The Inside-Outside Duality for Elastic Scattering Problems

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Abstract

In this article we derive the inside-outside duality for two time-harmonic, elastic scattering problems. First we consider a rigid scattering object inside an isotropic, homogeneous background medium and second, we consider a penetrable, inhomogeneous scattering object inside this background medium. For the first scattering problem we make use of the particular behavior or certain eigenvalues of the corresponding far field operator to characterize interior Dirichlet eigenvalues of the negative Navier operator. Then we adapt this technique to determine interior transmission eigenvalues that correspond to the second scattering problem.

1 Introduction

A typical problem in inverse, elastic scattering theory is to determine the shape of a rigid obstacle from far field measurements. An obvious attempt to approach this problem is to extend the available methods for acoustic scattering theory to the present case. Results of this approach are for example extensions of the linear sampling method and the factorization method to elastic scattering problems [2, 1, 13]. However these methods can fail at interior eigenvalues of the negative Navier-operator $-\Delta^*$. The Navier-operator can be considered as the extension of the Laplacian for elastic scattering models. Therefore it also shares important properties with the Laplacian. For example, if we assume Dirichlet boundary conditions, the Navier operator has a set of infinitely many, discrete eigenvalues which tend to infinity. As for the Laplacian, this property is an immediate consequence of the coercivity of these operators, see e.g. [22]. Due to the sensitivity of the above-mentioned reconstruction techniques at interior eigenvalues, there is a natural interest in determining these eigenvalues from far field data without knowledge of the scattering object. In the first part of this article we will show how the inside-outside duality for rigid obstacles can potentially be used to determine these eigenvalues or at least guarantee certain frequency bands that contain no interior eigenvalues. As in the case of acoustic scattering by impenetrable scattering objects [19], the inside-outside duality yields a full characterization of Dirichlet eigenvalues of the negative Navier operator.

In the second part of this article, we consider scattering by penetrable, inhomogeneous scattering objects. This scattering problem corresponds to an interior transmission eigenvalue problem, which has been examined in [7, 8, 6]. In these studies, the well-posedness of the interior transmission eigenvalue problem has been examined and the existence of at most a countable set of interior transmission eigenvalues has been shown under strict conditions for the material parameters. These results have been generalized in [5], where the existence of an infinite, discrete set of interior transmission eigenvalues has been shown for general settings that include the setting we consider in this article. The study of the interior transmission eigenvalue problem is interesting in relation to the application of the linear sampling method [9, 4, 11] and the factorization method [10]. As in the case of rigid obstacles, these methods can fail at interior transmission eigenvalues. We will show that the

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inside-outside duality can be used to provide a sufficient condition for the determination of interior transmission eigenvalues. As for other scattering scenarios that involve penetrable scattering objects [16, 19, 20, 21, 24], a full characterization of interior transmission eigenvalues by the inside-outside duality is to date only possible if we assume certain conditions for the material parameters.

We assume that the three-dimensional space $\mathbb{R}^3$ is filled with an isotropic, homogeneous background medium that is described by the constant Lamé parameter μ and λ and has normalized mass density $\rho = 1$. The propagation of time-harmonic elastic waves in this space is described by the Navier equation

$$\Delta^* u + \omega^2 u = 0, \tag{1}$$

where $\omega > 0$ is the frequency and the Navier operator Δ^* is given by

$$\Delta^* := \mu\Delta + (\lambda + \mu)\nabla \operatorname{div}. \tag{2}$$

Note that since the displacement field u is vector-valued, the Laplace operator Δ is applied componentwise and $\nabla u = (\nabla u_1, \nabla u_2, \nabla u_3)^T$ is the Jacobi matrix of u. To guarantee propagation of an elastic wave in this medium, we require the Lamé constants to satisfy $\mu > 0, \lambda + 2\mu > 0$. The displacement field u can be decomposed as $u = u_\mathrm{p} + u_\mathrm{s}$, where u_p describes its longitudinal (pressure) part and u_s describes its transversal (shear) part. Note that both of these parts solve the Helmholtz equations

$$(\Delta u_\mathrm{p} + k_\mathrm{p}^2) u_\mathrm{p} = 0, \qquad (\Delta u_\mathrm{s} + k_\mathrm{s}^2) u_\mathrm{s} = 0,$$

with positive wavenumbers

$$k_\mathrm{p}^2 = \frac{\omega^2}{\lambda + 2\mu}, \quad k_\mathrm{s}^2 = \frac{\omega^2}{\mu}. \tag{3}$$

Now we consider the exterior time-harmonic Dirichlet scattering problem. For an impenetrable scattering object $D \subset \mathbb{R}^3$ with Lipschitz boundary, we seek a solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3)$ to

$$\Delta^* u + \omega^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u = 0 \quad \text{on } \partial D. \tag{4}$$

The total field $u = u^s + u^i$ is the sum of a scattered field u^s and an incident plane wave u^i. To define the incident plane wave more precisely, we introduce longitudinal and transversal plane waves as incoming waves with direction of propagation $\theta \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ by

$$u_\mathrm{p}^i(x, \theta) = q_\mathrm{p} e^{ik_\mathrm{p} x\cdot\theta}, \qquad u_\mathrm{s}^i(x, \theta) = q_\mathrm{s} e^{ik_\mathrm{s} x\cdot\theta}, \qquad x \in \mathbb{R}^3. \tag{5}$$

Here q_p and q_s are polarization vectors that are parallel, or orthogonal, to θ respectively. Both plane waves are entire solutions of the Navier equation and so is the linear combination

$$u^i(x, \theta) = u_\mathrm{p}^i(x, \theta) + u_\mathrm{s}^i(x, \theta). \tag{6}$$

We require the scattered field u^s to fulfill the Kupradze radiation condition

$$\lim_{r\to\infty} \left(\frac{\partial u_\mathrm{p}^s}{\partial r} - ik_\mathrm{p} u_\mathrm{p}^s \right) = 0, \quad \lim_{r\to\infty} \left(\frac{\partial u_\mathrm{s}^s}{\partial r} - ik_\mathrm{s} u_\mathrm{s}^s \right) = 0, \quad r = |x|, \tag{7}$$

uniformly in all directions. Here the radiation condition is defined in terms of the longitudinal wave $u_\mathrm{p}^s = -k_\mathrm{p}^{-2} \nabla \operatorname{div} u^s$ and the transversal wave $u_\mathrm{s}^s = u^s - u_\mathrm{p}^s$. Note that solutions that fulfill Kupradze's radiation condition are in this article called radiating solutions. We now introduce two function spaces of longitudinal and transversal vector fields on $\mathbb{S}^2$ by

$$\begin{aligned} L_\mathrm{p}^2(\mathbb{S}^2) &:= \{g_\mathrm{p} : \mathbb{S}^2 \to \mathbb{C}^3 : g_\mathrm{p}(\theta) \times \theta = 0, |g_\mathrm{p}| \in L^2(\mathbb{S}^2)\}, \\ L_\mathrm{s}^2(\mathbb{S}^2) &:= \{g_\mathrm{s} : \mathbb{S}^2 \to \mathbb{C}^3 : g_\mathrm{s}(\theta) \cdot \theta = 0, |g_\mathrm{s}| \in L^2(\mathbb{S}^2)\}. \end{aligned}$$

Note that a function $g(\theta) \in L^2(\mathbb{S}^2)^3$ possesses a decomposition

$$g(\theta) = g_\mathrm{s}(\theta) + g_\mathrm{p}(\theta), \qquad g_\mathrm{s}(\theta) := \theta \times g(\theta) \times \theta, \quad g_\mathrm{p}(\theta) := (g(\theta)\cdot\theta)\theta, \tag{8}$$

such that $g_\mathrm{s} \in L^2_\mathrm{s}(\mathbb{S}^2)$ and $g_\mathrm{p} \in L^2_\mathrm{p}(\mathbb{S}^2)$. Radiating solutions to the Navier equation have the asymptotic behavior

$$u^s(x) = \frac{e^{ik_\mathrm{p}|x|}}{|x|} u^\infty_\mathrm{p}(\hat{x}) + \frac{e^{ik_\mathrm{s}|x|}}{|x|} u^\infty_\mathrm{s}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty, \tag{9}$$

uniformly in all directions $\hat{x} := x/|x|$. Here u^∞_p and u^∞_s are the longitudinal and transversal far fields and we will call the sum $u^\infty := u^\infty_\mathrm{p} + u^\infty_\mathrm{s}$ the far field of u. In order to introduce the far field operator we will first generalize the incident field and introduce the Herglotz wave field v^in_g for a function $g \in L^2(\mathbb{S}^2)^3$ by

$$v^\mathrm{in}_g(x) := \int_{\mathbb{S}^2} \left(e^{ik_\mathrm{p} x\cdot\theta} g_\mathrm{p}(\theta) + e^{ik_\mathrm{s} x\cdot\theta} g_\mathrm{s}(\theta) \right) \mathrm{d}s(\theta), \qquad x \in \mathbb{R}^3. \tag{10}$$

We now define the far field operator as the far field of the solution v_g to the exterior Dirichlet scattering problem, where the incident wave field is the the Herglotz wave function v^in_g, i.e. $F : L(\mathbb{S}^2)^3 \to L^2(\mathbb{S}^2)^3$ is given by

$$Fg := v^\infty_g, \tag{11}$$

where $v^\infty_g = v^\infty_{g,\mathrm{p}} + v^\infty_{g,\mathrm{s}}$. This far field operator has some crucial properties which are necessary to derive the inside-outside duality. We know from [1, Theorem 3.3, Theorem 3.4] that the far field operator F is compact and normal and that its eigenvalues λ_j lie on a circle in the complex plane with center $2\pi\mathrm{i}/\omega$ and radius $2\pi/\omega$. As we will show in Theorem 2, the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ converge to zero from the left side. We represent the eigenvalues λ_j of the far field operator F in polar coordinates, i.e.

$$\lambda_j = |\lambda_j| e^{i\vartheta_j}, \qquad \vartheta_j \in [0, \pi], \tag{12}$$

where we set $\vartheta_j = 0$ if $\lambda_j = 0$. By this representation, each eigenvalue λ_j corresponds to a phase ϑ_j and since the eigenvalues converge to zero from the left side, there is one distinct eigenvalue λ_* with a smallest phase

$$\vartheta_* := \min_{j\in\mathbb{N}} \vartheta_j. \tag{13}$$

Note that the eigenvalues $\lambda_j = \lambda_j(\omega)$ and their phases $\vartheta_j = \vartheta_j(\omega)$ depend on the frequency ω. The inside-outside duality now states that ω_0^2 is a Dirichlet eigenvalue of $-\Delta^*$ if, and only if, $\vartheta_*(\omega) \to 0$ as ω approaches ω_0, see Theorem 5 and Theorem 6 for a precise statement.

The second scattering problem we are going to consider is scattering by penetrable, inhomogeneous bodies. For a real-valued mass density $\rho \in L^\infty(\mathbb{R}^3)$ such that $\rho = 1$ in the exterior of D, we seek a solution $u \in H^1_\mathrm{loc}(\mathbb{R}^3, \mathbb{C}^3)$ to the equation

$$\Delta^* u + \omega^2 \rho u = 0 \qquad \text{in } \mathbb{R}^3, \tag{14}$$

such that

$$[u]_{\partial D} = 0 \qquad \text{and} \qquad [T_\nu u]_{\partial D} = 0,$$

where ν denotes the ourward normal to ∂D and $[\cdot]_{\partial D}$ the jump of a vector-valued function over the boundary ∂D. Finally T_ν is the stress tensor, given by

$$T_\nu := 2\mu\nu\cdot\nabla + \lambda\nu\,\mathrm{div} + \mu\nu \times \mathrm{curl}\,.$$

The total field $u = u^s + u^i$ is the sum of a scattered field u^s and the incident field u^i that has been defined in (6). The scattered field u^s is assumed to satisfy Kupradze's radiation condition (7). Then the scattered field u^s has a representation in terms of its far field u^∞ as in (9). Choosing the incident field to be the Herglotz wave field v_g, defined in (10) for a function $g \in L^2(\mathbb{S}^2)^3$, the far field operator F is defined in (11). The far field operator retains the properties that we have already mentioned for the exterior Dirichlet scattering problem, i.e. it is compact and normal and its eigenvalues lie on a circle in the complex plane with center $2\pi\mathrm{i}/\omega$ and radius $2\pi/\omega$, see [10]. The scattering problem is related to an interior transmission eigenvalue problem for elastic scattering. To state this problem, we define

$$H_0^2(D,\mathbb{C}^3) := \{u \in H^2(D,\mathbb{C}^3) : u = 0,\ T_\nu u = 0 \text{ on } \partial D\}.$$

Then the squared frequency ω^2 is called an interior transmission eigenvalue if there are non-trivial functions $u, w \in L^2(D,\mathbb{C}^3)$ such that $w - v \in H_0^2(D,\mathbb{C}^3)$ and

$$\begin{aligned} \Delta^* u(x) + \omega^2 \rho u(x) &= 0 \qquad \text{in } D, \\ \Delta^* w(x) + \omega^2 w(x) &= 0 \qquad \text{in } D, \\ u(x) - w(x) &= 0 \qquad \text{on } \partial D, \\ T_\nu u(x) - T_\nu w(x) &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{15}$$

It has been shown that there exists an infinite number of discrete interior transmission eigenvalues with infinity as the only possible accumulation point, see [5]. We want to determine these interior transmission eigenvalues by the inside-outside duality. To indicate our main result, note that for positive mass densities $\rho \in L^\infty(D)$ the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of the far field operator F converge to zero from the right, see Lemma 9 below. Then there is one distinct eigenvalue λ^* with a largest phase

$$\vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j. \tag{16}$$

Again, we denote the dependence of the phases on the frequency by $\vartheta_j = \vartheta_j(\omega)$. The first part of our main result now states the following: If ω_0^2 is an interior transmission eigenvalue and the expression $\alpha(\omega_0)$ in (45) does not vanish, then $\vartheta^*(\omega) \to \pi$ as $\omega \to \omega_0$. On the other hand, if $\vartheta^*(\omega) \to \pi$ for $\omega \to \omega_0$, then ω_0^2 is an interior transmission eigenvalue, see Theorem 12 and Theorem 13 for a precise statement.

Before we proceed with the discussion of the exterior Dirichlet scattering problem, we introduce some technical details. For a elastic wave equations we will later seek solutions in the space of vectorial Sobolev functions $H^1(D,\mathbb{C}^3)$. For our purpose we equip the space with the norm

$$\|u\|^2_{H^1(D,\mathbb{C}^3)} := \|u\|^2_{L^2(D,\mathbb{C}^3)} + \|\operatorname{div} u\|^2_{L^2(D,\mathbb{C})} + \|\nabla u\|^2_{L^2(D,\mathbb{C}^{3\times 3})}.$$

Using now Green's first theorem and Gauss' integral theorem for the operator Δ^* from (2), we obtain Betti's first formula, i.e. for two functions $u, \varphi \in H^1(D,\mathbb{C}^3)$ such that $\Delta^* u \in L^2(D,\mathbb{C}^3)$, we get that

$$\int_D \Delta^* u \cdot \overline{\varphi}\,\mathrm{d}x = -\int_D \left(\mu \nabla u : \nabla\overline{\varphi} + (\mu+\lambda)\operatorname{div} u \operatorname{div}\overline{\varphi}\right)\,\mathrm{d}x + \int_{\partial D} T_\nu u \cdot \overline{\varphi}\,\mathrm{d}s. \tag{17}$$

Here, $A : B$ denotes the Frobenius scalar product of the matrices A, B, defined by $A : B = \sum_{i,j} a_{ij} b_{ij}$. After this preliminary considerations, we will in the next section consider elastic scattering by an impenetrable scattering object with Dirichlet boundary conditions.

2 The Exterior Dirichlet Problem

In this section we assume the presence of an impenetrable scattering object $D \subset \mathbb{R}^3$ within the homogeneous background medium, such that the exterior of D is connected and the boundary ∂D

is Lipschitz. Then ω^2 is a Dirichlet eigenvalue of $-\Delta^*$ if there exists a solution $v \in H_0^1(D, \mathbb{C}^3)$ such that

$$\Delta^* v + \omega^2 v = 0 \quad \text{in } D \qquad \text{and} \qquad v = 0 \quad \text{on } \partial D.$$

This eigenvalue problem is understood in a weak sense, i.e. $v \in H_0^1(D, \mathbb{C}^3)$ needs to satisfy

$$\int_D \left(\mu \nabla v : \nabla \overline{\varphi} + (\lambda + \mu) \operatorname{div} v \ \operatorname{div} \overline{\varphi} - \omega^2 v \cdot \overline{\varphi}\right) \mathrm{d}x = 0$$

for all $\varphi \in H^1(D, \mathbb{C}^3)$. Closely related to this problem is the exterior Dirichlet boundary value problem (4) which is also understood in a weak sense, i.e. in the formulation for the scattered field, we seek a radiating solution $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3)$ to

$$\int_{\mathbb{R}^3 \setminus \overline{D}} \left(\mu \nabla u^s : \nabla \overline{\varphi} + (\lambda + \mu) \operatorname{div} u^s \ \operatorname{div} \overline{\varphi} - \omega^2 u^s \cdot \overline{\varphi}\right) \mathrm{d}x = 0 \tag{18}$$

for all test functions $\varphi \in H^1(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3)$ with compact support in $\mathbb{R}^3 \setminus \overline{D}$ such that $u^s = -u^i$ on the boundary ∂D, where $u^i(x, \theta)$ is the incident plane wave with direction $\theta \in \mathbb{S}^2$, defined in (6). In this section we will proceed as follows: First we will state a factorization of the far field operator F from (11) and examine the properties of the arising operators in Lemma 1. Then we will use these properties to show in Lemma 2 that the eigenvalues λ_j of the far field operator converge to zero only from the left side. Using a particular characterization of the smallest phase, we will then calculate the necessary auxiliary derivative in Lemma 4 in order to state the first part and the second part of the inside-outside duality in Theorem 5 and Theorem 6.

We start by discussing a factorization of the far field operator and introduce the elastic single layer potential

$$\mathrm{SL}\varphi(x) := \int_{\partial D} \Phi_N(x, y) \varphi(y) \, \mathrm{d}s(y), \qquad x \in \mathbb{R}^3 \setminus \partial D, \tag{19}$$

where Φ_N is the fundamental solution to the Navier equation,

$$\Phi_N(x, y) := \frac{k_{\mathrm{s}}^2}{4\pi\omega^2} \frac{e^{ik_{\mathrm{s}}|x-y|}}{|x-y|} I + \frac{1}{4\pi\omega^2} \nabla_x \nabla_x \left[\frac{e^{ik_{\mathrm{s}}|x-y|}}{|x-y|} - \frac{e^{ik_{\mathrm{p}}|x-y|}}{|x-y|}\right], \qquad x, y \in \mathbb{R}^3, \quad x \neq y, \tag{20}$$

and I denotes the identity matrix. This operator is linear and bounded from $H^{-1/2}(\partial D, \mathbb{C}^3)$ into $H^1(B_R, \mathbb{C}^3)$. Denoting by $[\cdot]|^{\pm}$ the trace of a function taken from the outside $(+)$ or the inside $(-)$, it holds that $\mathrm{SL}\varphi|^{\pm} = S\varphi$ in $H^{1/2}(\partial D, \mathbb{C}^3)$, where the elastic single layer operator $S : H^{-1/2}(\partial D, \mathbb{C}^3) \to H^{1/2}(\partial D, \mathbb{C}^3)$ is given by

$$(S\varphi)(x) := \int_{\partial D} \Phi_N(x, y) \varphi(y) \, \mathrm{d}s(y), \qquad x \in \partial D.$$

Furthermore for a function $\varphi \in H^{-1/2}(\partial D, \mathbb{C}^3)$, the jump relation

$$T_\nu \mathrm{SL}\varphi|^- - T_\nu \mathrm{SL}\varphi|^+ = \varphi \tag{21}$$

holds, see [17] for the mapping properties and [18, Ch.V, § 3, 4, 5] for the jump relations of the operator.

We denote the duality pairing $\langle H^{-1/2}(\partial D, \mathbb{C}^3), H^{1/2}(\partial D, \mathbb{C}^3)\rangle$ by $(\cdot, \cdot)$ and summarize the properties of S in the following lemma. For a proof, we refer to [1].

Lemma 1. *Let ω^2 be no Dirichlet eigenvalue of the Navier equation.*
(a) For all $\varphi \in H^{-1/2}(\partial D, \mathbb{C}^3)$ it holds that $\mathrm{Im}\,(\varphi, S\varphi) \leq 0$.
(b) It holds that $(\varphi, S\varphi) = 0$ if and only if $\varphi = 0$.
(c) Denote by S_{i} the single-layer operator for the frequency $\omega = \mathrm{i}$. Then S_{i} is compact, self-adjoint and positive definite, i.e. for a constant $c > 0$

$$(\varphi, S_{\mathrm{i}}\varphi) \geq c\|\varphi\|_{H^{-1/2}(\partial D, \mathbb{C}^3)} \qquad \forall \varphi \in H^{-1/2}(\partial D, \mathbb{C}^3).$$

(d) The difference $S - S_{\mathrm{i}}$ is compact from $H^{-1/2}(\partial D, \mathbb{C}^3)$ into $H^{1/2}(\partial D, \mathbb{C}^3)$.

As a second ingredient for a factorization we introduce the injective, bounded operator $A : H^{1/2}(\partial D, \mathbb{C}^3) \to L^2(\mathbb{S}^2)^3$ by $Af = v^\infty$, where v^∞ is the far field of the radiating solution $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ to the problem

$$\Delta^* v + \omega^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad v = f \quad \text{on } \partial D.$$

Using for example a boundary integral equation approach, see [17, 18], it can be shown that this problem is uniquely solvable. Before we state the factorization, note finally that the solution operator A has dense range in $L^2(\mathbb{S}^2)^3$. Now we can state a factorization of the far field operator. It holds that

$$F = -4\pi A S^* A^*. \tag{22}$$

A proof for this factorization and the properties of these operators can be found in [1]. Using this factorization and the properties of the operator S from Lemma 1, one can easily adapt the arguments from the proof of [19, Lemma 12] to show that the eigenvalues of the far field operator converge to zero only from the left.

Theorem 2. *Assume that ω^2 is no Dirichlet eigenvalue of $-\Delta^*$. Then the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of F converge to zero from the left side, i.e. $\mathrm{Re}\,\lambda_j < 0$ for $j \in \mathbb{N}$ large enough.*

Recall the representation of the eigenvalues λ_j in polar coordinates in (12) and the definition of the smallest phase ϑ_* in (13). Due to the compactness and normality of the far field operator and the distinct structure of the eigenvalues, a particular characterization of the cotangent of the smallest phase holds.

Lemma 3. *If ω^2 is no Dirichlet eigenvalue of $-\Delta^*$, then*

$$\cot \vartheta_* = \max_{g \in L^2(\mathbb{S}^2)^3} \frac{\mathrm{Re}\,(Fg, g)_{L^2(\mathbb{S}^2)^3}}{\mathrm{Im}\,(Fg, g)_{L^2(\mathbb{S}^2)^3}}.$$

The maximum is attained at any eigenvector g_* to the eigenvalue λ_* of F with smallest phase.

Proof. For the convenience of the reader, we include the proof idea. For a full proof, see see [19, Theorem 3]. Let $g_j \in L^2(\mathbb{S}^2)^3$ be the eigenfunctions of F corresponding to the eigenvalues λ_j and the phases ϑ_j. Since the eigenfunctions form a complete orthonormal basis of $L^2(\mathbb{S}^2)^3$, we can represent $g \in L^2(\mathbb{S}^2)^3$ as $g = \sum_{j\in\mathbb{N}} (g, g_j) g_j$. Since $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j$ this shows in particular that

$$(Fg, g) = \sum_{j\in\mathbb{N}} \lambda_j |(g, g_j)|^2, \tag{23}$$

and therefore

$$\mathrm{Re}\,(Fg, g) = \sum_{j\in\mathbb{N}} \mathrm{Re}\,(\lambda_j) |(g, g_j)|^2 \qquad \text{and} \qquad \mathrm{Im}\,(Fg, g) = \sum_{j\in\mathbb{N}} \mathrm{Im}\,(\lambda_j) |(g, g_j)|^2.$$

Setting $r_j = |\lambda_j|$ we furthermore have that $\mathrm{Re}\,(\lambda_j) = r_j \cos(\vartheta_j)$ and $\mathrm{Im}\,(\lambda_j) = r_j \sin(\vartheta_j)$. Now we can calculate

$$\frac{\mathrm{Re}\,(Fg,g)}{\mathrm{Im}\,(Fg,g)} = \frac{\sum_{j\in\mathbb{N}} \mathrm{Re}\,(\lambda_j)|(g,g_j)|^2}{\sum_{j\in\mathbb{N}} \mathrm{Im}\,(\lambda_j)|(g,g_j)|^2} = \frac{\sum_{j\in\mathbb{N}} \cos(\vartheta_j) r_j |(g,g_j)|^2}{\sum_{j\in\mathbb{N}} \sin(\vartheta_j) r_j |(g,g_j)|^2} \leq \frac{\cos(\vartheta_*)}{\sin(\vartheta_*)} = \cot(\vartheta_*),$$

where the inequality in the latter calculation is due to a monotonicity argument that relies on the special structure of the eigenvalues λ_j in the complex plane, see [19, Lemma 4]. Due to the orthonormality of the eigenfunctions, the inequality becomes an equality be choosing $g = g_*$. ■

Using the factorization $F = -4\pi A S^* A^*$ and the denseness of the range of A^* in $H^{-1/2}(\partial D, \mathbb{C}^3)$, this characterization can also be expressed using the single-layer operator S. Since $(Fg,g)_{L^2(\mathbb{S}^2)^3} = -(S^*A^*g,\, A^*g) = -(\varphi,\, S\varphi)$ for $\varphi = A^*g \in H^{-1/2}(\partial D, \mathbb{C}^3)$, it follows that

$$\cot\vartheta_* = \max_{\varphi\in H^{-1/2}(\partial D,\mathbb{C}^3)} \frac{\mathrm{Re}\,(\varphi, S\varphi)}{\mathrm{Im}\,(\varphi, S\varphi)}. \tag{24}$$

We will from now on indicate the dependency of relevant quantities on the frequency ω by writing $S = S_\omega$, $\mathrm{SL} = \mathrm{SL}_\omega$, $\lambda_j = \lambda_j(\omega)$, $\vartheta = \vartheta(\omega)$ and so on. In the next lemma we compute an auxiliary derivative that is important for the proof of the first part of our final result in Theorem 5.

Lemma 4. *Assume that ω_0^2 is a Dirichlet eigenvalue of $-\Delta^*$ in D. Then S_{ω_0} has a non-trivial kernel and for all elements φ_0 in this kernel it holds that $(\varphi_0, S_{\omega_0}\varphi_0) = 0$. Furthermore, the mapping $\omega \to (\varphi_0, S_\omega \varphi_0)$ is differentiable in ω_0 and*

$$\alpha(\omega_0) := \frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega\varphi_0)\Big|_{\omega=\omega_0} = 2\int_D |v_{\omega_0}|^2 \,\mathrm{d}x, \qquad \textit{where } v_{\omega_0} = \mathrm{SL}_{\omega_0}\varphi_0. \tag{25}$$

Proof. For arbitrary $\omega \in \mathbb{R}$, we have that $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ is a solution of $\Delta^* v_\omega + \omega^2 v_\omega = 0$ in $\mathbb{R}^3$. If $\omega = \omega_0$, the far field $v^\infty_{\omega_0}$ of v_{ω_0} vanishes as a consequence of the proof of Lemma 6.1 in [1] and by Rellich's Lemma, v_{ω_0} vanishes in the exterior of D such that $v_{\omega_0} \in H^2_0(D, \mathbb{C}^3)$ is a Dirichlet eigenfunction of $-\Delta^*$, i.e. $\Delta^* v_{\omega_0} + \omega_0^2 v_{\omega_0} = 0$ in D and $u = 0$ on ∂D. Furthermore by applying the chain rule, we have that the derivative $v'_{\omega_0} := (\,\mathrm{d}/\,\mathrm{d}\omega\; v_\omega)|_{\omega=\omega_0} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ solves

$$\Delta^* v'_{\omega_0} + \omega_0^2 v'_{\omega_0} + 2\omega_0 v_{\omega_0} = 0 \quad \text{in } \mathbb{R}^3 \tag{26}$$

in a weak sense. In particular Betti's formula in (17) shows that

$$\int_D \left(\mu \nabla v'_{\omega_0} : \nabla\overline{\phi} + (\mu+\lambda)\,\mathrm{div}\, v'_{\omega_0}\,\mathrm{div}\,\overline{\phi}\right)\,\mathrm{d}x = \int_D \left(\omega_0^2 v'_{\omega_0}\cdot\overline{\phi} + 2\omega_0 v_{\omega_0}\cdot\overline{\phi}\right)\,\mathrm{d}x \tag{27}$$

for all $\phi \in C_0^\infty(D, \mathbb{C}^3)$. Now we use the jump relation for the single layer potential from (21) to compute

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega\varphi_0) = \left(\varphi_0, \frac{\mathrm{d}}{\mathrm{d}\omega}S_\omega\varphi_0\right) = \left(\varphi_0, \frac{\mathrm{d}}{\mathrm{d}\omega}v_\omega\right) = \left(T_\nu v_\omega|^- - T_\nu v_\omega|^+, \frac{\mathrm{d}}{\mathrm{d}\omega}v_\omega\right).$$

Since v_{ω_0} vanishes in the exterior of D, the exterior surface traction also vanishes, so that $T_\nu v_{\omega_0}|^+ = 0$. Therefore we get

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega\varphi_0)\Big|_{\omega=\omega_0} = \left(T_\nu v_\omega|^- - T_\nu v_\omega|^+, \frac{\mathrm{d}}{\mathrm{d}\omega}v_\omega\right)\Big|_{\omega=\omega_0} = \left(T_\nu v_{\omega_0}|^-, v'_{\omega_0}\right),$$

where $v'_{\omega_0} := (\mathrm{d}/\mathrm{d}\omega\ v_\omega)|_{\omega=\omega_0}$. Since $v_{\omega_0} \in H_0^2(D, \mathbb{C}^3)$, we can use Betti's formula (17) and $\Delta^* v_{\omega_0} = -\omega_0^2 v_{\omega_0}$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega \varphi_0)\Big|_{\omega=\omega_0} &= \int_D \left(\Delta^* v_{\omega_0} \cdot \overline{v'_{\omega_0}} + \mu \nabla v_{\omega_0} : \nabla \overline{v'_{\omega_0}} + (\mu + \lambda) \operatorname{div} v_{\omega_0} \operatorname{div} \overline{v'_{\omega_0}}\right) \mathrm{d}x \\ &= \int_D \left(-\omega_0^2 v_{\omega_0} \cdot \overline{v'_{\omega_0}} + \mu \nabla v_{\omega_0} : \nabla \overline{v'_{\omega_0}} + (\mu + \lambda) \operatorname{div} v_{\omega_0} \operatorname{div} \overline{v'_{\omega_0}}\right) \mathrm{d}x \\ &= \int_D \left(-\omega_0^2 v_{\omega_0} \cdot \overline{v'_{\omega_0}} + \omega_0^2 \overline{v'_{\omega_0}} \cdot v_{\omega_0} + 2\overline{v_{\omega_0}} v_{\omega_0}\right) \mathrm{d}x = 2 \int_D |v_{\omega_0}|^2 \, \mathrm{d}x,\end{aligned}$$

where we used (27) for $\phi = v_{\omega_0}$ for the second to last equality. ■

We can now state the first and second part of the inside-outside duality. The first part makes use of the positivity of the derivative $\alpha(\omega_0)$, which we calculated in the last lemma, to set up a Taylor expansion for the characterization of the cotangent smallest phase.

Theorem 5 (Inside-Outside Duality - Part 1)**.** *Let ω_0^2 be a Dirichlet eigenvalue of $-\Delta^*$. Then it holds that* $\lim_{\omega \nearrow \omega_0} \vartheta_*(\omega) = 0$.

Proof. For the conveniece of the reader, we include the proof from [19, Lemma 7]. Since ω_0^2 is an interior Dirichlet eigenvalue, there exists a function $\varphi_0 \in H^{-1/2}(\partial D, \mathbb{C}^3)$ such that $(\varphi_0, S(\omega_0)\varphi_0) = 0$. Assume that $I = (\omega_0 - \varepsilon, \omega_0 + \varepsilon)$ is an interval that does not contain other Dirichlet eigenvalues. Recall the characterization

$$\cot \vartheta_*(\omega) = \max_{\varphi \in H^{-1/2}(\partial D, \mathbb{C}^3)} \frac{\mathrm{Re}\,(\varphi,\, S_\omega \varphi)}{\mathrm{Im}\,(\varphi,\, S_\omega \varphi)} \qquad \text{for } k \in I \setminus \{\omega_0\}.$$

Now we define $f(\omega) = (\varphi_0,\, S_\omega \varphi_0)$ for $\omega \in I$ and note that the last Lemma states that this function is differentiable at ω_0. Taylor's theorem states that

$$f(\omega) = f(\omega_0) + \alpha(\omega - \omega_0) + r(\omega),$$

where $f(\omega_0) = 0$ by construction and the remainder $r(\omega)$ satisfies $r(\omega) = o(|\omega - \omega_0|)$ as $\omega \to \omega_0$. Further, note that $\mathrm{Im}\,(r(\omega)) \le 0$ due to Lemma 1, because the last lemma shows that the derivative $\alpha = \mathrm{d}f/\mathrm{d}\omega$ at ω_0 is real-valued and $\mathrm{Im}\, f(\omega) \le 0$. Hence,

$$\cot \vartheta_*(\omega) = \max_{\varphi \in H^{-1/2}(\partial D, \mathbb{C}^3)} \frac{\mathrm{Re}\,(\varphi,\, S_\omega \varphi)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(\varphi,\, S_\omega \varphi)_{L^2(\mathbb{S}^2)}} \overset{\varphi=\varphi_0}{\ge} \frac{\alpha(\omega - \omega_0) + \mathrm{Re}\,(r(\omega))}{\mathrm{Im}\,(r(\omega))}. \tag{28}$$

Note is particular, that the choise $\varphi = \varphi_0$ is possible since the maximum is taken over the whole space $H^{-1/2}(\partial D, \mathbb{C}^3)$. Now we use that α is positive. Therefore $\omega \nearrow \omega_0$ implies that $\alpha(\omega - \omega_0) \le 0$ tends slower to zero than $0 < \mathrm{Im}\,(r(\omega)) = o(|\omega - \omega_0|)$, that is, $[\alpha(\omega - \omega_0) + \mathrm{Re}\,(r(\omega))]/\mathrm{Im}\,(r(\omega)) \to \infty$. Obviously, $\cot \vartheta_*(\omega) \to \infty$ for $\vartheta_*(\omega) \in (0, \pi)$ implies that $\vartheta_*(\omega) \to 0$. ■

Theorem 6 (Inside-Outside Duality - Part 2)**.** *Assume that the interval $I = (\omega_0 - \varepsilon, \omega_0)$ contains no ω such that ω^2 is a Dirichlet eigenvalue of $-\Delta^*$. If* $\lim_{\omega \nearrow \omega_0} \vartheta_*(\omega) \to 0$*, then ω_0^2 is a Dirichlet eigenvalue of $-\Delta^*$ in D.*

Proof. Arguing by contradiction, we assume that $\lim_{\omega \nearrow \omega_0} \vartheta_*(\omega) = 0$ but ω_0^2 is no Dirichlet eigenvalue of $-\Delta^*$. Using the characterization of the smallest phase from (24), this implies that

$$\max_{\varphi \in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S_\omega \varphi)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(\varphi,\, S_\omega \varphi)_{L^2(\mathbb{S}^2)}} \longrightarrow \infty \qquad \text{as } \omega \nearrow \omega_0.$$

Then it follows that there is a sequence $\omega_j \nearrow \omega_0 \in I$ and a sequence $\varphi_j \in H^{-1/2}(\partial D, \mathbb{C}^3)$ with $\|\varphi_j\|_{H^{-1/2}(\partial D,\mathbb{C}^3)} = 1$ such that

$$0 > \mathrm{Im}\,(\varphi_j, S_{\omega_j}\varphi_j) \to 0 \qquad \text{and} \qquad \mathrm{Re}\,(\varphi_j, S_{\omega_j}\varphi_j) \le 0 \tag{29}$$

as j becomes large. Since the sequence $(\varphi_j)_{j\in\mathbb{N}}$ is bounded, we find a subsequence, also denoted by $(\varphi_j)_{j\in\mathbb{N}}$, which weakly converges to a $\varphi_0 \in H^{-1/2}(\partial D, \mathbb{C}^3)$. From [1] we know that

$$\mathrm{Im}\,(\varphi_j, S_{\omega_j}\varphi_j) = -\omega_j \|v_j^\infty\|^2_{L^2(\mathbb{S}^2)^3}, \tag{30}$$

where $v_j = \mathrm{SL}_{\omega_j}\varphi_j$. Since the mapping from φ_j to v_j^∞ is compact, it follows that v_j^∞ converges strongly to a function v_0^∞, which is the far field of the function $v_0 = \mathrm{SL}_{\omega_0}\varphi_0$. The far field vanishes due to equation (30). Since we assumed that ω_0^2 is no Dirichlet eigenvalue of $-\Delta^*$, we conclude that $v_0 = 0$ everywhere and therefore φ_0 also vanishes so that $\varphi_j \rightharpoonup 0$. Now we can apply Betti's first formula for a ball $B_R := \{x \in \mathbb{R}^3 : |x| \le R\}$ which contains the scatterer D, to compute that

$$\begin{aligned}(\varphi_j, S_{\omega_j}\varphi_j) &= \int_{\partial D} \overline{v_j} \cdot (T_\nu v_j|_- - T_\nu v_j|_+)\,\mathrm{d}s \\ &= \int_{B_R} \left(\mu \nabla v_j : \nabla \overline{v_j} + (\lambda+\mu)\,\mathrm{div}\, v_j\, \mathrm{div}\, \overline{v_j} - \omega^2 |v_j|^2\right)\,\mathrm{d}x - \int_{\partial B_R} T_\nu v_j \cdot \overline{v_j}\,\mathrm{d}S.\end{aligned}$$

Note that the last integral tends strongly to zero as v_j converges strongly to zero on ∂B_R by elliptic regularity and compact embedding results, see also the proof of [19, Theorem 8] for the acoustic case. Note also that since v_j converges weakly to $v_0 = 0$ in $H^1(B_R, \mathbb{C}^3)$, it strongly converges to zero in $L^2(B_R, \mathbb{C}^3)$. Now we can use (29) and use the real part of the last equation to obtain

$$0 \ge \mathrm{Re}\,(\varphi_j, S_{\omega_j}\varphi_j) = \int_{B_R} \left(\mu \nabla v_j : \nabla \overline{v_j} + (\mu+\lambda)\,\mathrm{div}\, v_j\, \mathrm{div}\, \overline{v_j} - \omega_j^2 v_j \cdot \overline{v_j}\right)\,\mathrm{d}x - \mathrm{Re} \int_{|x|=R} T_\nu v_j \cdot \overline{v_j}\,\mathrm{d}S$$

or equivalently

$$\int_{B_R} (\mu \nabla v_j : \nabla \overline{v_j} + (\lambda+\mu)\,\mathrm{div}\, v_j\, \mathrm{div}\, \overline{v_j})\,\mathrm{d}x \le \omega_j^2 \int_{B_R} |v_j|^2\,\mathrm{d}x - \mathrm{Re} \int_{|x|=R} T_\nu v_j \cdot \overline{v_j}\,\mathrm{d}S \to 0$$

as $j \to \infty$. Therefore v_j converges strongly to zero in $H^1(B_R, \mathbb{C}^3)$ by our definition of the H^1-norm and also the trace $v_j|_{\partial D} = S_{\omega_j}\varphi_j$ tends strongly to zero in $H^{1/2}(\partial D, \mathbb{C}^3)$. Since ω_0^2 is no Dirichlet eigenvalue of $-\Delta^*$, the single layer boundary operator S is an isomorphism and therefore we conclude that $\varphi_j \to 0$ as $j \to \infty$. But this is a contradiction to our assumption that $\|\varphi_j\| = 1$ for all $j \in \mathbb{N}$. ■

3 Scattering by Penetrable Inhomogeneous Media

As in the previous section, we assume that the scattering object is embedded in an isotropic and homogeneous elastic background medium that is described by the Lamé constants λ, μ and has normalized constant mass density equal to one. Embedded in the medium is a penetrable, inhomogeneous scattering object $D \subset \mathbb{R}^3$ with Lipschitz boundary. The scattering object has the same Lamé parameter as the background medium and its mass density is given by a bounded function $\rho \in L^\infty(D)$ such that contrast $q = \rho - 1$ is positive and bounded away from zero, i.e. there exists a positive constant q_0 such that $q(x) \ge q_0$ almost everywhere in D. We consider a variational formulation of equation (14) and seek a radiating solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ to

$$\int_{\mathbb{R}^3} \left(\mu \nabla u : \nabla \overline{\varphi} + (\mu+\lambda)\,\mathrm{div}\, u\, \mathrm{div}\, \overline{\varphi} - \omega^2 \rho u \cdot \overline{\varphi}\right)\,\mathrm{d}x = 0 \tag{31}$$

for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ with compact support, where we extended $q = \rho - 1$ by zero outside of D. Recall from the introduction that the total field $u = u^i + u^s$ can be decomposed into the incoming plane wave u^i from (6) and a scattered field u^s that fulfills the radiation condition (7) and can therefore be represented in terms of its far field as in (9). Recall in this context also the definition of the far field operator in (11). Let us consider the equation for the scattered field and slightly generalize the scattering problem by allowing any source terms $f \in L^2(D, \mathbb{C}^3)$. We seek a radiating solution $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ to the problem

$$\int_{\mathbb{R}^3} \left(\mu \nabla v : \nabla \overline{\varphi} + (\mu + \lambda) \operatorname{div} v \operatorname{div} \overline{\varphi} - \omega^2 \rho v \cdot \overline{\varphi} \right) \mathrm{d}x = -\omega^2 \int_D q f \cdot \overline{\varphi} \, \mathrm{d}x \tag{32}$$

for all test functions $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ with compact support. Choosing $f = -u^i$ then yields the original scattering problem. Existence and uniqueness of a solution to this problem can for example be shown by an integral equation approach, see, e.g. [23, 25]. Recall the definition of interior transmission eigenvalues in (15). This eigenvalue problem is understood in a variational sense, i.e. ω^2 is an interior transmission eigenvalue if there is a pair $(u, w) \in L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$, such that $u - w \in H^2_0(D, \mathbb{C}^3)$ and

$$\int_D u \cdot \left(\Delta^* \varphi - \omega^2 \rho \varphi \right) \mathrm{d}x = 0, \qquad \int_D w \cdot \left(\Delta^* \varphi - \omega^2 \varphi \right) \mathrm{d}x = 0 \qquad \forall \varphi \in C^\infty_0(D, \mathbb{C}^3) \tag{33}$$

$$\int_D u \cdot \left(\Delta^* \varphi - \omega^2 \rho \varphi \right) \mathrm{d}x = \int_D w \cdot \left(\Delta^* \varphi - \omega^2 \varphi \right) \mathrm{d}x \qquad \varphi \in C^\infty(D, \mathbb{C}^3). \tag{34}$$

We know from [7] that there is only a discrete set of interior transmission eigenvalues. As with acoustic scattering, interior transmission eigenvalues are related to the properties of the far field operator F. Whenever ω^2 is no interior transmission eigenvalue, then the far field operator F is injective or conversely, when F is not injective, then ω^2 must be an interior transmission eigenvalue. From now on we proceed as follows: First we derive a factorization for the far field operator and examine the properties of the arising operators in Lemma 8. Then we will use these properties to show that the eigenvalues λ_j of F converge to zero from one specific side in Lemma 9. Using a characterization of the cotangent of the largest phase similar to the last section, we will then calculate a crucial auxiliary derivative in Lemma 11. Finally we will use this auxiliary derivative to prove the inside-outside duality in Theorem 12 and Theorem 13.

We will now show that the eigenvalues converge to zero from one specific side. To this end we first derive a factorization of the far field operator and examine the properties of the arising operators of this factorization. The definition of the Herglotz wave field in (10) implies the existence of a Herglotz wave operator $H : L^2(\mathbb{S}^2)^3 \to L^2(D, \mathbb{C}^3)$, which is given by

$$Hg = v_g \qquad \text{where } v_g(x) = \int_{\mathbb{S}^2} \left[g_{\mathrm{p}}(\theta) e^{ik_{\mathrm{p}} x \cdot \theta} + g_{\mathrm{s}}(\theta) e^{ik_{\mathrm{s}} x \cdot \theta} \right] \mathrm{d}s(\theta), \qquad x \in D,$$

where g_{p} and g_{s} have been defined in (8). The adjoint of the Herglotz operator $H^* : L^2(D, \mathbb{C}^3) \to L^2(\mathbb{S}^2)^3$ is then given by

$$H^* \varphi(\theta) = \int_D \left(\varphi_{\mathrm{p}}(x) e^{-ik^e_{\mathrm{p}} x \cdot \theta} \, \mathrm{d}x + \varphi_{\mathrm{s}}(x) e^{-ik^e_{\mathrm{s}} x \cdot \theta} \, \mathrm{d}x \right), \qquad \theta \in \mathbb{S}^2,$$

where $\varphi_{\mathrm{s}}(x) = \theta \times \varphi(x) \times \theta$ and $\varphi_{\mathrm{p}}(x) = (\varphi(x) \cdot \theta)\theta$ for $\theta \in \mathbb{S}^2$. Let us define a volume potential $V : L^2(D, \mathbb{C}^3) \to H^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ by

$$Vh(x) = \int_D \Phi_N(x, y) h(y) \, \mathrm{d}y,$$

where Φ_N is the fundamental solution of the Navier equation from (20) such that Vh solves

$$(\Delta^* + \omega^2)Vh = -h, \qquad \text{in } \mathbb{R}^3,$$

see [22]. We also know from the proof of [13, Lemma 3.1] that H^*h is the far field w^∞ of the function $w = Vh$. As the final ingredient for our factorization, we introduce the operator $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ by

$$Tf = \omega^2 q(f - v)$$

where v is the radiating solution of (32). Then we can prove the following factorization.

Theorem 7. *It holds that $F = H^*TH$.*

Proof. We follow the standard procedure and introduce an auxiliary operator $G : L^2(D, \mathbb{C}^3) \to L^2(\mathbb{S}^2)^3$ that maps a function f onto the far field v^∞ of the solution of v to (32). Then by the superposition principle, we have that $F = GH$. As we noted above, H^*h is the far field of the function $w = Vh$. Now we write (32) equivalently as

$$\int_{\mathbb{R}^3} \left(\mu \nabla v : \nabla \overline{\varphi} + (\mu + \lambda) \operatorname{div} v \operatorname{div} \overline{\varphi} - \omega^2 v \cdot \overline{\varphi}\right) \, \mathrm{d}x = -\int_D \omega^2 q(f - v) \cdot \overline{\varphi} \, \mathrm{d}x. \tag{35}$$

From the discussion above, it then follows that $G = H^*T$. Since $F = GH$, this implies the factorization of the far field operator. ■

Before we proceed we want to give a characterization of the closure of the range of the Herglotz wave operator. If we denote by $\overline{\mathcal{R}(H)}$ this closure in $L^2(D, \mathbb{C}^3)$, then it holds that

$$X := \overline{\mathcal{R}(H)} = \left\{ w \in L^2(D, \mathbb{C}^3) : \int_D w \cdot (\Delta^* \phi + \omega^2 \phi) \, \mathrm{d}x = 0 \qquad \phi \in C_0^\infty(D, \mathbb{C}^3) \right\} \tag{36}$$

as a consequence of, e.g., [3, Theorem 4.2]. Now we summarizes important properties of the middle operator T in the following lemma.

Lemma 8. *(a) For all $f \in L^2(D, \mathbb{C}^3)$ and $\omega > 0$ it holds that* $\operatorname{Im}(Tf, f)_{L^2(D, \mathbb{C}^3)} \geq 0$.
(b) If $\operatorname{Im}(Tw, w)_{L^2(D, \mathbb{C}^3)} = 0$ *for a non-trivial $w \in X$ and $\omega > 0$, then ω^2 is an interior transmission eigenvalue with corresponding transmission eigenpair $(w - v, w)$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ is the weak radiating solution to* (32).
(c) If $\omega^2 > 0$ is an interior transmission eigenvalue with corresponding transmission eigenpair (u, w), then $w \in X$ and $(Tw, w)_{L^2(D, \mathbb{C}^3)} = 0$.
(d) The operator T can be written as $T = \omega^2 q(\mathrm{Id} + C)$ for a compact operator $C : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$.

Proof. (a) In order to simplify notation below, we introduce a sesquilinear form Ψ by

$$\Psi_{\Omega, \tilde{\rho}}(u, \varphi) := \int_\Omega \left(\mu \nabla u : \nabla \overline{\varphi} + (\lambda + \mu) \operatorname{div} u \ \operatorname{div} \overline{\varphi} - \omega^2 \tilde{\rho} u \cdot \overline{\varphi}\right) \, \mathrm{d}x \tag{37}$$

for an open set $\Omega \subset \mathbb{R}^3$ and functions $\tilde{\rho} \in L^\infty(\Omega)$, $u, \varphi \in H^1(\Omega, \mathbb{C}^3)$. Now we start with an auxiliary calculation. We choose a cut-off function $\phi \in C^\infty(\mathbb{R}^3)$ with compact support such that $\phi = 1$ in a ball $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$, where the radius of the ball in chosen large enough so that D is contained in B_R. Then we set the test function $\varphi = \phi v$ in (35), where v is the solution to this problem. Then we get that

$$\Psi_{\mathcal{B}_R, 1}(v, v) + \Psi_{\mathbb{R}^3 \setminus B_R, 1}(v, v) = -\omega^2 \int_D q(f - v) \cdot \overline{v} \, \mathrm{d}x,$$

where Ψ was defined in (37). Note that v is a smooth solution of the Navier equation outside the ball B_R, i.e. $\Delta^* v + \omega^2 v = 0$. We apply Betti's formula to obtain that

$$\Psi_{\mathbb{R}^3\setminus B_R,1}(v,\varphi) = \int_{|x|=R} T_\nu v \cdot \overline{v} \ \mathrm{d}s$$

and therefore we have in total that

$$\Psi_{\mathcal{B}_R,1}(v,v) + \int_{|x|=R} T_\nu v \cdot \overline{v} \ \mathrm{d}s = -\omega^2 \int_D q(f-v)\cdot \overline{v} \ \mathrm{d}x. \tag{38}$$

After this preliminary considerations, we come to our main assert. Choose an arbitrary $f \in L^2(D,\mathbb{C}^3)$. We have by definition, that

$$(Tf,f)_{L^2(D,\mathbb{C}^3)} = \omega^2 (q(f-v),f)_{L^2(D,\mathbb{C}^3)}.$$

Define now $g \in L^2(D,\mathbb{C}^3)$ by $g := f - v$, and $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C}^3)$ solves (32). Then we get that

$$(Tf,f)_{L^2(D,\mathbb{C}^3)} = (\omega^2 g, g+v)_{L^2(D,\mathbb{C}^3)} = \omega^2 (qg,g)_{L^2(D,\mathbb{C}^3)} + \omega^2 \int_D qg\cdot\overline{\varphi} \ \mathrm{d}x.$$

Resubstituting g and then using equation (38) shows that

$$(Tf,f)_{L^2(D,\mathbb{C}^3)} = (q\omega^2 g, g)_{L^2(D,\mathbb{C}^3)} + \Psi_{\mathcal{B}_R,1}(v,v) + \int_{|x|=R} T_\nu v\cdot\overline{v} \ \mathrm{d}s \tag{39}$$

which implies that

$$\mathrm{Im}\,(Tf,f)_{L^2(D,\mathbb{C}^3)} = \mathrm{Im}\int_{|x|=R} T_\nu v\cdot\overline{v} \ \mathrm{d}s$$

since q and μ,λ are all real-valued. Now we can apply [25, Lemma 1] and get that

$$\mathrm{Im}\,(Tf,f)_{L^2(D,\mathbb{C}^3)} = 2\mathrm{i}\omega \|v^\infty\|_{L^2(D,\mathbb{C}^3)}. \tag{40}$$

(b) Assume there exists a non-trivial $w \in \overline{\mathcal{R}(H)}$ such that $\mathrm{Im}\,(Tw,w)_{L^2(D,\mathbb{C}^3)} = 0$ and let v be the solution of (35) for $f = w$. Then we conclude from the (a)-part of this proof that the far field v^∞ vanishes and by Rellich's Lemma v vanishes outside of D, which implies that $v \in H^2_0(D,\mathbb{C}^3)$. Setting $u = w + v$, we calculate for $\phi \in C_0^\infty(D,\mathbb{C}^3)$ that

$$\begin{aligned}\int_D u\cdot\left[\Delta^*\phi + \omega^2(1+q)\phi\right] \ \mathrm{d}x &= \int_D v\cdot\left[\Delta^*\phi + \omega^2(1+q)\phi\right] + \int_D w\cdot\left[\Delta^*\phi+\omega^2(1+q)\phi\right] \\ &= \int_D \left(\mu\nabla v : \nabla\phi + (\mu+\lambda)\,\mathrm{div}\, v\,\mathrm{div}\,\phi - \omega^2 v\cdot\phi\right) \mathrm{d}x - \omega^2\int_D qw\cdot\phi \ \mathrm{d}x = 0,\end{aligned}$$

where we used that $w \in X$ solves the Navier equation $\Delta^* w + \omega^2 w = 0$. From this calculation, we conclude that (u,w) fulfills (33) and substituting $u = w + v$ shows that (34) also holds, such that (u,w) is an transmission eigenvalue pair and ω^2 is the corresponding interior transmission eigenvalue.
(c) Let $\omega^2 > 0$ be a transmission eigenvalue with eigenpair $(u,w) \in L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)$. We will show that $(Tw,w)_{L^2(D,\mathbb{C}^3)} = 0$. Since $w \in X = \overline{\mathcal{R}(H)}$, there exists a sequence $g_j \in L^2(\mathbb{S}^2)$ such that the corresponding Herglotz wave functions w_j converge to w in $L^2(D,\mathbb{C}^3)$. Since ω^2 is a transmission eigenvalue, (34) implies that $v = u - w \in H^2_0(D,\mathbb{C}^3)$ satisfies

$$\int_D [\Delta^* v + \omega^2 v]\cdot\overline{\phi} \ \mathrm{d}x = \omega^2\int_D q(w-v)\cdot\overline{\phi} \ \mathrm{d}x$$

for all $\phi \in L^2(D,\mathbb{C}^3)$. Choosing $\phi = w$ yields

$$\int_D [\Delta^* v + \omega^2 v] \cdot \overline{w} \, \mathrm{d}x = \omega^2 \int_D q(w-v) \cdot \overline{w} \, \mathrm{d}x = (Tw, w)_{L^2(D,\mathbb{C}^3)}.$$

As $w \in \overline{\mathcal{R}(H)}$ there is a sequence $(w_j)_{j\in\mathbb{N}}$ of Herglotz wave functions such that $w_j \to w$ as $j \to \infty$. This implies that $\|w - w_j\|_{L^2(D,\mathbb{C}^3)} \to 0$ as $j \to \infty$. Since w_j solves the Navier equation and $v \in H_0^2(D,\mathbb{C}^3)$, we get

$$\int_D [\Delta^* v + \omega^2 v] \cdot \overline{w} \, \mathrm{d}x = \lim_{j\to\infty} \int_D [\Delta^* v + \omega^2 v] \cdot \overline{w_j} \, \mathrm{d}x = 0$$

by Betti's first identity. In consequence, $(Tw, w)_{L^2(D,\mathbb{C}^3)} = 0$.
(d) This is clear due to the compactness of the embedding of $H^1(D,\mathbb{C}^3) \to L^2(D,\mathbb{C}^3)$. ∎

The properties of the operator T and the specific structure of the eigenvalues λ_j of F imply that the eigenvalues converge to zero from the right side, see [16, Lemma 4] for a proof.

Theorem 9. *Assume that ω^2 is no interior transmission eigenvalue. Then the eigenvalues λ_j of F converge to zero from the right, i.e.* $\mathrm{Re}\,\lambda_j > 0$ *for $j \in \mathbb{N}$ large enough.*

Recall the representation of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ in polar coordinates in (12) and the definition of the largest phase $\vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j$ in (16). Since the far field operator retains normality and compactness and due to the distinct properties of its eigenvalues, we can easily adapt the proof of Lemma 3 to show that if ω^2 is no interior transmission eigenvalue, then

$$\cot \vartheta^* = \min_{g\in L^2(\mathbb{S}^2)^3} \frac{\mathrm{Re}\,(Fg, g)_{L^2(\mathbb{S}^2)^3}}{\mathrm{Im}\,(Fg, g)_{L^2(\mathbb{S}^2)^3}}, \tag{41}$$

see also [19, Theorem 13]. As in the previous section, we use the factorization of $F = H^*TH$ and rewrite the characterization of the largest phase in (41) to obtain

$$\cot \vartheta^* = \max_{f\in L^2(D,\mathbb{C}^3)} \frac{\mathrm{Re}\,(THf, Hf)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(THf, Hf)_{L^2(D,\mathbb{C}^3)}} = \max_{\varphi\in X} \frac{\mathrm{Re}\,(T\varphi, \varphi)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(T\varphi, \varphi)_{L^2(D,\mathbb{C}^3)}},$$

where $X = \overline{\mathcal{R}(H)}$ was defined in (36). From now on the dependency of all quantities on the frequency ω becomes important. We indicate it by writing $T = T_\omega, X = X_\omega, \lambda_j = \lambda_j(\omega), \Psi_{\dots} = \Psi_{\dots,\omega}$ etc. At this point we have done all the necessary preliminary work for the proof of the second part of the inside-outside duality. However, the proof of the first part of the inside-outside duality in Theorem 5 shows that the maximum in the latter expression is required to be taken over the whole space $L^2(D,\mathbb{C}^3)$ instead of just over the frequency-dependent subspace X_ω. To deal with this problem we introduce a projection operator. Therefore assume that there is a projection $P_\omega : L^2(D,\mathbb{C}^3) \to X_\omega$ that is differentiable with respect to ω. We can use this projection to rewrite the characterization for the largest phase as

$$\cot \vartheta^*(\omega) = \max_{w\in L^2(D,\mathbb{C}^3)} \frac{\mathrm{Re}\,(T_\omega P_\omega w, P_\omega w)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(T_\omega P_\omega w, P_\omega w)_{L^2(D,\mathbb{C}^3)}}.$$

To show that there exists a projection $P_\omega : L^2(D,\mathbb{C}^3) \to X_\omega$ we give an explicit representation of P_ω. First we denote by W the completion of $C_0^\infty(D,\mathbb{C}^3)$ with respect to the norm $\|\varphi\|_W := \|\Delta^*\varphi + \omega^2\varphi\|_{L^2(D,\mathbb{C}^3)}$. Note that this completion is well-defined, since if $\|\Delta^*\varphi + \omega^2\varphi\|_{L^2(D,\mathbb{C}^3)} = 0$

for $\varphi \in C_0^\infty$, the compact support of φ in D and representation formulas for solutions of the Navier-equation as in [12] imply that $\varphi = 0$. Now we define P_ω by

$$P_\omega w = w - (\Delta^* \hat{w} + \omega^2 \hat{w})$$

where $\hat{w} \in W$ solves the W-coercive variational problem

$$\int_D (\Delta^* \hat{w} + \omega^2 \hat{w}) \cdot (\Delta^* \varphi + \omega^2 \varphi) \,\mathrm{d}x = \int_D w \cdot (\Delta^* \varphi + \omega^2 \varphi) \,\mathrm{d}x \qquad \forall \varphi \in W.$$

If $w \in X_\omega$, then the right side of the last equation vanishes and the coercivity of the sesquilinear form on W implies that $\hat{w} = 0$, showing that $P_\omega w = w$. On the other hand for an arbitrary $w \in L^2(D, \mathbb{C}^3)$ we have that $P_\omega w \in X_\omega$ due to the definition of $\hat{w}$, which shows that P_ω is projection onto X_ω. The differentiability of this function is a consequence of the differentiability of the map $\omega \to \hat{w} = \hat{w}(\omega)$. Assume now that ω_0^2 is an interior transmission eigenvalue such that there exists a non-trivial function $w_0 \in X_{\omega_0}$ such that $(T_{\omega_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$. To prove the first part of the inside-outside duality as in the proof of [16, Lemma 5.1] we need to calculate the derivative

$$\alpha(\omega_0) := \frac{\mathrm{d}}{\mathrm{d}\omega} (T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)} \Big|_{\omega=\omega_0}. \tag{42}$$

We start by calculating an auxiliary derivative, which neglects the projection operator.

Lemma 10. *Let $\omega_0^2 > 0$ be an interior transmission eigenvalue with eigenpair $(u_0, w_0) \in L^2(D, \mathbb{C}^3) \times X_{\omega_0}$. Then $v_0 = u_0 - w_0 \in H_0^2(D, \mathbb{C}^3)$ is the radiating solution to*

$$\Delta^* v_0 + \omega_0^2 v_0 = -\omega^2 q w_0 \tag{43}$$

and the mapping $\omega \to (T_\omega w_0, w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable at ω_0 such that

$$\frac{\mathrm{d}}{\mathrm{d}\omega} (T_\omega w_0, w_0) \Big|_{\omega=\omega_0} = \frac{2}{\omega_0} \int_D \left(\mu \nabla v_0 : \nabla \overline{v_0} + (\lambda + \mu) \operatorname{div} v_0 \operatorname{div} \overline{v_0}\right) \mathrm{d}x$$

Proof. Note that (43) holds due to the properties of the eigenpair (u_0, w_0), see also the proof of Lemma 8 for details. For arbitrary $\omega > 0$ we define $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ as the radiating solution to

$$\int_{\mathbb{R}^3} \left(\mu \nabla v_\omega : \nabla \overline{\varphi} + (\mu + \lambda) \operatorname{div} v_\omega \operatorname{div} \overline{\varphi} - \omega^2 \rho v_\omega \cdot \overline{\varphi}\right) \,\mathrm{d}x = \omega^2 \int_D q w_0 \cdot \overline{\varphi} \,\mathrm{d}x \tag{44}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Note that if $\omega = \omega_0$ then $v_{\omega_0} = v_0 \in H_0^2(D, \mathbb{C}^3)$ is the radiating solution to (43) by Betti's formula.The map $\omega \mapsto v_\omega$ is Fréchet-differentiable and $v'_{\omega_0} := [\mathrm{d}v/\mathrm{d}\omega\ v_\omega]|_{\omega=\omega_0} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ solves

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(\mu \nabla v'_{\omega_0} \nabla \overline{\varphi} + (\mu + \lambda) \operatorname{div} v'_{\omega_0} \operatorname{div} \overline{\varphi} - \omega_0^2 \rho v'_{\omega_0} \cdot \overline{\varphi}\right) \,\mathrm{d}x &= -\int_D 2\omega_0 q w_0 \cdot \overline{\varphi} \,\mathrm{d}x + \int_D 2\omega_0 \rho v_{\omega_0} \cdot \overline{\varphi} \,\mathrm{d}x \\
&= \frac{2}{\omega_0} \int_D \left(\mu \nabla v_{\omega_0} : \nabla \overline{\varphi} + (\lambda + \mu) \operatorname{div} v_{\omega_0} \operatorname{div} \overline{\varphi}\right) \mathrm{d}x
\end{aligned}$$

for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with compact support. Moreover, for $\omega = \omega_0$ the solution $v_{\omega_0} \in H_0^2(D)$ has compact support and hence (44) holds in this case even for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$. Using that $(T_{\omega_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega} (T_\omega w_0, w_0)_{L^2(D,\mathbb{C}^3)} \Big|_{\omega=\omega_0} &= \int_D q \omega_0^2 v'_{\omega_0} \cdot \overline{w_0} \,\mathrm{d}x - \int_D 2\omega_0 q (w_0 - v_{\omega_0}) \overline{w_0} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left(\mu \nabla v'_{\omega_0} \nabla \overline{v_{\omega_0}} + (\mu + \lambda) \operatorname{div} v'_{\omega_0} \operatorname{div} \overline{v_{\omega_0}} - \omega_0^2 \rho v'_{\omega_0} \cdot \overline{v_{\omega_0}}\right) \mathrm{d}x \\
&= \frac{2}{\omega_0} \int_D \left(\mu \nabla v_{\omega_0} : \nabla \overline{v_{\omega_0}} + (\lambda + \mu) \operatorname{div} v_{\omega_0} \operatorname{div} \overline{v_{\omega_0}}\right) \mathrm{d}x
\end{aligned}$$

which shows the assertion. ■

Lemma 11. *Let ω_0^2 be an interior transmission eigenvalue with eigenpair $(u_0, w_0) \in L^2(D, \mathbb{C}^3) \times X_{\omega_0}$. Then the map $\omega \to (T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable in ω_0 such that*

$$\alpha(\omega_0) = \frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}\Big|_{\omega=\omega_0} = \frac{2}{\omega_0}\int_D \left(\mu \nabla v_0 : \nabla \overline{v_0} + (\lambda+\mu)\,\mathrm{div}\, v_0\,\mathrm{div}\,\overline{v_0}\right)\,\mathrm{d}x + 4\omega_0 \mathrm{Re}\int_D \overline{v_0}\cdot w_0\,\mathrm{d}x, \tag{45}$$

where $v_0 \in H_0^2(D, \mathbb{C}^3)$ is again the radiating solution to (43).

Proof. Let $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ be defined as in the proof of the last lemma, such that $v_0 = v_{\omega_0}$. By definition of the projection P_ω and the space X_ω, we have that $w_\omega := P_\omega w_0 \in X_\omega$ solves the Navier equation, i.e.

$$\int_D w_\omega \cdot [\Delta^*\varphi - \omega^2\varphi]\ \mathrm{d}x = 0 \qquad \forall \varphi \in C_0^\infty(D, \mathbb{C}^3).$$

Using the differentiability of the projection operator P_ω, the derivative P'_ω of P_ω with respect to ω is given by $\mathrm{d}/\mathrm{d}\omega(P_\omega w_0) = w'_\omega$, where $w'_\omega \in L^2(D, \mathbb{C}^3)$ solves

$$\int_D w'_\omega \cdot [\Delta^*\varphi - \omega^2\varphi]\ \mathrm{d}x = 2\omega \int_D \varphi \cdot w_\omega\,\mathrm{d}x \tag{46}$$

for all $\varphi \in C_0^\infty(D, \mathbb{C}^3)$. Applying the chain rule, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega P_\omega w_0, P_\omega w_0) &= (T'_\omega P_\omega w_0, P_\omega w_0) + (T_\omega P'_\omega w_0, P_\omega w_0) + (T_\omega P_\omega w_0, P'_\omega w_0)\\ &= (T'_\omega P_\omega w_0, P_\omega w_0) + \overline{(T^*_\omega P_\omega w_0, P'_\omega w_0)} + (T_\omega P_\omega w_0, P'_\omega w_0).\end{aligned}$$

Furthermore the symmetry of the sesquilinear form in (44) for the choice $\varphi = v_\omega$ implies that T is self-adjoint on the kernel of $w_0 \to (Tw_0, w_0)_{L^2(D,\mathbb{C}^3)}$ such that $T_{\omega_0} w_0 = T^*_{\omega_0} w_0$, for details see the proof of Theorem [16, Lemma] for acoustic scattering. Using the result of the last lemma, we obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}\right]\Big|_{\omega=\omega_0} = \int_D \left(\mu\nabla v_{\omega_0} : \nabla\overline{v_{\omega_0}} + (\lambda+\mu)\,\mathrm{div}\, v_{\omega_0}\,\mathrm{div}\,\overline{v_{\omega_0}}\right)\,\mathrm{d}x + 2\mathrm{Re}\,(T_{\omega_0} w_0, P'_{\omega_0} w_0)_{L^2(D,\mathbb{C}^3)}.$$

Now we can use that $v_{\omega_0} \in H_0^2(D, \mathbb{C}^3)$ and partial integration to get

$$\begin{aligned}2\mathrm{Re}\,(T_{\omega_0} w_0, P'_{\omega_0} w_0)_{L^2(D,\mathbb{C}^3)} &= 2\mathrm{Re}\left[\int_D \omega^2 q w_0 \cdot \overline{w'_{\omega_0}}\,\mathrm{d}x - \int_D \omega^2 q v_{\omega_0}\cdot\overline{w'_{\omega_0}}\,\mathrm{d}x\right]\\ &= 2\mathrm{Re}\left[\int_D [\Delta^* v_{\omega_0} + \omega_0^2(1+q)v_{\omega_0}]\cdot\overline{w'_{\omega_0}}\,\mathrm{d}x - \omega_0^2\int_D q v_{\omega_0}\cdot\overline{w'_{\omega_0}}\ \mathrm{d}x\right]\\ &= 2\mathrm{Re}\int_D [\Delta^* v_{\omega_0} + \omega_0^2 v_{\omega_0}]\cdot w'_{\omega_0}\,\mathrm{d}x = 2\mathrm{Re}\int_D 2\omega_0 \overline{v_{\omega_0}}\cdot w_0\,\mathrm{d}x,\end{aligned}$$

where we used (46). This shows our claim. ■

After this preliminary considerations, we can now state the first and second part of the inside-outside duality. The proof of the first part of the inside-outside duality again makes use of the derivative α to set up a Taylor expansion of the characterization of the cotangent of the largest phase ϑ^* as in the proof of Theorem 5. For a proof which includes a projection P_ω, we refer to the proof of [16, Lemma 5.1].

Theorem 12 (Inside-Outside Duality - Part 1)**.** *Let ω_0^2 be an interior transmission eigenvalue and $\alpha(\omega_0)$ be the expression in* (45)*. Then the following statement holds:*

$$\lim_{\omega\nearrow\omega_0} \vartheta^*(\omega) = \pi \qquad \text{if } \alpha(\omega_0) > 0 \qquad \text{and} \qquad \lim_{\omega\searrow\omega_0} \vartheta^*(\omega) = \pi \qquad \text{if } \alpha(\omega_0) < 0.$$

Theorem 13 (Inside-outside duality - Part 2)**.** *Assume that $\omega_0 > 0$ and that $I = (\omega_0-\varepsilon, \omega_0+\varepsilon)\setminus\{\omega_0\}$ does not contain interior transmission eigenvalues. If $\vartheta^*(\omega) \to \pi$ for $I \ni \omega \to \omega_0$, then ω_0^2 is an interior transmission eigenvalue.*

Proof. Assume that $\vartheta^*(\omega) \to \pi$ for $I \ni \omega \to \omega_0$. We have that

$$\cot(\vartheta^*) = \min_{w\in X_\omega} \frac{\operatorname{Re}(T_\omega w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_\omega w, w)_{L^2(D,\mathbb{C}^3)}} \to -\infty \qquad \text{for } I \ni \omega \to \omega_0.$$

Thus, there is a sequence $\{\omega_j\}_{j\in\mathbb{N}} \subset I$ such that $\omega_j \to \omega_0$ and $w_j \in X_{\omega_j}$ with $\|w_j\|_{L^2(D,\mathbb{C}^3)} = 1$ such that $0 < \operatorname{Im}(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \to 0$ as $j \to \infty$ and $\operatorname{Re}(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \le 0$ for j large enough. Let $v_j \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ be the corresponding radiating solution to

$$\int_{\mathbb{R}^3} \left(\mu\nabla v_j : \nabla\overline{\varphi} + (\mu+\lambda)\operatorname{div} v_j \operatorname{div}\overline{\varphi} - \omega_j^2 \rho v_j \cdot \overline{\varphi}\right) \mathrm{d}x = \omega_j^2 \int_D q w_j \cdot \overline{\varphi}\, \mathrm{d}x \tag{47}$$

for test functions φ in $H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ with compact support. Since the sequence w_j is bounded in $L^2(D, \mathbb{C}^3)$ there exists a weakly convergent subsequence $w_j \rightharpoonup w_0$ in $L^2(D, \mathbb{C}^3)$ as $j \to \infty$. In particular $w_0 \in X_{\omega_0}$ and $v_j \rightharpoonup v_0$ weakly in $H^1(B_R, \mathbb{C}^3)$ for all radii $R > 0$, where $v_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ is the corresponding weak radiating solution to (47) with ω_j, w_j replaced by ω_0, w_0. In the proof of Lemma 8 we have already shown that

$$\operatorname{Im}(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} = \frac{\omega_j}{4\pi^2} \|v_j^\infty\|^2_{L^2(\mathbb{S}^2)^3}, \qquad j \in \mathbb{N}.$$

The left hand side converges to zero and the right hand side to $\omega_0/(4\pi^2)\, \|v_0^\infty\|_{L^2(\mathbb{S}^2)^3}$. We conclude that $v_0^\infty = 0$ and v_0 vanishes in the exterior of D by Rellich's Lemma.

Assume now that $\omega_0^2 > 0$ is not an interior transmission eigenvalue. Then it follows from Lemma 8(b) that w_0 and v_0 vanish everywhere, such that w_j and v_j converge weakly to zero as $j \to \infty$. We define $g_j = w_j - v_j$ and recalling the arguments of the proof of Lemma 8, we get that

$$(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} = -\omega_j^2 (q g_j, g_j)_{L^2(D,\mathbb{C}^3)} + \Psi_{B_R,1,\omega_j}(v_j, v_j) + \int_{|x|=R} T_\nu v \cdot \overline{v}\, \mathrm{d}s$$

Now we can use (29) and use the real part of the last equation to obtain

$$0 \ge \operatorname{Re}(T_{\omega_j} w_j, w_j) = \Psi_{B_R,1,\omega_j}(v_j, v_j) + \operatorname{Re}\int_{|x|=R} T_\nu v_j \cdot \overline{v_j}\, \mathrm{d}S$$

or equivalently

$$\int_{B_R} (\mu\nabla v_j : \nabla\overline{v_j} + (\lambda+\mu)\operatorname{div} v_j \operatorname{div}\overline{v_j})\, \mathrm{d}x \le \omega_j^2 \int_{B_R} |v_j|^2\, \mathrm{d}x + \operatorname{Re}\int_{|x|=R} T_\nu v_j \cdot \overline{v_j}\, \mathrm{d}S, \qquad j \in \mathbb{N}.$$

As $\|v_j\|_{L^2(B_R,\mathbb{C}^3)} \to 0$ and $\|v_j\|_{H^{1/2}(\partial B_R,\mathbb{C}^3)} \to 0$ as $j \to \infty$ due to the compact embedding of $H^1(B_R, \mathbb{C}^3)$ in $L^2(B_R, \mathbb{C}^3)$ and the smoothness of v_j in a neighborhood of ∂B_R, the right-hand side of the latter inequality converges to zero as j tends to infinity. Therefore, v_j converges strongly to zero in $H^1(B_R, \mathbb{C}^3)$ due to the definition of the H^1-norm. Then it follows that $w_j \to 0$ in $L^2(D, \mathbb{C}^3)$. But this is a contradiction to our assumption that $\|w_j\| = 1$ for all $j \in \mathbb{N}$. ∎

Remark 14. *Up to this point, the theory could be generalized by allowing constant Lamé parameters that are different in the interior and the exterior of the scattering object by using vector-valued product spaces. We neglect this extension however as the derivation would be very technical while not contributing to the general understanding of the underlying arguments. Furthermore the existence proof of interior transmission eigenvalues with non-trivial derivative from the next section would not be possible anymore.*

4 Conditions for the Material Parameter

In this section we want to show that there exist interior transmission eigenvalues ω_0^2 for which the derivative $\alpha(\omega_0)$ in (45) is positive. While the results in this section are certainly not conclusive and only hold under severe restrictions for the density ρ, they mainly serve to show that there exist interior transmission eigenvalues at all for which the derivative α does not vanish. In this section we proceed as follows: Following [14, Section 2], we first we prove an existence results for interior transmission eigenvalues if the contrast $q = \rho - 1 \in L^\infty(D)$ is large enough. Then we show under which conditions the derivative $\alpha(\omega_0)$ does not vanish and finally we bring these two results together to show the existence of interior transmission eigenvalues with non-trivial derivative α.

We will start by showing an existence result for interior transmission eigenvalues, given that the contrast q is large enough. To this end we equip the space $H_0^2(D, \mathbb{C}^3)$ with the inner product $(\phi, \psi)_{H_0^2(D,\mathbb{C}^3)} = (1/q\ \Delta^*\phi, \Delta^*\psi)_{L^2(D,\mathbb{C}^3)}$. To see that this is indeed an inner product, we need to show definiteness. Assume that for any function $\phi \in H_0^2(D, \mathbb{C}^3)$ that

$$(\phi, \phi)_{H_0^2(D,\mathbb{C}^3)} = (1/q\ \Delta^*\phi, \Delta^*\phi)_{L^2(D,\mathbb{C}^3)} = 0.$$

Since $1/q > 0$ in D, we conclude that $\Delta^*\phi = 0$ almost everywhere in D. In particular it follows that $(\Delta^*\phi, \phi)_{L^2(D,\mathbb{C}^3)} = 0$, which by Betti's formula (17) implies that

$$\|\nabla\phi\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + \|\operatorname{div}\phi\|_{L^2(D,\mathbb{C}^3)} = 0.$$

Since ϕ has zero boundary conditions, this in turn implies that $\phi = 0$ and therefore shows the definiteness of the scalar product.
The interior transmission eigenvalue problem (15) can equivalently be written as a fourth-order equation for $v = u - w \in H_0^2(D, \mathbb{C}^3)$, which yields

$$(\Delta^* + \omega^2)\frac{1}{q}(\Delta^* + k^2\rho)v = 0,$$

which in its weak formulation reads

$$a_\omega(v, \psi) := \int_D \frac{1}{q}\left[\Delta^* v + \omega^2\rho v\right] \cdot \left[\Delta^*\psi + \omega^2\psi\right] \mathrm{d}x = 0 \qquad \forall\psi \in H_0^2(D, \mathbb{C}^3). \tag{48}$$

Arguing as in [14, Section 2], we have that ω^2 is an interior transmission eigenvalue if and only if there exists a non-trivial function $v \in H_0^2(D, \mathbb{C}^3)$ such that $a_\omega(v, \psi) = 0$ for all $\psi \in H_0^2(D, \mathbb{C}^3)$. To give an existence result, we define μ_1 as the smallest eigenvalue of the bi-Navier operator, i.e. $(\Delta^*)^2\hat{v} = \mu_1\hat{v}$ in D for an eigenfunction $\hat{v} \in H_0^2(D, \mathbb{C}^3)$. Furthermore let $\gamma = \gamma(\mu, \lambda)$ be a constant such that

$$\mu\|\nabla u\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda + \mu)\|\operatorname{div} u\|^2_{L^2(D,\mathbb{C})} \geq \gamma\|u\|^2_{L^2(D,\mathbb{C}^3)} \qquad \forall u \in H_0^2(D, \mathbb{C}^3).$$

It is clear that such a constant γ exists, since applying the Poincaré-inequality component-wise, we have that there is a constant γ_0 such that $\gamma_0\|u\|_{L^2(D,\mathbb{C}^3)} \leq \|\nabla u\|_{L^2(D,\mathbb{C}^{3\times 3})}$. Then we can show that an interior transmission eigenvalue exists if the contrast q is large enough. Recall for this purpose that $q(x) \geq q_0$ for a constant $q_0 > 0$ for almost all $x \in D$.

Theorem 15. *If $q \in L^\infty(D, \mathbb{C}^3)$ is large enough such that*

$$\mu_1 < \frac{(1 + q_0/2)^2\gamma^2}{1 + q_0}, \tag{49}$$

then there exists at least one transmission eigenvalue ω_0^2 in the intervall $(0, (1 + q/2)\gamma/(1 + q))$.

Proof. We will follow [14] to show existence of interior transmission eigenvalues. First we rewrite the bilinear form a_ω as

$$a_\omega(v, \psi) = \int_D \frac{1}{q}[\Delta^* v + \omega^2 v] \cdot [\Delta^* \psi + \omega^2 \psi] \,\mathrm{d}x + \omega^2 \int_D v \cdot [\Delta^* \psi + \omega^2 \psi] \,\mathrm{d}x \tag{50}$$

for all $\psi \in H_0^2(D, \mathbb{C}^3)$. We can rewrite a_ω as

$$a_\omega = a_0 + \omega^2 b_1 + \omega^4 b_2,$$

where b_1 and b_2 are bilinear forms that are given by

$$\begin{aligned} b_1(v, \psi) &= \int_D \frac{1}{q}[v\Delta^* \psi + \psi \Delta^* v] \,\mathrm{d}x + \int_D v\Delta^* \psi \,\mathrm{d}x, \\ b_2(v, \psi) &= \int_D \frac{q + 1}{q} v\psi \,\mathrm{d}x, \qquad v, \psi \in H_0^2(D, \mathbb{C}^3). \end{aligned}$$

and a_0 is the inner product on H_0^2 that we introduced above. We use Riesz' representation theorem and find bounded operators B_1, B_2 from $H_0^2(D, \mathbb{C}^3)$ into itself such that

$$b_j(v, \psi) = (B_j v, \psi)_{H_0^2(D,\mathbb{C}^3)} \qquad \forall v, \psi \in H_0^2(D, \mathbb{C}^3), j = 1, 2.$$

Therefore we can write the equation $a_\omega(v, \psi) = 0$ for all $\psi \in H_0^2(D, \mathbb{C}^3)$ equivalently as

$$v + \omega^2 B_1 v + \omega^4 B_2 v = 0.$$

From the symmetry of b_j we conclude that B_1, B_2 are self-adjoint. Furthermore these operators are also compact, since they represent differential operators of order less than four, see [15] for the corresponding acoustic case. Finally the operator B_2 is positive. Now we define

$$A_\omega = \mathrm{Id}_3 + \omega^2 B_1 + \omega^4 B_2$$

and notice that this operator is self adjoint due to the self-adjointness of the operators that constitute the operator. Furthermore its spectrum is real and discrete and due to the compactness of B_1 and B_2, we can know that the only possible accumulation point is 1. Furthermore the eigenvalues depend continuously on the frequency ω. Notice that the spectrum of the operator $A_0 = \mathrm{Id}$ only consists of $\{1\}$. If we now find a function $\hat{v} \in H_0^2(D, \mathbb{C}^3)$ and a corresponding value $\hat{\omega}$, such that $a_{\hat{\omega}}(\hat{v}, \hat{v}) < 0$, we know from the min-max principle that the smallest eigenvalue of $A_{\hat{\omega}}$ is negative. Since the smallest eigenvalue depends continuously on the frequency ω, it follows that there is a value ω between 0 and $\hat{\omega}$ such that the kernel of A_ω is non-trivial and therefore k^2 is a transmission eigenvalue. We will now construct such a function $\hat{v}$. First we use (50) to estimate

$$\begin{aligned} a_\omega(v, v) &\leq \frac{1}{q_0} \int_D [\Delta^* v + \omega^2 v]^2 \,\mathrm{d}x + \omega^2 \int_D v \cdot \Delta^* v \,\mathrm{d}x + \omega^4 \|v\|_{L^2(D,\mathbb{C}^3)} \\ &= \frac{1}{q_0} \int_D \left[(\Delta^* v)^2 + \omega^2(2 + q_0) v \cdot \Delta^* v\right] \,\mathrm{d}x + \frac{(1 + q_0)\omega^4}{q_0} \|v\|^2_{L^2(D,\mathbb{C}^3)} \\ &= \frac{1}{q_0} \int_D \left[(\Delta^* v)^2 - \omega^2(2 + q_0)(\mu|\nabla v|^2 + (\lambda + \mu)|\operatorname{div} v|^2)\right] \,\mathrm{d}x + \frac{(1 + q_0)\omega^4}{q_0} \|v\|^2_{L^2(D,\mathbb{C}^3)}, \end{aligned}$$

where we used Betti's formula. Let now $\hat{v}$ be an eigenfunction of the bi-Navier operator $(\Delta^*)^2$, corresponding to an eigenvalue μ_1, i.e. $(\Delta^*)^2\hat{v} = \mu_1\hat{v}$ in D. Therefore we obtain

$$a_\omega(\hat{v},\hat{v}) \leq \frac{\mu_1 + \omega^4(1+q_0)}{q_0}\|\hat{v}\|^2_{L^2(D,\mathbb{C}^3)} - \frac{\omega^2(2+q_0)}{q_0}\left[\mu\|\nabla\hat{v}\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda+\mu)\|\operatorname{div}\hat{v}\|^2_{L^2(D,\mathbb{C}^3)}\right].$$

We can continue to estimate

$$a_\omega(\hat{v},\hat{v}) \leq \frac{1}{q_0}\left[\mu_1 + \omega^4(1+q_0) - \omega^2(2+q_0)\gamma\right]\|\hat{v}\|^2_{L^2(D,\mathbb{C}^3)}.$$

Following [14], we have

$$\mu_1 + \omega^4(1+q_0) - \omega^2(2+q_0)\gamma = \left(\omega^2\sqrt{1+q_0} - \frac{(1+q_0/2)}{\sqrt{1+q_0}}\right)^2 + \mu_1 - \frac{(1+q_0/2)^2\gamma^2}{1+q_0}.$$

Choosing $\omega^2 = (1+q_0/2)\gamma/(1+q_0)$, the first bracket vanishes such that if q_0 is big enough such that

$$\mu_1 < \frac{(1+q_0/2)^2\gamma^2}{1+q_0},$$

we can conclude that $a_k(\hat{v},\hat{v}) < 0$ and therefore there exists an interior transmission eigenvalue ω_0^2 in the interval $(0,(1+q_0/2)\gamma/(1+q_0))$. ■

For the remainder of this section we assume a constant contrast $q = q_0$ in D. Recall that for the eigenpair $(u_0,w_0) \in L^2(D,\mathbb{C}^3)\times X_{\omega_0}$, corresponding to the interior transmission eigenvalue ω_0^2, the derivative $\alpha(\omega_0)$ is given by

$$\alpha(\omega_0) = \frac{2}{\omega_0}\int_D\left(\mu\nabla v_0 : \nabla\overline{v_0} + (\lambda+\mu)\operatorname{div}v_0\operatorname{div}\overline{v_0}\right)\mathrm{d}x + 4\omega_0\int_D\overline{v_0}\cdot w_0\,\mathrm{d}x,$$

where v_0 is the radiating solution to (43). Then $\tilde{\alpha}(\omega_0) := \frac{\omega_0}{2}\alpha(\omega_0)$ is given by

$$\begin{aligned}\tilde{\alpha}(\omega_0) &= \int_D\left(\mu\nabla v_0 : \nabla\overline{v_0} + (\lambda+\mu)\operatorname{div}v_0\operatorname{div}\overline{v_0}\right)\mathrm{d}x + 2\omega_0^2\int_D\overline{v_0}\cdot w_0\,\mathrm{d}x\\ &= \mu\|\nabla v_0\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda+\mu)\|\operatorname{div}v_0\|^2_{L^2(D,\mathbb{C})} + 2\omega_0^2\int_D\overline{v_0}\cdot w_0\,\mathrm{d}x.\end{aligned}$$

The following condition for the positivity of the derivative α holds.

Lemma 16. *Let ω_0^2 be an interior transmission eigenvalue and assume that*

$$\gamma\left(\frac{2}{q}+1\right) - 2\frac{q+1}{q}\omega_0^2 > 0. \tag{51}$$

Then $\alpha(\omega_0) > 0$.

Proof. We start by rewriting the integral

$$\begin{aligned}2\omega_0^2\int_D\overline{v_0}\cdot w_0\,\mathrm{d}x &= \frac{2}{q}\omega_0^2\int_D q\overline{v_0}\cdot w_0\,\mathrm{d}x\\ &= \frac{2}{q}\int_D\left(\mu\nabla v_0 : \nabla\overline{v_0} + (\lambda+\mu)\operatorname{div}v_0\operatorname{div}\overline{v_0} - \omega_0^2\rho v_0\cdot\overline{v_0}\right)\mathrm{d}x\\ &= \frac{2}{q}\left(\mu\|\nabla v_0\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda+\mu)\|\operatorname{div}v_0\|^2_{L^2(D,\mathbb{C})} - \rho\omega_0^2\|v_0\|^2_{L^2(D,\mathbb{C}^3)}\right).\end{aligned}$$

Using this expression, we obtain that

$$\tilde{\alpha}(\omega_0) = \left(\frac{2}{q}+1\right)\mu\|\nabla v_0\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + \left(\frac{2}{q}+1\right)(\lambda+\mu)\|\operatorname{div} v_0\|^2_{L^2(D,\mathbb{C})} - \frac{2}{q}\rho\omega_0^2\|v_0\|^2_{L^2(D,\mathbb{C}^3)}$$

Using $\rho = q+1$, we get that

$$\tilde{\alpha}(\omega_0) \geq \left[\gamma\left(\frac{2}{q}+1\right) - \frac{2}{q}(q+1)\omega_0^2\right]\|v_0\|_{L^2(D,\mathbb{C}^3)}$$

which yields the condition

$$\gamma\left(\frac{2}{q}+1\right) - 2\frac{q+1}{q}\omega_0^2 > 0$$

for the positivity of $\tilde{\alpha}(\omega)$ and $\alpha(\omega_0)$. ■

The condition in (51) shows that in our consideration transmission eigenvalues ω_0^2 must not be too large for the derivative $\alpha(\omega_0)$ to be positive. In the next corollary, we show that the derivative is positive for the interior transmission eigenvalue from Theorem 15.

Corollary 17. *Let the contrast q fulfill the condition* (49). *Then there exists at least one interior transmission eigenvalue $\omega_0^2 < (1+q/2)\gamma/(1+q)$ and for all interior transmission eigenvalues ω_0^2 that fulfill this bound, its holds that $\alpha(\omega_0) > 0$.*

Proof. From Lemma 16 we know that $\alpha(\omega_0) > 0$ if the condition

$$\gamma\left(\frac{2}{q}+1\right) - \frac{2}{q}(q+1)\omega_0^2 > 0$$

is fulfilled. Since $\omega_0^2 \in (0, (1+q/2)\gamma/(1+q))$, it suffices to show that

$$\gamma\left(\frac{2}{q}+1\right) - \frac{2}{q}(q+1)(1+q/2)\gamma/(1+q) = \gamma\left(\frac{2}{q}+1\right) - \frac{2}{q}(1+q/2)\gamma \geq 0.$$

Dividing by γ and multiplying by q yields as a sufficient condition that

$$2 + q - 2(1+q/2) \geq 0,$$

which is obviously true. This shows that for the transmission eigenvalue ω_0^2 the derivative is indeed positive. ■

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