The Inside-Outside Duality for Scattering Problems by Inhomogeneous Media

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Abstract: This paper investigates the relationship between interior transmission eigenvalues $k_0 > 0$ and the accumulation point 1 of the eigenvalues of the scattering operator $\mathcal{S}(k)$ when k approaches k_0. As it is well known, the spectrum of $\mathcal{S}(k)$ is discrete, the eigenvalues $\mu_n(k)$ lie on the unit circle in $\mathbb{C}$ and converge to 1 from one side depending on the sign of the contrast. Under certain (implicit) conditions on the contrast it is shown that interior transmission eigenvalues k_0 can be characterized by the fact that one eigenvalue of $\mathcal{S}(k)$ converges to 1 from the *opposite side* if k tends to k_0 from below. The proof uses the Cayley transform, Courant's maximum–minimum principle, and the factorization of the far field operator. For constant contrasts that are positive and large enough or negative and small enough, we show that the conditions necessary to prove this characterization are satisfied at least for the smallest transmission eigenvalue.

1 Introduction

Interior transmission eigenvalue problems occur in the study of scattering problems of plane time-harmonic fields by an inhomogeneous medium filling a bounded domain D. In scattering theory, they play about the same role as the eigenvalue problem for $-\Delta$ in D with respect to, e.g., Dirichlet boundary conditions for the scattering problem by a sound-soft obstacle. We refer to [13] for their first appearance and to [5, 9, 18, 15, 4] for an incomplete list of references. Moreover, interior transmission eigenvalue problems are interesting objects of research in themselves because they fail to be self-adjoint. There are a few results for the corresponding inverse spectral problem, see [17, 1, 2], where the task is to recover information about the index of refraction from the knowledge of the interior transmission eigenvalues. From the point of view of *inverse scattering problems* one would like to recover this information from the far field patterns of scattered waves or from the far field operator F, the linear integral operator whose kernel is formed by the far field patterns. It has been known for a long time that the injectivity of the far field operator F corresponding to the scattering by an inhomogeneous medium D is assured if the wavenumber k is not an eigenvalue of the corresponding interior transmission eigenvalue problem in D. (This is actually easy to prove, see Theorem 2.3 below.) In other words, the scattering of a superposition of plane waves of the form

$$u^i(x) \;=\; \int_{|\hat{\theta}|=1} g(\hat{\theta})\, e^{ik\,x\cdot\hat{\theta}}\, ds(\hat{\theta})\,, \quad x \in \mathbb{R}^3\,,$$

cannot result in a vanishing scattered field u^s if the wave number k is not an eigenvalue of the corresponding interior transmission eigenvalue problem in D. Therefore, there exists a relationship between the injectivity of the far field operator F – a property of the exterior of D – and the eigenvalues of an eigenvalue problem – a property of the interior of D. This relationship is sometimes called the *inside–outside duality* in scattering theory (see, e.g., [11, 12]) and has been used to determine the interior transmission eigenvalues from the far field patterns. The computation of these eigenvalues as those wavenumbers for which the far field operator fails to be injective suffers – at least theoretically – from the fact that injectivity is only a *necessary* but not a sufficient condition for k being no eigenvalue. It has been shown in [19] that it is very well possible that k is an eigenvalue but at the same time F is injective. Indeed, in Theorem 2.3 below we recall a result which states that non-injectivity of the far field operator F for some k is equivalent to the fact that k is an interior transmission eigenvalue plus an additional condition on the corresponding eigenfunctions. In [19] it has been shown for D being a rectanglus box that the far field operator is *always* one-to-one. In this case it is hence – at least theoretically – not possible to determine interior transmission eigenvalues merely by checking the injectivity of far field operators in an interval of frequencies. It is the goal of this paper to weaken the condition of non–injectivity of the far field operator F for wavenumber k into a condition which is exactly equivalent to k being an interior transmission eigenvalue.

In our proof we follow the paper [12] in which the scattering by an impenetrable, sound-soft obstacle has been studied. We (A. K. and A. L.) were not aware of this paper until recently and want to point out that this impressive work already contains the factorization of the far field operator in exactly the same form as in [14][1]. The notation and partly also the analytical techniques this paper uses differ from standard notation and tools used in the community interested in mathematical inverse scattering theory. This might be a possible explanation why the paper [12] and its results apparently remained largely unknown in this community.

The approach presented below strongly relies on the fact that the far field operator is normal and does not straightforwardly extend to, e.g., absorbing media. Analogous results for different scattering problems leading to normal far field operators will be presented in forthcoming papers.

The paper is organized as follows. In Section 2 we introduce the scattering problem by an inhomogeneous medium and the corresponding far field operator F and scattering operator $\mathcal{S}$. We recall properties of these operators, prove the mentioned relationship between non-injectivity of F to the interior transmission eigenvalue problem and recall the factorization of F as in [16]. In Section 3 we characterize the interior transmission eigenvalues by the operator T which appears in the factorization of F. Section 4 studies the phases δ_n of the eigenvalues μ_n of the scattering operator $\mathcal{S}$. The eigenvalues lie on the unit circle in $\mathbb{C}$ because of the unitarity of $\mathcal{S}$. We recall from [16] that μ_n tends to 1 from "above" or "below" as $n \to \infty$, depending on the sign of the contrast. We translate this into a condition on the phases δ_n and prove a characterization of the "first" eigenvalue[2] using the Cayley transform and Courant's maximum–minimum principle. Finally, in Sections 5 and 6 we consider this "first" eigenvalue as a function of the wave number k and show that it tends to 1, too, as k approaches an interior eigenvalue k_0. However, roughly speaking, this "first" eigenvalue approaches 1 from "below" or "above" if μ_n approaches 1 from "above" or "below" (see our main Theorem 6.3 for a precise

[1] On the other hand, the authors of [12] were apparently not aware of the existing literature on boundary integral operators as, e.g., the first edition of the monograph [7].

[2] See (4.21) for the precise definition of what we mean by first eigenvalue

formulation). We also give the corresponding result for the far field operator F instead of the scattering operator $\mathcal{S}$ (see Corollary 6.4) and finish with an illustrative numerical example for the case of scattering from a penetrable ball.

2 Transmission Eigenvalues and the Far Field Operator

We will model time-harmonic scattering from an inhomogeneous medium by the Helmholtz equation with a refractive index n that is equal to one outside the scattering object D (compare equation (2.2) below). This motivates to introduce the contrast function $q := n^2 - 1$. Our general assumptions on the contrast and on the scatterer D in the entire paper are as follows.

Assumption 2.1 *Let $D \subset \mathbb{R}^3$ be open and bounded such that the complement $\mathbb{R}^3 \setminus \overline{D}$ is connected. Furthermore, we assume that the boundary ∂D of D is smooth enough such that the imbedding of $H^1(D)$ into $L^2(D)$ is compact. Let $q \in L^\infty(D)$ be real-valued and satisfy*

(1) There exists $c_0 > 0$ with $1 + q(x) \geq c_0$ for almost all $x \in D$.

(2) Either $q(x) > 0$ for almost all $x \in D$ or $q(x) < 0$ for almost all $x \in D$.

(3) $|q|$ is locally bounded below, i.e. for every compact subset $M \subset D$ there exists $c > 0$ (depending on M) such that

$$|q(x)| \;\geq\; c \text{ for almost all } x \in M\,. \tag{2.1}$$

We extend q by zero outside of D.

We note that part (3) of this assumption is satisfied for continuous contrasts q that vanish at most on the boundary of D.

By $k > 0$ we denote the wave number. For any incident plane wave $u^i(x) = \exp(ik\hat{\theta} \cdot x)$ of direction $\hat{\theta} \in S^2$, the (direct) scattering problem is to determine the scattered field $u^s \in H^1_{loc}(\mathbb{R}^3)$ such that the total field $u := u^s + u^i$ satisfies the Helmholtz equation

$$\Delta u(x) \;+\; k^2\big(1 + q(x)\big)\, u(x) \;=\; 0, \quad \text{for } x \in \mathbb{R}^3\,, \tag{2.2}$$

and such that u^s satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s(x)}{\partial r} \;-\; iku^s(x) \;=\; \mathcal{O}\big(r^{-2}\big)\,, \quad r = |x| \to \infty\,. \tag{2.3}$$

From now on, we call any solution v of the Helmholtz equation $\Delta v + k^2 v = 0$ outside some ball containing D that satisfies the radiation condition a *radiating* solution. If appropriate, we indicate the dependence of all fields on the incident direction by writing $u^i(x,\hat\theta)$, $u^s(x,\hat\theta)$, and $u(x,\hat\theta)$.

It is well-known that this direct scattering problem is uniquely solvable (see, e.g., [7] or Lemma 2.4 below). Furthermore, any radiating solution v of the Helmholtz equation has the asymptotic behavior

$$v(x) \;=\; \frac{\exp(ik|x|)}{4\pi|x|}\, v^\infty(\hat x) \;+\; \mathcal{O}(|x|^{-2})\,, \quad |x| \to \infty\,, \tag{2.4}$$

uniformly with respect to $\hat x = x/|x| \in S^2$. The complex valued function v^∞ is called the far field pattern. In the special case where v is the scattered field u^s, the far field pattern depends on the direction $\hat x$ of observation and the direction $\hat\theta$ of the incident plane wave. We indicate this dependence by writing $u^\infty(\hat x,\hat\theta)$. Furthermore we define the far field operator F from $L^2(S^2)$ into itself as the integral operator whose kernel is this far field pattern; that is,

$$(Fg)(\hat x) \;=\; \int_{S^2} g(\hat\theta)\, u^\infty(\hat x,\hat\theta)\, ds(\hat\theta)\,, \quad \hat x \in S^2\,. \tag{2.5}$$

The far field operator F is closely related to the scattering operator (or scattering matrix) $\mathcal{S}$, namely

$$\mathcal{S} \;=\; I \;+\; \frac{ik}{8\pi^2}\, F\,. \tag{2.6}$$

We note that the factor $8\pi^2$ in the denominator – instead of the more common factor 2π – stems from our definition (2.4) of the far field pattern.

It is well-known (see, e.g., [16, Theorem 4.4] or [6]) that $\mathcal{S}$ is unitary (that is, $\mathcal{S}^*\mathcal{S} = \mathcal{S}\mathcal{S}^* = I$) and that F is normal (that is, F and its adjoint F^* commute).

Due to Assumption 2.1, $|q|$ is positive within D. This makes it convenient to introduce the weighted L^2-space $L^2(D,|q|dx)$ which is the completion of $C_0^\infty(D)$ with respect to the inner product

$$(u,v)_{L^2(D,|q|dx)} \;=\; \int_D u(x)\,\overline{v(x)}\,|q(x)|\,dx\,.$$

Obviously, $L^2(D,|q|dx)$ contains $L^2(D)$ as a dense subspace and coincides with $L^2(D)$ if $|q|$ is bounded below in D by a (global) positive constant.

The starting point of this paper is the following connection between the injectivity of F and the following interior transmission eigenvalue problem.

Definition 2.2 *$k > 0$ is an interior transmission eigenvalue if there exists a non-trivial pair $(u, w) \in L^2(D, |q|dx) \times L^2(D, |q|dx)$ such that $u - w \in H^2(D)$ and*

$$\Delta u + k^2(1 + q)u = 0 \text{ in } D\,, \qquad \Delta w + k^2 w = 0 \text{ in } D\,, \tag{2.7}$$

$$u = w \text{ on } \partial D \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \text{ on } \partial D\,. \tag{2.8}$$

The differential equations are understood in the ultra-weak sense; that is,

$$\int_D w\,(\Delta\psi + k^2\psi)\,dx \;=\; 0 \quad \text{for all } \psi \in C_0^\infty(D)\,,$$

and analogously for u. The boundary conditions can be reformulated as $u - w \in H_0^2(D)$.

Theorem 2.3 *F fails to be one-to-one if, and only if, $k > 0$ is an interior transmission eigenvalue such that the corresponding solution w of $\Delta w + k^2 w = 0$ can be extended to all of $\mathbb{R}^3$ as a Herglotz wave function,*

$$w(x) \;=\; w_g(x) \;=\; \int_{S^2} g(\hat\theta)\, e^{ikx\cdot\hat\theta} ds(\hat\theta)\,, \quad x \in \mathbb{R}^3\,, \tag{2.9}$$

for some $g \in L^2(S^2)$, $g \neq 0$.

Proof: We sketch the proof for the convenience of the reader. Let $g \in L^2(S^2)$ be in the null space of F; that is

$$(Fg)(\hat x) \;=\; \int_{S^2} g(\hat\theta)\, u^\infty(\hat x, \hat\theta)\, ds(\hat\theta) \;=\; 0 \quad \text{for all } \hat x \in S^2\,.$$

By linearity, Fg is the far field pattern of the scattered field u_g^s which corresponds to the incident field

$$u_g^i(x) \;=\; \int_{S^2} g(\hat\theta)\, e^{ikx\cdot\hat\theta}\, ds(\hat\theta)\,, \quad x \in \mathbb{R}^3\,.$$

Rellich's lemma (see [7]) and unique continuation implies that the scattered field u_g^s vanishes outside of D. (Here we use the fact that the exterior of D is connected.) Therefore, the total field u_g coincides with the incident field u_g^i outside of D. Therefore, the Cauchy data of u_g and u_g^i coincide on ∂D, and (u_g, u_g^i) solves (2.7), (2.8).

If, on the other hand, (u, w) solves (2.7), (2.8) and w has the form (2.9) for some $g \in L^2(S^2)$ then, by the same arguments, g is in the null space of F. The proof is finished by noting that $g \neq 0$ is equivalent to $w_g \neq 0$ (see, e.g., [7]). $\square$

It is the aim of this paper to characterize the interior transmission eigenvalues by the far field operator F – or, equivalently, the scattering operator $\mathcal{S}$. We note that injectivity of F is equivalent to the fact that 1 is not an eigenvalue of $\mathcal{S}$. Since $\mathcal{S}$ is unitary its eigenvalues lie on the unit circle in the complex plane. This translates into the fact that the eigenvalues λ_j of F lie on the circle of radius $8\pi^2/k$ centered at $(8\pi^2/k)i$ on the imaginary axes. They tend to zero because F is compact. We will show that they tend to zero from one side only, depending on the sign of q. To prove this we will need properties of the factorization method. We collect them in the following Theorem 2.5, after introducing some notation. We denote by $H : L^2(S^2) \to L^2(D, |q|dx)$ the linear and compact Herglotz operator, defined by

$$(H\psi)(x) \;=\; \int\limits_{S^2} \psi(\theta)\, e^{ik\,x\cdot\theta}\, ds(\theta)\,, \quad x \in D\,. \tag{2.10}$$

Moreover, we introduce the constant

$$\sigma \;:=\; \begin{cases} 1 & \text{if } q > 0 \text{ in } D, \\ -1 & \text{if } q < 0 \text{ in } D. \end{cases}$$

Finally, $T : L^2(D, |q|dx) \to L^2(D, |q|dx)$ is defined by $Tf = f + k^2 v|_D$, where $v \in H^1_{loc}(\mathbb{R}^3)$ is the radiating weak solution to

$$\Delta v + k^2(1+q)v \;=\; -q\,f \quad \text{in } \mathbb{R}^3\,, \tag{2.11}$$

that is,

$$\int_{\mathbb{R}^3} \big(\nabla v \cdot \nabla\overline{\varphi} - k^2(1+q)\, v\,\overline{\varphi}\big)\, dx \;=\; \int_{\mathbb{R}^3} q\, f\, \overline{\varphi}\, dx \tag{2.12}$$

for all $\varphi \in H^1(\mathbb{R}^3)$ with compact support and, additionally, v satisfies (2.3). Existence and uniqueness is formulated in the following lemma.

Lemma 2.4 *For all $k > 0$ and all $f \in L^2(D, |q|dx)$ there exists a unique radiating solution $v \in H^1_{loc}(\mathbb{R}^3)$ of (2.11). For every $R > 0$ such that $\overline{D} \subset B := B(0, R)$ and every compact set K with $K \cap \overline{D} = \emptyset$ the solution operators $f \mapsto v|_B$ and $f \mapsto (v|_K, \nabla v|_K)$ and $f \mapsto v^\infty$ are bounded from $L^2(D, |q|dx)$ into $H^1(B)$, into $C(K) \times C(K)^3$, and into $L^2(S^2)$, respectively, the latter two even compact. Furthermore, these operators depend continuously on k.*

There are several ways to prove this lemma, e.g., by reformulating the problem into the Lippmann-Schwinger integral equation. We omit the proof but refer to, e.g., [16].

Theorem 2.5 *(a) Let $F : L^2(S^2) \to L^2(S^2)$ be defined by (2.5). Then*

$$F \;=\; k^2 \sigma\, H^* \, T\, H. \tag{2.13}$$

(b) The mapping $f \mapsto v|_D$ is compact from $L^2(D, |q|dx)$ into itself. Therefore, T is a compact perturbation of the identity.

(c) $\sigma \,\mathrm{Im}\,(Tf, f)_{L^2(D,|q|dx)} \geq 0$ for all $f \in L^2(D, |q|dx)$.

For the **proof** we follow the presentation of [16], Theorems 4.5 and 4.8.

(a) Note that the scattered field u^s satisfies the differential equation

$$\Delta u^s + k^2(1+q)u^s \;=\; -k^2 q\, u^i \quad \text{in } \mathbb{R}^3 .$$

Define the operator G from $L^2(D, |q|dx)$ into $L^2(S^2)$ by $Gf = v^\infty$ where v solves (2.11). Then $F = k^2 GH$ by the superposition principle. The adjoint of H is given by

$$(H^*\psi)(\hat{x}) \;=\; \int_D |q(y)|\, \psi(y)\, e^{-ik\hat{x}\cdot y} dy\,, \quad \hat{x} \in S^2\,,$$

which is the far field pattern w^∞ of the volume potential

$$w(x) \;=\; \int_D |q(y)|\, \psi(y)\, \Phi(x,y)\, dy \;=\; \sigma \int_D q(y)\, \psi(y)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3 .$$

It satisfies $\Delta w + k^2 w = -\sigma q\, \psi$ in $\mathbb{R}^3$ and is radiating. From

$$\Delta v + k^2 v \;=\; -q\left[f + k^2\, v\right]$$

and the definition of T we conclude that $Gf = v^\infty = \sigma H^* T f$. Substituting this into $F = k^2\, G\, H$ yields the factorization (2.13).

(b) This is easily seen by a regularity argument.

(c) For $f \in L^2(D, |q|dx)$ and the corresponding field v we have with $Tf = f + k^2\, v$ that

$$\begin{aligned}
(Tf, f)_{L^2(D,|q|dx)} \;&=\; \int_D |Tf|^2\, |q|\, dx \;-\; k^2 \int_D (f + k^2\, v)\, \overline{v}\, |q|\, dx \\
&=\; \|Tf\|^2_{L^2(D,|q|dx)} \;+\; k^2\sigma \int_{|x|<R} (\Delta v + k^2 v)\, \overline{v}\, dx \\
&=\; \|Tf\|^2_{L^2(D,|q|dx)} \;+\; k^2\sigma \int_{|x|<R} \left[k^2|v|^2 - |\nabla v|^2\right] dx \\
&\quad\; + k^2\sigma \int_{|x|=R} \overline{v}\, \frac{\partial v}{\partial \nu}\, ds\,.
\end{aligned} \tag{2.14}$$

By the radiation condition we conclude that

$$
\begin{aligned}
(Tf,f)_{L^2(D,|q|dx)} \;&=\; \|Tf\|^2_{L^2(D,|q|dx)} \;+\; k^2\sigma \int\limits_{|x|<R} \left[k^2|v|^2 - |\nabla v|^2\right] dx \\
&\qquad + \; ik^3\sigma \int\limits_{|x|=R} |v|^2 ds \;+\; \mathcal{O}(R^{-1})
\end{aligned} \tag{2.15}
$$

as R tends to infinity. Taking the imaginary part and the limit as $R \to \infty$ yields

$$
\sigma \, \mathrm{Im}\,(Tf,f)_{L^2(D,|q|dx)} \;=\; k^3 \|v^\infty\|^2_{L^2(S^2)} \;\geq\; 0\,. \tag{2.16}
$$

$\square$

3 A Characterization of Transmission Eigenvalues

Now we want to characterize the interior transmission eigenvalues by the operator T. To this end, we denote the closure of the range of the Herglotz operator H in $L^2(D,|q|dx)$ by X; that is,

$$
\begin{aligned}
X \;&=\; \mathrm{closure}_{L^2(D,|q|dx)}\, \mathcal{R}(H) \\
&=\; \left\{ w \in L^2(D,|q|dx) : \int_D w(\Delta\psi + k^2\psi)\, dx = 0 \text{ for all } \psi \in C_0^\infty(D) \right\}.
\end{aligned} \tag{3.17}
$$

The last set equality is due to the density of Herglotz wave functions in the set of all solutions to the Helmholtz equation (see [8]). Then we have:

Theorem 3.1 *(a) Let $k>0$ be an interior transmission eigenvalue with corresponding non-trivial pair (u,w). Then $w \in X \setminus \{0\}$, and w satisfies $(Tw,w)_{L^2(D,|q|dx)} = 0$.*
(b) Let $k>0$ and let $w \in X \setminus \{0\}$ satisfy $\mathrm{Im}\,(Tw,w)_{L^2(D,|q|dx)} = 0$. Then there exists $u \in L^2(D)$ such that k is an interior transmission eigenvalue with corresponding pair (u,w). The function $u-w$ belongs to $H_0^2(D)$ and does not vanish. In particular, $\sigma \mathrm{Im}\,(Tw,w)_{L^2(D,|q|dx)} > 0$ for all $w \in X \setminus \{0\}$ if $k>0$ is not an interior transmission eigenvalue.

Proof: (a) Let $k>0$ be an interior transmission eigenvalue with corresponding $u,w \in L^2(D,|q|dx)$. We define $v = \frac{1}{k^2}(u-w)$ in D and note that $v \in H_0^2(D)$. We extend v by zero outside of D and observe that v is the radiating solution of

$$
\Delta v \;+\; k^2(1+q)v \;=\; -q\,w \quad \text{in } \mathbb{R}^3\,. \tag{3.18}
$$

Therefore, $v \in H^2_{loc}(\mathbb{R}^3)$ with compact support and $w \in X$ satisfies (2.11). Let $w_n \in C^\infty(\overline{D})$ such that $\Delta w_n + k^2 w_n = 0$ in D and $w_n \to w$ in $L^2(D, |q|dx)$. Then $Tw = w + k^2 v|_D$ and thus

$$\begin{aligned}(Tw, w_n)_{L^2(D,|q|dx)} &= \int_D (w + k^2 v)\,|q|\,\overline{w}_n\,dx \\ &= -\sigma \int_D (\Delta v + k^2 v)\,\overline{w}_n\,dx \;=\; 0\end{aligned}$$

by Green‘s second theorem. The left hand side converges to $(Tw, w)_{L^2(D,|q|dx)}$ which proves that $(Tw, w)_{L^2(D,|q|dx)} = 0$. Finally, $w \neq 0$ because otherwise also v vanishes as a radiating solution of (3.18), too.

(b) Let now $w \in X$ such that $\mathrm{Im}\,(Tw, w)_{L^2(D,|q|dx)} = 0$. We denote by $v \in H^2_{loc}(\mathbb{R}^3)$ the radiating solution of (2.11) for $f = w$. From (2.16) we conclude that $v^\infty = 0$. Rellich‘s lemma yields that v vanishes outside of D; that is, $v \in H^2_0(D)$. We set $u = w + k^2 v$. For $\psi \in C^\infty_0(D)$ we conclude that

$$\begin{aligned}\int_D u\,[\Delta\psi + k^2(1+q)\psi]\,dx &= k^2 \int_D v\,[\Delta\psi + k^2(1+q)\psi]\,dx \\ &\quad + \int_D w\,[\Delta\psi + k^2(1+q)\psi]\,dx \\ &= k^2 \int_D \psi\,[\Delta v + k^2(1+q)v]\,dx \;+\; k^2 \int_D w\,q\,\psi\,dx \\ &= 0\,.\end{aligned}$$

Therefore, (u, w) satisfies the conditions of the interior transmission eigenvalue problem. □

Remark 3.2 *From part (b) of the previous proof we observe that* $\mathrm{Im}\,(Tw, w)_{L^2(D,|q|dx)} = 0$ *for* $w \in X$ *implies that the radiating solution* $v \in H^2_{loc}(\mathbb{R}^3)$ *to* (3.18) *vanishes outside of* D *and thus* $v|_D \in H^2_0(D)$.

Corollary 3.3 *Assume that* $(Tw, w)_{L^2(D,|q|dx)} = 0$ *for some* $w \in X$*. Define again* $v \in H^2_{loc}(\mathbb{R}^3)$ *to be the radiating solution of* (3.18)*. Then*

$$k^2 \int_D \left(|\nabla v|^2 - k^2 |v|^2\right) dx \;=\; \sigma\,\|Tw\|^2_{L^2(D,|q|dx)}. \tag{3.19}$$

Proof: As we noted in the preceding Remark 3.2, v vanishes outside of D; that is, $v|_D \in H^2_0(D)$. Hence, (2.15) yields

$$0 \;=\; (Tw, w)_{L^2(D,|q|dx)} \;=\; \|Tw\|^2_{L^2(D,|q|dx)} \;+\; k^2\sigma \int_D \left[k^2|v|^2 - |\nabla v|^2\right] dx \;+\; \mathcal{O}(R^{-1}).$$

The claim follows from passing to the limit as $R \to \infty$. □

4 Phases of the Spectrum of the Scattering Operator

In this section, we collect well-known properties of the eigenvalues of the (normal) far field operator and the related scattering operator. These properties will be useful later on to prove an inside–outside duality between the interior transmission eigenvalues and the eigenvalues of the far field operator.

We recall that the eigenvalues λ_n of F lie on the circle of radius $8\pi^2/k$ centered at $(8\pi^2/k)i$ on the imaginary axes and converge to zero.

Lemma 4.1 *Let k be no interior transmission eigenvalue. Then λ_n converges from the right to zero if $\sigma = +1$ and from the left if $\sigma = -1$; that is, $\sigma \mathrm{Re}\, \lambda_n > 0$ for sufficiently large n.*

Proof: Let $\psi_n \in L^2(S^2)$ be the eigenfunctions of F corresponding to the eigenvalues λ_n such that $\{\psi_n : n \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(S^2)$. Define $\phi_n \in \mathcal{R}(H) \subset L^2(D, |q|dx)$ by

$$\phi_n \;=\; \frac{k}{\sqrt{|\lambda_n|}}\, H\psi_n\,, \quad n \in \mathbb{N}\,.$$

From the factorization we conclude that

$$\begin{aligned}
\sigma\,(T\phi_n, \phi_m)_{L^2(D,|q|dx)} \;&=\; \frac{k^2\sigma}{\sqrt{|\lambda_n||\lambda_m|}}\,(TH\psi_n, H\psi_m)_{L^2(D,|q|dx)} \\
&=\; \frac{k^2\sigma}{\sqrt{|\lambda_n||\lambda_m|}}\,(H^*TH\psi_n, \psi_m)_{L^2(S^2)} \\
&=\; \frac{1}{\sqrt{|\lambda_n||\lambda_m|}}\,(F\psi_n, \psi_m)_{L^2(S^2)} \;=\; \frac{\lambda_n}{|\lambda_n|}\,\delta_{n,m} \;=\; s_n\delta_{n,m}
\end{aligned}$$

with $s_n = \lambda_n/|\lambda_n|$ and $\delta_{n,m} = 0$ for $n \neq m$ and $\delta_{n,m} = 1$ for $n = m$. We note that $|s_n| = 1$ and $\mathrm{Im}\, s_n > 0$. Since λ_n tends to zero, the only accumulation points of s_n can be 1 or -1. From $T = I + C$ for some compact operator C we conclude that

$$\|\phi_n\|^2_{L^2(D,|q|dx)} \;+\; (C\phi_n, \phi_n)_{L^2(D,|q|dx)} \;=\; \sigma s_n\,. \tag{4.20}$$

First we show that the sequence ϕ_n is bounded. Otherwise there exists a subsequence such that $\|\phi_n\|_{L^2(D,|q|dx)} \to \infty$. The sequence $\hat\phi_n = \phi_n/\|\phi_n\|_{L^2(D,|q|dx)}$ satisfies

$$1 \;+\; (C\hat\phi_n, \hat\phi_n)_{L^2(D,|q|dx)} \;=\; \frac{\sigma s_n}{\|\phi_n\|^2_{L^2(D,|q|dx)}} \;\longrightarrow\; 0\,.$$

Since it is bounded there exists a weakly convergent subsequence $\hat\phi_n \rightharpoonup \hat\phi$. The compactness of C implies $(C\hat\phi_n, \hat\phi_n)_{L^2(D,|q|dx)} \to (C\hat\phi, \hat\phi)_{L^2(D,|q|dx)}$ and thus $1 + (C\hat\phi, \hat\phi)_{L^2(D,|q|dx)} = 0$.

Taking the imaginary part yields $\mathrm{Im}\,(C\hat{\phi},\hat{\phi})_{L^2(D,|q|dx)} = 0$ and thus also

$$\mathrm{Im}\,(T\hat{\phi},\hat{\phi})_{L^2(D,|q|dx)} = 0\,.$$

Theorem 3.1 shows that $\hat{\phi} = 0$ which contradicts $1 + (C\hat{\phi},\hat{\phi})_{L^2(D,|q|dx)} = 0$. Therefore, the sequence ϕ_n is bounded and contains a weakly convergent subsequence $\phi_n \rightharpoonup \phi$. Again, $(C\phi_n,\phi_n)_{L^2(D,|q|dx)}$ converges to $(C\phi,\phi)_{L^2(D,|q|dx)}$. Also, taking the imaginary part of (4.20), yields $\mathrm{Im}\,(C\phi_n,\phi_n)_{L^2(D,|q|dx)} \to 0$ and thus again $\mathrm{Im}\,(C\phi,\phi)_{L^2(D,|q|dx)} = 0$ and $\mathrm{Im}\,(T\phi,\phi)_{L^2(D,|q|dx)} = 0$. We conclude again $\phi = 0$ and thus $(C\phi_n,\phi_n)_{L^2(D,|q|dx)} \to 0$. Equation (4.20) yields $\|\phi_n\|^2_{L^2(D,|q|dx)} - \sigma s_n \longrightarrow 0$ and enforces σ to be the accumulation point of (s_n) rather than $-\sigma$. $\square$

We translate this into a condition on the eigenvalues of the scattering matrix $\mathcal{S}$.

Corollary 4.2 *Let k be no interior transmission eigenvalue and let $\mu_n \in \mathbb{C}$ be the eigenvalues of the scattering operator $\mathcal{S} = I + \frac{ik}{8\pi^2}F$. Then $|\mu_n| = 1$ for all n, they converge to 1 and $\sigma\mathrm{Im}\,\mu_n > 0$ for sufficiently large n.*

Let k be not an interior transmission eigenvalue. Then 1 is not an eigenvalue of $\mathcal{S}$ and the far field operator F is one-to-one with dense range. We define the Cayley transform $\mathcal{T}$ by

$$\mathcal{T} \;=\; i(I+\mathcal{S})(I-\mathcal{S})^{-1} \;:\; L^2(S^2) \supset \mathcal{R}(F) \;\longrightarrow\; L^2(S^2)$$

It is easily seen that $\mathcal{T}$ is selfadjoint, its spectrum is discrete, and $\mu_n = \exp(-2i\delta_n)$ is an eigenvalue of $\mathcal{S}$ for some $\delta_n \in (0,\pi)$ if, and only if, $\cot\delta_n \in \mathbb{R}$ is an eigenvalue of $\mathcal{T}$. Since (μ_n) tends to one with $\sigma\mathrm{Im}\,\mu_n > 0$ we conclude that

$$\lim_{n\to\infty}\delta_n \;=\; \begin{cases} 0 & \text{if } \sigma = -1\,, \\ \pi & \text{if } \sigma = +1\,, \end{cases} \qquad \lim_{n\to\infty}\cot\delta_n \;=\; \begin{cases} +\infty & \text{if } \sigma = -1\,, \\ -\infty & \text{if } \sigma = +1. \end{cases}$$

In particular, the numbers

$$\delta_+ = \max_{n\in\mathbb{N}}\delta_n \quad \text{if } \sigma = -1 \qquad \text{and} \qquad \delta_- = \min_{n\in\mathbb{N}}\delta_n \quad \text{if } \sigma = +1, \tag{4.21}$$

are well-defined and belong to $(0,\pi)$ if k is not an interior transmission eigenvalue. We call them the "first" eigenvalue.

Lemma 4.3 *Let k be no interior transmission eigenvalue.*

(a) Let first $\sigma = -1$; that is, $q < 0$ in D. Then

$$\cot\delta_+ \;=\; \inf_{w\in X}\frac{\mathrm{Re}\,(Tw,w)_{L^2(D,|q|dx)}}{-\mathrm{Im}\,(Tw,w)_{L^2(D,|q|dx)}}\,.$$

(b) Second, let $\sigma = +1$; that is, $q > 0$ in D. Then

$$\cot \delta_- \;=\; -\inf_{w\in X} \frac{\mathrm{Re}\,(Tw,w)_{L^2(D,|q|dx)}}{\mathrm{Im}\,(Tw,w)_{L^2(D,|q|dx)}}\,.$$

Note that in both cases the denominator is strictly positive because of the assumption on k and Theorem 3.1(b).

Proof: (a) Courant's max-min principle (see [10]) yields

$$\begin{aligned}
\cot \delta_+ \;&=\; \inf_{f\in\mathcal{R}(F)} \frac{(\mathcal{T}f,f)_{L^2(S^2)}}{\|f\|^2_{L^2(S^2)}} \;=\; \inf_{f\in\mathcal{R}(F)} \frac{i\big((I+\mathcal{S})(I-\mathcal{S})^{-1}f,f\big)_{L^2(S^2)}}{\|f\|^2_{L^2(S^2)}}\\
&=\; \inf_{g\in L^2(S^2)} \frac{i\big((I+\mathcal{S})g,(I-\mathcal{S})g\big)_{L^2(S^2)}}{\|(I-\mathcal{S})g\|^2_{L^2(S^2)}}\\
&=\; \inf_{g\in L^2(S^2)} \frac{i\big(\|g\|^2_{L^2(S^2)} + 2i\,\mathrm{Im}\,(\mathcal{S}g,g)_{L^2(S^2)} - \|\mathcal{S}g\|^2_{L^2(S^2)}\big)}{\|g\|^2_{L^2(S^2)} - 2\,\mathrm{Re}\,(\mathcal{S}g,g)_{L^2(S^2)} + \|\mathcal{S}g\|^2_{L^2(S^2)}}\\
&=\; \inf_{g\in L^2(S^2)} \frac{\mathrm{Im}\,(\mathcal{S}g,g)_{L^2(S^2)}}{\mathrm{Re}\,(\mathcal{S}g,g)_{L^2(S^2)} - \|g\|^2_{L^2(S^2)}}\,.
\end{aligned}$$

Now we make use of the form $\mathcal{S} = I + (ik)/(8\pi^2)F$ and the factorization (2.13). In the following we write (Th,h) instead of $(Th,h)_{L^2(D,|q|dx)}$. Then

$$\cot \delta_+ \;=\; \inf_{g\in L^2(S^2)} \frac{\mathrm{Re}\,(Fg,g)_{L^2(S^2)}}{-\mathrm{Im}\,(Fg,g)_{L^2(S^2)}} \;=\; \inf_{g\in L^2(S^2)} \frac{\mathrm{Re}\,(THg,Hg)}{-\mathrm{Im}\,(THg,Hg)} \;=\; \inf_{w\in X} \frac{\mathrm{Re}\,(Tw,w)}{-\mathrm{Im}\,(Tw,w)}$$

which proves the assertion. Part (b) follows the same lines by just replacing the infimum by the supremum and noting that

$$\sup_{w\in X} \frac{\mathrm{Re}\,(Tw,w)}{-\mathrm{Im}\,(Tw,w)} \;=\; -\inf_{w\in X} \frac{\mathrm{Re}\,(Tw,w)}{\mathrm{Im}\,(Tw,w)}\,.$$

$\square$

5 Spectral Behavior of the Scattering Operator at a Transmission Eigenvalue

In the following we study the dependence of the eigenvalues of the scattering operator on the wavenumber k and write $X(k)$, $T(k)$, $\delta_\pm(k)$, and so on to indicate this dependence. The eigenvalue characterization from Lemma 4.3 uses k-dependent spaces over which the infimum or supremum is taken. In a first step, we will transform this k-dependence into the quadratic forms by introducing the orthogonal projection operator $P(k)$ from $L^2(D,|q|dx)$ into $X(k)$. This step will hence eliminate the k-dependence of the considered function spaces.

For the sake of notational simplicity, we write $k \nearrow k_0$ and $k \searrow k_0$ to indicate that $k \in \mathbb{R}$ tends to $k_0 \in \mathbb{R}$ from below and above, respectively. More precisely, if $k \nearrow k_0$ ($k \searrow k_0$) the inequality $k < k_0$ ($k > k_0$) is always satisfied in the limiting process.

Recall that the numbers $\delta_\pm(k)$ were defined in (4.21). They allow us to state the following simple result.

Lemma 5.1 *Let $k_0 > 0$ and $w_0 \in X(k_0)$ such that $w_0 \neq 0$ and $\big(T(k_0)w_0, w_0\big)_{L^2(D,|q|dx)} = 0$ and assume that*

$$\alpha \ := \ \left[\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\right]\Bigg|_{k=k_0} \in \mathbb{R}\setminus\{0\}.$$

(a) Let $\sigma = -1$; that is, $q < 0$ in D. Then

$$\lim_{k\nearrow k_0}\delta_+(k) \ = \ \pi \ \text{if}\ \alpha > 0\,, \quad \lim_{k\searrow k_0}\delta_+(k) \ = \ \pi \ \text{if}\ \alpha < 0\,.$$

(b) Let $\sigma = +1$; that is, $q > 0$ in D. Then

$$\lim_{k\nearrow k_0}\delta_-(k) \ = \ 0 \ \text{if}\ \alpha > 0\,, \quad \lim_{k\searrow k_0}\delta_-(k) \ = \ 0 \ \text{if}\ \alpha < 0\,.$$

Note that we will show in Lemmas 5.2 and 5.3 that $k \mapsto \big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}$ is indeed a differentiable function in k_0 for every fixed $w_0 \in X(k_0)$.

Proof: (a) We note from Theorem 3.1 that $\big(T(k_0)w_0, w_0\big)_{L^2(D,|q|dx)} = 0$ implies that k_0 is an interior transmission eigenvalue. Let $I = (k_0 - \varepsilon, k_0 + \varepsilon)$ be an interval containing no other transmission eigenvalue. From the previous lemma we have for $k \in I \setminus \{k_0\}$

$$\cot\delta_+(k) \ = \ \inf_{g\in L^2(D,|q|dx)} \frac{\mathrm{Re}\,\big(T(k)P(k)g, P(k)g\big)_{L^2(D,|q|dx)}}{-\mathrm{Im}\,\big(T(k)P(k)g, P(k)g\big)_{L^2(D,|q|dx)}}\,.$$

Furthermore, from Taylor's theorem we have that

$$\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)} \ = \ \alpha(k-k_0) \ + \ r(k)$$

with $r(k) = o(|k-k_0|)$ and $\mathrm{Im}\, r(k) < 0$. Here we used that

$$\big(T(k_0)P(k_0)w_0, P(k_0)w_0\big)_{L^2(D,|q|dx)} \ = \ \big(T(k_0)w_0, w_0\big)_{L^2(D,|q|dx)} = 0\,.$$

Therefore, we have

$$\cot \delta_+(k) \;\leq\; \frac{\operatorname{Re}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}}{-\operatorname{Im}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}} \;=\; \frac{\alpha(k-k_0) + \operatorname{Re} r(k)}{-\operatorname{Im} r(k)}$$

which converges to $-\infty$ provided $k \to k_0$ such that $\alpha(k-k_0) < 0$.

Part (b) is proven in the same way. $\square$

Now we want to investigate the derivative of $k \mapsto \big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}$ at an eigenvalue k_0. We begin with the following result.

Lemma 5.2 *Let $k_0 > 0$ be an interior transmission eigenvalue. Due to Theorem 3.1 there exists $w_0 \in X(k_0)$ such that $w_0 \neq 0$ and $(T(k_0)w_0, w_0)_{L^2(D,|q|dx)} = 0$. Then $k \mapsto \big(T(k)w_0, w_0\big)_{L^2(D,|q|dx)}$ is differentiable in k_0 and*

$$\left[\frac{d}{dk}\big(T(k)w_0, w_0\big)_{L^2(D,|q|dx)}\right]\Bigg|_{k=k_0} \;=\; 2\sigma k_0 \int_D |\nabla v_0|^2 dx \;\neq\; 0 \tag{5.22}$$

where v_0 is the radiating solution of (2.11) for $k = k_0$ and $f = w_0$; that is,

$$\Delta v_0 + k_0^2(1+q)v_0 \;=\; -q\, w_0 \quad \textit{in } \mathbb{R}^3\,. \tag{5.23}$$

Proof: First we note that v_0 vanishes outside of D by Remark 3.2. In particular, $v_0 \in H_0^2(D)$. Let v' be the unique radiating solution of

$$\Delta v' \;+\; k_0^2(1+q)v' \;=\; -2k_0(1+q)v_0 \quad \text{in } \mathbb{R}^3\,, \tag{5.24}$$

and v_k the solution of (5.23) corresponding to k instead of k_0. Then it is easily seen that

$$\begin{aligned}
\frac{d}{dk}\big(T(k_0)w_0, w_0\big)_{L^2(D,|q|dx)} \;&=\; \frac{d}{dk}\int_D (w_0 + k^2 v_0)\,\overline{w_0}\,|q|\,dx\Bigg|_{k=k_0} \\
&=\; 2k_0 \int_D v_0\,\overline{w_0}\,|q|\,dx \;+\; k_0^2 \int_D v'\,\overline{w_0}\,|q|\,dx\,.
\end{aligned}$$

Now we eliminate w_0 from this equation by using (5.23)

$$\begin{aligned}
&\frac{d}{dk}\big(T(k_0)w_0, w_0\big)_{L^2(D,|q|dx)} \\
&= \; -2k_0\sigma \int_D v_0\,\big[\Delta\overline{v_0} + k_0^2(1+q)\overline{v_0}\big]\,dx \;-\; \sigma k_0^2 \int_D v'\,\big[\Delta\overline{v_0} + k_0^2(1+q)\overline{v_0}\big]\,dx \\
&= \; 2k_0\sigma \int_D \big[|\nabla v_0|^2 - k_0^2(1+q)|v_0|^2\big]\,dx \;-\; \sigma k_0^2 \int_D \overline{v_0}\,\big[\Delta v' + k_0^2(1+q)v'\big]\,dx \\
&= \; 2k_0\sigma \int_D \big[|\nabla v_0|^2 - k_0^2(1+q)|v_0|^2\big]\,dx \;+\; 2\sigma k_0^3 \int_D (1+q)\,|v_0|^2\,dx \\
&= \; 2\sigma k_0 \int_D |\nabla v_0|^2 dx
\end{aligned}$$

where we used the definition of v'. This term vanishes only for constant functions v_0; that is, for $v_0 = 0$ because of $v_0 \in H_0^2(D)$. $\square$

Now we study the dependence of the orthogonal projection $P(k) : L^2(D, |q|dx) \to X(k)$ on k. To give an explicit representation of $P(k)$, let us denote by W the completion of $C_0^\infty(D)$ with respect to the semi-norm $\|\psi\|_W := \big\|(\Delta\psi + k^2\psi)/\sqrt{|q|}\big\|_{L^2(D)}$. Then $P(k)$ is explicitly given by

$$P(k)g \;=\; g \;-\; \frac{1}{|q|}(\Delta\hat{w} + k^2\hat{w})\,,$$

where $\hat{w} \in W$ solves the 4th order system

$$(\Delta + k^2)\frac{1}{|q|}(\Delta + k^2)\hat{w} \;=\; (\Delta + k^2)g \tag{5.25}$$

in the variational sense; that is, $\hat{w}$ solves the W-coercive variational problem

$$\int_D \frac{1}{|q|}\,(\Delta\hat{w} + k^2\hat{w})\,(\Delta\psi + k^2\psi)\,dx \;=\; \int_D g\,(\Delta\psi + k^2\psi)\,dx \qquad \text{for all } \psi \in W.$$

Then we have the following extension of the previous lemma:

Lemma 5.3 *Let $k_0 > 0$ be an interior transmission eigenvalue and $w_0 \in X(k_0)$ such that $w_0 \neq 0$ and $(T(k_0)w_0, w_0)_{L^2(D,|q|dx)} = 0$. Then $k \mapsto \big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}$ is differentiable in k_0 and*

$$\left[\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\right]\Bigg|_{k=k_0} \;=\; 2\sigma k_0\left[\int_D |\nabla v_0|^2 dx + 2\,\mathrm{Re}\int_D v_0\,\overline{w_0}\,dx\right] \tag{5.26}$$

where v_0 is the solution of (5.23).

Proof: First we note that $k \mapsto P(k)w_0$ is Frechét–differentiable and, by differentiating the characterization of $P(k)w_0$,

$$P'(k)w_0 \;=\; -\frac{1}{|q|}\,(\Delta\hat{w}' + k^2\hat{w}' + 2k\hat{w})$$

where $\hat{w} \in W$ solves (5.25) for $g = w_0$ and $\hat{w}' \in W$ solves

$$(\Delta + k^2)\frac{1}{|q|}(\Delta + k^2)\hat{w}' \;=\; 2k\,w_0 \;-\; \frac{2k}{|q|}(\Delta\hat{w} + k^2\hat{w}) \;-\; 2k(\Delta + k^2)\frac{1}{|q|}\hat{w}\,.$$

By the chain rule we have

$$\begin{aligned}
&\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\\
&\quad= \big(T'(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)} + \big(T(k)P'(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\\
&\qquad+ \big(T(k)P(k)w_0, P'(k)w_0\big)_{L^2(D,|q|dx)}\\
&\quad= \big(T'(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)} + \overline{\big(T^*(k)P(k)w_0, P'(k)w_0\big)}_{L^2(D,|q|dx)}\\
&\qquad+ \big(T(k)P(k)w_0, P'(k)w_0\big)_{L^2(D,|q|dx)}.
\end{aligned}$$

The first term on the right hand side has been computed at $k=k_0$ in the previous lemma. The adjoint T^* of T is given by $T^*g = g + k^2\overline{v_g}$ where v_g is the radiating solution of

$$\Delta v_g \;+\; k^2(1+q)v_g \;=\; -q\,\overline{g} \quad \text{in } \mathbb{R}^3\,.$$

Indeed, if $f \in L^2(D,|q|dx)$ with corresponding v_f satisfying

$$\Delta v_f \;+\; k^2(1+q)v_f \;=\; -q\,f \quad \text{in } \mathbb{R}^3\,,$$

then

$$\begin{aligned}
(Tf,g)_{L^2(D,|q|dx)} &= (f,g)_{L^2(D,|q|dx)} \;+\; \sigma\,k^2\int_D v_f\,\overline{g}\,q\,dx \\
&= (f,g)_{L^2(D,|q|dx)} \;-\; \sigma\,k^2\int_D v_f\left(\Delta v_g + k^2(1+q)v_g\right)dx \\
&= (f,g)_{L^2(D,|q|dx)} \;-\; \sigma\,k^2\int_D v_g\left(\Delta v_f + k^2(1+q)v_f\right)dx \\
&= (f,g)_{L^2(D,|q|dx)} \;+\; \sigma\,k^2\int_D v_g\,f\,q\,dx \\
&= (f, g+k^2\overline{v_g})_{L^2(D,|q|dx)}\,.
\end{aligned}$$

From $0 = (T(k_0)w_0,w_0)_{L^2(D,|q|dx)} = (w_0, T^*(k_0)w_0)_{L^2(D,|q|dx)}$ one concludes as in the proof of Theorem 2.5 that v which corresponds to $T^*(k_0)w_0$ vanishes outside of D and, therefore, coincides with $\overline{v_0}$. This shows that $T^*(k_0)w_0 = T(k_0)w_0$ and thus

$$\begin{aligned}
\left[\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\right]\Bigg|_{k=k_0} \;&=\; 2\sigma k_0\int_D |\nabla v_0|^2 dx \\
&\quad + 2\,\mathrm{Re}\,\big(T(k_0)w_0, P'(k_0)w_0\big)_{L^2(D,|q|dx)}
\end{aligned}$$

because $P(k_0)w_0 = w_0$. Therefore also $\hat{w}=0$ and

$$P'(k_0)w_0 \;=\; -\frac{1}{|q|}(\Delta\hat{w}' + k_0^2\hat{w}')$$

and $\hat{w}' \in W$ solves

$$(\Delta + k_0^2)\frac{1}{|q|}(\Delta + k_0^2)\hat{w}' \;=\; 2k_0\,w_0\,.$$

We compute, noting that $P(k_0)w_0 = w_0$,

$$\begin{aligned}
&\big(T(k_0)w_0, P'(k_0)w_0\big)_{L^2(D,|q|dx)} \\
&\quad= -\int_D (w_0 + k_0^2 v_0)\,\overline{(\Delta\hat{w}' + k_0^2\hat{w}')}\,dx \;=\; \sigma\int_D \frac{1}{|q|}\,(\Delta v_0 + k_0^2 v_0)\,\overline{(\Delta\hat{w}' + k_0^2\hat{w}')}\,dx \\
&\quad= \sigma\int_D v_0\,\overline{(\Delta + k_0^2)\frac{1}{|q|}(\Delta + k_0^2)\hat{w}'\,dx} \;=\; 2k_0\sigma\int_D v_0\,\overline{w_0}\,dx\,.
\end{aligned}$$

This proves the assertion. $\square$

6 Inside-Outside Duality

We would like to find conditions on the contrast q ensuring that the derivative in (5.26) is either strictly positive or strictly negative, that is,

$$\left[\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\right]\Big|_{k=k_0} \;=\; 2\,k_0\,\sigma A \;\lessgtr\; 0 \tag{6.27}$$

$$\text{where}\quad A \;=\; \int_D \left[|\nabla v_0|^2 + 2\,\mathrm{Re}\,(v_0\,\overline{w_0})\right] dx\,, \tag{6.28}$$

for all $0 \neq w_0 \in X(k_0)$ such that $\big(T(k)w_0, w_0\big)_{L^2(D,|q|dx)} = 0$. Due to Lemma 5.1, such a sign property allows, generally speaking, to characterize when $k_0 > 0$ is an interior transmission eigenvalue, merely knowing the far field operators $F(k)$ in some interval around k_0. Unfortunately, we are able to prove this sign property under strong assumptions on the contrast q only, roughly speaking for constant contrast that is either positive and large enough or negative and small enough. Weakening these assumptions is an issue of ongoing research.

Theorem 6.1 *Let k_0 be the smallest interior transmission eigenvalue and $q(x) = q_0 > 0$ for $x \in D$ being constant such that*

$$q_0 \;>\; 2\left[\left(\frac{\hat\rho_1}{\rho_1^2} - 1\right) + \frac{\sqrt{\hat\rho_1}}{\rho_1}\sqrt{\frac{\hat\rho_1}{\rho_1^2} - 1}\,\right]. \tag{6.29}$$

Here, $\hat\rho_1$ and ρ_1 denote the smallest Dirichlet eigenvalues of Δ^2 and $-\Delta$, respectively, in D.[3] *Then*

$$\left[\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\right]\Big|_{k=k_0} \;>\; 0 \tag{6.30}$$

for all $0 \neq w_0 \in X(k_0)$ such that $\big(T(k)w_0, w_0\big)_{L^2(D,|q|dx)} = 0$.

Proof: Multiplication of (5.23) with $\overline{v_0}$, integration, and Green's first identity yields

$$k_0^2(1+q_0)\,\|v_0\|^2_{L^2(D)} \;-\; \|\nabla v_0\|^2_{L^2(D)} \;=\; -q_0\,(w_0, v_0)_{L^2(D)}\,;$$

that is, taken the real part,

$$2\,\mathrm{Re}\,(w_0, v_0)_{L^2(D)} \;=\; \frac{2}{q_0}\,\|\nabla v_0\|^2_{L^2(D)} \;-\; 2k_0^2\,\frac{1+q_0}{q_0}\,\|v_0\|^2_{L^2(D)}\,. \tag{6.31}$$

We write

$$\begin{aligned} A \;&=\; \|\nabla v_0\|^2_{L^2(D)} \;+\; 2\,\mathrm{Re}\,(w_0, v_0)_{L^2(D)} \;=\; \frac{2+q_0}{q_0}\,\|\nabla v_0\|^2_{L^2(D)} \;-\; 2k_0^2\,\frac{1+q_0}{q_0}\,\|v_0\|^2_{L^2(D)} \\ &=\; 2\,\frac{1+q_0}{q_0}\left[\frac{1+q_0/2}{1+q_0}\,\|\nabla v_0\|^2_{L^2(D)} \;-\; k_0^2\,\|v_0\|^2_{L^2(D)}\right]. \end{aligned}$$

[3] Note that $\hat\rho_1 > \rho_1^2$, see [18]

Now we use that $\rho_1 \, \|v_0\|^2_{L^2(D)} \leq \|\nabla v_0\|^2_{L^2(D)}$ and arrive at

$$A \;\geq\; 2\,\frac{1+q_0}{q_0}\left[\rho_1\,\frac{1+q_0/2}{1+q_0} \;-\; k_0^2\right]\|v_0\|^2_{L^2(D)}\,.$$

It has been shown in [15] that for q_0 satisfying (6.29) the smallest transmission eigenvalue k_0 satisfies

$$k_0^2 \;<\; \rho_1\,\frac{1+q_0/2}{1+q_0}\,.$$

Therefore, for this eigenvalue we have that $A > 0$. □

Theorem 6.2 *Let k_0 be the smallest interior transmission eigenvalue and $q(x) = q_0$ for $q_0 \in (-1, 0)$. Then there exists $\hat{q} \in (-1, 0)$ such that*

$$-1 < q_0 \leq \hat{q} \quad \textit{implies that} \quad \left[\frac{d}{dk}\big(T(k)P(k)w_0, P(k)w_0\big)_{L^2(D,|q|dx)}\right]\bigg|_{k=k_0} \;<\; 0 \qquad (6.32)$$

for all $0 \neq w_0 \in X(k_0)$ such that $\big(T(k)w_0, w_0\big)_{L^2(D,|q|dx)} = 0$.

Proof: We exploit again (6.31) together with the Poincaré inequality $\rho_1\|v_0\|^2_{L^2(D)} \leq \|\nabla v_0\|^2_{L^2(D)}$ involving the first Dirichlet eigenvalue ρ_1 of $-\Delta$ in D to estimate that

$$\begin{aligned}
A \;&=\; \|\nabla v_0\|^2_{L^2(D)} \;+\; 2\,\mathrm{Re}\int_D v_0\overline{w}_0\,dx \\
&=\; \left(1-\frac{2}{|q_0|}\right)\|\nabla v_0\|^2_{L^2(D)} \;+\; 2k_0^2\,\frac{1+q_0}{|q_0|}\,\|v_0\|^2_{L^2(D)} \\
&\leq\; \left(1-\frac{2}{|q_0|}\right)\|\nabla v_0\|^2_{L^2(D)} \;+\; 2k_0^2\,\frac{1+q_0}{|q_0|}\,\frac{1}{\rho_1}\|\nabla v_0\|^2_{L^2(D)} \\
&=\; \left[1-\frac{2}{|q_0|} \;+\; 2k_0^2\,\frac{1+q_0}{|q_0|\rho_1}\right]\|\nabla v_0\|^2_{L^2(D)}\,.
\end{aligned}$$

If R_+ denotes the radius of the smallest ball $B = B(0, R_+)$ containing $\overline{D}$ then

$$\rho_1 \;=\; inf_{\substack{u\in H_0^1(D)\\ u\neq 0}}\frac{\|\nabla u\|_{L^2(D)}}{\|u\|_{L^2(D)}} \;\geq\; \inf_{\substack{u\in H_0^1(B)\\ u\neq 0}}\frac{\|\nabla u\|_{L^2(B)}}{\|u\|_{L^2(B)}} \;=\; \rho_{1,B} \;=\; \left(\frac{\pi}{R_+}\right)^2$$

where $\rho_{1,B}$ denotes the smallest Dirichlet eigenvalue of the ball $B = B(0, R_+)$. Substituting this estimate into the previous estimate for A we arrive at

$$A \;\leq\; \left[1-\frac{2}{|q_0|} + \;2k_0^2\,\frac{1+q_0}{|q_0|}\,\frac{1}{\pi^2}R_+^2\right]\;\|\nabla v_0\|^2_{L^2(D)}.$$

The last term on the right is negative if, and only if,

$$1 \;+\; 2k_0^2\,\frac{1+q_0}{|q_0|}\,\frac{1}{\pi^2}\,R_+^2 \;\leq\; \frac{2}{|q_0|}\,; \quad \text{that is,} \quad k_0^2 \;\leq\; \pi^2\,\frac{1-|q_0|/2}{1+q_0}\,\frac{1}{R_+^2}\,.$$

From the proof of Theorem 3.2 and Corollary 3.1 in [3] we know that the smallest transmission eigenvalue k_0 of the obstacle D for the contrast q_0 satisfies

$$k_0 \;\le\; \frac{k_{B(0,1),q_0}}{R_-}$$

where $k_{B(0,1),q_0}$ is the *smallest* transmission eigenvalue of the unit ball for the contrast q_0 and R_- is the radius of the largest ball contained in $\overline{D}$. Hence, whenever

$$k^2_{B(0,1),q_0} \;<\; \pi^2 \, \frac{1-|q_0|/2}{1+q_0} \frac{R_-^2}{R_+^2} \tag{6.33}$$

we can conclude that A is negative at least for the smallest transmission eigenvalue.

For constant q_0, the smallest transmission eigenvalue of the unit ball can be estimated from above by the smallest positive zero of

$$W(\lambda) \;=\; \det \begin{pmatrix} j_0(\lambda) & j_0(\lambda\sqrt{1+q_0}) \\ -j_0'(\lambda) & -\sqrt{1+q_0}\, j_0'(\lambda\sqrt{1+q_0}) \end{pmatrix} \qquad \text{for } \lambda \ge 0\,.$$

It is well-known that positive roots of W are transmission eigenvalues. Setting $n_0 = \sqrt{1+q_0} \ge 0$, we find that

$$W(\lambda, n_0) \;=\; j_0'(\lambda)\, j_0(\lambda n_0) \;-\; j_0(\lambda)\, n_0\, j_0'(\lambda n_0), \qquad \lambda \ge 0,\; n_0 \ge 0.$$

We use that $j_0(\lambda) = \sin(\lambda)/\lambda$ and observe that $j_0(0) = 1$ and $j_0'(0) = 0$. Therefore,

$$W(\lambda, 0) \;=\; j_0'(\lambda) \;=\; \frac{\lambda\cos\lambda - \sin\lambda}{\lambda^2} \;=\; \frac{\phi(\lambda)}{\lambda^2}$$

with $\phi(\lambda) = \lambda\cos\lambda - \sin\lambda$. Elementary arguments (consider $\phi'(\lambda) = -\lambda\sin\lambda$) show that the first positive zero $\hat\lambda_1$ of $W(\cdot, 0)$ is in the interval $(\pi, 2\pi)$. Furthermore,

$$\left.\frac{\partial W(\lambda, n_0)}{\partial\lambda}\right|_{n_0=0} \;=\; \frac{\phi'(\lambda)}{\lambda^2} \;-\; 2\,\frac{\phi(\lambda)}{\lambda^3}$$

and thus

$$\frac{\partial W(\hat\lambda_1, 0)}{\partial\lambda} \;=\; -\frac{\sin\hat\lambda_1}{\hat\lambda_1} \;\ne\; 0\,.$$

The implicit function theorem assures existence of $\hat n_0 > 0$ and an intervall I around $\hat\lambda_1$ such that

$$W(\lambda_1(n_0), n_0) \;=\; 0$$

is uniquely solvable in $I \times [0, \hat n_0]$ and $\lambda_1(n_0) \to \hat\lambda_1$ as $n_0 \to 0$. Since the limit $n_0 \to 0$ corresponds to $q_0 \to -1$ we obtain in particular that the smallest transmission eigenvalue $k_{B(0,1),q_0}$ remains bounded as $q_0 \to -1$, since

$$0 \;<\; k_{B(0,1),q_0} \;\le\; \lambda_1(n_0) \;\to\; \hat\lambda_1 \qquad \text{as } q_0 \to -1.$$

In consequence, the left–hand side of (6.33) remains bounded as $q_0 \to -1$ while the right–hand side obviously tends to $+\infty$ as $q_0 \to -1$. This shows that there exists $\hat{q} \in (-1, 0)$ such that (6.33) is indeed satisfied. $\Box$

Now we are ready to prove the main result of this paper.

Theorem 6.3 (Inside-Outside Duality for $\mathcal{S}$)

Let $k_0 > 0$ and $I = (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$ such that no $k \in I$ is an interior transmission eigenvalue. For $k \in I$ let $\mu_n(k) = \exp\bigl(-2i\delta_n(k)\bigr)$ be the eigenvalues of $\mathcal{S}(k)$ with phases $\delta_n(k) \in (0, \pi)$.

(a) Let k_0 be the smallest interior transmission eigenvalue and let $q = q_0$ satisfy (6.29). Then

$$\lim_{k \in I,\ k \nearrow k_0} \delta_-(k) \;=\; 0 \tag{6.34}$$

where $\delta_-(k) = \min_n \delta_n(k)$.

(b) Let (6.34) hold and $q > 0$ in D. Then k_0 is an interior transmission eigenvalue.

(c) Let k_0 be the smallest interior transmission eigenvalue and let $q = q_0$ satisfy the assumption of (6.32) *in Theorem 6.2. Then*

$$\lim_{k \in I,\ k \nearrow k_0} \delta_+(k) \;=\; \pi \tag{6.35}$$

where $\delta_+(k) = \max_n \delta_n(k)$.

(d) Let (6.35) hold and $q < 0$ in D. Then k_0 is an interior transmission eigenvalue.

Proof: (a) Let k_0 be the smallest interior transmission eigenvalue and $w_0 \in X(k_0)$ with $w_0 \neq 0$ and $\bigl(T(k_0)w_0, w_0\bigr)_{L^2(D,|q|dx)} = 0$. We apply part (b) of Lemma 5.1 and note that $\alpha > 0$ by Theorem 6.1. This yields (6.34).

(b) Let now (6.34) hold and assume, on the contrary, that k_0 is not an interior transmission eigenvalue. By Lemma 4.3 the condition (6.34) is equivalent to

$$\inf_{w \in X(k)} \frac{\operatorname{Re}\,(T(k)w, w)_{L^2(D,|q|dx)}}{\operatorname{Im}\,(T(k)w, w)_{L^2(D,|q|dx)}} \;\longrightarrow\; -\infty \quad \text{as } k \nearrow k_0\,.$$

Therefore, there exist sequences $k_j \in I$, $k_j \nearrow k_0$, and $w_j \in X(k_j)$ with $\|w_j\|_{L^2(D,|q|dx)} = 1$ such that

$$\operatorname{Im}\,(T(k_j)w_j, w_j)_{L^2(D,|q|dx)} \longrightarrow 0\,,\ j \to \infty\,, \quad \text{and} \quad \operatorname{Re}\,(T(k_j)w_j, w_j)_{L^2(D,|q|dx)} \;\leq\; 0$$

for sufficiently large j. Let $v_j \in H^1_{loc}(\mathbb{R}^3)$ be the corresponding radiating solutions of (5.23). Then $w_j \rightharpoonup w_0$ weakly in $L^2(D,|q|dx)$ for some subsequence and some $w_0 \in L^2(D,|q|dx)$. It is easy to see that $w_0 \in X(k_0)$ and $v_j \rightharpoonup v_0$ weakly in $H^1\big(B(0,R)\big)$ for every ball $B(0,R)$ where $v_0 \in H^1_{loc}(\mathbb{R}^3)$ is the radiating weak solution to $\Delta v_0 + k_0^2(1+q)v_0 = -q\,w_0$ in $\mathbb{R}^3$ (see Lemma 2.4 for an existence proof). From (2.16) for $f = w_j$ we conclude that

$$\mathrm{Im}\,\big(T(k_j)w_j, w_j\big)_{L^2(D,|q|dx)} \;=\; k_j^3 \|v_j^\infty\|^2_{L^2(S^2)}\,.$$

The left hand side converges to zero, the right hand side to $k_0^3\|v_0^\infty\|^2_{L^2(S^2)}$ by Lemma 2.4 again. Therefore, $v_0^\infty = 0$ and thus v_0 vanishes outside of D by Rellich's lemma. Because k_0 is not an interior transmission eigenvalue we conclude that w_0 and v_0 vanish everywhere; that is, w_j and v_j converge weakly to zero. Now we compute, similarly to (2.14),

$$\begin{aligned}
(Tw_j, w_j)_{L^2(D,|q|dx)} \;&=\; \int\limits_D |w_j|^2\, q\, dx \;+\; k_j^2 \int_D v_j\, \overline{w_j}\, q\, dx \\
&=\; \|w_j\|^2_{L^2(D,|q|dx)} \;-\; k_j^2 \int\limits_{|x|<R} v_j\,(\Delta\overline{v_j} + k_j^2(1+q)\overline{v_j})\, dx \\
&=\; \|w_j\|^2_{L^2(D,|q|dx)} \;+\; k_j^2 \int\limits_{|x|<R} \big[|\nabla v_j|^2 - k_j^2(1+q)\,|v_j|^2\big]\, dx \\
&\quad\; - k_j^2 \int_{|x|=R} v_j\, \frac{\partial\overline{v_j}}{\partial\nu}\, ds
\end{aligned}$$

and thus, taking the real part,

$$\begin{aligned}
k_j^2 \int\limits_{|x|<R} |\nabla v_j|^2\, dx \;&=\; \mathrm{Re}\,\big(T(k_j)w_j, w_j\big)_{L^2(D,|q|dx)} \;-\; \|w_j\|^2_{L^2(D,|q|dx)} \\
&\quad\; + k_j^4 \int\limits_{|x|<R} (1+q)|v_j|^2\, dx \;+\; k_j^2 \int_{|x|=R} v_j\, \frac{\partial\overline{v_j}}{\partial\nu}\, ds \\
&\leq\; k_j^4 \int\limits_{|x|<R} (1+q)\,|v_j|^2\, dx \;+\; k_j^2 \int\limits_{|x|=R} v_j\, \frac{\partial\overline{v_j}}{\partial\nu}\, ds
\end{aligned}$$

which converges to zero by Lemma 2.4 and the compact imbedding of $H^1\big(B(0,R)\big)$ into $L^2\big(B(0,R)\big)$. Therefore, v_j converges to zero in $H^1\big(B(0,R)\big)$ for every $B(0,R)$ which implies $w_j \to 0$ in $L^2(D,|q|dx)$, a contradiction to $\|w_j\|_{L^2(D,|q|dx)} = 1$. This ends the proof of part (b).

(c) As for the proof of part (a), we apply Lemma 5.1(a), noting that $\alpha = 2\sigma k_0 A > 0$ by Theorem 6.1 and the assumption $q < 0$ (i.e., $\sigma = -1$). This yields (6.35). The proof of part (d) follows in the same way as the proof of part (b), with obvious adaptions due to the different sign of q. $\square$

The last theorem has been formulated for the scattering operator $\mathcal{S}$. Of course, there is an analogous result for the far field operator F. Let $\lambda_n(k)$ be the eigenvalues of $F(k)$. From Lemma 4.1 we recall that the projections $s_n(k) = \lambda_n(k)/|\lambda_n(k)|$ onto the unit circle in $\mathbb{C}$ satisfy $\operatorname{Im} s_n(k) > 0$ for all n and $\lim_{n\to\infty} s_n(k) = \pm 1$ provided $q \gtrless 0$. In particular, if $q > 0$ there exists a unique $s_-(k) \in \mathbb{C}$ with $|s_-(k)| = 1$ and $\operatorname{Im} s_-(k) > 0$ and

$$\operatorname{Re} s_-(k) \;=\; \min\{\operatorname{Re} s_n(k) : n \in \mathbb{N}\}\,, \tag{6.36}$$

while for $q < 0$ there exists a unique $s_+(k) \in \mathbb{C}$ with $|s_+(k)| = 1$ and $\operatorname{Im} s_+(k) > 0$ and

$$\operatorname{Re} s_+(k) \;=\; \max\{\operatorname{Re} s_n(k) : n \in \mathbb{N}\}\,. \tag{6.37}$$

Corollary 6.4 (Inside-Outside Duality for F)

Let $k_0 > 0$ and $I = (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$ such that no $k \in I$ is an interior transmission eigenvalue. Let $\lambda_n(k)$ be the eigenvalues of $F(k)$. Define $s_-(k)$ and $s_+(k)$ as above by (6.36) and (6.37), respectively. (That is, as the projection of λ_n onto the unit circle which is furthest to the left if $q > 0$ and furthest to the right if $q < 0$, respectively.)

(a) Let k_0 be the smallest interior transmission eigenvalue and let $q = q_0$ satisfy (6.29). Then

$$\lim_{k\in I,\; k\nearrow k_0} s_-(k) \;=\; -1\,. \tag{6.38}$$

(b) Let (6.38) hold and $q > 0$ in D. Then k_0 is an interior transmission eigenvalue.

(c) Let k_0 be the smallest interior transmission eigenvalue and let $q = q_0$ satisfy the assumption of (6.32) *in Theorem 6.2. Then*

$$\lim_{k\in I,\; k\nearrow k_0} s_+(k) \;=\; +1\,. \tag{6.39}$$

(d) Let (6.39) hold and $q < 0$ in D. Then k_0 is an interior transmission eigenvalue.

Proof: We only consider the case $q > 0$ of (a) and (b) since the case $q < 0$ of (c) and (d) can be shown analogously. By $\mathcal{S} = I + (ik)/(8\pi^2)\, F$ we observe that μ_n is an eigenvalue of $\mathcal{S}$ if, and only if, the number

$$\lambda_n \;=\; \frac{8\pi^2 i}{k}\big(1 - \mu_n\big) = \frac{8\pi^2 i}{k}\big(1 - \exp(-2i\delta_n)\big)$$

is an eigenvalue of F. Furthermore, we note that

$$\operatorname{Re} s_n(k) \;=\; -\frac{\sin(2\delta_n(k))}{\sqrt{\sin^2(2\delta_n(k)) + (1-\cos(2\delta_n(k)))^2}}$$

and this is minimal for minimal δ_n (provided $\delta_n \in (0, \pi/2)$). Furthermore, $\lim_{k\nearrow k_0} \delta_n(k) = 0$ is equivalent to $\lim_{k\nearrow k_0} \operatorname{Re} s_n(k) = -1$. Now the claim follows directly from Theorem 6.3. □

We want to illustrate the statements of Theorem 6.3 and of Lemma 4.1 with a simple and explicit numerical example. Consider the scattering from a penetrable ball $B_R \subset \mathbb{R}^3$ of radius $R > 0$, centered in the origin, and assume that the refractive index inside B_R equals $n_0 = (1 + q_0)^{1/2}$ for a constant $q_0 > -1$. For this setting it is well-known that one can compute $F(k) : L^2(S^2) \to L^2(S^2)$ semi-analytically. Indeed, explicit computations using spherical Bessel and Hankel functions j_l and h_l of order $l \in \mathbb{N}_0$, respectively, and spherical harmonics Y_l^m, one computes that the far field operator can in this special case be represented as

$$F(k)g \;=\; \frac{16\pi^2}{ik} \sum_{l=0}^{\infty} \sum_{|m|\le l} \frac{n_0\, j_l'(kn_0R)\, j_l(kR) - j_l(kn_0R)\, j_l'(kR)}{j_l(kn_0R)\, h_l'(kR) - n_0 h_l(kR)\, j_l'(kn_0R)}\, g_l^m\, Y_l^m\,, \tag{6.40}$$

with Fourier coefficients g_l^m of $g \in L^2(S^2)$ given by

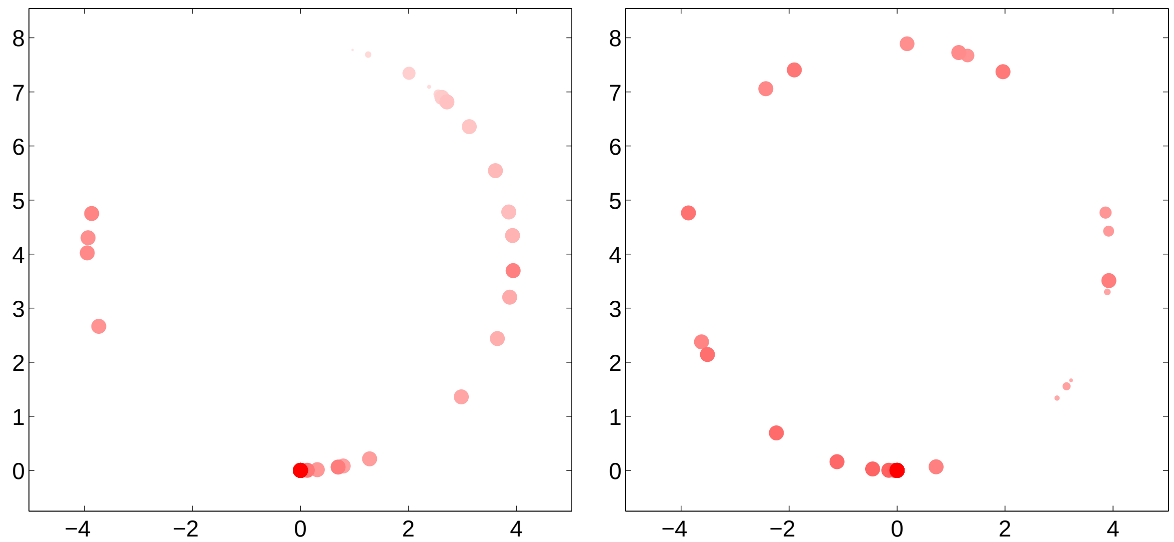
$$g_l^m \;=\; \int_{S^2} g(\hat{x}) \overline{Y_l^m}(\hat{x})\, ds(\hat{x}), \qquad l \in \mathbb{N}_0,\ |m| \le l,$$

see, e.g., [7] for similar computations in case of a sound-soft ball. Hence, the eigenvalues of $F(k)$ are

$$\lambda_l^{\mathrm{sph}}(k) \;=\; \frac{16\pi^2}{ik} \frac{n_0\, j_l'(kn_0R)\, j_l(kR) - j_l(kn_0R)\, j_l'(kR)}{j_l(kn_0R)\, h_l'(kR) - n_0\, h_l(kR)\, j_l'(kn_0R)}\,, \qquad l \in \mathbb{N}_0\,. \tag{6.41}$$

Interior transmission eigenvalues are precisely those $k_0 > 0$ for which the numerator in the last expression vanishes. Note that the order of the eigenvalues λ_l^{sph} comes from the series expression of the far field operator in (6.40); in particular, the order is *not* chosen in accordance with (6.36) or (6.37).

According to Lemma 4.1, as $l \to \infty$ the eigenvalues $\lambda_l^{\mathrm{sph}}(k)$ should approach zero from the right (left) if the contrast is positive (negative). We confirm this statement for the above-described spherical setting by plotting the eigenvalues of $F(k)$ for $k = 20$ and for the contrast $q_0 = \pm 0.5$. Figure 1 shows that for $q_0 = 0.5$ the eigenvalues converge from the right to zero, whereas for $q_0 = -0.5$, they converge from the left to zero. In this and in all the following examples, the radius R of the ball equals to one.



(a) (b)

Figure 1: The eigenvalues $\lambda_l^{\mathrm{sph}}(20)$ from (6.41) of $F(20)$ for the cases (a) $q_0 = 0.5$ and (b) $q_0 = -0.5$ in the complex plane. The size of the markers increases in $l = 1, \ldots, 50$ while the transparency of the markers decreases.

Figure 2 confirms the statement of Theorem 6.3: For $q_0 = 0.9 > 0$, the function $k \mapsto \delta_-(k)$ plotted in (a) tends to 0 as k approaches the smallest transmission eigenvalue from below (and all further ones occurring in the considered interval). For $q_0 = -0.9 < 0$, the function $k \mapsto \delta_+(k)$ plotted in (b) tends to π as k approaches the smallest transmission eigenvalue from below (and all further ones occurring in the considered interval). For each k, the minimum and the maximum used to define $\delta_-(k) = \min_{n\in\mathbb{N}} \delta_n$ and $\delta_+(k) = \max_{n\in\mathbb{N}} \delta_n$, respectively, is computed using the first 300 values of $\lambda_l^{\mathrm{sph}}(k)$.
Finally, we plot that the eigenvalue curves $k \mapsto \lambda_1^{\mathrm{sph}}(k)$ of $F(k)$ (see (6.41)) for the special spherical setting indicated above. To this end, we use 100 equidistributed wave numbers between $k_{\min}$ and $k_{\max}$. Again, we consider positive and negative constant contrasts $q_0 = 1.5$ and $q_0 = -0.9$, respectively. Figure 3(a) shows that for $q_0 = 1.5$ the eigenvalue $\lambda_1^{\mathrm{sph}}(k)$ approaches 0 twice from the left as k increases. For $q_0 = -0.9$, Figure 3(b) shows that $\lambda_1^{\mathrm{sph}}(k)$ approaches 0 twice from the right as k increases. Both observations fit to the statement of Corollary 4.1, since $s_+(k) \to -1$ and $s_-(k) \to 1$ as $k \nearrow k_0$ means that the corresponding eigenvalue of $F(k)$ approaches 0 from the left and right, respectively. In the special case of a spherical scatterer, zero is an eigenvalue of $F(k_0)$ for any transmission eigenvalue k_0 and, even more, the curves $\lambda_l^{\mathrm{sph}}(k)$ depend smoothly on $k > 0$. This is not valid in general. We already mentioned in the introduction that it may happen that the far field operator at a transmission eigenvalue is injective.

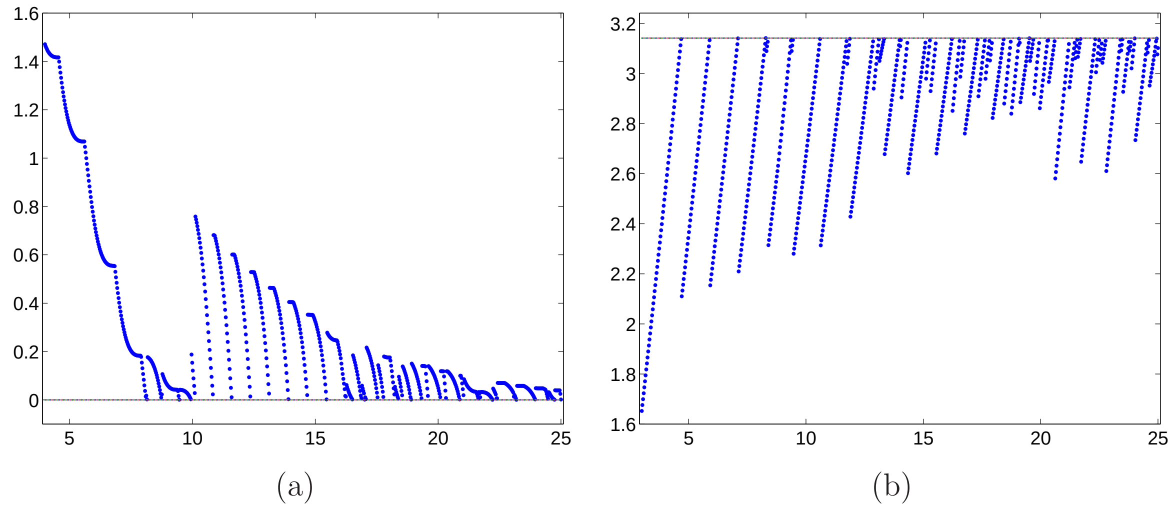


Figure 2: (a) The function $k \mapsto \delta_-(k)$ plotted for $k \in (4, 25)$ for contrast $q_0 = 0.9$. (b) The function $k \mapsto \delta_+(k)$ plotted for $k \in (3, 25)$ for contrast $q_0 = -0.9$. Both for (a) and (b) the domain D is the unit ball.

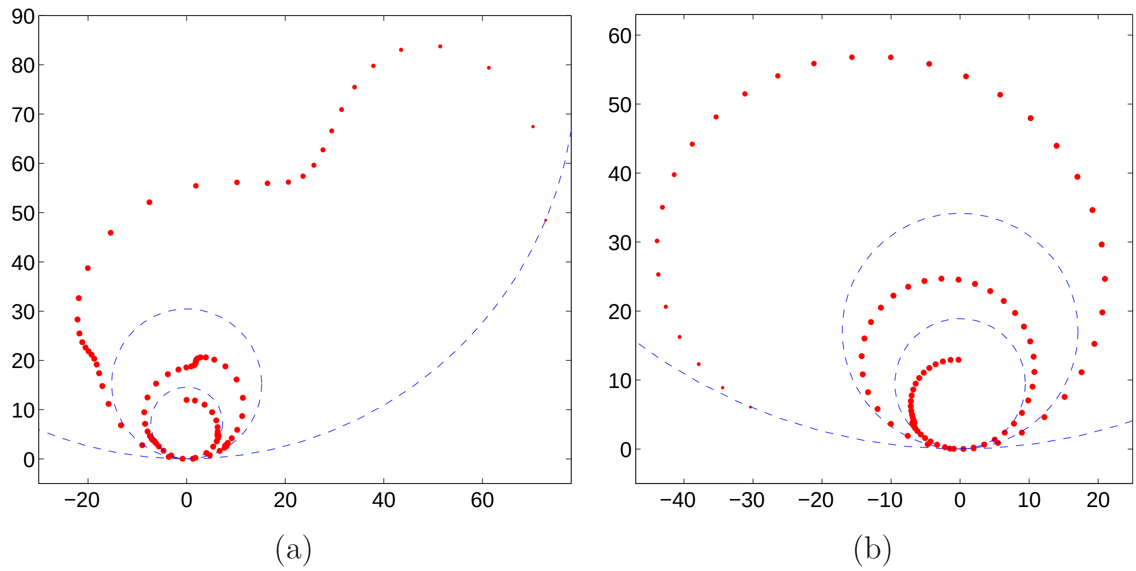


Figure 3: The eigenvalue $\lambda_1^{\mathrm{sph}}(k)$ from (6.41) for the contrasts (a) $q_0 = 1.5$ and (b) $q_0 = -0.9$ plotted in the complex plane for 100 equidistributed wavenumbers k between $k_{\min}$ and $k_{\max}$. In (a), $k_{\min} = 1$ and $k_{\max} = 13.2$ while in (b) $k_{\min} = 1$ and $k_{\max} = 12.2$. The size of the markers increases in k. The three dashed lines indicate circles containing the eigenvalues of $F(k)$ for $k = 1$ and the two interior transmission eigenvalues visible in the plots where $k \mapsto \lambda_1^{\mathrm{sph}}(k)$ vanishes.

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