Computing Interior Eigenvalues of Domains from Far Fields

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Abstract

Interior eigenvalues of bounded scattering objects can be rigorously characterised from multi-static and multi-frequency far field data, that is, from the behavior of scattered waves far away from the object. This characterisation, the so-called inside-outside duality, holds for various types of penetrable and impenetrable scatterers and is based on the behavior of a particular eigenvalue of the far field operator. It naturally leads to a numerical algorithm for computing interior eigenvalues of a scatterer that does not require shape or physical properties of the scatterer as input. Since the non-linear inverse problem to compute such interior eigenvalues from far field data is ill-posed, we propose a regularising algorithm that is shown to converge as the noise level of the far field data tends to zero. We illustrate feasibility and accuracy of our algorithm by numerical experiments where we compute interior transmission eigenvalues and Robin eigenvalues of the Laplacian in three-dimensional domains from scattering data of these domains due to plane incident waves.
Keywords: time-harmonic scattering, interior eigenvalue, inside-outside duality, regularisation.

1 Introduction

Time-harmonic scattering of incident waves from a domain $D \subset \mathbb{R}^3$ is always linked to a corresponding interior eigenvalue problem in D that arises naturally by asking when an incident wave does not scatter, that is, when the scattered wave for some incident wave vanishes entirely outside D. For example, seeking for non-scattering incident waves for domains with Dirichlet or Neumann boundary conditions leads to the interior eigenvalue problem for the Dirichlet or Neumann Laplacian [CK13] while for penetrable inhomogeneous media such incident waves are connected to the interior transmission eigenvalue problem [Kir86].

For several of these scattering problems interior eigenvalues of the scatterer can be rigorously characterised – or at least determined – by the eigenvalues of far field operators for a range of wave numbers using an inside-outside duality first shown in [EP95]: Roughly speaking, $k_0^2 > 0$ is an interior eigenvalue if and only if the eigenvalue of the far field operator $F = F(k)$ with the smallest or largest phase tends to zero as k tends to k_0. Since this condition can be explicitly checked if one possesses far field data for multiple frequencies, it is natural to use it for a numerical algorithm computing interior eigenvalues from far field data. Note, however, that the above-described criterion makes use of eigenvalues of F with small magnitude that are easily perturbed by measurement noise. In consequence, the computation of interior eigenvalues based on this criterion is an ill-posed non-linear problem and any inversion algorithm applied to this problem requires regularisation. In this paper, we prove convergence of such a regularisation technique, first proposed in [LP14], and demonstrate numerically that interior eigenvalues can be stably computed from scattering data. It is well-known

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that such interior eigenvalues yield information on, e.g., the size of the scattering object or the magnitude of material parameters [GN13, CCM07], which indicates the interest in algorithms for their computation from measured data in, e.g., non-destructive testing.

The algorithm we consider here is applicable to any scattering problem that satisfies the above-sketched inside-outside duality. Examples include acoustic scattering from obstacles with Dirichlet, Neumann, or Robin boundary condition [EP95, LP14], from isotropic or anisotropic penetrable acoustic media [KL13, LP15], and electromagnetic scattering from penetrable anisotropic dielectric media [LR15]. Since the algorithm always works in the same way for all these settings we concentrate in this paper on two representative problems, namely the computation of interior transmission eigenvalues from far field data of the corresponding isotropic inhomogeneous medium, and the computation of Robin eigenvalues of the Laplacian from far field data of the corresponding impenetrable obstacle with Robin boundary condition. Let us briefly indicate both scattering and eigenvalue problems here. A scattering problem from an inhomogeneous medium considers an entire solution u^i (the incident field) to the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$ with wave number $k > 0$ and a contrast function q supported in a domain $D \subset \mathbb{R}^3$ and seeks for a total field u, solution to

$$\Delta u + k^2(1+q)u = 0 \qquad \text{in } \mathbb{R}^3, \tag{1}$$

subject to a radiation condition for the difference $u^s = u - u^i$ (the scattered field). Whenever u^s vanishes outside of D for some non-trivial u^i, the pair $(u, u^i) =: (v, w)$ is an eigenpair to the following transmission eigenvalue problem with interior transmission eigenvalue $k^2 =: \mu$,

$$\Delta v + \mu(1+q)v = 0 \quad \text{and} \quad \Delta w + \mu w = 0 \text{ in } D, \qquad v = w \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D.$$

Next, scattering from a Robin obstacle involves an incident field u^i, a scatterer D, and a Robin coefficient τ, and seeks for a total field u solving

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \text{subject to} \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial D} + \tau\, u|_{\partial D} = 0 \quad \text{on } \partial D, \tag{2}$$

such that the scattered field $u^s = u - u^i$ again satisfies a radiation condition. If u^s vanishes outside of D, then the incident field $u^i =: v$ solves the following interior Robin eigenvalue problem for $-\Delta$ with eigenvalue $k^2 =: \mu$,

$$\Delta v + \mu v = 0 \quad \text{in } D, \qquad \left.\frac{\partial v}{\partial \nu}\right|_{\partial D} + \tau\, v|_{\partial D} = 0 \quad \text{on } \partial D.$$

A first link between the corresponding scattering and eigenvalue problems hence stems for the fact that a vanishing scattered field gives rise to an interior eigenvalue. (Unfortunately, this link is not easy to exploit for eigenvalue computations, see [KL13]).

The inside-outside duality instead offers a straightforward way to approximate interior eigenvalues from far field data via the dependence of the smallest of largest phase on the wave number, that does not require *any* knowledge on the nature of the scattering object: Compute this extremal phase of the eigenvalues of the far field operator for a sufficiently dense set of wave numbers in an interval of interest and check at which wave numbers this phase jumps; the resulting wave numbers are then roots of interior eigenvalues. In practice, this method requires multi-static and multi-frequency data for a sufficiently fine grid of wave numbers, which is an obvious drawback of the algorithm. (An adaptive algorithm that starts on a coarse grid of wave numbers and successively refines this grid merely close to wave numbers where the largest phase is close to π would be possible, but does not resolve the need to a-priori possess data for many wave numbers.) Note, however, that multi-frequency data for intervals of wave numbers is almost for free whenever the time-harmonic data are obtained from time-resolved measurements of time-dependent wave fields generated from incident pulses, simply by computing a Fourier transform in time of the measurements. Such time-resolved measurements are a common modality for, e.g., ultrasound waves.

Of course, an alternative way to compute interior eigenvalues is to determine first the scatterer from the given data and second the interior eigenvalues from the scatterer. While the advantage of this procedure certainly is its independence from multi-frequency data, its has at least three disadvantages, making the algorithm proposed below attractive whenever multi-frequency (or time-resolved) scattering data are at hand: First, to determine parameters of a scatterer one requires a model for those parameters that might not always be known. While Newton-like methods based on parameter-to-state mappings still are the workhorses for parameter identification tasks, these algorithms typically fail whenever the assumed setting is inaccurate. They further fail whenever the solution to the inversion problem is non-unique, which is always the case when material parameters are anisotropic. Second, algorithms for parameter identification typically require to solve many partial differential equations and, for this reason, are significantly more time-consuming than the technique analysed here, basically requiring to execute the QZ-algorithm once for each wave number under consideration for a matrix of small size. (Our numerical experiments later on involve between 35 and 130 wave numbers and the far field matrices are of size 120.) Moreover, the computation of interior eigenvalues for a given parameter setting further increases both the numerical workload and error and is far from trivial, in particular for the non-selfadjoint and non-linear interior transmission eigenvalue problem. Third, our method is a nice example of an algorithm for feature reconstruction in an inversion problem where one avoids to invert for the entire problem setting to extract the searched-for quantity of interest directly from the data. Such algorithms are prominent in inverse problems for reducing the ill-posedness of the inversion task and thus to increase accuracy of the solution while, at the same time, speeding up computation times.

This rest of the paper is structured as follows: In the next section we recall basic results on scattering from inhomogeneous media and obstacles, the associated far field operators and state the theoretical link between the scattering and eigenvalue problems. Section 3 provides approximation and interpolation results on the unit sphere that we use to construct discrete approximations to the far field operator in Section 4. Section 5 proves convergence of eigenvalues and phases of these approximations to those the exact far field operator. Finally, Section 6 proves the above-introduced main result on the largest regularised discrete phase. Feasibility and accuracy of the resulting algorithm is shown in Section 7.

2 Scattering and Eigenvalue Problems

We consider a bounded Lipschitz domain $D \subset \mathbb{R}^3$ with connected complement that plays to role of the scattering object and a wave number $k > 0$. When considering the transmission problem (1) we assume that the contrast function $q \in L^\infty(D)$ is essentially bounded and real-valued with support $\overline{D}$. When considering the Robin scattering problem (2) we assume that the coefficient $\tau \in L^\infty(\partial D, \mathbb{R})$ is real-valued. (Note that choosing $\tau = 0$ yields a Neumann boundary condition.)

Consider now an incident time-harmonic plane wave $u^i(x, \theta) = \exp(\mathrm{i}k\, \theta \cdot x)$ of direction $\theta \in \mathbb{S}^2 = \{z \in \mathbb{R}^3,\, |z| = 1\}$. For both Helmholtz equations (1) and (2) it is then well-known [CK13, KG08] that there exists a unique weak solution $u = u(\cdot, \theta) \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that the scattered field $u^s = u^s(\cdot, \theta) \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ defined by $u^s(\cdot, \theta) = u(\cdot, \theta) - u^i(\cdot, \theta)$ satisfies Sommerfeld's radiation condition,

$$\left(\frac{\partial u^s}{\partial |x|} - \mathrm{i}k u^s \right) = \mathcal{O}\left(\frac{1}{|x|^2} \right) \text{ as } |x| \to \infty, \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \in \mathbb{S}^2. \tag{3}$$

As a consequence of this radiation condition, the scattered wave behaves like an outgoing spherical wave,

$$u^s(x,\theta) = \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}\left(u^\infty(\hat{x},\theta) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \quad \text{as } |x| \to \infty,$$

with a far field pattern $u^\infty(\cdot,\theta) \in L^2(\mathbb{S}^2)$. The far field pattern describes the behavior of the scattered field far away from the obstacle and is roughly speaking the only information one can stably measure far away from the obstacle. The far field operator is then defined by

$$F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2), \qquad Fg(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x},\theta) g(\theta)\, \mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}^2, \tag{4}$$

that is, Fg is a linear combination of far fields using a density $g \in L^2(\mathbb{S}^2)$. For both governing equations (1) and (2) it is well-known that the associated far field operator is compact and normal as an operator in $L^2(\mathbb{S}^2)$, that is, there exists a complete orthonormal eigensystem $(\lambda_j, g_j)_{j\in\mathbb{N}}$ with eigenvalues λ_j tending to zero such that

$$Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j \qquad \text{for all } g \in L^2(\mathbb{S}^2).$$

It is moreover well-known that each eigenvalue λ_j lies on the circle of radius $8\pi^2/k$ centered at $8\pi^2\mathrm{i}/k$ in the complex plane, see, e.g., [KG08]. For this reason, the continuity of $k \mapsto F(k)$ (that can be shown using integral equation techniques) allows to choose the ordering of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ decreasing in magnitude such that additionally all functions $k \mapsto \lambda_j(k)$ are continuous. Later on, it will be convenient to represent the eigenvalues in polar coordinates,

$$\lambda_j = r_j \exp(\mathrm{i}\vartheta_j) \qquad \text{with } r_j \geq 0 \text{ and } \vartheta_j \in [0,\pi). \tag{5}$$

Whenever $r_j = 0$ we set $\vartheta_j = 0$. As $\lambda_j \in \{z \in \mathbb{C},\, |z - 8\pi\mathrm{i}/k| = 8\pi^2/k\}$ it holds that $0 \leq \vartheta_j \leq \pi$.

Concerning the far field operator for the Robin scattering problem (2,3) one can show that merely a finite number of its eigenvalues λ_j possesses a negative real part, that is, $\mathrm{Re}\,(\lambda_j) > 0$ for all $j \in \mathbb{N}$ large enough, see [LP14]. Since $\lambda_j \to 0$ as $j \to \infty$ the phases ϑ_j hence tend to zero as $j \to \infty$. In consequence, the largest phase $\vartheta^* = \max_{j\in\mathbb{N}} \vartheta_j$ of the eigenvalues is well defined and attained by some eigenvalue λ^* that does not vanish. The dependence of this largest phase on the wave number characterises the interior Robin eigenvalues of D (see Theorem 1 below). Concerning the transmission scattering problem, it follows from [KL13] that whenever the contrast function q in (1) is positive within D, the same properties hold for the eigenvalues of the far field operator for the transmission scattering problem (1,3).

To keep our notation compact we indeed restrict ourselves to positive contrast functions q such that $q \geq c_0 > 0$ in D, since then interior eigenvalues to both problems under investigation are determined by the largest phase of eigenvalues of the far field operator. Considering negative contrasts would require to work with the smallest phase of the eigenvalues to F; interior Dirichlet eigenvalues of D are characterised via the smallest phase as well. It is, however, not difficult to transfer all results shown below to such settings by, roughly speaking, exchanging the largest phase and its limit π by the smallest phase with limit zero, see [KL13, LP15, LR15].

While the above-introduced scattering problems are posed in the exterior of D the corresponding interior eigenvalue problems are posed inside D: First, the interior transmission eigenvalue problem corresponding to (1) is to find a transmission eigenvalue $\mu \in \mathbb{C}$ and an associated eigenpair $(v_\mu, w_\mu) \in L^2(D) \times L^2(D)$ such that $v_\mu - w_\mu \in H_0^2(D)$ satisfies

$$\Delta v_\mu + \mu(1+q)v_\mu = 0 \quad \text{in } D, \qquad \text{and} \qquad \Delta w_\mu + \mu w_\mu = 0 \quad \text{in } D. \tag{6}$$

Both equations are understood in the weak sense, that is,

$$\int_D v_\mu[\Delta\overline{\psi} + \mu(1+q)\overline{\psi}]\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_D w_\mu[\Delta\overline{\psi} + \mu\overline{\psi}]\,\mathrm{d}x = 0 \qquad \text{for all } \psi \in H_0^2(D).$$

This eigenvalue problem is non-selfadjoint and non-linear and, despite the importance of transmission eigenvalues in inverse scattering theory, existence of eigenvalues has only been shown recently, see [PS08, CGH10]. The numerical approximation of such eigenvalues when D and q are given poses significant difficulties due to the non-standard structure of the problem, see [Sun11, MS12, Kle13].

Second, the Robin eigenvalue problem corresponding to (2) is to find an eigenvalue $\mu \in \mathbb{R}$ and a corresponding eigenfunction $v_\mu \in H^1(D)$ such that

$$-\Delta v_\mu = \mu\, v_\mu \quad \text{in } D \qquad \text{and} \qquad \frac{\partial v_\mu}{\partial \nu} + \tau v_\mu = 0 \quad \text{on } \partial D \tag{7}$$

holds in the weak sense, that is,

$$\int_D \nabla v_\mu \cdot \nabla\overline{\psi}\,\mathrm{d}x + \int_{\partial D} \tau v_\mu \overline{\psi}\,\mathrm{d}S = \mu \int_D v_\mu\,\overline{\psi}\,\mathrm{d}x \qquad \text{for all } \psi \in H^1(D).$$

This eigenvalue problem is rather standard due to its linear and selfadjoint structure.

The following theorem states the link between the interior eigenvalues to (6) and (7) and the largest phase $\vartheta^*(k)$ of the eigenvalues $\lambda_j(k) = r_j(k)\exp(\mathrm{i}\vartheta_j(k))$ of the far field operator $F = F(k)$ to the scattering problems (1,3) and (2,3), respectively. For interior transmission eigenvalues, the duality statement is only a partial one, as it cannot be guaranteed that all eigenvalues are characterised by ϑ^*. This reflects the non-selfadjoint structure of the eigenvalue problem, possibly leading to complex eigenvalues not contained in $\mathbb{R}$.

Theorem 1 (Th. 6.3 in [KL13] & Th. 17 in [LP14])**.** *(a) If $q \in L^\infty(D)$ satisfies $q \geq c_0 > 0$ in D for some $c_0 > 0$ and if the largest phase satisfies $\lim_{i\to\infty} \vartheta^*(k_i) = \pi$ for some sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \neq k$ and $k_i \to k > 0$ as $i \to \infty$, then k^2 is an interior transmission eigenvalue.*

Further, for some $c_+ > 0$ and all $q \in W^{1,\infty}(D)$ with $\|\nabla q\|_{L^\infty(D)}$ small enough, the lower bound $q \geq c_+$ implies that $\lim_{i\to\infty} \vartheta^*(k_i) = \pi$ if $k^2 > 0$ is the smallest positive transmission eigenvalue and if $\{k_i\}_{i\in\mathbb{N}}$ with $k_i < k$ tends to k from below.

(b) If $\lim_{i\to\infty} \vartheta^*(k_i) = \pi$ for some sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \neq k$ and $k_i \to k > 0$ as $i \to \infty$, then k^2 is an interior Robin eigenvalue of $-\Delta$ in D. If $k^2 > 0$ is a Robin eigenvalue of $-\Delta$ in D then $\lim_{i\to\infty} \vartheta^*(k_i) = \pi$ for any sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i > k$ that tends to k from above.

Remark 2. *Since all phases ϑ_j of eigenvalues of F are contained in $[0, \pi)$ and since $k \mapsto \lambda_j(k)$ is continuous except at wave numbers where λ_j equals 0 or π, the last theorem implies that discontinuities of the largest phase can only happen at interior eigenvalues. For positive Robin eigenvalues, there always holds the converse statement as well.*

Proof. (a) Theorem 6.3(b) in [KL13] shows the first part of this statement, see also Corollary 6.4 formulated in terms of the far field instead of the scattering operator. Note that the proof of Theorem 6.3(b) in [KL13] is valid for any sequence of wave numbers k_i different from k but tending to k as $i \to \infty$, that is, the assumption $k_i < k$ of that proof is not necessary (see Theorem 15 in [LR15] for a corresponding proof without this assumption). The second part of Theorem 1(a) on variable contrasts follows as in Theorem 6.3 of [LP15] or Lemma 24 of [LR15].

(b) This statement follows from Theorem 17 in [LP14]; again, the assumption $k_i > k$ in that proof can be skipped. □

From now on we consider $F = F(k)$ to be the far field corresponding to one of the scattering problems (1,3) or (2,3) and will use this operator to compute the corresponding interior transmission or Robin eigenvalues without distinguishing these two cases explicitly, because the algorithm we rely on is independent of the underlying setting.

Theorem 1 motivates to numerically approximate interior eigenvalues by computing the eigenvalues of a finite-dimensional approximation to the far field operator F corresponding to either (1,3) or (2,3) for a grid of wave numbers by checking where the largest phase of these eigenvalues jumps. A crucial difficulty here is the ill-posedness of the underlying inverse eigenvalue problem: As the essential spectrum of F is the origin, any accurate finite-dimensional approximation F_N to F is likely to possess many small eigenvalues with arbitrary phases in a ball around zero whose radius equals the approximation error $\varepsilon = \|F_N - F\|$. Thus, a crucial regularisation step for the accurate computation of interior eigenvalues consists, roughly speaking, in neglecting eigenvalues of F_N that are too small to provide accurate phase information. (In the inverse problems language, this is a regularisation strategy.) The maximal phase of the remaining eigenvalues of F_N is called the largest regularised discrete phase and denoted by $\vartheta^{\divideontimes}(k, N)$.

Assuming that the sequence k_i tends to $k > 0$ as $i \to \infty$ and that the approximation error $\varepsilon = \varepsilon_{N_i}$ tends to zero as $i \to \infty$ we prove in Theorem 20 that k^2 is an interior eigenvalue if the largest regularised discrete phase $\vartheta^{\divideontimes}(k_i, N_i)$ tends to π as $i \to \infty$. Numerically checking for jumps of $\vartheta^{\divideontimes}$ on a finite grid of wave numbers K using discrete derivatives thus yields a simple and fast algorithm to approximate interior eigenvalues:

(1) Approximate the eigenvalues of $F_N(k)$ for $k \in K$ using the QZ-algorithm.

(2) Compute the largest phase $\vartheta^{\divideontimes}(k, N)$ of all eigenvalues of $F_N(k)$ with magnitude larger than ε.

(3) Check where the absolute value of the discrete derivative $\vartheta^{\divideontimes}(k, N)$ is larger than a fixed constant $\gg 1$ times the mean of all absolute values of this discrete derivative.

The resulting wave numbers approximate roots of interior eigenvalues. The accuracy of these approximations of course depends on the step size of the grid K and the approximation error of $F_N(k)$ for $k \in K$. Note, however, that the smoothness of the eigenvalue curves can be exploited to improve the eigenvalue estimates considerably by an extrapolation procedure (see Section 7).

3 Interpolation on the Sphere

The regularised algorithm for the computation of interior eigenvalues from far field data will be formulated and analysed in terms of approximate and discrete far field data. For its analysis we link this discrete data with the exact far field operator F, that is, to construct norm convergent finite-dimensional approximations to F. To this end, we consider in this section sequences of interpolation operators $\{\mathcal{I}_N\}_{N\in\mathbb{N}}$ possessing suitable approximation properties.

We start with a sequence of sets of pairwise different directions $\Theta_N := \{\theta_N^{(j)}\}_{j=1}^N \subset \mathbb{S}^2$ such that

$$h_N := \inf\left\{\left|\theta - \theta_N^{(j)}\right|,\, \theta \in \mathbb{S}^2,\, 1 \le j \le N\right\} \to 0 \qquad \text{as } N \to \infty. \tag{8}$$

Without loss of generality we suppose that $h_N \le 1$ for all $N \in \mathbb{N}$. To the directions Θ_N we associate continuous functions $\Phi_N := \{\phi_N^{(j)},\, 1 \le j \le N\} \in C^0(\mathbb{S}^2)$ and suppose that the matrix

$$A_{\Phi_N} = \left(\phi_N^{(\ell)}(\theta_N^{(j)})\right)_{j,\ell=1}^N \in \mathbb{C}^{N\times N}$$

is invertible and functions in Φ_N are linearly independent. This allows to define the interpolation operator

$$\mathcal{I}_N : C^0(\mathbb{S}^2) \to L^2(\mathbb{S}^2), \qquad \mathcal{I}_N[g] = \sum_{j=1}^N \left[A_{\Phi_N}^{-1}\left(g(\theta_N^{(j)})\right)_{j=1}^N\right](j)\,\phi_N^{(j)}, \tag{9}$$

such that $\mathcal{I}_N[g](\theta_N^{(j)}) = g(\theta_N^{(j)})$ for $j = 1, \dots, N$. Since functions in the Sobolev space $H^s(\mathbb{S}^2)$ with $s > 1$ are continuous due to Sobolev's embedding theorem, $\mathcal{I}_N$ is in particular well-defined and bounded from $H^s(\mathbb{S}^2)$, $s > 1$, into $L^2(\mathbb{S}^2)$. A crucial hypothesis on $\mathcal{I}_N$ is the following interpolation estimate on the latter function spaces: We assume that there is $\sigma = \sigma_{\mathcal{I}} \geq 0$ and $C_{\mathcal{I}} = C_{\mathcal{I}}(s, \sigma) > 0$ such that for some $s > 1 - \sigma$ there holds

$$\|g - \mathcal{I}_N[g]\|_{L^2(\mathbb{S}^2)} \leq C_{\mathcal{I}} h_N^s \|g\|_{H^{s+\sigma}(\mathbb{S}^2)} \qquad \text{for all } g \in H^{s+\sigma}(\mathbb{S}^2) \text{ and all } N \in \mathbb{N}. \tag{A1}$$

Of course, $\mathcal{I}_N$ is then bounded from $H^{s+\sigma}(\mathbb{S}^2)$ into $L^2(\mathbb{S}^2)$, $\|\mathcal{I}_N[g]\|_{L^2(\mathbb{S}^2)} \leq (1 + C_{\mathcal{I}})\|g\|_{H^{s+\sigma}(\mathbb{S}^2)}$.

Example 3 (Interpolation formulas)**.** *Several interpolation formulas satisfy Assumption* (A1)*.*

(a) Assume that $\{\Gamma_N^{(j)}, 1 \leq j \leq N\} \subset \mathbb{S}^2$ *are disjoint and relatively open subsets of* $\mathbb{S}^2$ *with Lipschitz boundary* $\partial\Gamma_N^{(j)}$ *such that the union of their closures is dense in* $\mathbb{S}^2$*, the numbers* $h_N = \sup_{j=1,\dots,N}\{|\hat{x} - \hat{y}|, \hat{x}, \hat{y} \in \Gamma_N^{(j)}\}$ *tend to zero as* $N \to \infty$*, and each direction* $\theta_N^{(j)} \in \Theta_N$ *belongs to* $\Gamma_N^{(j)}$*,* $j = 1, \dots, N$*. Recall from the Sobolev embedding theorem that functions in* $H^s(\mathbb{S}^2)$ *with* $s > 1$ *are Hölder continuous with index* $\min(s-1, 1)$*, that is,*

$$\|g\|_{C^{0,\min(s-1,1)}(\mathbb{S}^2)} = \sup_{\hat{x},\hat{y} \in \mathbb{S}^2} \left\{ \frac{|g(\hat{x}) - g(\hat{y})|}{|x - y|^{\min(s-1,1)}} \right\} \leq C\|g\|_{H^s(\mathbb{S}^2)}$$

with $C = C(s)$ *independent of* $g \in H^s(\mathbb{S}^2)$*, see [HÿS7, Prop. 8.6.10]. Denote the indicator function of the* j*th surface patch of* Γ_N *by* $\mathbf{1}_{\Gamma_N^j}$*, set* $V_{\Phi_N} = \mathrm{span}\{\mathbf{1}_{\Gamma_N^j}, j = 1, \dots, N\}$ *and define*

$$\mathcal{I}_N : C^0(\mathbb{S}^2) \to V_{\Phi_N} \subset L^2(\mathbb{S}^2), \qquad \mathcal{I}_N[g](\hat{x}) = \sum_{j=1}^N g(\theta_N^{(j)})\mathbf{1}_{\Gamma_N^j}(\hat{x}) \quad \textit{for } \hat{x} \in \mathbb{S}^2. \tag{10}$$

This projection is bounded from $H^s(\mathbb{S}^2)$ *into* $L^2(\mathbb{S}^2)$ *and for* $\hat{x} \in \Gamma_N^j$ *there holds*

$$|\mathcal{I}_N[g](\hat{x}) - g(\hat{x})| = |g(\theta_N^{(j)}) - g(\hat{x})| \leq \|g\|_{C^{0,s-1}(\mathbb{S}^2)} |\theta_N^{(j)} - \hat{x}|^{\min(s-1,1)} \leq C\, h_N^{\min(s-1,1)} \|g\|_{H^s(\mathbb{S}^2)}.$$

Integrating this estimate over $\mathbb{S}^2$ *shows that* $\mathcal{I}_N$ *from* (10) *satisfies* (A1) *with* $\sigma = 1$ *for all* $s \in (1, 2]$*.*

(b) If $\{\Gamma_N^{(j)}, 1 \leq j \leq N\}$ *is shape-regular and quasi-uniform family of curved surface panels on* $\mathbb{S}^2$ *(see [SS13, Ch. 4.1]), considering piecewise polynomial and globally continuous functions allows to construct interpolation formulas of higher order than* (10) *by suitably enlarging the set of interpolation points, see [SS13, Section 4.1.7, Theorem 4.3.21].*

(c) Reference [JSW99] discusses several interpolation schemes constructed via a strictly positive definite kernel $\kappa \in C^0(\mathbb{S}^2 \times \mathbb{S}^2)$*. Under appropriate assumptions on the nodal points* Θ_N *and the kernel* κ *several error estimates for the interpolation error in terms of Sobolev norms are shown. More details on interpolation on the sphere can be found in, e.g., [FGS98].*

We remark that functions in the N-dimensional subspace

$$V_{\Phi_N} = \mathrm{span}\{\phi_N^{(j)}, j = 1, \dots, N\} \subset L^2(\mathbb{S}^2) \tag{11}$$

spanned by the basis functions in Φ_N are interpolated exactly by $\mathcal{I}_N$, that is,

$$\mathcal{I}_N[\phi_N^{(j)}] = \phi_N^{(j)}, \quad j = 1, \dots, N, \qquad \text{and} \qquad \mathcal{I}_N[g] = g \quad \text{for all } g \in V_{\Phi_N}. \tag{12}$$

To compute the L^2-adjoint of $\mathcal{I}_N$ we define for $g \in L^2(\mathbb{S}^2)$ an associated coefficient vector $g_{\Phi_N} \in \mathbb{C}^N$,

$$g_{\Phi_N}(j) = (g, \phi_N^{(j)})_{L^2(\mathbb{S}^2)}, \qquad j = 1, \dots, N, \tag{13}$$

and introduce Dirac distributions $\delta_{\theta_N^{(j)}}$ at the directions $\theta_N^{(j)} \in \mathbb{S}^2$, defined by $\delta_{\theta_N^{(j)}}(g) = g(\theta_N^{(j)})$ for $g \in C^0(\mathbb{S}^2)$. Then

$$\begin{aligned}
(\mathcal{I}_N[f],\, g)_{L^2(\mathbb{S}^2)} &= \Big(\sum_{j=1}^N \big[A_{\Phi_N}^{-1} f_{\Theta_N}\big](j)\, \phi_N^{(j)},\, g \Big)_{L^2(\mathbb{S}^2)} = \sum_{j=1}^N \big[A_{\Phi_N}^{-1} f_{\Theta_N}\big](j)\, (\phi_N^{(j)},\, g)_{L^2(\mathbb{S}^2)} \\
&= \sum_{j=1}^N f_{\Theta_N}(j)\, \overline{\big[(A_{\Phi_N}^*)^{-1} g_{\Phi_N}\big]}(j) = \Big\langle f,\, \sum_{j=1}^N [(A_{\Phi_N}^*)^{-1} g_{\Phi_N}](j)\, \delta_{\theta_N^{(j)}} \Big\rangle_{H^s(\mathbb{S}^2)\times H^{-s}(\mathbb{S}^2)},
\end{aligned}$$

such that the adjoint $\mathcal{I}_N^* :\, L^2(\mathbb{S}^2) \to H^{-s}(\mathbb{S}^2)$, $s > 1$, satisfies

$$\mathcal{I}_N^*[g] = \sum_{j=1}^N \big[(A_{\Phi_N}^*)^{-1} g_{\Phi_N}\big](j)\, \delta_{\theta_N^{(j)}}. \tag{14}$$

This operator is well-defined and bounded from $L^2(\mathbb{S}^2)$ into $H^{-s}(\mathbb{S}^2)$ if $s > 1$ as A_{Φ_N} is invertible,

$$\begin{aligned}
\|\mathcal{I}_N^*[g]\|_{H^{-s}(\mathbb{S}^2)} &\le \sup_{\|f\|_{H^s(\mathbb{S}^2)}=1} \sum_{j=1}^N [(A_{\Phi_N}^*)^{-1} g_{\Phi_N}(j)]\, f(\theta_N^{(j)}) \\
&\le \big\|(A_{\Phi_N}^*)^{-1}\big\|_2 \sup_{\|f\|_{H^s(\mathbb{S}^2)}=1} \sum_{j=1}^N |(g,\, \phi_N^{(j)})_{L^2(\mathbb{S}^2)}| \|f\|_{H^s(\mathbb{S}^2)} \le \big\|A_{\Phi_N}^{-1}\big\|_2 \sum_{j=1}^N \|\phi_N^{(j)}\|_{L^2(\mathbb{S}^2)} \|g\|_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

Choosing $s > 1 - \sigma$ with $\sigma = \sigma_{\mathcal{I}}$ from (A1), the latter estimate implies that

$$\begin{aligned}
\|g - \mathcal{I}_N^*[g]\|_{H^{-(s+\sigma)}(\mathbb{S}^2)} &= \sup_{\|f\|_{H^{s+\sigma}(\mathbb{S}^2)}=1} |\langle f,\, g - \mathcal{I}_N^*[g]\rangle| = \sup_{\|f\|_{H^{s+\sigma}(\mathbb{S}^2)}=1} |\langle f - \mathcal{I}_N[f],\, g\rangle| \\
&\le \sup_{\|f\|_{H^{s+\sigma}(\mathbb{S}^2)}=1} \|f - \mathcal{I}_N[f]\|_{L^2(\mathbb{S}^2)} \|g\|_{L^2(\mathbb{S}^2)} \le C_{\mathcal{I}} h_N^s \|g\|_{L^2(\mathbb{S}^2)}, \qquad g \in L^2(\mathbb{S}^2).
\end{aligned} \tag{15}$$

Again, this implies that $\|\mathcal{I}_N^*[g]\|_{H^{-(s+\sigma)}(\mathbb{S}^2)} \le (1 + C_{\mathcal{I}})\|g\|_{L^2(\mathbb{S}^2)}$ for $g \in L^2(\mathbb{S}^2)$.

We finally associate to $\mathcal{I}_N$ and $\mathcal{I}_N^*$ their discrete counterparts $Q_N :\, \mathbb{C}^N \to L^2(\mathbb{S}^2)$ and $Q_N^* :\, L^2(\mathbb{S}^2) \to \mathbb{C}^N$, defined by

$$Q_N g_N = \sum_{j=1}^N \big[A_{\Phi_N}^{-1} g_N\big](j)\, \phi_N^{(j)} \quad \text{for } g_N \in \mathbb{C}^N \quad \text{and} \quad Q_N^* g = \big(\big[(A_{\Phi_N}^*)^{-1} g_{\Phi_N}\big](j)\big)_{j=1}^N \quad \text{for } g \in L^2(\mathbb{S}^2).$$

(Recall from (13) that g_{Φ_N} has entries $g_{\Phi_N}(j) = (g,\, \phi_N^{(j)})_{L^2(\mathbb{S}^2)}$.) Due to

$$(Q_N g_N,\, g)_{L^2(\mathbb{S}^2)} = \sum_{j=1}^N \big[A_{\Phi_N}^{-1} g_N\big](j)\ (\phi_N^{(j)}, g)_{L^2(\mathbb{S}^2)} = \sum_{j=1}^N g_N(j)\, \big(\overline{(A_{\Phi_N}^*)^{-1} g_{\Phi_N}}\big)(j) = \langle g_N,\, Q_N^* g\rangle_{\mathbb{C}^N}$$

both operators are adjoint to each other. Further, $Q_N^* g \in \mathbb{C}^N$ is the vector used in (14) to define $\mathcal{I}_N^*$,

$$\mathcal{I}_N^* g = \sum_{j=1}^N [Q_N^* g](j) \delta_{\theta_N^{(j)}}, \tag{16}$$

and, by definition of $\mathcal{I}_N$ in (9), the equality

$$Q_N g_{\Theta_N} = \mathcal{I}_N[g] \qquad \text{holds for } g_{\Theta_N} = \big(g(\theta_N^{(j)})\big)_{j=1}^N \in \mathbb{C}^N. \tag{17}$$

The operator norm of Q_N is bounded by

$$\|Q_N g_N\|^2_{L^2(\mathbb{S}^2)} \leq \|A^{-1}_{\Phi_N}\|_2^2 \sum_{j=1}^{N} \|\phi_N^{(j)}\|^2_{L^2(\mathbb{S}^2)} \|g_N\|^2_{\mathbb{C}^N} = C_Q(N)\|g_N\|^2_{\mathbb{C}^N} \qquad \text{for } g_N \in \mathbb{C}^N, \tag{18}$$

with $C_Q(N) = \|A^{-1}_{\Phi_N}\|_2^2 \sum_{j=1}^{N} \|\phi_N^{(j)}\|^2_{L^2(\mathbb{S}^2)}$, and $\|Q_N^* g\|^2_{\mathbb{C}^N} \leq C_Q(N)\|g\|^2_{L^2(\mathbb{S}^2)}$ holds for Q_N^*.

Remark 4 (Bounds for $C_Q(N)$). *(a) For the interpolation scheme of Example 3(a) the approximation space $V_{\Phi_N} = \mathrm{span}\{\mathbf{1}_{\Gamma_N^j}, j = 1, \dots, N\}$ is spanned by indicator functions of the surface patches $\Gamma_N^{(j)}$. Thus, $\sum_{j=1}^{N} \|\phi_N^{(j)}\|^2_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} 1 \,\mathrm{d}S = 4\pi$. Since the interpolation $\mathcal{I}_N(g)$ equals $g(\theta_N^{(j)})$ on the jth patch, A_{Φ_N} is the unit matrix and $C_N(Q) = 4\pi$ for all $N \in \mathbb{N}$.*

(b) Without going into details we remark that the interpolation operators mentioned in Example 3(b) lead to constants $C_Q(N)$ that are uniformly bounded in N as well. However, interpolation schemes relying on, e.g., spherical harmonics do not necessarily yield constants $C_Q(N)$ that are uniformly bounded in N.

4 Far Field Operator Approximation from Discrete Far Field Data

Assume that we have inexact time-harmonic scattering data $u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)})$ for incoming plane waves with pairwise different directions $\theta_N^{(\ell)} \subset \mathbb{S}^2$, measured in the direction $\theta_N^{(j)}$ for $1 \leq j, \ell \leq N$. The noise level in these data is $\delta > 0$ and measured in the spectral matrix norm,

$$\left(\sum_{j,\ell=1}^{N} \left| u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) - u^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) \right|^2 \right)^{1/2} \leq \delta. \tag{19}$$

Our algorithm for the computation of interior eigenvalues from far field data is formulated and analysed in terms of the eigenvalues of the matrix

$$\mathbb{F}_N^\delta := \big(u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)})\big)_{j,\ell=1}^{N} \in \mathbb{C}^{N\times N}, \tag{20}$$

multiplied by a suitable weight matrix. For the analysis of our algorithm we link this discrete data with the far field operator F, that is, we construct from $\mathbb{F}_N^\delta$ an approximation F_N^δ that converges to F as $N \to \infty$ and $\delta \to 0$, such that the eigenvalues of the matrix $\mathbb{F}_N^\delta$ equal the non-zero eigenvalues of F_N^δ. The idea is to exploit the operator Q_N from the last section that, roughly speaking, maps scattering data associated to the directions $\theta_N^{(j)}$ to an interpolating function in $L^2(\mathbb{S}^2)$.

We consider a second $N \times N$-matrix $\mathbb{F}_N := \big(u^\infty(\theta_N^{(j)}, \theta_N^{(\ell)})\big)_{j,\ell=1}^{N}$ defined via the exact discrete scattering data, such that $\|\mathbb{F}_N^\delta - \mathbb{F}_N\|_2 \leq \delta$ due to (19). Using these matrices we define finite-dimensional approximations F_N and F_N^δ to the exact far field operator F by

$$F_N^\delta g = Q_N \mathbb{F}_N^\delta Q_N^* g = \sum_{j=1}^{N} \left[A^{-1}_{\Phi_N} \left(\mathbb{F}_N^\delta \big[(A^*_{\Phi_N})^{-1} g_{\Phi_N} \big] \right) \right] (j)\ \phi_N^{(j)} \qquad \text{for } g \in L^2(\mathbb{S}^2), \tag{21}$$

and an analogous definition for F_N with $\mathbb{F}_N^\delta$ replaced by $\mathbb{F}_N$. Both operators map $L^2(\mathbb{S}^2)$ into $L^2(\mathbb{S}^2)$; as the range of Q_N equals $V_{\Phi_N} = \mathrm{span}\{\phi_N^{(j)}, j = 1, \dots, N\}$, see (11), their ranges are included in V_{Φ_N} and their null spaces equal the orthogonal complement $V_{\Phi_N}^\perp$. Due to (16) and (17), F_N can also be represented as

$$F_N g = Q_N \mathbb{F}_N Q_N^* g = Q_N \left[j \mapsto (F \mathcal{I}_N^* g)(\theta_N^{(j)}) \right] = Q_N \left(F \mathcal{I}_N^* g\right)_{\Theta_N} = \mathcal{I}_N F \mathcal{I}_N^* g \qquad \text{for } g \in L^2(\mathbb{S}^2).$$

Lemma 5. *Assume that* $\{\mathcal{I}_N\}_{N\in\mathbb{N}}$ *satisfies Assumption* (A1) *for* $\sigma = \sigma_{\mathcal{I}} \geq 0$ *and* $s+\sigma > 1$. *Then* $\|F - F_N\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C h_N^s \to 0$ *for* $N \to \infty$.

Proof. By the approximation estimates (A1) and (15) for $\mathcal{I}_N$ and $\mathcal{I}_N^*$, respectively, and Lemma 21,

$$\begin{aligned}
\|(F - F_N)g\|_{L^2(\mathbb{S}^2)} &= \|Fg - \mathcal{I}_N F \mathcal{I}_N^* g\|_{L^2(\mathbb{S}^2)} \\
&\leq \|F(g - \mathcal{I}_N^* g)\|_{L^2(\mathbb{S}^2)} + \|(F - \mathcal{I}_N F)\mathcal{I}_N^* g\|_{L^2(\mathbb{S}^2)} \\
&\leq C_{\mathcal{I}} h_N^s \|F\|_{H^{-(s+\sigma)}(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \|g\|_{L^2(\mathbb{S}^2)} \\
&\qquad + \|I - \mathcal{I}_N\|_{H^{s+\sigma}(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \|F\|_{H^{-(s+\sigma)}(\mathbb{S}^2)\to H^{s+\sigma}(\mathbb{S}^2)} \|\mathcal{I}_N^* g\|_{H^{-(s+\sigma)}(\mathbb{S}^2)} \\
&\leq 2C_{\mathcal{I}} \|F\|_{H^{-(s+\sigma)}(\mathbb{S}^2)\to H^{s+\sigma}(\mathbb{S}^2)} (2 + C_{\mathcal{I}}) h_N^s \|g\|_{L^2(\mathbb{S}^2)} \leq C h_N^s \|g\|_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

□

Lemma 6. *If the error bound* (19) *holds, that is, if* $\|\mathbb{F}_N - \mathbb{F}_N^\delta\|_2 \leq \delta$, *then* $\|F_N - F_N^\delta\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C_Q(N)\,\delta$ *where* $C_Q(N)$ *is the constant from* (18).

Proof. As $\|F_N - F_N^\delta\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} = \|Q_N(\mathbb{F}_N - \mathbb{F}_N^\delta)Q_N^*\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)}$, the triangle inequality implies the claim. □

Theorem 7. *Assume that* $\{\mathcal{I}_N\}_{N\in\mathbb{N}}$ *satisfies Assumption* (A1) *for* $\sigma = \sigma_{\mathcal{I}} \geq 0$ *and* $s > 1 - \sigma > 0$. *Further, assume that the discrete, perturbed far field data* $\mathbb{F}_N^\delta$ *satisfy the error bound* (19). *Then*

$$\|F - F_N^\delta\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C h_N^s + C_Q(N)\delta, \qquad N \in \mathbb{N},$$

with C *independent of* $N \in \mathbb{N}$.

As mentioned above, for the analysis of our inversion method to compute interior eigenvalues from far field data, we additionally have to require a connection between the eigenvalues of the finite-dimensional operator F_N^δ and the eigenvalues of the matrix $\mathbb{F}_N^\delta$. To this end, we introduce weights $w_N(j) > 0$ for $1 \leq j \leq N \in \mathbb{N}$, defined by

$$w_N(j) := \left(\int_{\mathbb{S}^2} |\phi_N^{(j)}|^2 \,\mathrm{d}S \right)^{1/2} > 0, \qquad j = 1, \dots, N.$$

The corresponding diagonal matrix $\mathbb{W}_N = \operatorname{diag}(w_N(j))_{j=1}^N \in \mathbb{R}^{N\times N}$ is positive definite and invertible.

The following lemma proves the above-mentioned link between the spectra of $\mathbb{F}_N^\delta$ and of F_N^δ, assuming that the following condition linking the directions Θ_N with the basis functions Φ_N is satisfied:

$$A_{\Phi_N}^* \left(\mathbb{W}_N^2\, g_{\Theta_N} \right) = g_{\Phi_N} \qquad \text{for all } g \in V_{\Phi_N} = \operatorname{span}\{\phi_N^{(1)}, \dots, \phi_N^{(j)}\}. \tag{A2}$$

Recall that $g_{\Phi_N} \in \mathbb{C}^N$ is defined by $g_{\Phi_N}(j) = (g, \phi_N^{(j)})_{L^2(\mathbb{S}^2)}$ and that $g_{\Theta_N}(j) = g(\theta_N^{(j)})$, $j = 1, \dots, N$.

Theorem 8. *If Assumption* (A2) *is satisfied, then all eigenvalues of* $\mathbb{W}_N \mathbb{F}_N \mathbb{W}_N$ *and* $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ *are eigenvalues of* F_N *and* F_N^δ, *respectively, and any additional eigenvalue of* F_N *and* F_N^δ *must vanish.*

Proof. We first note that F_N^δ and F_N both map $V_{\Phi_N} \subset L^2(\mathbb{S}^2)$ into V_{Φ_N} and vanish on the orthogonal complement $V_{\Phi_N}^\top$. Hence, any eigenfunction of F_N^δ and F_N for a non-zero eigenvalue belongs to V_{Φ_N}.

Assume that $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N g_{\Theta_N} = \lambda g_{\Theta_N}$ for $\lambda \in \mathbb{C}$ and $0 \neq g_{\Theta_N} \in \mathbb{C}^N$. Since $\Phi_N = \{\phi_N^{(1)}, \dots, \phi_N^{(j)}\}$ is a basis of V_{Φ_N} and since A_{Φ_N} is invertible, there exists a unique $g \in V_{\Phi_N}$ such that $\mathbb{W}_N g_{\Theta_N} =$

$(A^*_{\Phi_N})^{-1} g_{\Phi_N} = Q^*_N g$. Hence, $\mathbb{F}^\delta_N Q^*_N(g) = \mathbb{F}^\delta_N \mathbb{W}_N g_{\Theta_N} = \lambda \mathbb{W}_N^{-1} g_{\Theta_N}$ and

$$\begin{aligned} F^\delta_N g &= Q_N(\mathbb{F}^\delta_N Q^*_N(g)) = \lambda Q_N(\mathbb{W}_N^{-1} g_{\Theta_N}) = \lambda \sum_{j=1}^N \big[A^{-1}_{\Phi_N} \mathbb{W}_N^{-2} (A^*_{\Phi_N})^{-1} g_{\Phi_N}\big](j)\, \phi_N^{(j)} \\ &\overset{(A2)}{=} \lambda \sum_{j=1}^N [A^{-1}_{\Phi_N} g_{\Theta_N}](j)\, \phi_N^{(j)} \overset{(9)}{=} \lambda \mathcal{I}_N[g] = \lambda g \qquad \text{because } g \in V_{\Phi_N}. \end{aligned}$$

If $g \in L^2(\mathbb{S}^2)$ is an eigenfunction of F^δ_N for the eigenvalue $\lambda \in \mathbb{C}$, then either $\lambda = 0$ or the eigenfunction $0 \neq g$ must belong to V_{Φ_N}. Thus, this eigenfunction is of the form

$$g = \sum_{j=1}^N \big[A^{-1}_{\Phi_N} \mathbb{W}_N g_{\Theta_N}\big](j)\, \phi_N^{(j)} = Q_N(\mathbb{W}_N^{-1} g_{\Theta_N}) \qquad \text{for some } g_{\Theta_N} \in \mathbb{C}^N,\ g \neq 0.$$

Since Q_N maps vectors in $\mathbb{C}^N$ to their unique interpolating function in V_{Φ_N}, it holds that

$$\begin{aligned} \lambda Q_N(\mathbb{W}_N^{-1} g_{\Theta_N}) = \lambda g = F^\delta_N g &= Q_N(\mathbb{F}^\delta_N Q^*_N(g)) = Q_N\big[\mathbb{F}^\delta_N (A^*_{\Phi_N})^{-1} g_{\Phi_N}\big] \\ &\overset{(A2)}{=} Q_N\big[\mathbb{F}^\delta_N W_N^2 g_{\Theta_N}\big] = Q_N\big[\mathbb{F}^\delta_N \mathbb{W}_N g_{\Theta_N}\big]. \end{aligned}$$

Moreover, $Q_N : \mathbb{C}^N \to V_{\Phi_N}$ is injective, which is sufficient to conclude that $\mathbb{F}^\delta_N \mathbb{W}_N g_{\Theta_N} = \lambda \mathbb{W}_N^{-1} g_{\Theta_N}$, that is, $\mathbb{W}_N \mathbb{F}^\delta_N \mathbb{W}_N g_{\Theta_N} = \lambda g_{\Theta_N}$. Setting $\delta = 0$ shows the same results for F_N and $\mathbb{F}_N$. □

Remark 9. *The approximation $\mathbb{W}_N \mathbb{F}^\delta_N \mathbb{W}_N$ mimics a usual numerical practice to use the matrix $\big(u^\infty_\delta(\theta_N^{(j)}, \theta_N^{(\ell)})\big)$, scaled by a diagonal weight matrix, as a numerical approximation of F.*

Corollary 10. *Under assumption* (A2)*, the spectral radius of $\mathbb{W}_N \mathbb{F}^\delta_N \mathbb{W}_N$ and of $\mathbb{W}_N \mathbb{F}_N \mathbb{W}_N$ equals the spectral radius of F^δ_N and of F_N, respectively. Furthermore, the operator norm $\|\mathbb{F}^\delta_N Q^*_N\|_{L^2(\mathbb{S}^2)\to\mathbb{C}^N}$ and $\|\mathbb{F}_N Q^*_N\|_{L^2(\mathbb{S}^2)\to\mathbb{C}^N}$ equals the operator norm $\|F^\delta_N\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)}$ and $\|F_N\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)}$, respectively.*

Proof. The equality of the spectral radii follow directly from the statement of Theorem 8. Moreover, the operator norm of an operator $T : X \to Y$ between Hilbert spaces X and Y equals the largest singular value of T, that is, the square root of the largest eigenvalue of T^*T. Due to Assumption (A2),

$$Q^*_N Q_N g_{\Theta_N} = (A^*_{\Phi_N})^{-1} \big(Q_N g_{\Theta_N}\big)_{\Phi_N} = (Q_N g_{\Theta_N})_{\Theta_N} = g_{\Theta_N} \qquad \text{for all } g_{\Theta_N} \in \mathbb{C}^N.$$

The operator norm equality now follows from $F^*_N F_N = (Q_N \mathbb{F}_N Q^*_N)^* Q_N \mathbb{F}_N Q^*_N = Q_N \mathbb{F}^*_N \mathbb{F}_N Q^*_N$. □

Example 11. *The interpolations from Example 3(a,b) satisfy Assumption* (A2)*: Recall that the piecewise constant interpolation relied on the indicator functions $\phi_N^{(j)} = \mathbf{1}_{\Gamma_N^j}$ of the surface patches $\Gamma_N^{(j)}$ from a surface partition of $\mathbb{S}^2$ such that $w_N^2(j) = \int_{\mathbb{S}^2} \phi_N^{(j)}\, \mathrm{d}S = \int_{\Gamma_N^{(j)}} 1\, \mathrm{d}S$. For $g \in V_{\Phi_N}$, the vector $g_{\Phi_N} \in \mathbb{C}^N$ consists of the entries $g_{\Phi_N}(j) = (g, \phi_N^{(j)})_{L^2(\mathbb{S}^2)} = \int_{\Gamma_N^{(j)}} g\, \mathrm{d}S = w_N^2(j) g(\theta_N^{(j)})$, that is, $g_{\Phi_N} = \mathbb{W}_N^2 g_{\Theta_N}$. As $\phi_N^{(\ell)}(\theta_N^{(j)}) = \delta_{j,\ell}$, A_{Φ_N} is the unit matrix and the equations $A_{\Phi_N} = A^{-1}_{\Phi_N} = A^*_{\Phi_N} = I_{\mathbb{C}^N}$ and $A^*_{\Phi_N}(\mathbb{W}_N^2 g_{\Theta_N}) = g_{\Phi_N}$ hold.*

More generally, whenever the interpolation formula $\mathcal{I}_N$ uses basis functions $\phi_N^{(j)}$ that equal one at the jth direction $\theta_N^{(j)}$ and vanish at all other directions from Θ_N, the matrix $A^*_{\Phi_N}$ is diagonal and Assumption (A2) *holds. This holds in particular, the pth order interpolation formulas from [SS13, Section 4.1].*

5 Approximating Phases of Eigenvalues of F from Far Field Data

In this section we provide estimates of the eigenvalues and their phases of the discrete far field operator approximation F_N^δ compared to those of the exact far field operator F. Recall from the last section that we defined incomplete and inexact far field data $\mathbb{F}_N^\delta = (u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}))_{j,\ell=1}^N \in \mathbb{C}^{N\times N}$ that we used to construct an approximation $F_N^\delta = Q_N \mathbb{F}_N^\delta Q_N^*$ to F via projection operators Q_N, see (21). We also showed in Theorem 8 that, roughly speaking, the eigenvalues of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ equal those of F_N^δ if (A2) is satisfied. Considering a sequence $\mathbb{F}_N^{\delta_N}$ of perturbed far field matrices with noise level $\delta_N > 0$, Theorem 7 shows that the following assumption on the noise level,

$$C_Q(N)\delta_N \to 0 \qquad \text{as } N \to \infty, \tag{22}$$

together with Assumption (A1) on the convergence of the interpolation scheme $\mathcal{I}_N$ implies that

$$\|F_N^{\delta_N} - F\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C h_N^s + C_Q(N)\delta_N \to 0 \qquad \text{as } N \to \infty. \tag{23}$$

Remark 12. *For the sake of convenience we do not explicitly denote the dependence of the noise level $\delta = \delta_N$ on N whenever there is no danger of confusion.*

As the determining criterion for interior eigenvalues in Theorem 1 relies on the phases of the eigenvalues of the far field operator F, we will first investigate how the eigenvalues of

$$\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N = \Big(w_N(j) u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) w_N(\ell) \Big)_{j,\ell=1}^N \in \mathbb{C}^{N\times N} \tag{24}$$

approximate the eigenvalues of F as $N \to \infty$ and then pass in a second step to their phases.

Due to the convergence $\|F - F_N^\delta\| \to 0$ as $N \to \infty$ from (23), perturbation results (see, e.g., [Kat95, Osb75] or [Cha81]) show that the spectrum $\sigma(F_N^\delta)$ of F_N^δ converges, in a sense explained below, to $\sigma(F) = \{\lambda_j,\, j \in \mathbb{N}\} \cup \{0\}$ of F. We need to derive quantitative estimates for this convergence. To this end, recall that $F_N^\delta : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ has a finite-dimensional image and is hence non-injective and compact. Thus, $\sigma(F_N^\delta)$ consists of a finite sequence of eigenvalues $\{\lambda_\ell^N\}_{\ell=1}^{J_N} \subset \mathbb{C}$, counted without multiplicities. Note that we do not explicitly denote the noise level $\delta > 0$ contained in the eigenvalues λ_ℓ^N of F_N^δ, to simplify notation. We order these eigenvalues according to their magnitude in decreasing order, such that $\lambda_{J(N)}^N = 0$. Bounds for the distance between the spectra of F and F_N^δ rely on the Bauer-Fike theorem [BF60]. A proof of the following variant can be found in [Bue, Theorem 11].

Theorem 13. *Let H be a separable Hilbert space and $A : H \to H$ a bounded normal operator such that there exists a bounded $W : \ell^2 \to H$ with bounded inverse and a bounded diagonal operator $\Lambda : \ell^2 \to \ell^2$ such that $A = W\Lambda W^{-1}$. If $B : H \to H$ is bounded and if μ is an eigenvalue of B, then*

$$\min_{\lambda\in\sigma(A)} |\mu - \lambda| \leq \|W^{-1}\|_{H\to\ell^2} \, \|B - A\|_{H\to H} \, \|W\|_{\ell^2\to H}.$$

Corollary 14. *For all eigenvalues λ_ℓ^N of F_N^δ it holds that*

$$\min_{j\in\mathbb{N}} |\lambda_\ell^N - \lambda_j| \leq \|F_N^\delta - F\|, \qquad \ell = 1, \dots, \mathcal{J}(N). \tag{25}$$

Proof. Apply Theorem 13 with $H = L^2(\mathbb{S}^2)$, $A = F = \sum_j \lambda_j (\cdot, g_j) g_j$ and $W : \ell^2 \to L^2(\mathbb{S}^2)$ defined by $W((c_j)) = \sum_{j\in\mathbb{N}} c_j g_j$. Orthonormality of the eigenbasis $\{g_j\}_{j\in\mathbb{N}}$ of F implies that W is an isometry and that $W^{-1}W = WW^{-1} = I$. Hence, Theorem 13 implies (25). □

Obviously, Corollary 14 implies that all eigenvalues λ_ℓ^N of F_N^δ have a distance to $\sigma(F) = \{\lambda_j\}_{j\in\mathbb{N}} \cup \{0\}$ of at most $\|F - F_N^\delta\|$. This statement does in general not hold when the roles of F_N^δ and F are exchanged.

Lemma 15. *Assume that $\|F - F_N^\delta\| < \varepsilon$ and that $\sigma_1 \subset \sigma(F)$ is a finite subset of $\sigma(F)$ such that the Hausdorff distance $\mathrm{dist}_H(\{\sigma_1\}, \sigma(F) \setminus \{\sigma_1\})$ between σ_1 and the rest of the spectrum of F satisfies $\mathrm{dist}_H(\{\sigma_1\}, \sigma(F) \setminus \{\sigma_1\}) > 2\varepsilon$.*

(a) All eigenvalues of F_N^δ are contained in an ε-neighborhood of $\sigma(F) = \{\lambda_j,\, j \in \mathbb{N}\} \cup \{0\}$.

(b) The ε-neighborhood $\sigma_1^\varepsilon = \{z \in \mathbb{C},\, \mathrm{dist}(z, \sigma_1) < \varepsilon\}$ contains at least one eigenvalue of F_N^δ.

(c) The closure of the linear hull of the eigenspaces corresponding to all eigenvalues of F in σ_1 is isomorphic to the closure of the linear hull of all eigenvalues of F_N^δ contained in σ_1^ε.

Proof. Parts (a) and (b) follow from Corollary 14. Part (c) follows from Theorem 3.16 in [Kat95, Chapter IV-§4]. □

For the next result, we recall that the set of eigenvalues λ_j^N of F_N^δ is precisely the set of eigenvalues of the matrix $\sigma(\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N)$ together with $\{0\}$. Depending on whether zero is an eigenvalue of the latter matrix or not, we choose $J_N^* \in \{J_N, J_N - 1\}$ such that $\{\lambda_j^N\}_{j=1}^{J_N^*}$ are the eigenvalues of the matrix $\sigma(\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N)$. Recall that we ordered the λ_j^N according to their magnitude, $|\lambda_1^N| \geq |\lambda_2^N| \geq \cdots \geq |\lambda_{J_N^*}^N|$.

Theorem 16. *Assume that $\mathcal{I}_N$, Φ_N, Θ_N and δ_N satisfy the assumptions* (A1), (A2) *and* (22).

(1) For $j \in \mathbb{N}$ there is $N_0 = N_0(j)$ such that the jth eigenvalue λ_j^N of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ is well-defined for $N \geq N_0(j)$.

(2) If $\lambda_j \neq 0$ is a non-zero eigenvalue of F, then there exists $\{j'(N)\}_{N\in\mathbb{N}}$, $1 \leq j'(N) \leq J(N)^*$, such that

$$|\lambda_{j'(N)}^N - \lambda_j| \leq \|F_N^\delta - F\| \leq C(h_N^s + C_Q(N)\delta_N) \overset{N\to\infty}{\to} 0 \qquad \text{as } N \to \infty.$$

The sum of the dimensions of the eigenspaces corresponding to all eigenvalues of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ contained in $B(\lambda_j, \varepsilon_N)$ equals the multiplicity of λ_j.

Proof. (1) Consider $N_0 = N_0(j) \in \mathbb{N}$ so large that $\varepsilon_N := \|F_N^\delta - F\|$ is strictly less than $\mathrm{dist}(\lambda_j, \sigma(F) \setminus \{\lambda_j\})/2$ for $N \geq N_0$. This choice is always possible due to our assumptions and Theorems 7 and 8, compare (23). According to Corollary 14 there exists $j'(N) \in \mathbb{N}$ such that $|\lambda_{j'(N)}^N - \lambda_j| \leq \varepsilon_N$. Since $\lambda_j \neq 0$ and since $0 \in \sigma(F)$ we moreover obtain that

$$|\lambda_{j'(N)}^N| \geq |\lambda_j| - \varepsilon_N \geq |\lambda_j - 0| - \varepsilon_N \geq \mathrm{dist}(\lambda_j, \sigma(F) \setminus \{\lambda_j\}) - \varepsilon_N > \varepsilon_N$$

due to our assumption on $N_0(j)$. Thus, $\lambda_{j'(N)}^N$ cannot vanish and hence must be an eigenvalue of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ according to Theorem 8. This implies that $1 \leq j'(N) \leq J(N)^*$.

Since $\varepsilon_N \to 0$ as $N \to \infty$ and since there exists an infinite number of different eigenvalues λ_j of F this shows that the number of non-zero eigenvalues of F_N^δ and hence also of the matrix $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ tends to infinity as $N \to \infty$.

(2) The given estimate of $|\lambda_{j'(N)}^N - \lambda_j|$ stems from (23). The sum of the dimensions of the eigenspaces corresponding to eigenvalues λ_ℓ^N included in $B(\lambda_j, \varepsilon_N)$ equals the multiplicity of λ_j according to Lemma 15. □

We noted in the introduction that the eigenvalues λ_j satisfy $\mathrm{Re}\,(\lambda_j) > 0$ for $j \in \mathbb{N}$ large enough and that hence there exists some eigenvalue λ^* with largest phase ϑ^* among all eigenvalues $\lambda_j = r_j \exp(\mathrm{i}\vartheta_j)$. The phase error of the eigenvalue approximation is investigated next. To this end, we write $\lambda_\ell^N = r_j^N \exp(\mathrm{i}\vartheta_j^N)$ with radius $r_j^N \geq 0$ and phase $\vartheta_j^N \in [0, 2\pi)$ and we set $\vartheta_j^N = 0$ in case that $\lambda_j^N = 0$.

As any eigenvalue λ_j of F with magnitude less than $\|F_N^\delta - F\|$ can be perturbed into an eigenvalue λ_ℓ^N with arbitrary phase, eigenvalues of F_N^δ close to the origin cannot carry reliable information on the phase of the corresponding exact eigenvalue of F.

Lemma 17. *(1) If $|\lambda_j| = r_j > \varepsilon_N := \|F_N^\delta - F\|$, then the phase of any eigenvalue $\lambda_{j'}^N \in B(\lambda_j, \varepsilon_N)$ of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ satisfies $|\vartheta_j - \vartheta_{j'}| < \arcsin(\varepsilon_N / r_j)$.*

(2) Under the assumptions of Theorem 16, assume that $N_0 = N_0(j)$ is so large that $\varepsilon_N < r_j$ for $N \geq N_0(j)$ and that $\{j'(N)\}_{N\in\mathbb{N}}$ is the sequence from Theorem 16(2) such that $\lambda_{j'(N)}^N \to \lambda_j$ as $N \to \infty$. Then

$$|\vartheta_{j'(N)}^N - \vartheta_j| < \arcsin(\varepsilon_N/r_j) \leq \frac{\pi}{2r_j}\|F - F_N^\delta\| \leq \frac{C\pi}{2r_j}(h_N^s + C_Q(N)\delta_N) \stackrel{N\to\infty}{\longrightarrow} 0$$

Proof. (1) Assume that λ_j is an eigenvalue of F and if $\lambda_{j'}^N$ is an eigenvalue of F_N^δ contained in the ball $B(\lambda_j, \varepsilon_N)$, compare Figure 1. The difference of the phases ϑ_j and $\vartheta_{j'}^N$ of λ_j and $\lambda_{j'}^N$ is bounded by the angle α between the line $[0, \lambda_j]$ and one of the tangents to the circle $\{|z - \lambda_j| = \varepsilon_N\}$. Elementary triangular geometry shows that $0 < \sin\alpha = \varepsilon_N/r_j < 1$, that is, $\alpha = \arcsin(\varepsilon_N/r_j)$. In consequence, $\vartheta_{j'}^N$ is included between the two numbers $\vartheta_j \pm \arcsin(\varepsilon_N/r_j)$.

(2) The second statement follows from $\arcsin(\varepsilon_N/r_j) \leq \pi\varepsilon_N/(2r_j)$ and the estimate of $\varepsilon_N = \|F - F_N^\delta\|$ from Theorem 16. $\square$

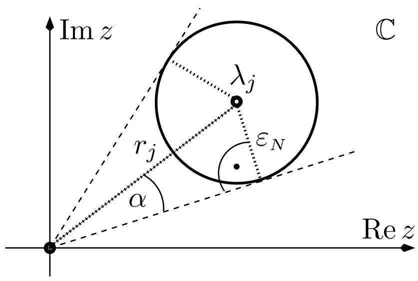


Figure 1: The maximal difference between ϑ_j and $\vartheta_{j'}^N$ is bounded by $\alpha = \arcsin(\varepsilon_N/r_j)$.

The last two results indicate two essential problems in view of our aim to stably detect interior eigenvalues of the scatterer D from discrete far field data: First, eigenvalues λ_ℓ^N of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ close to zero cannot provide accurate phase information on the corresponding eigenvalue λ_j of F due to the factor $1/r_j$ in the estimates of the last corollary or of Lemma 17. Second, Lemma 17 merely bounds the phase error in terms of the radius r_j of the exact eigenvalue λ_j. To derive analogous bounds in terms of the perturbed eigenvalues of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ we define $\varepsilon_N := \|F_N^\delta - F\|$ and consider an eigenvalue λ_ℓ^N of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ that satisfies

$$|\lambda_\ell^N| > 4\pi\,(\varepsilon_N/k)^{1/2} + \varepsilon_N. \tag{26}$$

(Recall that $k > 0$ is the wave number of the scattering problem used to define the far field operator F.) If the latter condition is satisfied then Corollary 14 states the existence of $\lambda_j \in \sigma(F)$ with $|\lambda_\ell^N - \lambda_j| \leq \varepsilon_N$. Then, necessarily,

$$r_j = |\lambda_j| > 4\pi\,(\varepsilon_N/k)^{1/2}.$$

Recall that λ_j lies on the circle $\{|8\pi^2 \mathrm{i}/k - z| = 8\pi^2/k\}$ in the complex plane, that is, real and imaginary part of $\lambda_j = a + \mathrm{i}b$ satisfy

$$a^2 + \left(b - \frac{8\pi^2}{k}\right)^2 = \left(\frac{8\pi^2}{k}\right)^2, \qquad \text{that is,} \qquad |\lambda_j|^2 = a^2 + b^2 = \frac{16\pi^2}{k}b.$$

The condition that $|\lambda_j|^2 = a^2 + b^2 > 16\pi^2 \varepsilon_N / k$ hence implies that $b = \mathrm{Im}\,(\lambda_j) > \varepsilon_N$. Hence, if λ_ℓ^N satisfies (26) then $\mathrm{Im}\,\lambda_\ell^N > 0$ such that its phase necessarily belongs to $(0, \pi)$.

Assume now additionally that N is so large that $\varepsilon_N^{1/2} < 4\pi/k^{1/2}$. By Lemma 17 applied to λ_ℓ^N and λ_j we obtain

$$|\vartheta_\ell^N - \vartheta_j| \leq \frac{\pi}{2} \frac{\varepsilon_N}{r_j} \leq \frac{1}{8} (k\varepsilon_N)^{1/2}. \tag{27}$$

Lemma 18. *Assume that λ_ℓ^N is an eigenvalue of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ that satisfies* (26) *and assume further that $\varepsilon_N^{1/2} < 4\pi/k^{1/2}$. Then the phase of λ_ℓ^N belongs to $(0, \pi)$. Further, there is an eigenvalue λ_j of F such that $|\lambda_\ell^N - \lambda_j| \leq \varepsilon_N$ and the phase difference $|\vartheta_\ell^N - \vartheta_j|$ between these two eigenvalues is bounded as in* (27).

6 Regularisation Theory for Computing Interior Eigenvalues

Theorem 1 states that k_0^2 is an interior eigenvalue if the largest phase $\vartheta^*(k)$ of the eigenvalues of $F(k)$ tends to π as k tends to k_0. To compute interior eigenvalues from discrete and noisy far field data we are hence interested in stably approximating the largest phase from the eigenvalues $\lambda_j^N(k)$ of $\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k) \mathbb{W}_N$, which basically means to ignore eigenvalues of $\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k) \mathbb{W}_N$ close to zero as those do not provide reliable phase information.

To model experimentally measured multi-frequency far field data we choose a discrete set of wave numbers

$$K = \{k_i\}_{i \in \mathbb{N}} \subset [k_{\min}, k_{\max}] \subset \mathbb{R}_{>0} \qquad \text{such that} \quad \overline{K} = [k_{\min}, k_{\max}]. \tag{A3}$$

Suppose that we know inexact discrete far field matrices $\mathbb{F}_N^{\delta_N}(k) \in \mathbb{C}^{N \times N}$ (see (20)) at wave numbers $k \in K$ such that the noise level

$$\max_{k \in K} \|\mathbb{F}_N^{\delta_N}(k) - \mathbb{F}_N(k)\|_2 \leq \delta_N, \quad N \in \mathbb{N}, \tag{A4}$$

is bounded by δ_N uniformly for all $k \in K$. Mimicking assumption (22), we moreover suppose that δ_N tends sufficiently fast to zero to ensure that

$$C_Q(N)\delta_N \to 0 \quad \text{as } N \to \infty. \tag{A5}$$

Further adopting the assumptions of Theorem 16, we suppose for all $N \in \mathbb{N}$ that the interpolation scheme $\mathcal{I}_N$ is convergent (compare (A1)) and that assumption (A2) holds, such that the eigenvalues $\lambda_\ell^N(k)$ of the matrix $\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k) \mathbb{W}_N$, roughly speaking, match those of the interpolated operator $F_N^{\delta_N}(k)$ for all $k \in K$ due to Theorem 8. Under these assumptions,

$$\varepsilon_N := \max_{k \in K} \|F_N^{\delta_N}(k) - F(k)\|_{L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)} \leq C h_N^s + C_Q(N)\delta_N \overset{N \to \infty}{\longrightarrow} 0. \tag{28}$$

We recall from Lemma 18 that for $k \in K$ and any eigenvalue $\lambda_\ell^N(k)$ of $F_N^{\delta_N}(k)$ the condition

$$|\lambda_\ell^N(k)| > 4\pi\, (\varepsilon_N / k)^{1/2} + \varepsilon_N, \tag{29}$$

together with the assumption $\varepsilon_N^{1/2} < 4\pi/k^{1/2}$ implies the existence of an eigenvalue $\lambda_j(k)$ of $F(k)$ such that $|\lambda_\ell^N(k) - \lambda_j(k)| \leq \varepsilon_N$ and such that the phases $\vartheta_\ell^N(k)$ and $\vartheta_j(k)$ of these two eigenvalues satisfy

$$|\vartheta_\ell^N(k) - \vartheta_j(k)| \leq \frac{1}{8} (k\varepsilon_N)^{1/2}. \tag{30}$$

To determine interior eigenvalues of D we need to construct a quantity that approximates the largest phase $\vartheta^*(k)$ of the eigenvalues to $F(k)$. To this end, we denote for $k \in K$ the largest phase of all eigenvalues of $\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k) \mathbb{W}_N$ that satisfy (29) by

$$\vartheta^{\circledast}(k,N) = \max \left\{ \vartheta_j^N, \ \lambda_j^N \in \sigma(\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k) \mathbb{W}_N), \ |\lambda_j^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N \right\}, \quad k \in K. \tag{31}$$

(The maximum of the empty set is set to zero.) The quantity $\vartheta^{\circledast}(k,N)$ is called the largest regularised discrete phase at discretisation level $N \in \mathbb{N}$ for the wave number $k \in K$.

Remark 19. *(1) The condition $|\lambda_\ell^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N$ in (31) implies by Lemma 18 that $\vartheta_\ell^N \in (0,\pi)$ if, additionally, $\varepsilon_N^{1/2} < 4\pi/k^{1/2}$. Thus, under the latter condition, $\vartheta^{\circledast}(k,N)$ is the maximum of finitely many phases $\vartheta_\ell^N \in (0,\pi)$ such that $\vartheta^{\circledast}(k,N) \in (0,\pi)$, too.*

(2) $\vartheta^{\circledast}(k,N)$ is computable whenever one possesses approximate far field data $\mathbb{F}_N^{\delta_N}(k)$ for some $N \in \mathbb{N}$ and $k \in K$ together with an upper estimate for the noise level ε_N from (28).

Theorem 20. *Assume that (A2), (A3), (A4), and (A5) hold, that $\{k_i\}_{i\in\mathbb{N}} \subset K$, $k_i \neq k$, is a sequence of wave numbers that tends to $k \in (k_{\min}, k_{\max})$, and denote the eigenvalue of $F(k_i)$ with largest phase by $\lambda^*(k_i)$ and its phase by $\vartheta^*(k_i)$.*

(a) If $\vartheta^{\circledast}(k_i, N_i) \to \pi$ for any sequence $\{N_i\}_{i \geq i_0} \subset \mathbb{N}$ with $N_i \to \infty$ as $i \to \infty$, then $\vartheta^*(k_i) \to \pi$ as $i \to \infty$ and k^2 is an interior eigenvalue of D.

(b) If $\vartheta^*(k_i) \to \pi$ there is $i_0 \in \mathbb{N}$ and a sequence $\{N_i\}_{i \geq i_0} \subset \mathbb{N}$ with $N_i \to \infty$ as $i \to \infty$ such that

$$4\pi \left(\frac{\varepsilon_{N_i}}{k_i} \right)^{1/2} + 2\varepsilon_{N_i} \leq \min \left\{ |\lambda^*(k_i)|, \operatorname{dist}\left[\lambda^*(k_i), \sigma\big(F(k_i)\big) \setminus \{\lambda^*(k_i)\} \right] \right\}, \qquad i \geq i_0, \tag{32}$$

and for any such sequence it holds that $\vartheta^{\circledast}(k_i, N_i) \to \pi$ as $i \to \infty$.

Proof. (a) Assume that $\{k_i\}_{i\in\mathbb{N}}$ is a sequence of wave numbers with $k_i \to k$ such that $\vartheta^{\circledast}(k_i, N_i) \to \pi$ as $i \to \infty$ for some sequence $\{N_i\}$ with $N_i \to \infty$ as $i \to \infty$. Recall from the definition of the largest regularised discrete phase $\vartheta^{\circledast}(k_i, N_i)$ in (31) that the eigenvalue $\lambda_{\ell_i}^{N_i}(k_i)$ of $T_{N_i} := \mathbb{W}_{N_i} \mathbb{F}_{N_i}^{\delta_{N_i}}(k_i) \mathbb{W}_{N_i}$ whose phase equals $\vartheta^{\circledast}(k_i, N_i)$ satisfies

$$|\lambda_{\ell_i}^{N_i}(k_i)| > 4\pi(\varepsilon_{N_i}/k_i)^{1/2} + \varepsilon_{N_i}, \quad i \in \mathbb{N}.$$

The assumptions (A2), (A3), (A4), and (A5) imply by (28) that the noise level ε_N tends to zero. Thus, if $i_0 \in \mathbb{N}$ is large enough, the assumption $\varepsilon_{N_i} < 16\pi^2/k_{\max} \leq 16\pi^2/k_i$ from Lemma 18 is satisfied for all $i \geq i_0$. Due to this lemma we deduce that for each $i \geq i_0$ there is an eigenvalue $\lambda_{\ell_i'}(k_i) \neq 0$ of $F(k_i)$ with phase $\vartheta_{\ell_i'}(k_i) \in (0,\pi)$ such that $|\lambda_{\ell_i}^{N_i}(k_i) - \lambda_{\ell_i'}(k_i)| \leq \varepsilon_{N_i} \to 0$ as $i \to \infty$. Moreover,

$$|\vartheta^{\circledast}(k_i, N_i) - \vartheta_{\ell_i'}(k_i)| = |\vartheta_{\ell_i}^{N_i}(k_i) - \vartheta_{\ell_i'}(k_i)| \leq \frac{1}{8}\big(k_i \varepsilon_{N_i}\big)^{1/2}, \quad i \in \mathbb{N}. \tag{33}$$

Finally, we exploit that $\vartheta_{\ell_i'}(k_i) \in (0,\pi)$ is less than or equal to the largest phase $\vartheta^*(k_i)$,

$$\pi \geq \vartheta^*(k_i) \geq \vartheta_{\ell_i'}(k_i) \geq \vartheta^{\circledast}(k_i, N_i) - \frac{1}{8}\big(k_i \varepsilon_{N_i}\big)^{1/2} \xrightarrow{i \to \infty} \pi.$$

Thus, $\vartheta^*(k_i) \to \pi$ as $i \to \infty$ and Theorem 1 implies that k^2 is an interior eigenvalue of D.

(b) If $\vartheta^*(k_i) \to \pi$ as $i \to \infty$, then Theorem 1 shows that k^2 is an interior eigenvalue, such that the discreteness of the interior transmission and Robin eigenvalues implies that there is $\beta > 0$ such that $(k^2 - \beta, k^2 + \beta)$ contains no other interior eigenvalue. Consequently, for some $i_0 \in \mathbb{N}$ it holds that $\{k_i\}_{i \geq i_0} \subset (k^2 - \beta, k^2 + \beta) \setminus \{k^2\}$. Recall that $\lambda^*(k_i) = r^*(k_i) \exp(\mathrm{i}\vartheta^*(k_i))$ denotes the eigenvalue of

$F(k_i)$ with largest phase $\vartheta^*(k_i)$. As k^2 is the only interior eigenvalue in $(k^2-\beta, k^2+\beta)$, Theorem 1 implies that $0 < \vartheta^*(k_i) < \pi$ and $r^*(k_i) > 0$ for all $i \geq i_0$. Further, the distance of $\lambda^*(k_i)$ to the rest of the spectrum of $F(k_i)$ is positive,

$$\rho_i := \text{dist}\left(\lambda^*(k_i),\, \sigma(F(k_i)) \setminus \{\lambda^*(k_i)\}\right) > 0 \quad \text{for } i \in \mathbb{N}.$$

As in part (a) of this proof, the assumptions of the theorem imply that the noise level ε_N tends to zero. Thus, there is a monotonously increasing sequence $\{N_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ such that (32) is satisfied, that is,

$$4\pi \left(\frac{\varepsilon_{N_i}}{k_i}\right)^{1/2} + 2\varepsilon_{N_i} < \min\left\{r^*(k_i),\ \rho_i\right\} \quad \text{for } i \geq i_0. \tag{34}$$

The last bound implies in particular that $2\varepsilon_{N_i} < \rho_i$ and hence Lemma 15 applied to $\lambda^*(k_i)$ and $F(k_i)$ shows that there exists at least one eigenvalue $\lambda^{N_i}_{\ell_i}(k_i)$ of $T_{N_i} := \mathbb{W}_{N_i}\mathbb{F}^{\delta_{N_i}}_{N_i}(k_i)\mathbb{W}_{N_i}$ such that $|\lambda^{N_i}_{\ell_i}(k_i) - \lambda^*(k_i)| \leq \varepsilon_{N_i}$ for $N \in \mathbb{N}$. As (34) implies that $4\pi(\varepsilon_{N_i}/k_i)^{1/2} + 2\varepsilon_{N_i} < r^*(k_i)$, we deduce that

$$|\lambda^{N_i}_{\ell_i}(k_i)| \geq r^*(k_i) - \varepsilon_{N_i} > 4\pi(\varepsilon_{N_i}/k_i)^{1/2} + \varepsilon_{N_i} \quad \text{for } i \geq i_0.$$

In particular, the definition of the largest regularised discrete phase

$$\vartheta^{⁂}(k_i, N_i) = \max\left\{\vartheta^{N_i}_j(k_i),\ \lambda^{N_i}_j(k_i) \in \sigma(T_{N_i}),\ |\lambda^{N_i}_j(k_i)| > 4\pi(\varepsilon_{N_i}/k_i)^{1/2} + \varepsilon_{N_i}\right\}$$

directly implies that $\vartheta^{⁂}(k_i, N_i)$ is larger than or equal to the phase $\vartheta^{N_i}_{\ell_i}(k_i)$ of $\lambda^{N_i}_{\ell}(k_i)$. As all eigenvalues of $F(k_i)$ lie on the circle $\{|8\pi^2\mathrm{i}/k_i - z| = 8\pi^2/k_i\}$ it holds that $\rho_i < 16\pi^2/k_i$ and (34) further implies that $\varepsilon^{1/2}_{N_i} < 4\pi/k_i^{1/2}$. Thus, we deduce from Lemma 18 first that $\vartheta^{N_i}_{\ell_i}(k_i) \in (0, \pi)$ and second that

$$|\vartheta^{N_i}_{\ell}(k_i) - \vartheta^*(k_i)| \leq \frac{1}{8}\left(k_i\varepsilon_{N_i}\right)^{1/2} \quad \text{for } N \in \mathbb{N}. \tag{35}$$

Hence, the inequalities

$$\vartheta^*(k_i) - \frac{1}{8}\left(k_i\varepsilon_{N_i}\right)^{1/2} \leq \vartheta^{N_i}_{\ell_i}(k_i) \leq \vartheta^{⁂}(k_i, N_i) < \pi \quad \text{for } i \geq i_0,$$

together with the convergence of $\vartheta^*(k_i)$ to π and of ε_{N_i} to zero, imply that $\vartheta^{⁂}(k_i, N_i) \to \pi$ as $i \to \infty$. $\square$

7 Numerical Examples

In this section we provide numerical examples illustrating the theoretical results on the computation of interior eigenvalues of D from the last section. To this end, we use synthetic far field data at various wave numbers scattered from two the unit ball $\{x \in \mathbb{R}^3, |x| < 1\} \subset \mathbb{R}^3$ or the unit cube $(0,1)^3 \subset \mathbb{R}^3$ which either is a penetrable media modelled by (1) with a constant refractive index $q = q_0 = 3$, or a Robin obstacle modelled by (2) with constant impedance function $\tau = \tau_0 = 1$. For these settings, both scattering problems (1,3) and (2,3) can be solved using boundary integral equation methods. For the transmission scattering problem, we use a 2×2 system of integral equations first given in [KM88], while the Robin scattering problem is tackled by restricting the representation $u^s = \mathrm{DL}(\partial u^s/\partial\nu) - \mathrm{SL}(u^s)$ of a the scattered field from (2,3) to ∂D. While the first integral equation is known to be uniquely solvable for all positive wave numbers, the second fails at interior Dirichlet eigenvalues of $-\Delta$ in D.

In our experiments we used the software package BEM++, see [SBA+14], to numerically compute far fields of scattered fields caused by plane incident waves via a Galerkin method based on piecewise linear and globally continuous basis functions (see [SBA+14, SS13] for details). The triangular surface mesh of D contains 1288 nodes for the unit ball and 611 nodes for the unit cube, corresponding to a mesh width of $h = 0.1$ for the ball and $h = 0.05$ for the cube. To construct the matrix $\mathbb{F}_N^\delta(k)$ we choose a surface mesh of $\mathbb{S}^2$ presented in [Ces96, Section II.2.3.2.1] that provides a partition of the sphere into $N = 48$, 80 or 120 quadrangles of equal area. Incident and far field directions are the centres of the quadrangles. We use piecewise constant interpolation on the quadrangles which yields weight matrices $\mathbb{W}_N^2 = w_N \operatorname{Id}_N$ that reduce to scalars $w_N = 4\pi/N$, such that $\mathbb{W}_N \mathbb{F}_N^\delta(k)\mathbb{W}_N = (4\pi/N)\mathbb{F}_N^\delta(k)$.

If squared wave numbers k^2 are too close to an interior Dirichlet eigenvalue of D then the far field data for the Robin scattering problem (2,3) provided by the numerical solver becomes inaccurate. This phenomenon did not consistently occur in our experiments and turned out not to perturb the results presented below since the interior Dirichlet eigenvalues were sufficiently far from the Robin eigenvalues. However, for safeguarding we did not consider simulated far field data for wave numbers k whenever $\mathbb{F}_N^\delta(k)$ possessed a relative normality error of more than 10%.

For some experiments we perturbed the simulated data $\mathbb{F}_N^\delta(k)$ by adding a random matrix that contains normally distributed random numbers with mean zero and variance one and is scaled to yield a relative noise level of either 1 or 5 percent individually for each k in the grid of wave numbers. Depending on the number N of directions we choose the step size $\triangle k$ for this grid as $\triangle k = 0.2$ for $N = 48$, $\triangle k = 0.1$ for $N = 80$, and $\triangle k = 0.1$ for $N = 120$.

Finally, after fixing a threshold $\varepsilon > 0$ corresponding to the noise level of the far field data we compute the eigenvalues of $(4\pi/N)\mathbb{F}_N^\delta(k)$ using the QZ algorithm and cut off those with magnitude larger than ε. Cutting off all eigenvalues with magnitude larger than $4\pi(\varepsilon/k)^{1/2} + 2\varepsilon$ as proposed by (31) turned out to be overcautious in practice such that the resulting scheme required very small noise levels to be accurate while the relaxed criterion showed to be both robust and accurate. In our experiments we set $\varepsilon = (8\pi^2/k)\varepsilon_r$, where the quantity $8\pi^2/k$ corresponds to half the upper bound of the exact far field operator norm $\|F(k)\|$, and we chose $\varepsilon_r = 0.005$ when no artificial noise is added to the far field data and ε_r equals the relative noise level 0.01 or 0.05 when $\mathbb{F}_N^\delta(k)$ is artificially perturbed.

After computing the phases of the remaining eigenvalues we check for jumps of the largest regularised discrete phase $\vartheta^{*}(k, N)$ to compute lower and upper estimates of square roots of the interior transmission and Robin eigenvalues. More precisely, we determine estimates for these roots by checking where the absolute value of the discrete derivative of the vector $(\vartheta^{*}(k_{\min}+n\triangle k, N))_{n=0}^{n_{\max}}$ is smaller than 10 times the mean value of the absolute value of its entries.

Due to Theorems 1 and 20 one expects that at small noise level the largest phase of the remaining eigenvalues approaches to π from the right or from the left at a couple of wave numbers in $[k_{\min}, k_{\max}]$ that are square roots of interior eigenvalues, such that one could in principle also check where the largest phase is close to π. (Recall that Theorem 1 implies that the exact largest phase $\vartheta^*(k)$ tends to π from the right and from the left at the square roots of the interior transmission and Robin eigenvalues of D, respectively.) Since for coarse wave number grids the largest phase is typically significantly smaller than π when the wave number is close to an interior eigenvalue, it turned out that checking for jumps of the largest phase yields, however, a significantly more reliable criterion than checking where the largest phase is close to π.

Since interior Robin eigenvalues can be computed explicitly both for the cube and the ball, see [GN13], we start with results for the Robin scattering problem (2,3). When D is the unit ball, the introduced algorithm detects five Robin eigenvalue roots in $[0.5, 5.5]$, see Table 1, obtained from the jumps of the largest regularised discrete phase for $N = 48$, 80 and 120. (Note that multiplicities of eigenvalues are not counted.) Since the largest phase approaches π from above at a Robin eigenvalue due to Theorem 1(b), Table 1 indicates for each detected jump of the largest discrete phase between two wave numbers the larger one of these two wave numbers as an upper bound for the square root of the exact Robin eigenvalue. Without artificial noise on the data, these upper bounds for the square roots of the Robin eigenvalues values are optimal for the chosen step size of the wave number grid. The results of Table 1 for $\triangle k = 0.05$ were computed from 101 matrix approximations to the far field operator of dimension 120×120 and the entire computation for either noise level took about 1.9 seconds using Matlab on a desktop computer with eight cores. The corresponding experiment for the cube $D = (0,1)^3$ yields the very same result, cf. Table 2.

Roots of interior eigenvalues (ball)	1st	2nd	3rd	4th	5th
Exact value (4 digits)	1.571	2.743	3.870	4.712	4.973
$N = 48$, $\triangle k = 0.2$, no artificial noise	1.70	2.90	3.90	4.90	5.10
$N = 80$, $\triangle k = 0.1$, no artificial noise	1.60	2.80	3.90	4.80	5.00
$N = 120$, $\triangle k = 0.05$, no artificial noise	1.60	2.75	3.90	4.75	5.00
$N = 120$, $\triangle k = 0.05$, 1% artificial noise	1.60	2.80	3.90	4.75	5.00
$N = 120$, $\triangle k = 0.05$, 5% artificial noise	1.60	2.80	3.95	4.75	5.05

Table 1: Estimates of the square roots of the first five Robin eigenvalues of $-\Delta$ in the unit ball from far field data for different levels of artificial additive noise.

Roots of interior eigenvalues (cube)	1st	2nd	3rd	4th	5th
Exact value (4 digits)	2.263	4.112	5.357	6.362	6.839
$N = 48$, $\triangle k = 0.2$, no artificial noise	2.30	4.30	5.50	6.50	6.90
$N = 80$, $\triangle k = 0.1$, no artificial noise	2.30	4.20	5.40	6.40	6.90
$N = 120$, $\triangle k = 0.05$, no artificial noise	2.30	4.15	5.40	6.40	6.85
$N = 120$, $\triangle k = 0.05$, 1% artificial noise	2.30	4.15	5.40	6.40	6.90
$N = 120$, $\triangle k = 0.05$, 5% artificial noise	2.35	4.20	5.45	6.50	6.95

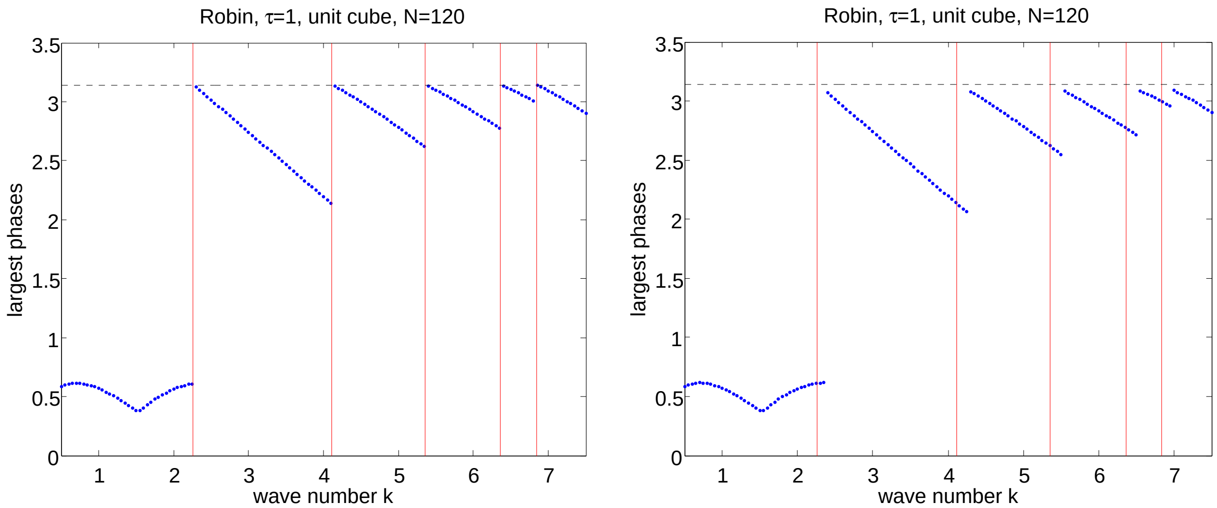
Table 2: Estimates of the square roots of the first five Robin eigenvalues of $-\Delta$ in the cube $(0,1)^3$ from far field data for different levels of artificial additive noise.

This situation changes, however, when perturbing the simulated data by artificial noise as the obtained bounds start to shift. The reason for this shift becomes obvious when considering the plot of the largest regularised discrete phase $k \mapsto \vartheta^{\divideontimes}(k,120)$ for the case of the cube with and without artificial noise on the data, plotted in Figure 2. By neglecting eigenvalues with magnitude smaller than the noise level when computing $k \mapsto \vartheta^{\divideontimes}(k,120)$, the latter function cannot reach π anymore but at most $\pi - \arcsin(\varepsilon k/(8\pi^2)) = \pi - \arcsin(\varepsilon_r)$, a function that decreases in the relative noise level $\varepsilon_r > 0$.

As a possible numerical remedy when working with noisy data we propose the following heuristic post-processing: If the largest regularised discrete phase jumps between the wave numbers $k_* - \triangle k$ and k_*, then compute the value $k_{\mathrm{appr}}^{\mathrm{rob}}$ where the straight line in the (k,ϑ)-plane through $(k_*, \vartheta^{\divideontimes}(k_*,N))$ and $(k_* + \triangle k, \vartheta^{\divideontimes}(k_* + \triangle k, N))$ intersects the line $\{\vartheta = \pi\}$,

$$k_{\mathrm{appr}}^{\mathrm{rob}} = k_* + \frac{\pi - \vartheta^{\divideontimes}(k_*,N)}{\vartheta^{\divideontimes}(k_* + \triangle k, N) - \vartheta^{\divideontimes}(k_*,N)} \triangle k, \tag{36}$$

to obtain a better approximation of the square root of an interior Robin eigenvalue. Table 3 indicates the improvement of the resulting approximations for $N = 120$ and an artificial noise level of 5% over those from Tables 1 and 2.



(a) Largest regularised discrete phases, no art. noise

(b) Largest regularised discrete phases, 5% art. noise

Figure 2: The largest regularised discrete phases for the artificial noise levels 0% (a) and 5% (b). Red vertical lines mark the location of the square roots of the exact Robin eigenvalues.

Roots of interior eigenvalues	1st	2nd	3rd	4th	5th
Ball – Exact value (4 digits)	1.571	2.743	3.870	4.712	4.973
Extrapolation (36)	1.575	2.763	3.884	4.719	4.988
Relative error	0.25%	0.73%	0.36%	0.14%	0.30%
Cube – Exact value (4 digits)	2.263	4.112	5.357	6.362	6.839
Extrapolation (36)	2.267	4.133	5.307	6.406	6.827
Relative error	0.17%	0.51%	0.93%	0.69%	0.17%

Table 3: Estimates of the square roots of the first five Robin eigenvalues by the extrapolation procedure. The relative artificial noise level equals 5%. Fixed parameters are $N = 120$, $\triangle k = 0.05$

We finally show results for the computation of positive interior transmission eigenvalues from far fields, using the same algorithm as explained above but, for simplicity, merely for $N = 120$ and $\triangle k = 0.05$. To the best of our knowledge, the transmission eigenvalues for the unit cube with constant contrast q_0 can not be computed analytically and no numerical computations have been published for this case in the literature yet, such that we merely know exact values in case that D is the ball (see [KL13] for the corresponding formula). The extrapolation formula (36) has to be adapted to transmission eigenvalues, since the largest phase tends to π from below at an interior transmission eigenvalue with $q_0 > 0$: If the largest regularised discrete phase jumps between k_* and $k_*+\triangle k$, we compute the intersection of the straight line through the points $(k_*-\triangle k,\ \vartheta^{\divideontimes}(k_*-\triangle k, N))$ and $(k_*,\ \vartheta^{\divideontimes}(k_*, N))$ with $\{\vartheta = \pi\}$,

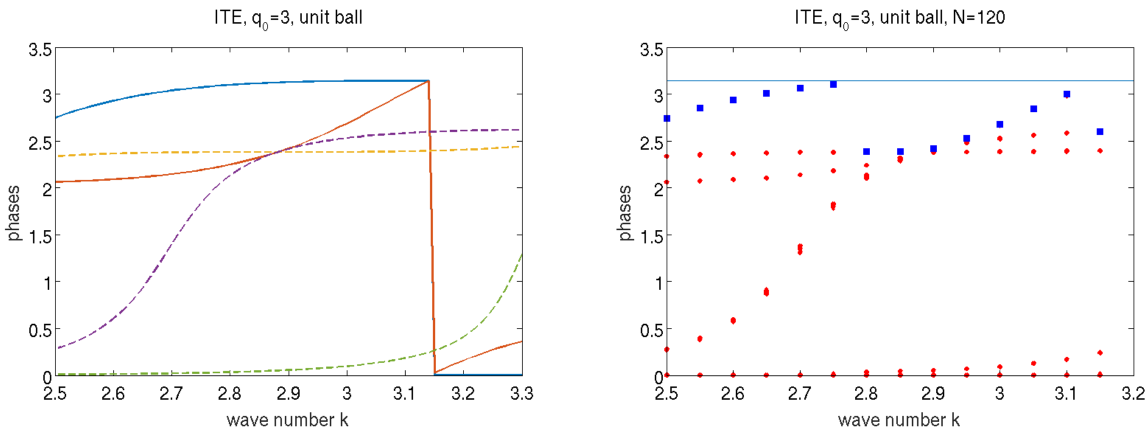
$$k_{\text{appr}}^{\text{ite}} = k_* + \frac{\pi - \vartheta^{\divideontimes}(k_*, N)}{\vartheta^{\divideontimes}(k_* + \triangle k, N) - \vartheta^{\divideontimes}(k_*, N)} \triangle k, \tag{37}$$

to increase accuracy of the eigenvalue estimate. Except for the smallest positive transmission eigenvalue, Table 4 shows results of about the same quality as for Robin eigenvalues. Indeed, the smallest positive transmission eigenvalue (which has multiplicity four for this setting) is poorly approximated for noisy data. Figure 3 hints why: For $q_0 = 3$, the two phase curves with largest phase intersect at $k = \pi$ (see the solid lines in Figure 3(a), but one of two phase curve is extremely flat when it reaches the value π. As Figure 3(b) shows, the largest discrete phase (marked by bold squares) for this reason indicates two distinct eigenvalues, underestimating the smallest positive transmission eigenvalue π. The rather steep dashed curves indicating the second and third interior transmission eigenvalue are, by the same argument, also estimated rather accurately for noisy data. As no lower bound for the derivative of the phase curves at values close to π is known, this phenomenon explains why, e.g., convergence rates for the eigenvalue approximations are difficult to obtain without further assumptions.

Roots of interior eigenvalues (ball)	1st		2nd	3rd	4th
Exact value (4 digits)	3.142	–	3.692	4.262	4.832
Estimate, no artificial noise	3.00	–	3.65	4.25	4.85
Extrapolation (37), no artificial noise	3.116	–	3.697	4.268	4.857
Relative error (extrapolation)	0.81%	–	0.14%	0.14%	0.52%
Estimate, 1% artificial noise	2.90	3.10	3.65	4.25	4.85
Extrapolation (37), 1% artificial noise	2.953	3.145	3.697	4.268	4.858
Relative error (extrapolation)	6.00%	0.10%	0.14%	0.14%	0.54%
Estimate, 5% artificial noise	2.75	3.10	3.65	4.25	4.80
Extrapolation (37), 5% artificial noise	2.803	3.144	3.698	4.268	4.845
Relative error (extrapolation)	10.78%	0.08	0.16%	0.14%	0.27%

Table 4: Estimates of the square roots of the first four interior transmission eigenvalues of the unit ball for various noise levels. Fixed parameters are $N = 120$, $\triangle k = 0.05$, $q_0 = 3$.

Table 5 finally shows the corresponding results for the cube $(-1, 1)^3$.



(a) Exact phases of F.

(b) Computed phases, 5% art. noise.

Figure 3: The phases of the five eigenvalues of $F(k)$ with largest phase (a) and the numerically computed regularised discrete phases (b) for the unit ball and contrast $q_0 = 3$. (Phases on the vertical axes vs. wave numbers on the horizontal axes.)

Roots of interior eigenvalues (cube)	1st	2nd	3rd	4th	5th
Estimate, no artificial noise	4.95	5.35	5.65	6.00	6.60
Extrapolation (37), no artificial noise	4.983	5.432	5.697	6.023	6.673
Estimate, 1% artificial noise	4.90	5.35	5.65	5.95	6.60
Extrapolation (37), 1% artificial noise	4.980	5.464	5.721	5.973	6.673
Estimate, 5% artificial noise	4.85	5.25	5.55	5.90	6.50
Extrapolation (37), 5% artificial noise	4.964	5.334	5.575	5.955	6.552

Table 5: Estimates of the square roots of the first five interior transmission eigenvalues of the unit cube for various noise levels. Fixed parameters are $N = 120$, $\triangle k = 0.05$, $q_0 = 3$.

A Auxiliary Results

The Sobolev spaces $H^s(\mathbb{S}^2)$ can, for $s \in \mathbb{R}$, be defined using the complete orthonormal system of spherical harmonics $Y_n^m \in L^2(\mathbb{S}^2)$ as the completion of $C^\infty(\mathbb{S}^2)$ in the norm

$$\|f\|_{H^s}^2 = \sum_{n=0}^{\infty} \sum_{|m|\leq n}^{n} (1+n^2)^s |f_n^m|^2, \qquad f_n^m = \int_{\mathbb{S}^2} f\, \overline{Y_n^m}\, \mathrm{d}S, \quad n \in \mathbb{N}_0,\, m \in \mathbb{Z},\, s \in \mathbb{R}. \tag{38}$$

Lemma 21. *The far field operator F is bounded from $H^{-s}(\mathbb{S}^2)$ into $H^s(\mathbb{S}^2)$ for all $s \geq 0$.*

Proof. The kernel u^∞ of the linear integral operator F is C^∞-smooth in both variables (see [CK13]). Thus, F is a bounded linear operator from $L^2(\mathbb{S}^2)$ into any function space $C^\ell(\mathbb{S}^2)$, $\ell \in \mathbb{N}$, of continuously differentiable functions on the sphere and thus also from $L^2(\mathbb{S}^2)$ into any Sobolev space $H^\ell(\mathbb{S}^2)$. Since $H^\ell(\mathbb{S}^2)$ can be defined equivalently via local charts (see, e.g., [McL00]) or via spherical harmonics, the boundedness of F implies that $|(Fg)_n^m|^2 \leq C(t)(1+n^2)^{-t}$ for any $t > 0$ and any $g \in L^2(\mathbb{S}^2)$ with $C(t)$ independent of g. Choosing $g = Y_n^m$ and $t = 2s$ shows the claim. ☐

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