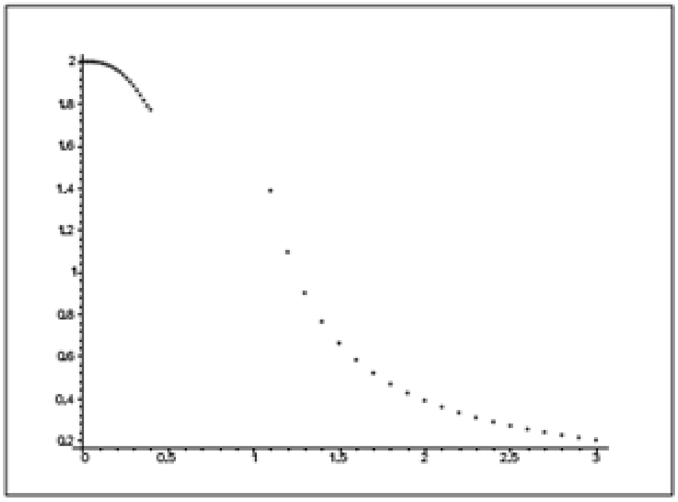

Figure 6: The amplitude of the solutions provided in the Figure 4 and Figure 5.

instead have derived it as follows (for $z \neq 0$)

$$\begin{aligned} \sec h^2 z &= \frac{4}{(e^z + e^{-z})^2} = \frac{4e^{-2|z|}}{(1 + e^{-2|z|})^2} \\ &= 2 \text{ sgn } \frac{d}{dz}\Big(\sum_{n=0}^{\infty}(-e^{-2|z|})^n\Big) = 2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{-2n|z|} \end{aligned} \tag{33}$$

This accounts for the convergence of the geometric series used in the derivation. Namely, in the original derivation, one should have noted that $|e^{2z}| < 1$ or $z < 0$.

This new derivation accounts for the case $z > 0$. Konotop and Vazquez [9] used this in their review of Wadati's derivation. They presented the infinite series result as

$$\langle u(x,t)\rangle = 8\eta^2 \sum_{n=0}^{\infty}(-1)^{n+1} n e^{-n|a|+n^2 b}. \tag{34}$$

There also appeared to be a problem with the derivation of the average, where Wadati should actually have computed

$$\langle u(x,t)\rangle = 8\eta^2 \sum_{n=0}^{\infty}(-1)^{n+1} n \Big\langle \exp\Big(- 2n\eta|x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'|\Big)\Big\rangle. \tag{35}$$

One could get around this problem by computing the average for space-time regions where $x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'$ is definitely of one sign. Another approach would instead be directly expanded as

$$u(x,t) = 2\eta^2 \sec h^2\Big(\eta(x - 4\eta^2 t - x_0 - 4\eta^2 t) - 6\eta\int_0^t \omega(t')dt'\Big) = 2\eta^2 \sec h^2(\theta + \sigma). \tag{36}$$

In the case when $\sigma = 0$ for $\sigma = -6\eta \int_0^t \omega(t')dt'$, we have

$$2\eta^2 \sec h^2(\theta + \sigma) = 2\eta^2 \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^n}{\partial \theta^n} \sec h^2 \theta. \tag{37}$$

The average can now be computed as

$$\langle u(x,t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \frac{\langle \sigma^n \rangle}{n!} \frac{\partial^n}{\partial \theta^n} \sec h^2 \theta \tag{38}$$

provided that we can compute $\langle \sigma^n \rangle$ as

$$\langle \sigma^n \rangle = \Big\langle \Big(-6\eta \int_0^t \omega(t')dt' \Big)^n \Big\rangle. \tag{39}$$

Herman [8] showed that such averages can be computed based upon the nature of the Gaussian noise as

$$\langle \sigma^n \rangle = \begin{cases} 0, & \text{when n is odd,} \\ (2l-1)!\langle \sigma^2 \rangle, & \text{when n=2l is even.} \end{cases}$$

Thus, we just need to compute $\langle \sigma^2 \rangle$ which can be completed by the following calculation

$$\begin{aligned} \langle \sigma^2 \rangle &= \Big\langle 36\eta^2 \int_0^t \omega(t_1)dt_1 \int_0^t \omega(t_2)dt_2 \Big\rangle = 72\varepsilon\eta^2 \int_0^t \int_0^t \min(t_1,t_2)dt_1dt_2 \\ &= 72\varepsilon\eta^2 \int_0^t \Big(\int_0^{t_2} \min(t_1,t_2)dt_1 + \int_{t_2}^t \min(t_1,t_2)dt_1 \Big) dt_2 \\ &= 72\varepsilon\eta^2 \int_0^t \Big(\frac{{t_2}^2}{2} + t_2(t-t_2) \Big) dt_2 = 24\varepsilon\eta^2 t^3. \end{aligned} \tag{40}$$

Now inserting this result in equation (38) we obtain

$$\langle u(x,t) \rangle = 2\eta^2 \sum_{t=0}^{\infty} \frac{\langle 12\varepsilon\eta^2 t^3 \rangle}{l!} \frac{\partial^{2l}}{\partial \theta^{2l}} \sec h^2 \theta. \tag{41}$$

In order to see the agreement with Wadati's result for small $b = 48\varepsilon\eta^2 t^3$, we need to set $\theta = \frac{a}{2}$. We also note that $\frac{\partial^{2l}}{\partial \theta^{2l}} = 2^{2l} \frac{\partial^{2l}}{\partial a^{2l}}$. Thus we obtain

$$\langle u(x,t) \rangle = 2\eta^2 \sum_{t=0}^{\infty} \frac{\langle 48\varepsilon\eta^2 t^3 \rangle}{l!} \frac{\partial^{2l}}{\partial a^{2l}} \sec h^2 \frac{a}{2} = 2\eta^2 \sum_{t=0}^{\infty} \frac{b^l}{l!} \frac{\partial^{2l}}{\partial a^{2l}} \sec .h^2 \frac{a}{2} \tag{42}$$

We further note that this solution again satisfies the heat equation and that for $b = 0$ this solution reduces to the soliton initial condition. Thus, we have seemingly bypassed any problem with the computing the average with an absolute value. However, this series is divergent for $b > 1$.

4 Conclusion

Soliton theory has been a challenging area of research over the years, especially since the mid-1970s, the soliton concept has become established in several areas of applied science and its applications in the diverse fields of science and engineering such as nonlinear analysis, water waves, relativistic and quantum field theory, control and system theory as well as electrical and communication engineering have made this theory more attractive. In this study the soliton solution and some of its large scale applications are studied with simulations. The mathematical derivation of soliton is shown by using the Korteweg-de Vries equation and Kadomtsev–Petviashvili equation in the form of nonlinear partial differential equation. It is necessary to mention, however, that not all nonlinear partial differential equations have soliton solutions. Those that do are generic and belong to a class for which the general initial-value problem can be solved by a technique called the inverse scattering transform, a brilliant scheme developed by Kruskal and his coworkers in 1965. We also investigate the result of the solution which is generated by doing a simulation of the stochastic KdV equation and the exact solution and a comparison between them is shown graphically. Also the solutions based on small and large times are represented graphically.

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Existence and Uniqueness of Solutions to Quasilinear Integro-differential Equations by the Method of Lines

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Abstract: In this work we consider a class of quasilinear integro-differential equations. We apply the method of lines to establish the wellposedness for a strong solution. The method of lines is a powerful tool for proving the existence and uniqueness of solutions to evolution equations. This method is oriented towards the numerical approximations.

Keywords: *method of lines; integro-differential equation; semigroups; contractions; strong solution.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

Let X and Y be two real reflexive Banach spaces such that Y is densely and compactly embedded in X. In the present analysis we are concerned with the following quasilinear integro-differential equation

$$\begin{cases} \frac{du}{dt}(t) + A(t,u(t))u(t) = \int_0^t k(t,s)A(s,u(s))u(s)ds + f(t,u_t),\ 0 < t \leq T, \\ u_0 = \phi \in C([-T,0],X), \end{cases} \tag{1}$$

where $A(t,u)$ is a linear operator in X, depending on t and u, defined on an open subset W of Y. We denote by $J=[0,T]$, k is a real valued function defined on $J \times J \to \mathbb{R}$ and f is defined from $J \times C([-T,0],X)$ into Y. Here $C([a,b],Z)$, for $-\infty \leq a \leq b < \infty$, is the

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Banach space of all continuous functions from $[a,b]$ into Z endowed with the supremum norm

$$\|\chi\|_{C([a,b],Z)} := \sup_{a \le s \le b} \|\chi(s)\|_Z, \quad \chi \in C([a,b],Z).$$

For $u \in C([-T,t],X)$, we denote by $u_t \in C([-T,0],X)$ a history function defined by

$$u_t(\theta) = u(t+\theta), \quad \theta \in [-T,0].$$

By a strong solution to (1) on $[0,T'],\ 0 < T' \le T$, we mean an absolutely continuous function u from $[-T,T']$ into X such that $u(t) \in W$ with $u_0 = \phi$ and satisfies (1) almost everywhere on $[0,T']$.

Kato [8] has proved the existence of a unique continuously differentiable solution to the quasilinear evolution equation in X

$$\frac{du}{dt} + A(u)u = f(u), \quad 0 < t \le T, \quad u(0) = u_0, \tag{2}$$

under the assumptions that there exists an open subset W of Y such that for each $w \in W$ the operator $A(w)$ generates a C_0-semigroup in X, $A(\cdot)$ is locally Lipschitz continuous on W from X into X, f defined from W into Y, is bounded and globally Lipschitz continuous from Y into Y, and there exists an isometric isomorphism $S : Y \to X$ such that

$$SA(w)S^{-1} = A(w) + B(w), \tag{3}$$

where $B(w)$ is in the set $B(X)$ of all bounded linear operators from X into X.

Crandall and Souganidis [6] have established the existence of a unique continuously differentiable solution to the quasilinear evolution equation (2) with $f = 0$ under more general assumptions on $A(w)$. Kato [10] has proved the existence of a strong solution to the quasilinear evolution equation

$$\frac{du}{dt} + A(t,u)u = f(t,u), \quad 0 < t \le T, \quad u(0) = u_0, \tag{4}$$

under similar conditions on $A(t,u)$ and $f(t,u)$ as considered by Crandall and Souganidis [6].

Recently Oka [11] has dealt with the abstract quasilinear Volterra integrodifferential equation

$$\begin{cases} \frac{du}{dt}(t) + A(t,u(t))u(t) = \int_0^t b(t-s)A(s,u(s))u(s)ds + f(t),\ t \in [0,T], \\ u(0) = \phi, \end{cases} \tag{5}$$

in a pair of Banach spaces $X \supset Y$, where $b : [0,T] \to \mathbb{R}$ is a scalar kernel and $A(t,w)$ is a linear operator in X, depending on t and w, defined on an open subset W of Y. Oka has proved the existence, uniqueness and continuous dependence on the data.

Our analysis is motivated by the work of Bahuguna [1]. In [1] the author considered the following quasilinear integrodifferential equation in a Banach space

$$\frac{du(t)}{dt} + A(u(t))u(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t),\ 0 < t < T,\ u(0) = u_0, \tag{6}$$

by using the application of Rothe's method, the author has established the existence and uniqueness of a strong solution which depends continuously on the initial data.

We shall use Rothe's method to establish the existence and uniqueness results. Rothe's method, introduced by Rothe [15] in 1930, is a powerful tool for proving the existence and uniqueness of a solution to a linear, nonlinear parabolic or a hyperbolic problem of higher order. This method is oriented towards the numerical approximations. For instance, we refer to Rektorys [14] for a rich illustration of the method applied to various interesting physical problems. It has been further developed for nonlinear differential and Volterra integro-differential equations (VIDEs) see [1–4, 7, 14] and references cited in these papers.

In the present study we extend the application of the method of lines to a class of nonlinear VIDEs. In earlier works on the application of the method of lines to integro-differential equations, only bounded perturbations to the heat equation in the integrands have been dealt with. In the problem considered in our paper we have a differential operator appearing in the integrand and hence we have the case of unbounded perturbation.

2 Preliminaries

Let X and Y be as in the first section. Let Z be either X or Y. We use $\| \|_Z$ to denote the norm of Z and by $B(X,Y)$ the set of all bounded linear maps on X to Y, with associated norm $\| \|_{B(X,Y)}$. We write $B(X)$ for $B(X,X)$ and corresponding norm by $\| \|_{B(X)}$. The domain of the operator T is denoted by $D(T)$. We denote by $\mathrm{C}(J_0, Z)$ and $\mathrm{Lip}(J_0, Z)$ the sets of all continuous and Lipschtz continuous functions from a subinterval J_0 of J into Z, respectively. Let $B_r(z_0, r)$ be the Z-ball of radius r at $z_0 \in Z$, i.e. the set $\{z \in Z \mid \ \|z - z_0\|_Z \leq r\}$.

For a real number $\beta, N(Z, \beta)$ represents the set of all densely defined linear operators L in Z such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I + \lambda L)$ is one to one with a bounded inverse defined everywhere on Z and

$$\|(I + \lambda L)^{-1}\|_{B(Z)} \leq (1 + \lambda\beta)^{-1},$$

where I is the identity operator on Z. The Hille-Yosida theorem states that $L \in N(Z, \beta)$ if and only if $-L$ is the infinitesimal generator of a strongly continuous semigroup $e^{-tL}, \ t \geq 0$ on Z satisfying $\|e^{-tL}\|_{B(Z)} \leq e^{\beta t}, \ t \geq 0$.

A linear operator L on $D(L) \subseteq Z$ into Z is said to be accretive in Z if for every $u \in D(L)$

$$\langle Lu, u^* \rangle \geq 0 \quad \text{for some} \quad u^* \in F(u),$$

where $F : Z \to 2^{Z^*}$, Z^* is the dual of Z

$$F(z) = \{z^* \in Z \mid \langle z, z^* \rangle = \|z\|^2 = \|z^*\|^2\},$$

and $\langle z, f \rangle$ is the value of $f \in Z^*$ at $z \in Z$. If $L \in N(Z, \beta)$ then $(L + \beta I)$ is m-accretive in Z, i.e. $(L + \beta I)$ accretive and the range $R(L + \lambda I) = Z$ for some $\lambda > \beta$. (see corollary 1.3.8 and the remarks preceding it in Pazy [13], p.12). If Z^* is uniformly convex then F is single-valued and uniformly continuous on bounded subsets of Z.

In most of this paper X and Y will be related via a linear isometric isomorphism $S : Y \to X$. We assume, in addition, that the embedding of Y in X is compact and the dual of X^* is uniformly convex. Further, we make the following hypotheses.

(A1) There exists an open subset W of Y and $u_0 \in W$. Furthermore, there exists $\beta \geq 0$ such that $A : [0, T] \times W \to N(X, \beta)$.

(A2) $Y \subseteq D(A(t,w))$, for each $(t,w) \in [0,T] \times W$, which implies that $A(t,w) \in B(Y,X)$ by the closed graph theorem. For each $w \in W$, $t \to A(t,w)$ is continuous in $B(Y,X)$-norm, and for each $t \in [0,T]$, $t \to A(t,w)$ is Lipschitz continuous in the sense that

$$\|(A(t_1,w_1) - A(t_2,w_2))v\|_{B(Y,X)} \leq \mu_A[|t_1 - t_2| + \|w_1 - w_2\|_X]\|v\|_Y,$$

where μ_A is a constant and there exists a constant γ_A such that

$$\|A(t,w)v\|_{B(Y,X)} \leq \gamma_A \|v\|_Y,$$

for all $v \in Y$ and $(t,w) \in [0,T] \times W$.

(A3) There is a family $\{S\}$ of isometric isomorphism Y onto X such that

$$SA(t,w)S^{-1} = A(t,w) + P(t,w),$$

where $P : [0,T] \times W \to B(X)$, $\|P(t,w)\|_{B(X)} \leq \gamma_P$ for $(t,w) \in [0,T] \times W$, with $\gamma_P > 0$, is a constant and

$$\|P(t,w_1) - P(t,w_2)\|_{B(X)} \leq \mu_P \|w_1 - w_2\|_Y, \quad \forall\, w_1, w_2 \in W,$$

where μ_P is a positive constant.

(A4) The function $k : J \times J \to \mathbb{R}$ and $f : J \times C([-T,0],X) \to Y$ satisfy the Lipschitz conditions

$$\begin{aligned} |k(t_2,s) - k(t_1,s)| &\leq L_k|t_2 - t_1|, \\ \|f(t,u) - f(s,v)\|_X &\leq L_f[|t-s| + \|u - v\|_{C([-T,0],X)}], \end{aligned}$$

where L_k and L_f are Lipschitz constant.

For all $u, v \in B_X(u_0, R)$. Let $R > 0$ be such that $W_R = B_Y(u_0, R) \subseteq W$ and let

$$\begin{aligned} R_0 &= \frac{R}{6}(1 + e^{2\theta T})^{-1}, &(7)\\ M_1 &= Tk_T(\gamma_A + \gamma_P C_e)R + L_f[T + \|\tilde{u}_0 - \phi\|_{\mathcal{C}([-T,0],X)}] + \|f(0,\phi)\|_X, &(8)\\ M_2 &= Tk_T(\gamma_A + \gamma_P C_e)R + L_f[T + \|\tilde{u}_{j-1} - \phi\|_{\mathcal{C}([-T,0],X)}] + \|f(0,\phi)\|_X, &(9) \end{aligned}$$

where C_e is a positive embedding constant, $\theta = \beta + \|P\|_X$ and $k_T = \sup_{s,t \in J} |k(t,s)|$. Let $z_0 \in Y$ and $T_0, 0 < T_0 \leq T$ be such that for $i = 1, 2$

$$\begin{aligned} \|Su_0 - z_0\|_X &\leq R_0, &(10)\\ T_0[\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_i] &\leq R_0. &(11) \end{aligned}$$

We notice that (10) and (11) imply that

$$(1 + e^{2\theta T})[\|Su_0 - z_0\|_X + T_0\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_i\}] \leq \frac{R}{3}. \tag{12}$$

We shall use later the following lemma due to Crandall and Souganidis [6].

Lemma 2.1 *Let $S : Y \to X$ be a linear isometric isomorphism, $Q \in N(X, \beta)$, $Y \subset D(Q)$, domain of Q, $P \in B(X)$, the space of all bounded linear operators on X and $SQ = QS + PS$. Set $\theta = \beta + \|P\|_{B(X)}$. Then for every $y \in X$ and $\lambda > 0$ such that $\lambda\theta < 1$, the problem*

$$x + \lambda Qx = y, \quad \tilde{x} + \lambda(Q\tilde{x} + P\tilde{x}) = y,$$

has a unique solution x and $\tilde{x}$ in X. Moreover

$$\|x\|_X \le (1 - \lambda\theta)^{-1}\|y\|_X, \quad \|\tilde{x}\|_X \le (1 - \lambda\theta)^{-1}\|y\|_X,$$

and if $y \in Y$, then $x \in Y$ and

$$\|x\|_Y \le (1 - \lambda\theta)^{-1}\|y\|_Y.$$

We have the following main result.

Theorem 2.1 *Suppose that (A1)-(A4) hold. Then there exists a unique strong solution u to (1) such that $u \in Lip(J_0, X), J_0 = [0, T_0]$. Furthermore, if $v_0 \in B_Y(u_0, R_0)$ then there exists a strong solution v to (1) on $[0, T_0]$ with the initial point $v(0) = \psi$ such that*

$$\|u(t) - v(t)\|_X \le C\|u_0 - v_0\|_X, \quad t \in [0, T_0], \tag{13}$$

where C is positive constant.

3 Construction of the Scheme and the Convergence

To apply Rothe's method, we use the following procedure. For any positive integer n we consider a partition t_j^n defined by $t_j^n = jh$; $h = \frac{T_0}{n}$, $j = 0, 1, 2, \dots, n$. We set $u_0^n = \phi(0)$ for all $n \in N$. Let $w_0^n = Su_0^n$ for $n \ge N$ where N is a positive integer such that $\theta(\frac{T_0}{N}) < \frac{1}{2}$. We consider the following scheme

$$\delta u_j^n + A(t_{j-1}^n, u_{j-1}^n)u_j^n = h\sum_{i=0}^{j-1} k_{ji}^n\, A(t_i^n, u_i^n)u_i^n + f_j, \tag{14}$$

where

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad k_{ji}^n = k(t_j^n, t_i^n) \quad \text{and} \quad f_j^n = f(t_j^n, \tilde{u}_{j-1}^n), \quad 1 \le i \le j \le n.$$

We define $\tilde{u}_0^n(t) = \phi(t)$ for $t \in [-T, 0]$, $\tilde{u}_0^n(t) = \phi(0)$ for $t \in [0, T_0]$ and for $2 \le j \le n$

$$\tilde{u}_{j-1}^n(\theta) = \begin{cases} \phi(t_j^n + \theta), & \theta \le -t_j^n, \\ u_{i-1}^n + (t - t_{j-1})\delta u_i^n, & \theta \in [-t_{j+1-i}^n, -t_{j-i}^n],\ 1 \le i \le j. \end{cases} \tag{15}$$

For notational convenience, we occasionally suppress the superscript n, throughout, C will represent a generic constant independent of j, h and n. Our first result is concerned with the solvability of (14) in W_R.

Lemma 3.1 *For each $n \ge N$, there exists a unique $u_j, j = 1, 2, \dots, n$, in W_R satisfying (14).*

Proof Lemma 2.1 implies that there exists a unique $u_1 \in Y$ such that

$$u_1 + hA(t_0, u_0)u_1 = u_0 + h^2 k_{10} A(t_0, u_0)u_0 + hf_1. \tag{16}$$

Applying S on both the sides in (16) using (A3) and letting $w_1 = Su_1$, we have

$$\begin{aligned}(w_1 - z_0) + hA(t_0, u_0)(w_1 - z_0) &+ hP(t_0, u_0)(w_1 - z_0) \\ &= (w_0 - z_0) - hA(t_0, u_0)z_0 - hP(t_0, u_0)z_0 \\ &\quad + h^2 k_{10}[A(t_0, u_0) + P(t_0, u_0)]w_0 + hSf_1.\end{aligned}$$

The estimates in Lemma 2.1 imply that

$$\|w_1 - z_0\|_X \le (1 - h\theta)^{-1}[\|w_0 - z_0\|_X + h\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_1\}].$$

Since $h\theta < \frac{1}{2}$, we have

$$\|w_1 - z_0\|_X \le e^{2h\theta}[\|w_0 - z_0\|_X + h\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_1\}].$$

Therefore,

$$\|w_1 - z_0\|_X \le (1 + e^{2h\theta})[\|w_0 - z_0\|_X + h\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_1\}] \le R,$$

in view of the estimates (12). Hence, $u_1 \in W_R$. Now, suppose that $u_j \in W_R$ for $i = 1, 2, \ldots, j-1$. Again, Lemma 2.1 implies that for $2 \le j \le n$, there exists a unique $u_j \in Y$ such that

$$u_j + hA(t_{j-1}, u_{j-1})u_j = u_{j-1} + h^2 \sum_{i=0}^{j-1} k_{ji} A(t_i, u_i)u_i + hf_j. \tag{17}$$

Proceeding as before and letting $w_j = Su_j$, we get the estimate

$$\|w_j - z_0\|_X \le e^{2h\theta}[\|w_{j-1} - z_0\|_X + h\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_2\}].$$

Reiterating the above inequality, we get

$$\|w_j - z_0\|_X \le e^{2jh\theta}[\|w_1 - z_0\|_X + jh\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_2\}].$$

Hence

$$\|w_j - z_0\|_X \le (1 + e^{2T\theta})[\|w_1 - z_0\|_X + T_0\{\gamma_A \|z_0\|_Y + \gamma_P \|z_0\|_X + M_2\}] \le R.$$

The above inequality and equations (16) and (17) imply that $u_j \in W_R$ satisfy (14) for $1 \le j \le n, n \ge N$. This completes the proof of the lemma. □

Lemma 3.2 *There exists a positive constant C, independent of j, h and n such that*

$$\|\delta u_j\|_X \le C, \qquad j = 1, 2, \ldots, n;\ n \ge N.$$

Proof In (14) for $j = 1$, we get

$$\delta u_1 + hA(t_0, u_0)\delta u_1 = -A(t_0, u_0)u_0 + hk_{10} A(t_0, u_0)u_0 + f_1.$$

Using Lemma 2.1 we have

$$\|\delta u_1\|_X \le e^{2hT}[(1+hk_T)\gamma_A\|u_0\|_Y + f_T] := C_1,$$

where $f_T = L_f[T + \|\tilde{u}_0 - \phi\|_{\mathcal{C}([-T,0],X)}] + \|f(0,\phi)\|_X$. Now, from (14) for $2 \le j \le n$, we have

$$\begin{aligned}\delta u_j + hA(t_{j-1},u_{j-1})\delta u_j &= \delta u_{j-1} - [A(t_{j-1},u_{j-1}) - A(t_{j-2},u_{j-2})]u_{j-1} \\ &\quad + hk_{jj-1}A(t_{j-1},u_{j-1})u_{j-1} \\ &\quad + h\sum_{i=0}^{j-2}[k_{ji} - k_{j-1i}]A(t_i,u_i)u_i + f_j - f_{j-1}.\end{aligned}$$

Applying Lemma 2.1 and using (A2) and (A4) we get

$$\begin{aligned}\|\delta u_j\|_X &\le e^{2h\theta}\big[(1+\mu_A hR)\|\delta u_{j-1}\|_X + \mu_A hR + h\gamma_A R\{|k_{jj-1}| \\ &\quad + \sum_{i=0}^{j-2}|k_{ji} - k_{j-1j}|\} + \|f_j - f_{j-1}\|_Y\big] \\ &\le e^{2h\theta}\big[(1+\mu_A hR)\|\delta u_{j-1}\|_X + M_3 h + L_f h\|\delta\tilde{u}_{j-1}\|_{\mathcal{C}([-T,0],X)}\big],\end{aligned}$$

where $M_3 = \mu_A R + \gamma_A R(k_T + L_k T) + L_f T$. Denoting by $C_2 = \mu_A R + L_f$, we have

$$\max_{1\le i\le j}\|\delta u_i\|_X \le e^{2h\theta}\big[(1+C_2 h)\max_{1\le i\le j-1}\|\delta u_i\|_X + M_3 h\big].$$

Reiterating the above inequality, we get

$$\max_{1\le i\le j}\|\delta u_i\|_X \le e^{2jh\theta}(1+C_2 h)^j\big[\|\delta u_1\|_X + M_3 T\big],$$

hence

$$\|\delta u_j\|_X \le e^{2(\theta + C_2)T}[C_2 + M_3 T] := C.$$

This completes the proof of the lemma. □

Definition 3.1 We define the Rothe sequence $\{U^n\} \in C([-T,T],Y)$ given by

$$U^n(t) = \begin{cases}\phi(t), & t \in [-T,0],\\ u_{j-1} + \frac{u_j - u_{j-1}}{h}(t - t_{j-1}), & t \in [t_{j-1},t_j],\ j = 1,2,\dots,n.\end{cases} \tag{18}$$

Further, we define a sequence of functions $\{X^n\}$ from $[-T,T]$ into Y given by

$$X^n(t) = \phi(t) \quad \text{for} \quad t \in (-T,0], \quad X^n(t) = u_j \quad \text{for} \quad t \in (t_{j-1},t_j]. \tag{19}$$

Remark 3.1 Each of the functions $\{X^n(t)\}$ lies in W_R for all $t \in (-h,T_0]$ and $\{U^n\}$ is Lipschitz continuous with uniform Lipschitz constant, i.e.,

$$\|U^n(t) - U^n(s)\|_X \le C|t-s|, \quad t,s \in J_0.$$

Furthermore, $\|U^n(t) - X^n(t)\|_X \le \frac{C}{n}$. Also, we define

$$K^n(t) = h\sum_{i=0}^{j-1}k_{ji}A(t_i,u_i)u_i, \quad t \in (t_{j-1},t_j], \tag{20}$$

$$f^n(t) = f(t_j,\tilde{u}^n_{j-1}), \quad t \in (t_{j-1},t_j]. \tag{21}$$

$$A^n(t,u) = A(t_{j-1},u), \quad t \in (t_{j-1},t_j),\ j = 1,2,\dots,n. \tag{22}$$

Lemma 3.3 *Under the given assumptions we have*
(*a*) $\{K^n(t)\}$ *is uniformly bounded;*
(*b*) $\int_0^t A^n(s, X^n(s-h))X^n(s)ds = u_0 - U^n(t) + \int_0^t K^n(s)ds + \int_0^t f^n(s)ds;$
(*c*) $\frac{d^-}{dt}U^n(t) + A^n(t, X^n(t-h))X^n(t) = K^n(t) + f^n(t), \quad t \in (0, T_0],$
where $\frac{d^-}{dt}$ *is the left-derivative.*

Proof (a) This is a direct consequence of the assumptions (A2)-(A4).
(b) For $2 \le j \le n$ and $t \in (t_{j-1}, t_j]$, by Definition 3.1, we have

$$\begin{aligned}
&\int_0^t A^n(s, X^n(s-h))X^n(s)ds \\
&= \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} A^n(s, X^n(s-h))X^n(s)ds + \int_{t_{j-1}}^{t} A^n(s, X^n(s-h))X^n(s)ds \\
&= -\sum_{i=1}^{j-1}(u_i - u_{i-1}) - \frac{1}{h}(t - t_{j-1})(u_j - u_{j-1}) + \ h\sum_{i=1}^{j-1}\left[h\sum_{p=0}^{i-1} k_{ip}A(t_p, u_p)u_p\right] \\
&\quad + (t - t_{j-1})[h\sum_{p=0}^{j-1} k_{ip}A(t_p, u_p)u_p] + \ h\sum_{i=0}^{j-1} f_i^n - (t - t_{j-1})f_j^n \\
&= u_0 - U^n(t) + \int_0^t K^n(s)ds + \int_0^t f^n(s)ds.
\end{aligned}$$

When $j = 1$, $t \in (0, t_1]$, we have

$$\begin{aligned}
\int_0^t A^n(s, X^n(s-h))X^n(s)ds &= tA(t_0, u_0)u_1 \\
&= -\frac{t}{h}(u_1 - u_0) + thk_{10}A(t_0, u_0)u_0 + tf_1^n \\
&= u_0 - U^n(t) + \int_0^t K^n(s)ds + \int_0^t f^n(s)ds.
\end{aligned}$$

(c) for $t \in (t_{j-1}, t_j]$,

$$A^n(t, X^n(t-h))X^n(t) = A(t_{j-1}, u_{j-1})u_j \quad \text{and} \quad \frac{d^-u^n}{dt}(t) = \frac{1}{h}(u_j - u_{j-1}).$$

Therefore,

$$\begin{aligned}
\frac{d^-u}{dt}(t) - A^n(t, X^n(t-h))X^n(t) &= \frac{1}{h}(u_j - u_{j-1}) - A(t_{j-1}, u_{j-1})u_j \\
&= h\sum_{i=0}^{j-1} k_{ji}A(t_i, u_i)u_i + f_j^n \\
&= K^n(t) + f^n(t).
\end{aligned}$$

This completes the proof of the lemma. □

In the next lemma we prove the local uniform convergence of the Rothe sequence.

Lemma 3.4 *There exists a subsequence* $\{U^{n_k}\}$ *of the sequence* $\{U^n\}$ *and a function* u *in* $Lip(J_0, X)$ *such that*

$$U^{n_k} \to u \quad in \quad C(J_0, X),$$

with supremum norm as $k \to \infty$.

Proof Since $\{X^n(t)\}$ is uniformly bounded in Y, the compact imbedding of Y implies that there exists a subsequence $\{X^{n_k}\}$ of $\{X^n\}$ and a function $u : J_0 \to X$ such that $X^{n_k}(t) \to u(t)$ in X as $k \to \infty$. The reflexivity of Y implies that $u(t)$ is the weak limit of $X^{n_k}(t)$ in Y hence $u(t) \in Y$ in fact in W_R since $X^{n_k}(t) \in W_R$. Now, $X^{n_k}(t) - U^{n_k}(t) \to 0$ in X, $U^{n_k}(t) \to u(t)$ as $k \to \infty$. The uniform Lipschitz continuity of $\{U^{n_k}\}$ on J_0 implies that $\{U^{n_k}\}$ is an equicontinuous family in $C(J_0, X)$ and the strong convergence of $U^{n_k}(t)$ to $u(t)$ in X implies that $\{U^{n_k}(t)\}$ is relatively compact in X. We use the Ascoli-Arzela theorem to assert that $U^{n_k} \to u$ in $C(J_0, X)$ as $k \to \infty$. Since U^{n_k} are in $\text{Lip}(J_0, X)$ with uniform Lipschitz constant, $u \in \text{Lip}(J_0, X)$. This completes the proof of the lemma. □

Lemma 3.5 *Let* $\psi : [0, T] \to X$ *be given by* $\psi(t) = A(t, u(t))u(t)$. *Then* ψ *is Bochner integrable on* $[0, T]$.

Proof Proof of this lemma can be established in similar way as that of Lemma 4.6 in Kato [9]. □

Lemma 3.6 *Let* $\{K^n(t)\}$ *be the sequence of functions defined by (20) and*

$$K(\psi)(t) = \int_0^t k(t,s)\psi(s)ds.$$

We have $K^{n_k}(t) \to K(\psi)(t)$, *uniformly on* $[0, T_0]$ *as* $k \to \infty$.

Proof For notational conveneince, we shall use the index n in place of n_k for the subsequence n_k of n. We first show that $K^n(t) - K(\psi_n)(t) \to 0$ uniformly on $[0, T_0]$ as $n \to \infty$ where $\psi_n : [0, T_0] \to X$ is given by $\psi_n(t) = A(t, X^n(t))X^n(t)$. For $t \in (t_{j-1}, t_j]$, we have

$$\begin{aligned} K^n(t) - K(\psi_n)(t) &= h\sum_{i=0}^{j-1} k_{ji}^n A(t_i, u_i)u_i - \int_0^t k(t,s)A(s, X^n(s))X^n(s)\, ds \\ &= \sum_{i=1}^{j-1}\left[\int_{t_{i-1}}^{t_i} [k_{ji}A(t_i,u_i) - k(t,s)A(s, X^n(s))]\, ds\right] u_i \\ &+ \; hk(t_j, t_0)A(t_0, u_0)u_0 - \left[\int_{t_{j-1}}^t k(t,s)A(s,u_j)ds\right] u_j. \end{aligned}$$

Since $\|A(t, u_j)u_j\|_X \le \gamma_A R$, and $k : [0, T_0] \to \mathbb{R}$ being Lipschitz continuous imply that the last two terms on the right hand side tend to zero strongly and uniformly on $[0, T_0]$ as $n \to \infty$ we have

$$\|K^n(t) - K(\psi_n)(t)\|_X \le \gamma_A R\left[\sum_{i=0}^{j-2}\int_{t_i}^{t_{i+1}} |k_{ji} - k(t,s)|ds\right].$$

Now, since k satisfies (A4), $k(t,s)$ is uniformly continuous in t as well as in s on $[0,T_0]$. Hence for each $\epsilon > 0$ we can choose n sufficiently large such that for $|t_1-t_2|+|s_1-s_2| < h = \frac{T}{n},\ t_i, s_i \in [0,T_0],\ i=1,2,$ we have

$$|k(t_1,s_1)-k(t_2,s_2)| < \frac{\epsilon}{\gamma_A RT}.$$

Then for sufficiently large n, we have

$$\|K^n(t)-K(\psi_n)(t)\|_X \leq \frac{\epsilon}{\gamma_A RT}\gamma_A Rjh < \epsilon,$$

Which show that $K^n(t) - K(\psi_n)(t) \to 0$ as $n \to \infty$, uniformly on $[0,T_0]$. Now we show that $K(\psi_n)(t) \to K(\psi)(t)$ uniformly as $n \to \infty$. For any $v \in X$, We note that $\langle A(t,u(t))u(t), v\rangle$ is continuous hence we may write

$$\Big\langle K(\psi)(t), v\Big\rangle = \int_0^t k(t,s)\langle A(s,u(s))u(s), v\rangle\, ds.$$

Now, for any $v \in X$,

$$\begin{aligned}\langle K(\psi_n)(t), v\rangle &= \sum_{i=0}^{j-2}\int_{t_i}^{t_{i+1}} k(t,s)\langle A(s,u_{i+1})u_{i+1}, v\rangle ds \\ &\quad + \int_{t_{j-1}}^{t} k(t,s)\langle A(t,u_j)u_j, v\rangle ds.\end{aligned}$$

This implies that $\langle K(\psi_n)(t), v\rangle \to \langle K(\psi)(s), v\rangle$, as $n \to \infty$. This completes the proof of the lemma. □

3.1 Proof of Theorem 2.1.

Proof First we show that $A^m(t, X^m(t-h))X^m(t) \rightharpoonup A(t,u(t))u(t)$ in X as $m \to \infty$, where ‘ $\rightharpoonup$’ denotes the weak convergence in X,

$$\begin{aligned}&A(t_{j-1}, X^m(t-h))X^m(t) - A(t,u(t))u(t)\\ &= [A(t_{j-1}, X^m(t-h)) - A(t,u(t))]X^m(t) + A(t,u(t))[X^m(t)-u(t)].\end{aligned}$$

Since,

$$\|[A(t_{j-1}, X^m(t-h)) - A(t,u(t))]X^m(t)\|_X \leq \mu_A R[|t_{j-1}-t| + \ \|X^m(t-h)-u(t)\|_X],$$

as $m \to \infty$ the right hand side of the above equation tends to zero. Since $X^m(t) \to u(t)$ in X uniformly on J_0 and $A(t,u(t)) \in N(X,\beta), \beta I + A(t,u)$ is m-accretive in X. We use Lemma 2.5 due to Kato [9] and the fact that

$$\|A(t,u(t))[X^m(t-h)-u(t)]\|_X \leq 2\mu_A R,$$

to assert that $A(t,u(t))X^m(t) \rightharpoonup A(t,u(t))u(t)$ in X and, hence, $A^m(t, X^m(t-h))X^m(t) \rightharpoonup A(t,u(t))u(t)$ in X as $m \to \infty$. Now we show that $A(t,u(t))u(t)$ is weakly continuous on J_0, let $\{t_p\} \subset J_0$ be a sequence such that $t_p \to t$, as $p \to \infty$. Then $u(t_p) \to u(t)$ in X as $p \to \infty$ and we can follow the same arguments as above to prove that $A(t_p, u(t_p))u(t_p) \rightharpoonup A(t, u(t))u(t)$ in X as $p \to \infty$. Now from Lemma 3.3 for each $x^* \in X^*$ we have

$$\langle U^m(t), x^* \rangle = \langle u_0, x^* \rangle + \int_0^t \langle -A^m(s, X^m(s-h))X^m(s) + K^m(s) + f^m(s), x^* \rangle ds.$$

Letting $m \to \infty$ using bounded convergence theorem and Lemma 3.6 we get

$$\langle U(t), x^* \rangle = \langle u_0, x^* \rangle + \int_0^t \langle -A(s, u(s))u(s) + K(\psi)(s) + f(s, u_s), x^* \rangle ds.$$

Continuity of the integrand implies that $\langle u(t), x^* \rangle$ is continuously differentiable on J_0. The Bochner integrability of $A(t, u(t))u(t)$ implies that the strong derivative of $u(t)$ exists $a.e.$ on J_0 and

$$\frac{du}{dt} + A(t, u(t))u(t) = \int_0^t k(t,s)A(s, u(s))u(s)ds + f(t, u_t), \quad a.e \quad \text{on} \quad J_0.$$

Since $u(0) = u_0$, u is a strong solution to (1). Now for the uniqueness of the solution of (1). Let v be another strong solution to (1) on J_0. Let $U = u - v$, then for $a.e.$ $t \in J_0$

$$\begin{aligned}
&\left\langle \frac{dU}{dt}(t), F(U(t)) \right\rangle + \langle \beta I + A(t, u(t))U(t), F(U(t)) \rangle \\
&= \beta \|U(t)\|_X^2 + \langle (A(t, u(t)) - A(t, v(t)))v(t), F(U(t)) \rangle \\
&+ \left\langle \int_0^t k(t,s)[A(s, u(s)) - A(s, v(s))]u(s)ds, F(U(t)) \right\rangle \\
&+ \left\langle \int_0^t k(t,s)A(s, v(s))[u(s) - v(s)]ds, F(U(t)) \right\rangle \\
&+ \langle f(t, u_t) - f(t, v_t), F(U) \rangle.
\end{aligned}$$

Using m-accretivity of $\beta I + A(t, u(t))u(t)$ and Assumptions (A2) and (A4) we get

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_X^2 \le C_T \|U\|_{C([0,t],X)}^2,$$

where $C_T = \beta + \mu_A R + k_T(\gamma_A C_e + \mu_A R) + L_f$. Integrating the above inequality on $(0, t)$ and taking the supremum we get

$$\frac{1}{2}\|U(t)\|_{C([0,t],X)}^2 \le C_T \int_0^t \|U\|_{C([0,s],X)}^2 ds.$$

Applying the Gronwall's inequality we get $U = 0$ on J_0.

Continuous dependence. Let $v_0 \in B_Y(u_0, R_0)$. Then

$$\|Sv_0 - z_0\|_X \le \|Sv_0 - Su_0\|_X + \|Su_0 - z_0\|_X \le 2R_0.$$

Hence

$$(1 + e^{2\theta T})[\|Sv_0 - z_0\|_X + T_0\{\gamma_A \|z_0\|_Y + \gamma_A \|z_0\|_X + M\}] \le 3R_0 = \frac{R}{2}.$$

We can proceed as before to prove the existence of $v_j^n \in W_R$ satisfying scheme (14) with u_j^n and u_0 replaced by v_j^n and v_0 respectively. Convergence of v_j^n to $v(t)$ can be proved in a similar manner. Let $U = u - v$ then following the steps used to prove the uniqueness, we have for *a.e.* $t \in J_0$

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_X^2 \leq C_T\|U\|^2_{C([0,t],X)}.$$

Integrating the above inequality on $(0, t)$ and taking the supremum we get

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{C([0,t],X)} \leq \frac{1}{2}T\|U(0)\|_X^2 + C_T\int_0^t \|U(t)\|^2_{C([0,s],X)}.$$

Applying the Gronwall's inequality we get

$$\|U(t)\|^2_{C([0,t],X)} \leq C\|U(0)\|_X^2,$$

where C is a positive constant. This completes the proof of the theorem. □

4 Application

For illustration, we consider the existence and uniqueness of a solutions for the following model

$$\begin{cases} a_0(x,u)\frac{\partial u}{\partial t} + \sum_{j=1}^m a_j(t,x,u)\frac{\partial u}{\partial x_j} = \int_{-T}^0 g(t, u(t+\theta, x)d\theta, \\ + \sum_{j=1}^m \int_0^t k(t-s)a_j(s,x,u)\frac{\partial u}{\partial x_j}ds, \quad 0 < t \leq T,\ x \in \mathbb{R}^m, \\ u(\theta, x) = \phi_0(\theta, x) \quad \text{for} \quad \theta \in [-T, 0] \quad \text{and} \quad x \in \mathbb{R}^m, \end{cases} \tag{23}$$

where the unknown $u = (u_1, \ldots, u_N)$ is an N-vector, a_0 and a_j, $j = 1, 2, \ldots, m$, are $N \times N$ symmetric matrix-valued smooth functions on $\Omega \times \mathbb{R}^N$ and $[0, T] \times \Omega \times \mathbb{R}^N$, respectively, where $\Omega \subset \mathbb{R}^m$ is a bounded domain with sufficiently smooth bounday. We set

$$\begin{aligned} &Y = H^s(\Omega, \mathbb{R}^N), \quad Z = H^{s-1}(\Omega, \mathbb{R}^N), \quad X = H^0(\Omega, \mathbb{R}^N), \quad W = B_r(Y), \\ &S = (1-\Delta)^{s/2}, \quad s > m/2 + 1, \\ &A(t, w) = a_0(x, w)^{-1}\sum_{j=1}^m a_j(t, x, w)\frac{\partial}{\partial x_j}, \end{aligned}$$

and use the variable norm

$$\|v\|_w^2 = \int_\Omega a_0(x, w)v.vdx.$$

We suppose that for $j = 1, 2, \ldots, m$, $a_j(t, x, u)$ are simultaneously diagonalizable by a common nonsingular C^1 matrix $q(t, x, w)$ and $a_0(x, w)$ is positive-definite. The function $g : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is continuous and Lipschitzian with respect to the second argument, the function $\phi_0 : [-r, 0] \times \Omega \to \mathbb{R}$ will be specified later.

Note that $A(t, w) \in G(X_w, 1, \beta)$ with β depending on $\|w\|_Y$, and $G(X_w, 1, \beta)$ denotes the set of all (negative) generators A of C_0-semigroups on X_w such that $\|e^{-tA}\| \leq Me^{\beta t}$ for $t > 0$. Again verification of the conditions is straightforward, except that we have to prove that $-A(t, w)$ is the generator of C_0-semigroup (for details see [8]).

Let $f : [0, T] \times C([-T, 0], X) \to Y$ be defined by

$$f(t, \chi)(x) = \int_{-T}^{0} g(t, \chi(\theta)(x)d\theta, \quad t \geq 0.$$

The initial data $\phi \in C([-T, 0], X)$ is defined by

$$\phi(\theta)(x) = \phi_0(\theta, x) \quad \text{for} \quad \theta \in [-T, 0].$$

Then (23) takes the following abstract form

$$\begin{cases} \frac{d}{dt}u(t) + A(t, u(t))u(t) = \int_0^t k(t-s)A(s, u(s))u(s)ds + f(t, u_t), \quad 0 < t \leq T, \\ u_0 = \phi \in C([-T, 0], X). \end{cases} \tag{24}$$

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Backstepping for Nonsmooth MIMO Nonlinear Volterra Systems with Noninvertible Input-Output Maps and Controllability of Their Large Scale Interconnections ◇

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Abstract: We prove the global controllability for a class of nonlinear MIMO Volterra systems of the triangular form as well as for their bounded perturbations. In contrast to the related preceeding work [15], we replace the condition of C^1 smoothness, which was essentially used before, with that of local Lipschitzness. Furthermore, we remove the assumption of the invertibility of the input-output interconnections, which was also essential in these preceeding results. In order to solve the problem, we revise the backstepping procedure proposed in these works, and combine it with another method of constructing discontinuous feedbacks proposed for the so-called "generalized triangular form" in the case of ODE [16, 21].

Keywords: *backstepping; Volterra nonlinear control systems; controllability, large scale systems.*

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1 Introduction

During the last two decades such recursive procedures as backstepping-like designs became very popular when solving various problems of adaptive and robust nonlinear control [5, 9, 17, 18, 23]. It is worth mentioning that, despite of the fruitfulness of the backstepping-like algorithms, the most works devoted to them address the triangular or pure-feedback form systems [13]

$$\begin{cases} \dot{x}_i = f_i(x_1, ..., x_{i+1}), \quad i = 1, \dots, n-1; \\ \dot{x}_n = f_n(x_1, ..., x_n, u) \end{cases} \tag{1}$$

that are feedback linearizable, i.e., to those which satisfy the condition $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i = 1, \dots, n$; or even have the strict-feedback form

$$\begin{cases} \dot{x}_i = b_i x_{i+1} + \theta_i \varphi_i(x_1, ..., x_i), \quad i = 1, \dots, n-1; \\ \dot{x}_n = b_n u + \theta_n \varphi_n(x_1, ..., x_n) \end{cases}$$

(with $b_i \neq 0$). Indeed, whatever the problem is (Lyapunov stabilization, adaptive stabilization etc.), the classical version of the backstepping requires system (1) to satisfy the following two properties:

(A) The virtual control $x_{i+1} = \alpha_i(t, x_1, ..., x_i)$ obtained at the i-th step $(i = 1, \dots, n)$ should be well-defined as an implicit function obtained from some nonlinear equation of the form $f_i(x_1, ..., x_{i+1}) = F_i(t, x_1, ..., x_i)$ to be resolved w.r.t. x_{i+1}, where $F_i(t, x_1, ..., x_i)$ is some function of the previous coordinates $x_1, ..., x_i$ (and maybe of t).

(B) Each virtual control $x_{i+1} = \alpha_i(t, x_1, ..., x_i)$ obtained at the i - th step should be smooth enough because one needs to take its derivatives at the next steps $i = 1, \dots, n$.

This necessarily leads to the conditions like $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i = 1, \dots, n$, (to comply with (A)) and like $f_i \in C^n$ or $f_i \in C^{n-i+1}$ (to comply with (B)).

Works [3, 4, 18, 22, 25, 26] were devoted to the issue of how to obviate the first restriction $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, at least for some special cases: when $f_i(x_1, ..., x_{i+1})$ are polynomials w.r.t. x_{i+1} of odd degree (see work [22]); when $f_i = x_{i+1}^p + \varphi_i(x_1, ..., x_i)$ (see works [18,26] devoted to the problem of global stabilization of such systems into the origin as well as further works by some of these authors devoted to various adaptive and robust control problems for this class); partial-state stabilization under the assumption that the "controllable part" satisfies some additional "growth conditions" (see work [25] and conditions (A3),(i),(ii),(iii)); the problem of feedback triangulation under the assumption that the set of regular points is open and dense in the state space (see [3]).

A natural generalization of these cases is the so-called "generalized triangular form" (GTF), when the only assumption is that $f_i(t, x_1, ..., x_i, \cdot)$ is a surjection whereas x_i and u are vectors not necessarily of the same dimension (and the dynamics is of class C^1 or C^n depending of the problem to be explored). In works [16, 21] it was proved that, first, the systems of this class are globally robustly controllable, in particular, their bounded perturbations are globally controllable as well (see [16]) and, second, they are globally asymptotically stabilizable into every regular point (see [21]). Note that, although the methods proposed in [14–16, 21] are called "backstepping", their only common feature with the classical backstepping designs is the induction over the dimension of the system and treatment x_{i+1} as the virtual control at the i-th step. As to the construction, the approach proposed in [14–16, 21] is absolutely different. This especially applies to [16] and to the preceeding related works [14, 15] devoted to the problem of global robust controllability.

It is worth mentioning that, despite of the importance of the Volterra equations in applications, the controllability problem for the Volterra systems was investigated in few works only. Works [1, 2] are devoted to the complete controllability of perturbations of linear Volterra systems. In these papers, some natural analogs of the integral criterion of the controllability for linear ODE systems were obtained.

In works [14, 15] the problem of global robust controllability was successively solved for the nonlinear Volterra systems of the triangular form

$$\dot{x}_i = f_i(t, x_1, ..., x_{i+1}) + \int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_{i+1}(s))ds, \quad i = 1, \ldots, n,$$

(where $x_{n+1} = u$ is the control, and $(x_1, ..., x_n)$ is the state) including the global controllability of their bounded perturbations. Although, as we highlighted above, the inductive construction proposed in these works differs totally from the classical backstepping designs, the following two assumptions, which are similar to (A) and (B), are essential in this construction:

(A') For every $x_1(\cdot), ..., x_i(\cdot)$ of class C^1 the integral equation

$$\dot{x}_i = f_i(t, x_1(t), ..., x_{i+1}(t)) + \int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_{i+1}(s))ds,$$

should be resolvable w.r.t. $x_{i+1}(\cdot)$ on the whole time interval $[t_0, T]$.

(B') The properties of the linearized control systems (and those of the Frechet derivative of the input-output map) are essential, which is why f_i and g_i should be of class C^1 at least.

The goal of the current paper is to remove these restrictions (A') and (B') and to show how a modification of the methods proposed in [16, 21] can be applied to the problem of global controllability of the Volterra systems. In many modern applications one has to deal with large scale interconnected systems - see, for instance [6, 10, 19]. Developing our technique, we solve the problem of global controllability for large scale interconnections of generalized triangular non-smooth Volterra systems.

2 Preliminaries

The first result of the current paper (Theorem 3.1 below) is concerned with the control systems of the Volterra integro-differential equations:

$$\dot{x}(t) = f(t, x(t), u(t)) + \int_{t_0}^{t} g(t, s, x(s))ds, \quad t \in I = [t_0, T], \tag{2}$$

where $u \in \mathbb{R}^m = \mathbb{R}^{m_{\nu+1}}$ is the control, $x = (x_1, ..., x_\nu)^T \in \mathbb{R}^n$ is the state with $x_i \in \mathbb{R}^{m_i}$, $m_i \leq m_{i+1}$ and $n = m_1 + \ldots, +m_\nu$, functions f and g have the form

$$f(t, x, u) = \begin{pmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2, x_3) \\ \cdots \\ f_\nu(t, x_1, ..., x_\nu, u) \end{pmatrix} \quad \text{and} \quad g(t, s, x) = \begin{pmatrix} g_1(t, s, x_1) \\ g_2(t, s, x_1, x_2) \\ \cdots \\ g_\nu(t, s, x_1, ..., x_\nu) \end{pmatrix} \tag{3}$$

with $f_i \in \mathbb{R}^{m_i}$, $g_i \in \mathbb{R}^{m_i}$ and satisfy the conditions:

(i) $f \in C(I \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $g \in C(I^2 \times \mathbb{R}^n; \mathbb{R}^n)$,

(ii) *f and g satisfy the* local *Lipschitz condition w.r.t.* (x, u), *i.e., for every compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there is $l_K > 0$ such that, for every $(x^1, u^1) \in K$ and every $(x_2, u_2) \in K$ it holds*

$$|f(t, x^1, u^1) - f(t, x^2, u^2)| \leq l_K(|x^1 - x^2| + |u^1 - u^2|) \quad \text{and}$$

$$|g(t, s, x^1) - g(t, s, x^2)| \leq l_K |x^1 - x^2| \ \text{ for all } t \in I, \ s \in I$$

(iii) *For each $i = 1, \dots, \nu$, each $t \in I$ and each $(x_1, ..., x_i)^T$ in $\mathbb{R}^{m_1 + \dots + m_i}$, we have* $f_i(t, x_1, ..., x_i, \mathbb{R}^{m_{i+1}}) = \mathbb{R}^{m_i}$.

Given $x^0 \in \mathbb{R}^n$, and $u(\cdot) \in L_\infty(I; \mathbb{R}^m)$, let $t \mapsto x(t, x^0, u(\cdot))$ denote the trajectory of (2), defined by this control $u(\cdot)$ and by the initial condition $x(t_0) = x^0$ on the maximal interval $J \subset I$ of the existence of the solution. As in [15], we say that a system of the Volterra integro-differential equations is globally controllable in time $I = [t_0, T]$ in class $C^\mu(I; \mathbb{R}^m)$ $(\mu \geq 0)$, iff for each initial state $x^0 \in \mathbb{R}^n$ and each terminal state $x^T \in \mathbb{R}^n$ there is a control $u(\cdot)$ in $C^\mu(I; \mathbb{R}^m)$ which "steers x^0 into x^T w.r.t. the system", i.e., the trajectory $x(\cdot)$ of the system with this control $u(\cdot)$ such that $x(t_0) = x^0$ is well-defined on I and satisfies $x(T) = x^T$.

In our second result (Theorem 3.2 in the next Section) we consider a large scale interconnection of systems like (2) in the form

$$\dot{X}_i(t) = F_i(t, X_i(t), U_i(t)) + \int_{t_0}^{t} G_i(t, s, X_i(s)) ds + H(t, X(t), U(t)) +$$

$$+ \int_{t_0}^{t} R(t, s, X(s), U(s)) ds, \quad i = 1, \dots, q, \quad t \in I = [t_0, T], \tag{4}$$

where $X = [X_1, \dots, X_q]^T \in \mathbb{R}^N$ is the state with $X_i = [x_{i,1}, \dots, x_{i,\nu_i}]^T \in \mathbb{R}^{n_i}$ and with $x_{i,j} \in \mathbb{R}^{m_{i,j}}$ and $U = [U_1, \dots, U_q]^T \in \mathbb{R}^M$ is the control with $U_i \in \mathbb{R}^{m_{i,\nu_i+1}}$ (and $N = \sum\limits_{i=1}^{q} n_i = \sum\limits_{i=1, j=1}^{q, \nu_i} m_{i,j}$; $M = \sum\limits_{i=1}^{q} \nu_{i+1}$).

We assume that functions F_i and G_i have the form

$$F_i(t, X_i, U_i) = \begin{pmatrix} F_{i,1}(t, x_{i,1}, x_{i,2}) \\ F_{i,2}(t, x_{i,1}, x_{i,2}, x_{i,3}) \\ \dots \\ F_{i,\nu_i}(t, x_{i,1}, ..., x_{i,\nu_i}, U_i) \end{pmatrix},$$

$$G_i(t, s, X_i) = \begin{pmatrix} G_{i,1}(t, s, x_{i,1}) \\ G_{i,2}(t, s, x_{i,1}, x_{i,2}) \\ \dots \\ G_{i,\nu_i}(t, s, x_{i,1}, ..., x_{i,\nu_i}) \end{pmatrix}. \tag{5}$$

We define

$$F(t, X, U) = \begin{pmatrix} F_1(t, X_1, U_1) \\ F_2(t, X_2, U_2) \\ \dots \\ F_q(t, X_q, U_q) \end{pmatrix}, \quad G(t, s, X) = \begin{pmatrix} G_1(t, s, X_1) \\ G_2(t, s, X_2) \\ \dots \\ G_q(t, s, X_q) \end{pmatrix},$$

and assume that the following conditions hold:

(I) $F \in C(I \times \mathbb{R}^N \times \mathbb{R}^M; \mathbb{R}^N)$, $G \in C(I^2 \mathbb{R}^N; \mathbb{R}^N)$.

(II) *There exists* $L > 0$ *such that, for every* $(X^1, U^1) \in K$ *and every* $(X_2, U_2) \in K$ *it holds*

$$|F(t, X^1, U^1) - F(t, X^2, U^2)| \leq L(|X^1 - X^2| + |U^1 - U^2|),$$
$$|G(t, s, X^1) - G(t, s, X^2)| \leq L|X^1 - X^2| \text{ for all } t \in I, \ s \in I$$

*(*global *Lipschitz property with respect to* (X, U)*).*

(III) *For each* $i = 1, \ldots, q$, *each* $j = 1, \ldots, \nu_i$, *each* $t \in I$ *and each* $(x_{i,1}, ..., x_{i,j})^T$ *in* $\mathbb{R}^{m_{i,1}+\ldots+m_{i,j}}$, *we have* $F_{i,j}(t, x_{i,1}, ..., x_{i,j}, \mathbb{R}^{m_{i,j+1}}) = \mathbb{R}^{m_{i,j}}$.

Also we assume that functions H and R satisfy the conditions:

(IV) $H \in C(I \times \mathbb{R}^N \times \mathbb{R}^M; \mathbb{R}^N)$, $R \in C(I^2 \times \mathbb{R}^N \times \mathbb{R}^M; \mathbb{R}^N)$, *and for each compact set* $Q \subset \mathbb{R}^N \times \mathbb{R}^M$, *there exists* $L_Q > 0$ *such that, for all* $(t, s) \in I^2$, $(X^1, U^1) \in Q$, $(X^2, U^2) \in Q$, *we have:*

$$|H(t, X^1, U^1) - H(t, X^2, U^2)| \leq L_Q(|X^1 - X^2| + |U^1 - U^2|),$$
$$|R(t, s, X^1, U^1) - R(t, s, X^2, U^2)| \leq L_Q(|X^1 - X^2| + |U^1 - U^2|),$$

(V) *There exists* $H_0 > 0$ *such that* H *and* R *satisfy the inequalities* $|H(t, X, U)| \leq H_0$ *and* $|R(t, s, X, U)| \leq H_0$ *for all* $(t, s, X, U) \in I^2 \times \mathbb{R}^N \times \mathbb{R}^M$.

Note that F_i and G_i have the "general traingular form", while H and R have an arbitrary form and are "cross terms", which characterize the interconnections of the isolated X_i-subsystems.

3 Main Results

Theorem 3.1 *Suppose that system (2) has the form (3) and satisfies conditions (i),(ii),(iii). Then system (2) is globally controllable in class* $C^\infty(I; \mathbb{R}^m)$.

Theorem 3.2 *Suppose that functions* F_i *and* G_i *have the form (5), satisfy (I),(II),(III), and suppose that* H *and* R *satisfy (IV), (V). Then system (4) is globally controllable in time* I *by means of controls of class* $C^\infty(I; \mathbb{R}^M)$.

Remark 3.1 Let us compare the results of [15] with our Theorems 3.1 and 3.2. First, in [15], functions f and g are required not only to be continuous but also to have all their partial derivatives, w.r.t. x and u, which are required to be continuous whereas we require (i) and (ii) only ((I) and (II) respectively for Theorem 3.2); (ii) or (II) being the standard condition needed to guarantee the existence and the uniqueness of the solution of the "Cauchy problem" for the Volterra systems. Second, our system (2) is MIMO and furthermore x_i and u are vectors of different dimensions whereas, in [15], the system is SISO (i.e., x_i and u are scalar) or at least x_i and u should be of the same dimension (see Remark 3.1 from [15]). Third (and this is essential), our current Assumption (iii) is much more general than the corresponding Assumption (ii) (or (II), p. 747) from [15]. In this sense, our current Theorem 3.1 and Theorem 3.2 generalize Theorem 3.3 and Theorem 3.2 from [15] respectively. However: firstly, in our case, function g has a bit more specific form than function g from [15] (g_i does not depend on x_{i+1} in the current paper); secondly, since we replace the assumption of C^1 smoothness with that of local Lipschitzness, we do not obtain stronger results on robustness (Theorem 3.1 from [15]).

Example 3.1 Consider the system given by

$$\begin{cases} \dot{x}_1(t) = (x_2(t) + x_1(t))|\sin x_2(t)| + \int\limits_0^t \sqrt{s^2 x_1^2(s) + 1}ds, \\ \dot{x}_2(t) = u(t)|\cos u(t)| + \int\limits_0^t \sqrt{e^{ts}(x_1^2(s) + x_2^2(s)) + 1}ds, \end{cases} \tag{6}$$

$t \in [0, T]$. It is clear that system (6) satisfies our Assumptions (i)-(iii) and therefore is globally controllable by Theorem 3.1. On the other hand, system (6) does not satisfy the Assumptions from [15] and the results of [15] are not applicable to system (6).

Remark 3.2 Note that, if $g = 0$ in (2), then (2) is reduced to the class of the so-called "generalized triangular form" of ODE control systems considered in [16, 20, 21]. However, in the case of ODE, stronger results were obtained in these works: global robust controllability (Theorem 3.1 from [16]), global asymptotic stabilization by means of smooth controls (Theorem 2.1 from [21]), and global discontinuous stabilization in the sense of Clarke-Ledyaev-Sontag-Subbotin (Theorem 3.4 from [16]).

4 Backstepping in the Non-smooth Case

Let us first reduce Theorem 3.1 to a backstepping process which can be compared with that from [16].

Let p be in $\{1, \ldots, \nu\}$. Define $k := m_1 + \ldots + m_p$ and consider the following k - dimensional control system

$$\dot{y}(t) = \varphi(t, y(t), v(t)) + \int\limits_{t_0}^{t} \psi(t, s, y(s))ds, \quad t \in I = [t_0, T], \tag{7}$$

where $y := (x_1, \ldots, x_p)^T \in \mathbb{R}^k = \mathbb{R}^{m_1 + \ldots + m_p}$ is the state, $v \in \mathbb{R}^{m_{p+1}}$ is the control and

$$\varphi(t, y, v) = \begin{pmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2, x_3) \\ \ldots \\ f_p(t, x_1, ..., x_p, v) \end{pmatrix}, \qquad \psi(t, s, y) = \begin{pmatrix} g_1(t, s, x_1) \\ g_2(t, s, x_1, x_2) \\ \ldots \\ g_p(t, s, x_1, ..., x_p) \end{pmatrix}, \tag{8}$$

for all (t, y, v) in $I \times \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$. Given $y^0 \in \mathbb{R}^k$, and $v(\cdot) \in L_\infty(I; \mathbb{R}^{m_{p+1}})$, let $t \mapsto y(t, y^0, v(\cdot))$ denote the trajectory, of (7), defined by the control $v(\cdot)$ and by the initial condition $y(t_0, y^0, v(\cdot)) = y^0$ on the maximal interval $J \subset I$ of the existence of the solution. We reduce the proof of Theorems 3.1 to the following theorem.

Theorem 4.1 *Let p be in $\{1, ..., \nu\}$. Suppose for each $y^0 \in \mathbb{R}^k$ and each $\delta > 0$, there is a family of functions $\{y(\xi, \cdot) = (x_1(\xi, \cdot), ..., x_p(\xi, \cdot))\}_{\xi \in \mathbb{R}^k}$ such that:*

1) The map $\xi \mapsto y(\xi, \cdot)$ is of class $C(\mathbb{R}^k; C^1(I; \mathbb{R}^k))$

2) For each $\xi \in \mathbb{R}^k$, each $t \in I$ and each $1 \le i \le p - 1$ we have:

$$\dot{x}_i(\xi, t) = f_i(t, x_1(\xi, t), \ldots, x_{i+1}(\xi, t)) + \int\limits_{t_0}^{t} g_i(t, s, x_1(\xi, s), \ldots, x_i(\xi, s))ds$$

(if $p = 1$, then, the set of equalities is empty and, by definition, Condition 2) holds true)

3) $y(\xi, t_0) = y^0$ and $|y(\xi, T) - \xi| < \delta$ for all $\xi \in \mathbb{R}^k$

Then, for each $(y^0, y^0_{p+1}) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$, and each $\varepsilon > 0$, there exists a family of controls $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^k\times\mathbb{R}^{m_{p+1}}}$ such that

4) The map $(\xi, \beta) \mapsto \hat{v}_{(\xi,\beta)}(\cdot)$ is of class $C(\mathbb{R}^k \times \mathbb{R}^{m_{p+1}}; C^\infty(I; \mathbb{R}^{m_{p+1}}))$

5) For each $(\xi, \beta) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$, we have $\hat{v}_{(\xi,\beta)}(T) = \beta$ and $\hat{v}_{(\xi,\beta)}(t_0) = y^0_{p+1}$.

6) $|y(T, y^0, \hat{v}_{(\xi,\beta)}(\cdot)) - \xi| < \varepsilon$ for all $(\xi, \beta) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$.

Let us prove that Theorem 3.1 follows from Theorem 4.1. Indeed, suppose Theorem 4.1 holds true.

Suppose $p = 1$ and $k = m_1$, and take an arbitrary $y^0_1 \in \mathbb{R}^{m_1}$. Given an arbitrary $\delta > 0$, find any family $\{y(\eta, \cdot)\}_{\eta\in\mathbb{R}^{m_1}} = \{x_1(\eta, \cdot)\}_{\xi\in\mathbb{R}^{m_1}}$ such that Conditions 1)-3) of Theorem 4.1 hold. Then, for $p = 1$, we have: for every $\varepsilon > 0$ and every $(y^0_1, y^0_2) \in \mathbb{R}^{m_1+m_2}$, there exists a family of controls $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}$ such that Conditions 4), 5), 6) of Theorem 4.1 hold with $p = 1$.

Suppose $p = 2$. Given any $y^0 = (y^0_1, y^0_2) \in \mathbb{R}^{m_1+m_2}$, and any $\delta > 0$, define $\varepsilon := \delta$, and for this $\varepsilon > 0$ find the family $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}$ obtained at the previous step (with $p = 1$). From Conditions 4)-6) applied to $p = 1$ it follows that the family $\{y(\xi, \cdot)\}_{\xi=(\eta,\beta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}$ defined by

$$y(\eta, \beta, t) := (y(t, y^0_1, \hat{v}_{(\eta,\beta)}(\cdot)), \hat{v}_{(\eta,\beta)}(t)) \text{ for all } \quad t \in I, \quad \xi = (\eta, \beta) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$$

satisfies the Conditions 1), 2), 3) of Theorem 4.1 with $p = 2$. Then we can apply Theorem 4.1 to $p = 2$, etc. Arguing by induction over $p = 1, \ldots, \nu$, we obtain for $p = \nu$ that for each $\varepsilon > 0$, each $x^0 \in \mathbb{R}^n$, and each $\alpha = y^0_{\nu+1} \in \mathbb{R}^{m_{\nu+1}}$ there exists a family of controls $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^n\times\mathbb{R}^{m_{\nu+1}}}$ such that Conditions 4), 5), 6) of Theorem 4.1 hold for $p = \nu$. Fix an arbitrary $\beta \in \mathbb{R}^{m_{\nu+1}}$ and define the family of controls $\{u_\xi(\cdot)\}_{\xi\in\mathbb{R}^n}$ as follows: $u_\xi(t) := \hat{v}_{(\xi,\beta)}(t)$ for all $t \in I$, $\xi \in \mathbb{R}^n$. Then $\{u_\eta(\cdot)\}_{\eta\in\mathbb{R}^n}$ satisfies the conditions:

(a) $\xi \mapsto u_\xi(\cdot)$ is of class $C(\mathbb{R}^n; C^\infty(I; \mathbb{R}^{m_{\nu+1}}))$

(b) For each $\xi \in \mathbb{R}^n$, the trajectory $t \mapsto x(t, x^0, u_\xi(\cdot))$ is well-defined and $|x(T, x^0, u_\xi(\cdot)) - \xi| < \varepsilon$.

Given any $\varepsilon > 0$, an arbitrary $x^0 \in \mathbb{R}^n$, and an arbitrary $x^T \in \mathbb{R}^n$, let $\{u_\xi(\cdot)\}_{\xi\in\mathbb{R}^n}$ be a family of controls such that (a), (b) hold. By conditions (a),(b) the map $\xi \mapsto \xi - x(T, x^0, u_\xi(\cdot)) + x^T$ is well-defined and of class $C(\mathbb{R}^n; \mathbb{R}^n)$. From condition (b), it follows that this continuous function maps the compact convex set $\overline{B_\varepsilon(x^T)}$ into $\overline{B_\varepsilon(x^T)}$. Then, by the Brouwer fixed-point theorem, there exists $\xi^* \in \overline{B_\varepsilon(x^T)} \subset \mathbb{R}^n$ such that $\xi^* = \xi^* - x(T, x^0, u_{\xi^*}(\cdot)) + x^T$, i.e., $x(T, x^0, u_{\xi^*}(\cdot)) = x^T$. Thus, for every $x^0 \in \mathbb{R}^n$, and every $x^T \in \mathbb{R}^n$, there is a control $u_{\xi^*}(\cdot) \in C^\infty(I; \mathbb{R}^{m_{\nu+1}})$ such that $x^T = x(T, x^0, u_{\xi^*}(\cdot))$, i.e., Theorem 3.1 follows from Theorem 4.1.

Let us prove Theorem 3.2. Given any $U(\cdot) = [U_1(\cdot), \ldots, U_q(\cdot)]^T$ in $L_\infty(I; \mathbb{R}^N)$ and $X^0 \in \mathbb{R}^N$ let $t \mapsto X(t, X^0, U(\cdot))$ denote the trajectory of system

$$\dot{X}_i(t) = F_i(t, X_i(t), U_i(t)) + \int_{t_0}^{t} G_i(t, s, X_i(s))ds \quad i = 1, \ldots, q, \quad t \in I = [t_0, T],$$

defined by the initial condition $X(t_0) = X^0$ and by the control $U = U(\cdot)$. Then arguing as above (for each X_i-subsystem separately), we construct a family $\{U_\xi(\cdot)\}_{\xi\in\mathbb{R}^N}$ such that the following conditions hold:

(c) $\xi \mapsto U_\xi(\cdot)$ is of class $C(\mathbb{R}^N; C^\infty(I;\mathbb{R}^M))$

(d) For each $\xi \in \mathbb{R}^N$, the trajectory $t \mapsto X(t, x^0, U_\xi(\cdot))$ is well-defined and $|X(T, X^0, U_\xi(\cdot)) - \xi| < \varepsilon$.

For each $\xi \in \mathbb{R}^N$, by $X(\xi,\cdot)$ denote the trajectory, of (4), defined by the control $U_\xi(\cdot)$ and by the initial condition $X(\xi, t_0) = X^0$. Using the Gronwall-Bellmann lemma, we easily obtain that $t \mapsto X(\xi, t)$ is well-defined for all $t \in I$, $\xi \in \mathbb{R}^N$ and there exists $D > 0$ such that $|X(\xi, t) - X(t, X^0, u_\xi(\cdot))| \leq D$ for all $t \in I$ and $\xi \in \mathbb{R}^N$, and therefore, by condition (d), we obtain: $|X(\xi, T) - \xi| \leq D + \varepsilon$ for all $\xi \in \mathbb{R}^N$. Taking an arbitrary $X^T \in \mathbb{R}^N$ and applying the Brouwer fixed-point theorem to the map $\xi \mapsto \xi - X(\xi, T) + X^T$, which maps the closed ball $\overline{B_{D+\varepsilon}(X^T)}$ into $\overline{B_{D+\varepsilon}(X^T)}$, we obtain the existence of $\xi^* \in \overline{B_{D+\varepsilon}(X^T)} \subset \mathbb{R}^N$ such that $X^T = X(\xi^*, T)$, which means that the control $U_{\xi^*}(\cdot) \in C^\infty(I;\mathbb{R}^M)$ steers X^0 into X^T in time I w.r.t. system (4). Since X^0 and X^T are chosen arbitrarely, the proof of Theorem 3.2 is complete.

5 Proof of Theorem 4.1

Fix an arbitrary p in $\{1,\dots,\nu\}$, an arbitrary $(y^0, y^0_{p+1}) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$, and an arbitrary $\varepsilon > 0$. Define $\delta := \frac{\varepsilon}{4}$ and assume that $\{y(\xi,\cdot)\}_{\xi\in\mathbb{R}^k}$ satisfies Assumptions 1)-3) of Theorem 4.1.

To prove Theorem 4.1, we change the approach from [15] and [16] as follows. Along with system (7), we consider the following k-dimensional control system of the Volterra equations

$$\begin{cases} \dot{x}_i(t) = f_i(t, x_1(t), ..., x_{i+1}(t)) + \int\limits_{t_0}^{t} g_i(t, s, x_1(s), ..., x_i(s))ds, \quad i = 1, \dots, p-1, \\ \dot{x}_p(t) = w(t) + \int\limits_{t_0}^{t} g_p(t, s, x_1(s), ..., x_p(s))ds, \end{cases} \quad t \in I \tag{9}$$

with states $y = (x_1, ..., x_p)^T \in \mathbb{R}^k$ and controls $w \in \mathbb{R}^{m_p}$. Given $y \in \mathbb{R}^k$, and $w(\cdot) \in L_\infty(I;\mathbb{R}^{m_p})$, let $t \mapsto z(t, y, w(\cdot))$ denote the trajectory, of (9), defined by the control $w(\cdot)$ and by the initial condition $z(t_0, y, w(\cdot)) = y$ on some maximal interval $J \subset I$ of the existence of the solution.

For all $\xi \in \mathbb{R}^k$, define

$$\omega(\xi, t) = \dot{x}_p(\xi, t) - \int\limits_{t_0}^{t} g_p(t, s, x_1(\xi, s), ..., x_p(\xi, s))ds, \ t \in I. \tag{10}$$

Then

$$y(\xi, t) = z(t, y^0, \omega(\xi,\cdot)) \text{ for all } \quad t \in I, \ \ \xi \in \mathbb{R}^k. \tag{11}$$

Then, using the Gronwall-Bellmann lemma, we get the existence of $\delta(\cdot)$ in $C(\mathbb{R}^k;]0, +\infty[)$ such that, for each $\xi \in \mathbb{R}^k$ and each $w(\cdot) \in L_\infty(I;\mathbb{R}^{m_p})$, we have:

$$\forall\, t \in I \quad |z(t, y^0, w(\cdot)) - y(\xi, t)| < \delta,$$

$$\text{whenever} \quad \| w(\cdot) - \omega(\xi,\cdot)\|_{L_\infty(I;\mathbb{R}^{m_p})} < \delta(\xi). \tag{12}$$

In order to complete the proof of Theorem 4.1, it suffices to prove the following Proposition, which is similar to Lemma 5.1 from [16].

Proposition 5.1 *Assume that $\{y(\xi,\cdot)\}_{\xi\in\mathbb{R}^k}$ is a family such that Conditions 1)-3) of Theorem 4.1 hold. Then, for system (7), there exist functions $M(\cdot) \in C(\mathbb{R}^k;]0,+\infty[)$ and a family $\{u(\xi,\cdot)\}_{\xi\in\mathbb{R}^k}$ of controls defined on I such that:*

1) For each $\xi \in \mathbb{R}^k$, the control $u(\xi,\cdot)$ is a piecewise constant function on I and the map $\xi \mapsto u(\xi,\cdot)$ is of class $C(\mathbb{R}^k; L_1(I;\mathbb{R}^{m_{p+1}}))$.

2) For each $\xi \in \mathbb{R}^k$, the trajectory $t \mapsto y(t,y^0,u(\xi,\cdot))$ is defined for all $t \in I$, and for each $\xi \in \mathbb{R}^k$ we have

$$|\omega(\xi,t) - f_p(t,y(t,y^0,u(\xi,\cdot)),u(\xi,t))| < \delta(\xi), \ \ t \in I$$

3) For each $\xi \in \mathbb{R}^k$, we have: $\| u(\xi,\cdot)\|_{L_\infty(I;\mathbb{R}^{m_{p+1}})} \le M(\xi)$.

Indeed, if Proposition 5.1 is proved, then, combining (10), (11), (12) with the form of the dynamics of (7),(9), we get

$$|y(t,y^0,u(\xi,\cdot)) - y(\xi,t)| < \delta \quad \text{for all} \ \ t \in I, \ \xi \in \mathbb{R}^k. \tag{13}$$

Using partitions of unity and arguing as in [15], [16], we get the existence of a family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^k\times\mathbb{R}^{m_{p+1}}}$ of controls such that Conditions 4) and 5) of Theorem 4.1 hold and such that for each $(\xi,\beta) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$ we have

$$|y(t,y^0,\hat{v}_{(\xi,\beta)}(\cdot)) - y(t,y^0,u(\xi,\cdot))| < \delta \ \text{ for all } t \in I, \tag{14}$$

($t \mapsto y(t,y^0,\hat{v}_{(\xi,\beta)}(\cdot))$ being defined on I for all (ξ,β) in $\mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$). Since $\delta = \frac{\varepsilon}{4}$, from (13), (14) and from Assumption 3) of Theorem 4.1 it follows that the family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^k\times\mathbb{R}^{m_{p+1}}}$ also satisfies Condition 6) of Theorem 4.1. This completes the proof of Theorem 4.1.

Remark 5.1 The main distinctions of the proof of Proposition 5.1 in comparison with that of Lemma 5.1 from [16] are as follows:

($\star$) In the current paper, we deal with the Volterra systems whereas [16] is devoted to the case of ODE.

($\star\star$) In the current work, the parameter ξ characterizes the terminal state and the system should be steered to starting from the initial point $y^0 \in \mathbb{R}^k$. In [16], the construction starts with the initial condition $z(\xi,T) = \xi$ given at the *terminal* instant T, and then the control strategy is adjusted inductively ([16], Lemma 6.1) while time is decreasing (from $t = T$ until the initial instant $t = t_1$) in order to reach a certain small neighborhood of the initial state. However, for the Volterra systems, such an invertion of time is not possible in general (and one cannot consider the Cauchy initial condition at terminal instant T). Therefore the direct repetition of the argument from [16], Section 6 would not suit.

($\star\star\star$) In the current work, we consider the non-smooth case (the right-hand side of (2) satisfies the local Lipschitz condition only).

5.1 Proof of Proposition 5.1

Following [16], choose any sequence $\{R_q\}_{q=1}^{\infty}\subset\mathbb{N}$ such that $R_1=1$, $R_{q+1}>R_q+1$, $q\in\mathbb{N}$. Define

$$\delta_q := \frac{1}{2}\min_{\xi\in\overline{B_{R_{q+1}}(0)}}\delta(\xi), \qquad M_q := \max_{\xi\in\overline{B_{R_q}(0)}}\| y(\xi,\cdot)\|_{C(I;\mathbb{R}^k)} + 4\delta + 1, \quad q\in\mathbb{N}; \tag{15}$$

$$K_q := \{y\in\mathbb{R}^k \mid |y|\le M_q\}; \;\; d_q := M_{q+2}+1, \;\; q\in\mathbb{N}; \tag{16}$$

$$W_q := \{\omega\in\mathbb{R}^{m_p} \mid |\omega|\le \max_{\xi\in\overline{B_{R_q}(0)}}\|\omega(\xi,\cdot)\|_{C(I;\mathbb{R}^{m_p})}+1\}, \quad q\in\mathbb{N}; \tag{17}$$

$$\Xi_1 := \overline{B_{R_1}(0)}; \;\; \Xi_{q+1} = \overline{B_{R_{q+1}}(0)}\backslash\overline{B_{R_q}(0)}, \;\; q\in\mathbb{N}; \tag{18}$$

$$E_1 := \overline{B_{R_1}(0)}\times I\times K_1;$$

$$E_{q+1} := E_q\bigcup\left(\left(\overline{B_{R_{q+1}}(0)}\setminus B_{R_q}(0)\right)\times I\times K_{q+1}\right), \;\; q\in\mathbb{N}; \tag{19}$$

$$E := \bigcup_{q=1}^{\infty}E_q. \tag{20}$$

Given an arbitrary $q\in\mathbb{N}$, and arbitrary $N\in\mathbb{N}$, define

$$\Lambda_N^q := \{(t,y,v)\in I\times K_{q+1}\times\mathbb{R}^{m_p} \mid \exists\overline{v}\in\mathbb{R}^{m_{p+1}}(|\overline{v}|\le N)\wedge(|\omega-f_p(t,y,\overline{v})|<\frac{\delta_q}{3}\}.$$

Then every Λ_N^q is open as a subset of the metric space $I\times K_{q+1}\times\mathbb{R}^{m_p}$ whose metric is generated by the norm of $\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^{m_p}$. Since $I\times K_{q+1}\times W_q$ is compact w.r.t. this metric space, using condition (iii) and the inclusions $\Lambda_N^q\subset\Lambda_{N+1}^q$ and $I\times K_{q+1}\times W_q\subset\bigcup_{N=1}^{\infty}\Lambda_N^q$, we obtain the existence of $N_0(q)\in\mathbb{N}$ such that $I\times K_{q+1}\times W_q\subset\Lambda_{N_0(q)}^q$. Without loss of generality, we assume that $N_0(q)\le N_0(q+1)$.

Define

$$U_q := \{v\in\mathbb{R}^{m_{p+1}} \mid \; |v|\le N_0(q)\}. \tag{21}$$

Then $U_q\subset U_{q+1}$, $q\in\mathbb{N}$ and, by the construction, for each $(t,y,\omega)\in I\times K_{q+1}\times W_q$ there exists $v\in U_q$ such that $|\omega-f_p(t,y,v)|<\frac{\delta_q}{3}$. Let $\{L_q\}_{q=1}^{\infty}\subset\mathbb{R}$ and $L(\cdot)\in C(\mathbb{R}^k;]0,+\infty[)$ be such that $0<L_{q+1}\le L_q$, $q\in\mathbb{N}$ and

$$2L_q(|\varphi(t,y,v)|+(T-t_0)|\psi(t,s,y,v)|+1)\le 1 \;\; \forall(t,s,y,v)\in I^2\times\overline{B_{d_q}(0)}\times U_{q+2}, \; q\in\mathbb{N}, \tag{22}$$

$$L_{q+1}\le L(\xi)\le L_q, \qquad \text{whenever } \xi\in\Xi_q, \;\; q\in\mathbb{N}. \tag{23}$$

Then we denote by $\digamma$ the following semi-ring of sets ([12, vol. 2, p. 17])

$$\Sigma_{\Theta(\cdot),\vartheta(\cdot),A_\Theta,A_\vartheta} := \{(\eta,s,z)\in\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^k \mid \vartheta(\eta,z)\le s\le\Theta(\eta,z)\}\backslash\{(\eta,s,z)\in\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^k$$
$$|(s=\Theta(\eta,z))\wedge((\eta,z)\in A_\Theta) \text{ or } (s=\vartheta(\eta,z))\wedge((\eta,z)\in A_\vartheta)\},$$

where $\Theta(\cdot)$, and $\vartheta(\cdot)$ range over the set of all the functions from class $C(\mathbb{R}^k\times\mathbb{R}^k;I)$ such that for all $(\xi,y,z)\in\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^k$

$$|\Theta(\xi,y)-\Theta(\xi,z)|\le L(\xi)|y-z| \;\text{ and }\; |\vartheta(\xi,y)-\vartheta(\xi,z)|\le L(\xi)|y-z|,$$

and $A_\Theta \subset \mathbb{R}^k \times \mathbb{R}^k$, $A_\vartheta \subset \mathbb{R}^k \times \mathbb{R}^k$ range over the set of all subsets of $\mathbb{R}^k \times \mathbb{R}^k$.

For each $(\xi, t, y) \in E$, let $q \in \mathbb{N}$ be such that $\xi \in \Xi_q$. From (18)-(20) it follows that $y \in K_{q+1}$. By (17), and by the definition of U_q, there exists $v_{\xi,t,y} \in U_q$ such that $|\omega(\xi, t) - f_p(t, y, v_{\xi,t,y})| < \frac{\delta_q}{3}$.

Using the compactness of each E_q in $\mathbb{R}^k \times I \times \mathbb{R}^k$ and the properties of semirings of sets (see Lemma 2 in [12, vol.2, p. 18]), we repeat the construction from [16, p.1435-1436] and obtain the existence of a sequence $\{(\xi_r, t_r, y_r)\}_{r=1}^{\infty}$, sequences $\{S_r\}_{r=1}^{\infty} \subset \digamma$ and $\{\Sigma_l\}_{l=1}^{\infty} \subset \digamma$ of sets from $\digamma$ and sequences of natural indices $1 \le r_1 < r_2 < \ldots < r_q < \ldots$ and $1 \le l_1 < l_2 < \ldots < l_q < \ldots$ such that first

$$(\xi_r, t_r, y_r) \in S_r \quad \text{and} \quad \forall (\eta, s, z) \in S_r \ \ (|\eta - \xi| < \frac{1}{4}) \wedge (|z - y| < \frac{1}{4}), \tag{24}$$

$$\forall (\eta, s, z) \in S_r \quad |\omega(\eta, s) - f_p(s, z, v_{\xi_r,t_r,y_r})| < \delta(\eta), \tag{25}$$

(this group of inequalities characterizes the size of S_r and the properties of the feedback controller to be constructed), second

$$E \subset \bigcup_{r=1}^{\infty} S_r; \ \text{ and } \ E_q \subset \bigcup_{r=1}^{r_q} S_r, \text{ for all } q \in \mathbb{N}, \tag{26}$$

$$S_r \bigcap E_1 \neq \emptyset, \qquad \text{if } 1 \le r \le r_1; \quad \text{and} \quad S_r \bigcap \Big(\Big(\overline{B_{R_{q+1}}(0)} \backslash B_{R_q}(0) \Big) \times I \times K_{q+1} \Big) \neq \emptyset,$$
$$\text{if } r_q + 1 \le r \le r_{q+1}, \tag{27}$$

$$S_r \bigcap \left(\bigcup_{j=1}^{r_q} S_j \right) = \emptyset, \quad \text{if } r \ge r_{q+1} + 1, \ \ q \in \mathbb{N}. \tag{28}$$

(this group of inclusions and inequalities characterizes the local finiteness of the countable covering $\{S_r\}_{r=1}^{\infty}$ of E), and third

(A_1) $\bigcup\limits_{r=1}^{r_q} S_r = \bigcup\limits_{l=1}^{l_q} \Sigma_l$ for all $q \in \mathbb{N}$ (which implies that $\bigcup\limits_{l=1}^{\infty} \Sigma_l = \bigcup\limits_{r=1}^{\infty} S_r$);
(A_2) $\Sigma_{l'} \bigcap \Sigma_{l''} = \emptyset$ for all $l' \neq l''$;
(A_3) for each $r \in \mathbb{N}$, there is a finite set of indices $P(r) \subset \mathbb{N}$ such that $S_r = \bigcup\limits_{l \in P(r)} \Sigma_l$.

This group of conditions characterizes the relationship between the original countable covering $\{S_r\}_{r=1}^{\infty}$ of E and its derivative covering $\{\Sigma_l\}_{l=1}^{\infty} \subset \digamma$, of E by mutually disjoint sets Σ_l, obtained by using the properties of semiring $\digamma$ [12, vol. 2, p. 18].

From (A_1), (A_2), (A_3), it follows that for every $l \in \mathbb{N}$ there exists $r(l) \in \mathbb{N}$ such that $\Sigma_l \subset S_{r(l)}$, and such that, if $1 \le l \le l_1$, then $1 \le r(l) \le r_1$, and if $l_q + 1 \le l \le l_{q+1}$ ($q \in \mathbb{N}$), then $r_q + 1 \le r(l) \le r_{q+1}$. Since $\Sigma_l \subset S_{r(l)}$, we obtain from (24), (26), (27):

$$\Big(B_{\frac{1}{2}}(\xi) \times I \times \mathbb{R}^k \Big) \bigcap \Sigma_l = \emptyset, \quad \text{whenever} \quad l \notin \Omega(\xi), \ \ l \in \mathbb{N}, \ \ \xi \in \mathbb{R}^k, \tag{29}$$

where $\Omega(\xi)$ is the finite number of indices given by

$$\Omega(\xi) := \begin{cases} \{l\}_{l=1}^{l_3}, & \text{if } \xi \in \Xi_1 \cup \Xi_2; \\ \{l\}_{l=l_{q-1}+1}^{l_{q+2}}, & \text{if } \ \xi \in \Xi_{q+1}, \ q \ge 2. \end{cases} \tag{30}$$

Define

$$v(\xi,t,y)=v_{\xi_{r(l)},t_{r(l)},y_{r(l)}},\quad \text{whenever } (\xi,t,y)\in\Sigma_l,\ l\in\mathbb{N}. \tag{31}$$

Then, from (25), (31), and from the inclusion $\Sigma_l \subset S_{r(l)}$, we obtain:

$$|\omega(\eta,s)-f_p(s,z,v(\eta,s,z))|<\delta(\eta) \text{ for all } (\eta,s,z)\in\bigcup_{l=1}^{\infty}\Sigma_l \tag{32}$$

Lemma 5.1 *1) For every $\xi\in\mathbb{R}^k$, there are a unique $z(\xi,\cdot)\in C(I;\mathbb{R}^k)$ such that*

$$z(\xi,t_0)=\xi, \tag{33}$$

a unique finite sequence of indices $\{\nu_j(\xi)\}_{j=1}^{N(\xi)}=\{\nu_j\}_{j=1}^{N(\xi)}\subset\Omega(\xi)$ such that $N(\xi)\le|\Omega(\xi)|$, and $\nu_\mu\ne\nu_j$ whenever $\mu\ne j$, and a unique finite sequence $t_0=\tau_1^*(\xi)<\tau_2^*(\xi)<\ldots<\tau_{N(\xi)}^*(\xi)<\tau_{N(\xi)+1}^*(\xi)=T$ such that:

1.a) $\dot z(\xi,t)$ is defined and continuous at each t in $I\setminus\{\tau_1^*(\xi),...,\tau_{N(\xi)}^*(\xi)\}$, and

$$(\xi,t,z(\xi,t))\in E \text{ and } |\omega(\xi,t)-f_p(t,z(\xi,t),v(\xi,t,z(\xi,t)))|<\delta(\xi),\quad t\in I \tag{34}$$

1.b) for each $j=1,\ldots,N(\xi)$, we have:

$$(\xi,t,z(\xi,t))\in\Sigma_{\nu_j}\quad \text{for all } t\in]\tau_j^*(\xi),\tau_{j+1}^*(\xi)[, \tag{35}$$

$$\dot z(\xi,t)=\varphi(t,z(\xi,t),v(\xi,t,z(\xi,t)))+\int_{t_0}^{t}\psi(t,s,z(\xi,s),v(\xi,s,z(\xi,s)))ds$$
$$\text{for all } t\in]\tau_j^*(\xi),\tau_{j+1}^*(\xi)[, \tag{36}$$

$$\tau_{j+1}^*(\xi)=\Theta_{\nu_j}(\xi,z(\xi,\tau_j^*(\xi))),\quad \text{and}\quad \tau_j^*(\xi)=\vartheta_{\nu_j}(\xi,z(\xi,\tau_{j+1}^*(\xi))) \tag{37}$$

2) Given any $\xi\in\mathbb{R}^k$, and any $l\in\mathbb{N}$, define $t\mapsto s_l(\xi,t)$ and $t\mapsto t_l(\xi,t)$ by

$$s_l(\xi,t)=t-\vartheta_l(\xi,z(\xi,t)),\quad t_l(\xi,t)=t-\Theta_l(\xi,z(\xi,t))\quad \text{for all } t\in I. \tag{38}$$

Then, for every $\xi\in\mathbb{R}^k$, and every $l\in\mathbb{N}$, first,

$$\frac{3(t-\tau)}{2}\ge s_l(\xi,t)-s_l(\xi,\tau)\ge\frac{t-\tau}{2}\ \text{whenever } t>\tau,\ l\in\mathbb{N}, \tag{39}$$

$$\frac{3(t-\tau)}{2}\ge t_l(\xi,t)-t_l(\xi,\tau)\ge\frac{t-\tau}{2}\ \text{whenever } t>\tau,\ l\in\mathbb{N}, \tag{40}$$

for all $t\in I$ and $\tau\in I$, and, second, there are unique $s_l^*(\xi)\in I$ and $t_l^*(\xi)\in I$ such that $s_l(\xi,s_l^*(\xi))=0$ and $t_l(\xi,t_l^*(\xi))=0$. Moreover, $t_0=s_{\nu_1}^*(\xi)$; $\tau_i^*(\xi)=t_{\nu_{i-1}}^*(\xi)=s_{\nu_i}^*(\xi)$ for every $i=2,...,N(\xi)$; and $T=t_{\nu_{N(\xi)}}^*(\xi)$.

The proof of the current Lemma 5.1, which is omitted, is by induction on $i\in\{1,\ldots,N(\xi)\}$ and is similar to that of Lemma 6.1 from [16]. The only difference is that the induction argument starts with the initial instant $t_0=\tau_1^*(\xi)$ whereas in [16] it starts with $T=\tau_1^*(\xi)$ down to t_0. Having proved Lemma 5.1 one combines it with the implicit function theorem and proves Lemma 5.2 (again by induction on $i\in\{1,\ldots,N(\xi)\}$).

Lemma 5.2 *For all* $i \in \{1, ..., N(\xi)\}$, *functions* $\eta \mapsto s^*_{\nu_i}(\eta)$, $\eta \mapsto t^*_{\nu_i}(\eta)$, $\eta \mapsto z(\eta, s^*_{\nu_i}(\eta))$, *and* $\eta \mapsto z(\eta, t^*_{\nu_i}(\eta))$ *defined in the previous Lemma 5.1 are continuous at every* $\xi \in \mathbb{R}^k$

The only differences of the proof of the current Lemma 5.2. in comparison with that of Lemma 6.2 from [16] are as follows: first one should use the implicit function theorem for the continuous monothone functions instead of C^1 - case (due to nonsmoothness), and second one needs to invert the time again in comparison with [16].

Finally, define the desired family of controls $\{u(\xi, \cdot)\}_{\xi \in \mathbb{R}^k}$ by

$$u(\xi, t) = v(\xi, t, z(\xi, t)) \quad \text{whenever} \quad t \in I, \;\; \xi \in \mathbb{R}^k. \tag{41}$$

From Lemmas 5.1 and 5.2 it immediately follows that the family $\{u(\xi, \cdot)\}_{\xi \in \mathbb{R}^k}$ defined by (41) satisfies all the Conditions 1),2),3) of Proposition 5.1. The proof of Proposition 5.1 is complete. This completes the proof of Theorem 4.1 and respectively those of Theorems 3.1 and 3.2.

6 Conclusion

The problem of global controllability of triangular integro-differential Volterra equations with noninvertible input-output links and with nonsmooth (Lipschitz continuous) dynamics has been solved. In addition we proved the global controllability of large scale interconnections of such systems when the cross-terms are bounded and Lipschitz continuous. The main distinctions of the current work in comparison with the thechniques used in preceeding works [15, 16] are as follows. First, in contrast to [15, 16], since the dynamics is not differentiable (but satisfies the local Lipschitz condition only) we cannot refer to the properties of the Frechet derivative of the input-state map $u(\cdot) \mapsto x(\cdot)$ that was essential in [15, 16] and cannot consider the linearized control system around a trajectory (which characterizes this Frechet derivative). Second, in contrast to [15] the input-output links $x_{i+1}(\cdot) \mapsto x_i(\cdot)$ are not invertible, which is why each virtual control needeed at each step of the backstepping procedure cannot be obtained as in [15] by solving the corresponding Volterra equations. To handle the second problem, we update some auxilary construction from [16] to the case of Volterra nonsmooth systems and to handle the first one we develop a backstepping design which is different from that from [15, 16]. All the arguments that are similar to those from [15, 16] are omitted and only essential changes are highlighted.

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Improved Multimachine Multiphase Electric Vehicle Drive System Based on New SVPWM Strategy and Sliding Mode — Direct Torque Control

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Abstract: This paper presents a Sliding Mode Direct Torque Control (SM-DTC) of a multiphase Induction Machine (IM) supplied with multiphase voltage source inverter (VSI) controlled by a new algorithm of Space Vector Pulse Width Modulation (SVPWM) for a high-performance multi-machine electric vehicle (EV) drive system. The SM-DTC is one of the effective nonlinear robust control approaches; it provides better dynamic performances of considered system. The new SVPWM algorithm develops a new analysis of voltage vectors to synthesize required phase voltages for driving multiphase IM with a minimum switch stress. Theoretical developments are verified for EV with two-separate-wheel-drives based on two pentaphase induction motors. The obtained results illustrate the effectiveness of the proposed drive system. Moreover, this system can be easily extended to an n-phase multi-machine drive system.

Keywords: *multiphase multimachine drive system; multiphase SVPWM; multiphase VSI; sliding mode; direct torque control.*

Mathematics Subject Classification (2000): 93C10, 93C85.

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1 Nomenclatures

s, r	: Stator, rotor indexes.
α, β	: Fixed stator reference frame indexes.
ref	: Reference index.
V, i, Φ	: Voltage, current, flux.
L	: Inductance.
R	: Resistance.
M	: Mutual inductance.
σ	: Total leakage coefficient.
n_p	: Pole pair number.
v	: Vehicle speed.
r	: Wheel radius.
$2d$	: Traction axle Length.
ρ	: Radius of way curvature.
ω	: Vehicle rotation speed.
Ω	: Motor speed.
Ω_R, Ω_L	: Speed of right and left motors.
Ω_R^*, Ω_L^*	: Reference speed of right and left motors.
T_{eR}^*, T_{eL}^*	: Reference torque of right and left motors.
T_i	: Simple time.
T_L	: Load torque.
y_1, y_2	: estimated stator flux components

2 Introduction

The research on development of electrical road vehicles aims to solve environment and energy problems caused by using the internal combustion engine vehicles (ICV). The first ones present many advantages as compared with the ICV ([1]– [8]).

The principal advantage of EV is the electric motor drive system. Thus, the trend within EV technology today is to develop Alternative Current (AC) motor drive systems for the next generation of such vehicles due to reduced size, weight, volume and maintenance.

The induction motors (IM) are relatively of a high reliability, high efficiency even in high speed range and low production cost. Therefore much attention is given to their control for various applications with different control requirements [2].

Recently, The Direct Torque Control (DTC) is more frequent in IM control. It is based on the decoupled control of stator flux and torque providing a quick and robust response with a simple control construction in AC drive ([5]– [9]). However, the conventional DTC presents a serious problem in low speed and in variation of motor parameters sensivity [10].

This paper presents a traction drive system for EV with two-independent-wheel-drives. This system includes two pentaphase induction motor drives controlled using hybrid control (SM-DTC). This control is one of the most effective nonlinear robust control approaches; it provides good dynamic performances of considered system [15]. In addition, this paper presents a new analysis of multiphase SVPWM for whatever number of phases. In order to synthesize an arbitrary phase voltage in terms of the times applied to the available switching vectors, the concept of orthogonal multi-dimensional vector space is used. An appropriate vector sequences are chosen to minimize switching losses.

Throughout this paper, the proposed algorithm is applied for supplying the pentaphase multimachine system proposed. The basic concepts can be easily extended to an n-phase system. The computational cost of the proposed strategy is low; it is well suited for real-time hardware implementation. The obtained results illustrate that with this configuration of EV Drive, it is possible to improve the stability of the vehicle under road conditions without any complicated mechanical components. Replacing the usual mechanical differential by an Electrical Differential (ED) is the solution to face the disadvantage of mechanical differential. This possibility is taken into consideration in this paper; the solution of ED is tested under different ways: straight-line, left and right turning.

3 Vehicle with Two-Separate-Wheel-Drives

The proposed EV Drive control can be used for all electric traction systems with two separate wheels drives. This system includes the elements represented in Figure 1. In this structure we find: two induction motors, two PWM inverters, the mechanic transmission system (motor to wheel), batteries, and control unit [3]. It is clearly noted that this topology of structure reduces the mechanic transmission components (mechanical differential operation is assured by an adequate control strategy of the two motors). Besides, this configuration offers the following advantages:

- Relatively, a good maneuverability: the torques of the two motors can be controlled independently precisingly and quickly.
- Elimination of mechanical differential.
- A good repartition of drive power.
- With other castor wheels used, the drive wheels can be the directional wheels.

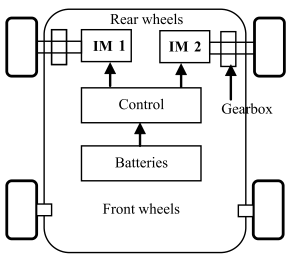

Figure 1: Traction system for electric vehicle with two-independent-wheel-drives.

4 Traction Drive System Proposed

Figure 2 illustrates the general scheme of the traction system proposed. The control of induction motors 1 and 2 are assured by SM-DTC. The torque references are generated by speed control of the two wheels, using SM controller. The speed references are generated by speed and direction orders. The speed and the direction orders are obtained, respectively, by the accelerator or brake pedals, and the steering wheel.

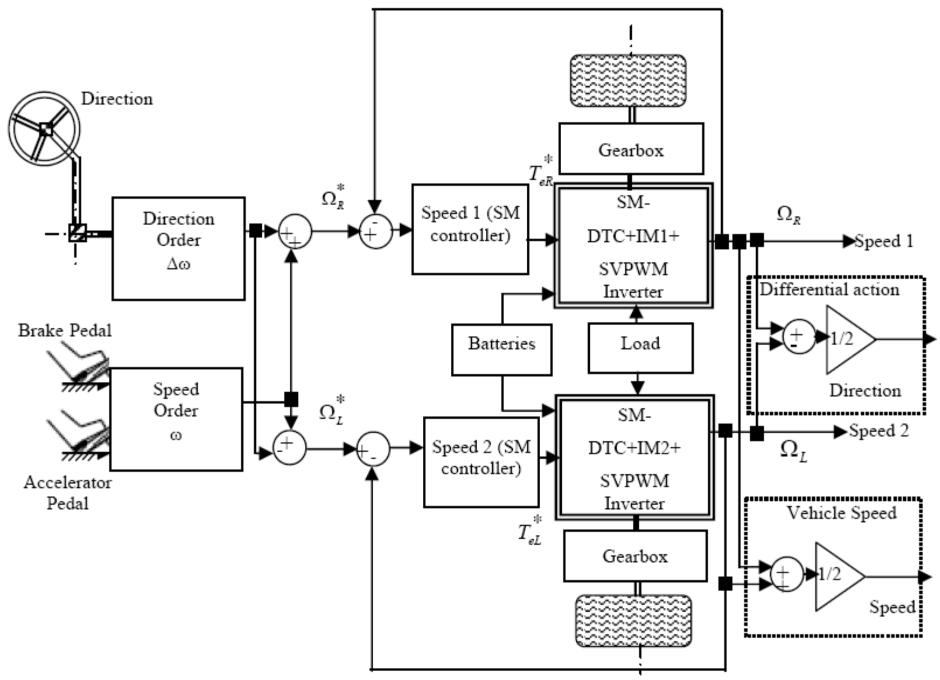

Figure 2: Scheme of the proposed drive system.

4.1 Electric differential

Figure 2 assumes that the linear speed of the vehicle v is imposed. The rotation speed for each motor depends on the type of driving regime selected.

For the straight road regime, the rotation speed for each motor becomes:

$$\Omega_L = \Omega_R = \frac{v}{r}. \tag{1}$$

For the turning regime, the angular speeds for each motor are different, for example in the left turning these speeds are expressed as [4]:

$$\Omega_L = \frac{2v}{\left(1+\dfrac{\rho+d}{\rho-d}\right)r} = \frac{v}{r} - \Delta\omega, \quad \Omega_R = \frac{2v}{\left(1+\dfrac{\rho+d}{\rho-d}\right)r} = \frac{v}{r} + \Delta\omega, \quad \Delta\omega = d.\frac{v}{\rho.r}, \tag{2}$$

where $\triangle\omega$ is imposed when the vehicle crosses a turning way.

4.2 Sliding mode – direct torque control

The Classic DTC presents the advantage of a very simple control scheme of stator flux and torque by two hysteresis controllers, which give the input voltage of the motor by selecting the appropriate voltage vectors of the inverter through a look-up-table in order to keep stator flux and torque within the limits of two hysteresis bands [5]. Figure 3 illustrates the general scheme for the classic DTC.

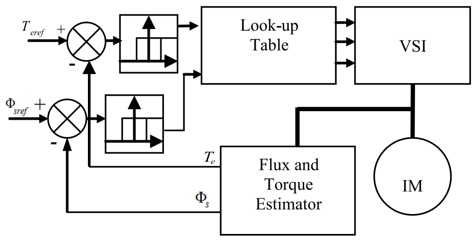

Figure 3: Basic of the Direct Torque Control Scheme.

The fast dynamic response of the classic DTC was entirely preserved, while the steady-state response was significantly improved even at a low switching frequency, but it was very sensitive to parameter uncertainties due to depending upon motor parameters.

The sliding mode control (SMC) is a very effective approach to solve the problem thanks to its well established design criteria, easy implementation, fast dynamic response, and robustness to parameter variations. Figure 4 illustrates the general scheme for the SM- DTC [11–14].

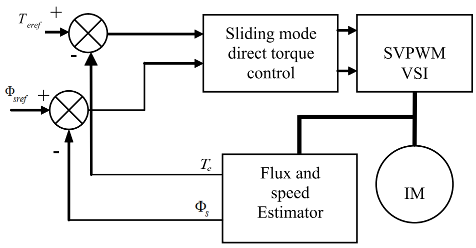

Figure 4: Structure of sliding mode DTC without switching table.

4.2.1 The induction motor model

The only difference between the five-phase motor model and the corresponding three-phase motor model is the presence of x-y component equations. Rotor x-y components are fully decoupled from d-q components and one from the other. Since rotor winding is short-circuited, x-y components cannot appear in the rotor winding. Zero sequence component equations for both stator and rotor can be omitted from further consideration due to short-circuited rotor winding and star connection of the stator winding. Finally, since stator x-y components are fully decoupled from d-q components and one from the other, the equations for x-y components can be omitted from further consideration as well. This means that the model of the five-phase induction motor in an arbitrary reference frame becomes identical to the model of a three-phase induction motor.

The induction motor model, developed in the reference frame (α, β) is described by (3). This model contains: four electrical variables (currents and flux), one mechanical variable and two control variables (stator voltages):

$$\begin{cases} \dot{x}_1 = \gamma.x_1 + \dfrac{\Gamma}{T_r}.x_3 + n_p.\Gamma.x_4.x_5 + \delta.V_{s\alpha}, \\ \dot{x}_2 = \gamma.x_2 + \dfrac{\Gamma}{T_r}.x_4 - n_p.\Gamma.x_3.x_5 + \delta.V_{s\beta}, \\ \dot{x}_3 = \dfrac{M}{T_r}.x_1 - \dfrac{1}{T_r}.x_3 - n_p.x_4.x_5, \\ \dot{x}_4 = \dfrac{M}{T_r}.x_2 - \dfrac{1}{T_r}.x_4 + n_p.x_3.x_5, \\ \dot{x}_5 = \eta.\left(x_2.x_3 - x_1.x_4\right) - \dfrac{T_L}{J}, \end{cases} \tag{3}$$

where the stator voltages and the states variables are:

$$V_s^T = \left[V_{s\alpha}, V_{s\beta}\right]^T, \qquad X^T = \left[x_1, x_2, x_3, x_4, x_5\right],^T = \left[i_{s\alpha}, i_{s\beta}, \Phi_{r\alpha}, \Phi_{r\beta}, \Omega\right]^T, \tag{4}$$

$$\begin{cases} \delta = \dfrac{1}{\sigma L_s}, \eta = \dfrac{n_p M}{J L_r}, \gamma = -\left(\dfrac{1}{\sigma T_s} + \dfrac{1-\sigma}{\sigma T_r}\right), \Gamma = \dfrac{1-\sigma}{\sigma M}, \\ \sigma = 1 - \dfrac{M^2}{L_s L_r}, T_s = \dfrac{L_s}{R_s}, T_r = \dfrac{L_r}{R_r}. \end{cases} \tag{5}$$

4.2.2 Switching surfaces selection

It is well known that the squared norm of the stator flux plays an important role in the performance of a motor and is also closely related to the electromagnetic torque. Therefore, we choose the control of the active torque u_T and the square of the flux norm $u_\Phi = \Phi^2$, which are defined as:

$$u_T = x_2.y_1 - x_1.y_2, \quad u_\Phi = y_1^2 + y_2^2. \tag{6}$$

Let's define the errors as:

$$e_1 = u_T - u_{T_{ref}}, \quad e_2 = u_\Phi - u_{\Phi_{ref}}, \tag{7}$$

where $u_{T_{ref}}$ and $u_{\Phi_{ref}}$ are the reference values of the active torque and the square of the flux norm, respectively.

The sliding-mode control is first used to find the sliding surface $S = 0$. In the present case, we adopt the integral function of the active torque and the square of the flux norm errors to obtain:

$$S_1 = e_1 + K_1 \int e_1 dt, \quad S_2 = e_2 + K_2 \int e_2 dt, \tag{8}$$

with K_1 and K_2 are positive constants.

4.2.3 Convergence conditions

So that control variables converge exponentially to their reference values, it is necessary for the surfaces to be null.

In addition, the realization of the sliding mode control is conditioned by checking the Lyaponov condition ([16]– [20]):

$$S_i.\dot{S}_i < 0, \;\; i = 1, 2, \tag{9}$$

and the invariance condition

$$\dot{S}_i = 0, \;\; i = 1, 2. \tag{10}$$

4.2.4 Switching function synthesis

Our goal is to generate a control law using the sliding mode control theory.

The derivative of the surfaces S_1 and S_2 will be:

$$\dot{S} = F + D.V, \tag{11}$$

where

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}; D = \begin{bmatrix} \frac{L_r.y_2}{\alpha.M} + x_2 & -\frac{L_r.y_1}{\alpha.M} - x_1 \\ 2.y_1 & 2.y_2 \end{bmatrix}, \tag{12}$$

$$\begin{cases} F_1 = \left(\frac{\beta}{\alpha + K_1}\right).u_T + n_p.x_5.\left(\phi_d + \frac{L_r}{\alpha.M}.u_\Phi\right) - K_1.u_{T_{ref}} - \dot{u}_{T_{ref}}, \\ F_2 = 2.R_s.\phi_d - \dot{u}_{\Phi_{ref}} - K_2.u_{\Phi_{ref}} + K_2.u_\Phi, \end{cases} \tag{13}$$

$$\alpha = M - \frac{L_s L_r}{M}, \quad \beta = \frac{L_s R_r + L_r R_s}{M}, \quad \phi_d = x_1.y_1 + x_2.y_2 \tag{14}$$

and to check the stability condition of Lyaponov, it is necessary to have:

$$\dot{S} = \mu.Sgn\,(S)\,. \tag{15}$$

By equalizing (15) and (11), we have the general control law:

$$V = -D^{-1}.\mu.Sgn\,(S) - D^{-1}.F. \tag{16}$$

We can write it as:

$$\begin{bmatrix} V_{s\alpha} \\ V_{s\beta} \end{bmatrix} = \begin{bmatrix} V_{eq\alpha} \\ V_{eq\beta} \end{bmatrix} + \begin{bmatrix} V_{c\alpha} \\ V_{c\beta} \end{bmatrix} \tag{17}$$

with definition of the equivalent control as:

$$\begin{bmatrix} V_{eq\alpha} \\ V_{eq\beta} \end{bmatrix} = -D^{-1}.\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{18}$$

and the commutation control as:

$$\begin{bmatrix} V_{c\alpha} \\ V_{c\beta} \end{bmatrix} = -D^{-1}.\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.\begin{bmatrix} Sgn\,(S_1) \\ Sgn\,(S_2) \end{bmatrix}. \tag{19}$$

Because the commutation control is included in the general control, it is necessary to choose μ_1 and μ_2 large enough: $\mu_1 > |F_1|\,, \quad \mu_2 > |F_2|\,.$

4.2.5 Chattering problem

It is well known that sliding-mode technique generates undesirable chattering; this problem can be solved by replacing the switching function with the saturation function [10]:

$$Sat\,(S_i) = \begin{cases} 1, & S_i > \lambda_i, \\ -1, & S_i < -\lambda_i, \\ \frac{S_i}{\lambda_i}, & |S_i| < \lambda_i, \end{cases} \tag{20}$$

where $\lambda_i > 0$ is a smooth factor.

5 SVPWM Algorithm Development

5.1 Principal of reference vector approximation

Similar to the SVM algorithm for three phase inverters, the reference space vector is used to select the corresponding set of nearest adjacent voltage space vectors. The adjacent vectors selected can synthesize a desired reference voltage vector using averaged approximation.

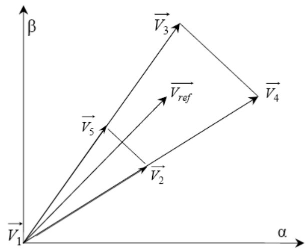

Figure 5: Generation of the reference vector by using five vectors.

If the reference vector lies in the sector connecting the tips of vectors $\overrightarrow{V_1}, \overrightarrow{V_2}, ..., \overrightarrow{V_n}$ (Figure 5), the average reference vector can be obtained with:

$$\overrightarrow{V}_{ref} = \frac{t_1}{T_i}\overrightarrow{V}_1 + \frac{t_2}{T_i}\overrightarrow{V}_2 + \frac{t_3}{T_i}\overrightarrow{V}_3 + ... + \frac{t_n}{T_i}\overrightarrow{V}_n, \tag{21}$$

where $t_1, t_2, ..., t_n$ must satisfy the condition $t_1 + t_2 + ... + t_n = T_i$.

5.2 Switching vector sets selection

From the vectors limiting one sector, we choose the sequence of vectors achieving one switch transition; there are (n+1) vectors. The sets of corresponding vectors are selected to be used in SVPWM algorithm; there are (n+1)/2 sets. For example, for seven phase inverter, Figures 6, 7, 8 present the sets selected with respect to the criteria of one switch transition, in the three plans. It is clear that the sets covered all range of reference vectors.

5.3 Applying time of switching vectors calculation

The duty cycles corresponding to voltage vectors are proportional to their distance from the reference vector.

5.4 Switching sequence arrangement

The row of applied nearest vectors depends on the sector number (even or odd). The row is showed by the arrow in Figures 9, 10 for five and seven phase inverters respectively.

For one vector approximation, the row of the elements in the sequence is reversed in the next half of the modulation period, as shown in Figures 11.

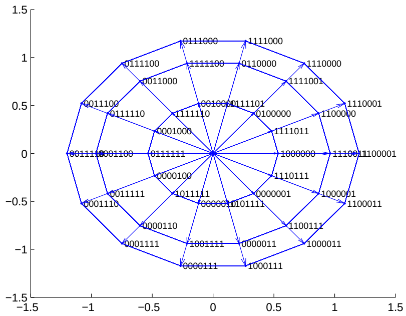

Figure 6: The sets selected for a seven phase inverter in the first plan.

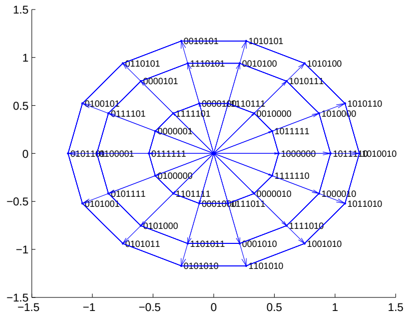

Figure 7: The sets selected for a seven phase inverter in the second plan.

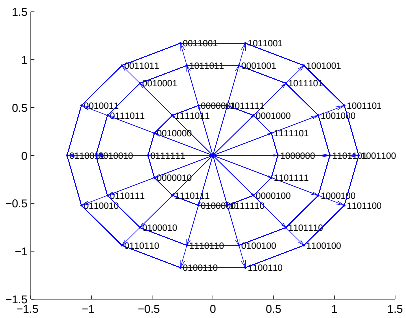

Figure 8: The sets selected for a seven phase inverter in the third plan.

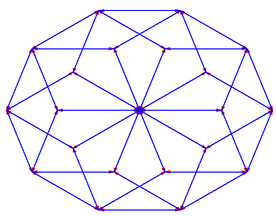

Figure 9: Vector sequence arrangement in the five phase inverter.

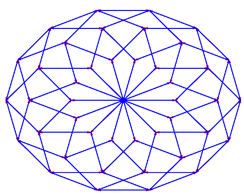

Figure 10: Vector sequence arrangement in the seven phase inverter

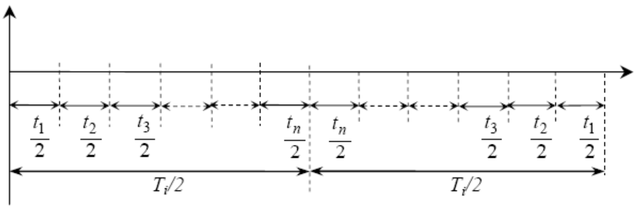

Figure 11: Switching sequence arranged in symmetrical pattern.

6 Simulation Results

6.1 Test of SVPWM strategy

To validate the proposed algorithm, simulation examples are realized for 5 and 7 phases inverters to indicate the simplicity of this algorithm for any number of phases as shown in Figures 12 to 19, which represent respectively the reference vector location (Figures 12, 16), the phase voltage (Figures 13, 17), switching sequence arrangement (Figures 14, 18), phase voltage spectrum (Figures 15, 19). Higher phases of SVM will be simulated with the same simplicity.

A deeper analysis of the resulting PWM voltage harmonics spectrum shows that the low order harmonics remain relatively weak and the increase of the phase number has an effect on the reduction of the harmonics content.

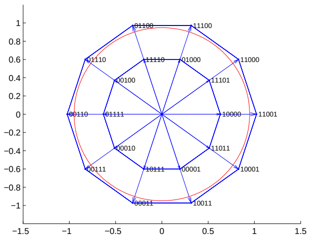

Figure 12: The reference vector location in the first plan for a five phase inverter .

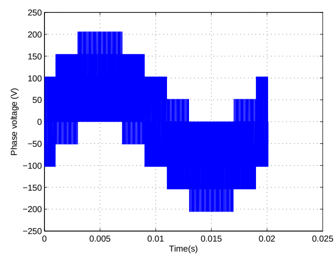

Figure 13: Phase voltage of a five-phase-VSI.

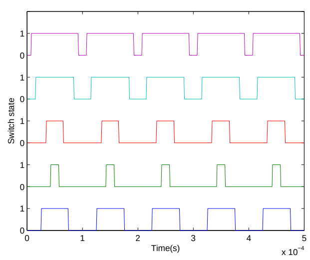

Figure 14: Switch state of a five-phase-VSI.

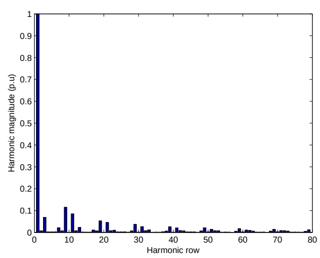

Figure 15: Phase voltage spectrum.

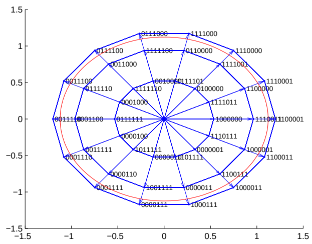

Figure 16: The reference vector location in the first plan of a seven-phase-inverter.

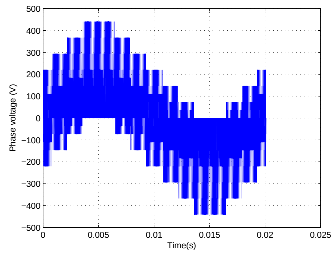

Figure 17: Phase voltage of a seven-phase-VSI.

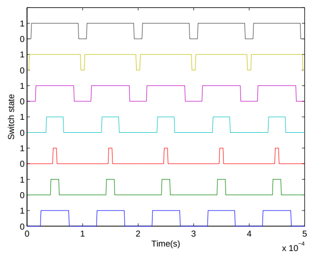

Figure 18: Switch state of a seven-phase-VSI.

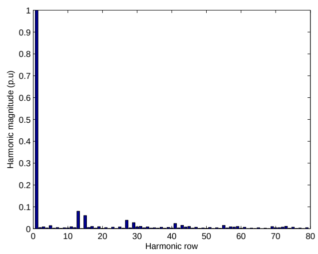

Figure 19: Phase voltage spectrum.

6.2 Test of speed and torque controls

Simulation results are obtained for a two identical squirrel cage pentaphase IM with parameters shown in the appendix. The reference speed represents the motion that the vehicle will have to cross. A trapezoidal form of speed is choosen, which allows simple calculations and also represents a realizable form. This form includes three phases:

- Phase 1: Constant acceleration; speed increases linearly.
- Phase 2: Null acceleration: constant speed.
- Phase 3: Constant acceleration; speed decreases linearly.

Figures 20, 22 and 24 represent the speed, torque and flux responses for the pentaphase induction motor respectively. The inverter phase current and the phase voltage of IM1 are shown in Figures 25, 26.

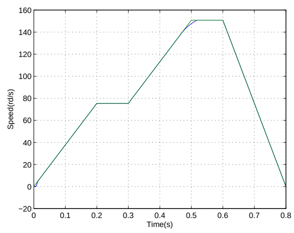

Figure 20: Motor speed.

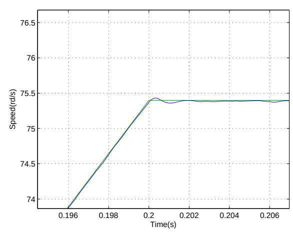

Figure 21: Motor speed zoom.

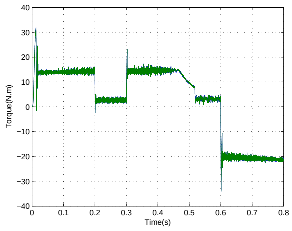

Figure 22: Motor torque.

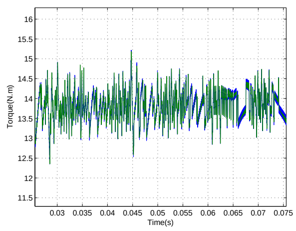

Figure 23: Motor torque zoom.

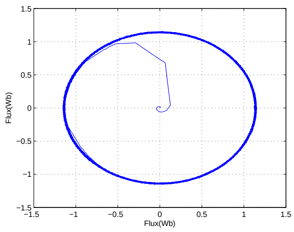

Figure 24: Motor flux.

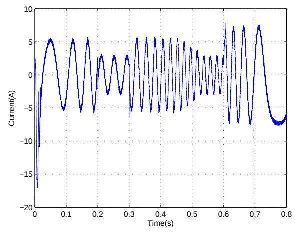

Figure 25: Stator phase current of IM1.

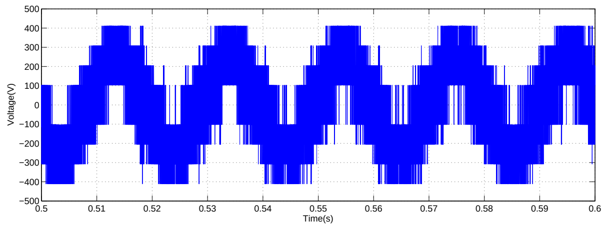

Figure 26: Inverter phase voltage.

It can be seen from Figures 20 that the motor speed tracks the reference very well and a small deviation appears only at the begining of the transient.

6.3 Test of electric differential

In order to test the performance of electrical differential used in electric vehicle, we have two interesting situations:

- The straight road regime, where both motors operate at the same speed.
- The turn regime, where each motor operates at a different speed.

Figures 27, 29 and 28, 30 represent the speed and torque responses for the two vehicle induction motors in a straight road, right and left turn. They show the follow up of the speed references and the motor speeds.

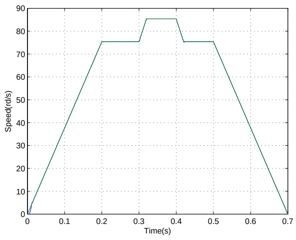

Figure 27: Right Motor speed.

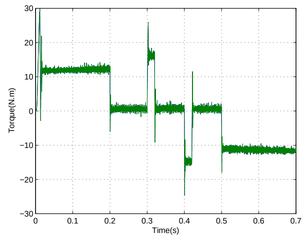

Figure 28: Right motor torque.

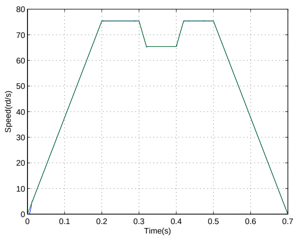

Figure 29: Left Motor speed.

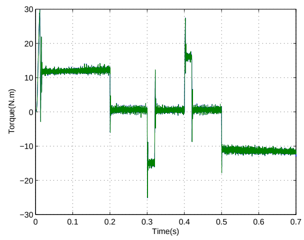

Figure 30: Left motor torque.

7 Conclusion

This paper describes a SM-DTC for a high-performance electric vehicle with two-separate-wheel-drive based on two pentaphase induction motors supplied by two pentaphase SVPWM VSIs. The proposed system aims at the elimination of hard mechanic devices (mechanic differential), and replacing it by soft ones (electric differential). Simulation tests have been carried out on a pentaphase induction motor drive. The obtained results illustrate that the sliding mode control provides a simple implementation in terms of time calculation with high performance of speed and torque response.

Appendix

Induction motors data:

Rated power	: 1Kw
Stator resistance	: 4.85 Ω
Rotor resistance	: 3.805 Ω
Stator inductance	: 0.274 H
Rotor inductance	: 0.274 H
Mutual inductance	: 0.258 H
Motor-Load inertia	: 0.031 Kg.m^2
Pole pairs	: 2

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Uncertain Dynamical Systems: Stability and Motion Control

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