Modelling and stability analysis of autonomous controlled production networks

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Abstract We present methods and tools for modelling autonomous controlled production networks and investigation of their stability properties. Production networks are described by dynamical systems of two types: systems of ordinary differential equations and time-delay systems. In particular with the help of time-delays we incorporate transportation processes and implement an autonomous control method, namely the queue length estimator (QLE). For the stability analysis we utilize Lyapunov functions from mathematical systems theory, where by stability we mean, roughly speaking, boundedness of the state of a system (e.g., the inventory level or the work in progress) over the time under bounded external inputs.

1 Introduction

Production, supply networks and other logistic structures are typical examples of complex systems with a nonlinear and sometimes chaotic behavior.

Their dynamics is subject to many different perturbations due to changes on market, changes in customer behavior, information and transport congestions, unreliable elements of the network etc. One of the approaches to handle such complex systems is to shift from centralized to decentralized or autonomous control, i.e., to allow the entities of a network to make their own decisions based on some given rules and available local information. However a system emerging in this way may become unstable and hence be not effective.

Typical examples of unstable behavior are unbounded growth of unsatisfied orders or unbounded growth of the queue of the workload to be processed by a machine. This causes high inventory costs and loss of customers. To avoid instability of

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a network it is worth to investigate its behavior in advance. In particular mathematical modelling and analysis provide helpful tools for design, optimization and control of such networks and for deeper understanding of their dynamical properties.

1.1 Production networks

The term production network is used to describe company or cross-company owned networks with geographically dispersed plants. The primary objective of production networks is to achieve economies of scale through joint planning of production processes, a mutual use of common resources and integrated planning value added processes [26]. These types of networks may react quickly on perturbations due to redundancies of common resources. But high flexibility causes interdependencies between production processes in different plants, e.g., allocation problems for products or planning of transports and transport capacity [17, 1]. Therefore production planning and control (PPC) of production networks has to cover these tasks and also has to provide methods for an integrated planning and synchronization within the network, including planning of sales and inventory [26]. Under highly dynamic and complex conditions current PPC methods cannot cope with disturbances or unforeseen events in an appropriate manner [14]. This may cause uncertainties of lead times, inconstancy of schedules or may also lead to instability or even chaos.

1.2 Autonomous control

The main idea of autonomous cooperating logistic processes is to enable intelligent logistic objects to route themselves through a logistic network according to their own objectives and to make and execute decisions, based on local information [27, 28]. In this context intelligent logistic objects may be physical or material objects, e.g., parts or machines, as well as immaterial objects (e.g., production orders, information). It has been already shown that different autonomous control methods can help to increase the logistics performance and robustness of single production systems [18, 20]. Due to the high structural and dynamical complexity of production networks one may expect that autonomous control has a positive effect on the dynamical behavior of these networks. This was confirmed by investigations of the performance of autonomously controlled production networks [19]. On the other hand autonomously controlled production networks may show a sudden change of the dynamical systems behavior in dependence of varying start parameters and the logistic performance collapses in the sense of unpredictable and increasing throughput times and growing inventory [21]. Thus investigations of autonomously controlled production networks stability are essential to identify such turning points of dynamical systems behavior.

The autonomous control to be modeled and used in this contribution is based on the queue length estimator (QLE), which was investigated in previous papers besides other existing autonomous control methods [20, 2, 22]. The QLE enables parts to choose the next transportation way to an entity of the network according to the local information about their current amount of the queuing workload.

1.3 Mathematical modelling and stability analysis

Roughly speaking, for production networks stability means that the state of the network remains bounded over time under bounded external inputs.

The state is the set of parameters, which we are interested in, for example the queue length of the workload to be processed by a machine, the work in progress (WIP) or the number of unsatisfied orders. In this contribution we identify the state as the number of unprocessed parts, which is the sum of the queue length and the WIP. Thus stable behavior of the network is decisive for the performance and vitality of a network. To design stable logistic networks we are going to apply tools from mathematical systems theory. In this context mathematical models describing network's behaviour are needed.

For manufacturing systems parameters assuring stable behavior can be found by using fluid models [3], re-entrant lines [4] or manufacturing systems with different job types [5]. An approach with flows of multiple fluids was used to analyse the stability region of an autonomously controlled shop floor scenario [24]. Scholz-Reiter et al. [23] presented a fluid model of a production network and obtained a stability region for a scenario with two locations and three types of products. First approaches have been already done to derive stability conditions of autonomously controlled production networks [6].

In this contribution a production network is described as an interconnection of many dynamical subsystems that are logistic locations. To cope with different dynamical characteristics of the network we develop two types of models: systems, based on ordinary differential equations (ODEs) and time-delay systems. Time-delay systems are described by functional differential equations and take transportation times into account in contrast to models, based on ODEs. Furthermore the QLE is modeled for both types of models. All the models are a basis for a stability analysis provided in this contribution.

Our stability analysis is based on the Lyapunov function theory and small-gain theorems. At the first step we describe the network's behavior by a mathematical model according to the type of its dynamics. Then we are looking for Lyapunov functions and the corresponding Lyapunov gains to establish stability of each subsystem. If all subsystems are stable we apply the so-called small-gain condition, that takes into account the interconnection structure of the network. If this condition is satisfied the stability of the network is proved, otherwise we cannot conclude whether the network is stable or not. But we can repeat the stability analysis choosing another Lyapunov function and/or gains. This framework is described in

Figure 1. This procedure can be applied to general nonlinear large-scale systems to perform a stability analysis and to derive bounds for parameters of a logistic system for which its behavior is stable.

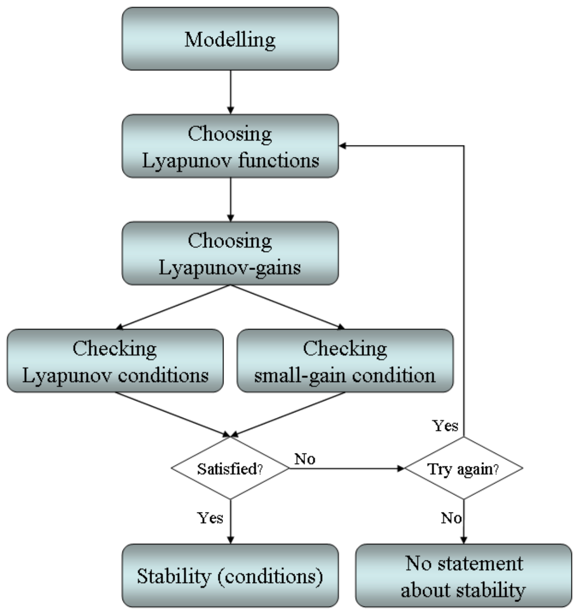

Fig. 1 Scheme of the stability analysis procedure

The structure of the contribution is as follows. In Section 2 we give the necessary notions of the dynamical systems and review the stability results for them, namely ODE systems are considered in Subsection 2.1 and time-delay systems in Subsection 2.2. These results will be used in Section 3 for modelling and analysis of the behavior of logistics networks with and without time-delays. The application will be supplemented by numerical simulations in Matlab for a certain scenario of a production network in Section 4. Section 5 concludes the contribution and outlines some approaches for the future work.

2 Modelling methods and mathematical stability theory

In this section we introduce two different methods to model dynamical networks such as production networks. Furthermore, the stability theory for these methods is presented.

2.1 Ordinary differential equations

One possibility to model production networks are ordinary differential equations (ODEs), see for example [13]. An ODE is of the form

$$\dot{x}(t) = f(x(t), u(t)),\ t \in \mathbb{R}_+, \tag{1}$$

where $x \in \mathbb{R}^N$ denotes the state of the system, $u \in \mathbb{R}^M$ is the essentially bounded measurable external input and $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ describes the system dynamics. ODEs describe the evolution of the state of the system with continuous time $t \in \mathbb{R}_+$, where $\mathbb{R}_+ := [0,\infty)$.

To have existence and uniqueness of a solution of a system of the form (1) the function f is assumed to be a locally Lipschitz continuous function. The solution is denoted by $x(t;x_0,u)$ or $x(t)$ for short, where $x_0 := x(0)$ is the initial condition.

In general, production networks consist of $n \in \mathbb{N}$ interconnected systems of the form

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u_i(t)),\ t \in \mathbb{R}_+,\ i = 1, \ldots, n, \tag{2}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$ and $f_i : \mathbb{R}^{\sum_{j=1}^{n} N_j + M_i} \to \mathbb{R}^{N_i}$ are locally Lipschitz continuous functions. Here, $x_j,\ j \neq i$ can be interpreted as internal inputs of the i-th subsystem and the solution is denoted by $x_i(t;x_i^0, x_j, j \neq i, u_i)$ or $x_i(t)$ for short, where $x_i^0 := x_i(0)$ is the initial condition.

If we define $N := \sum_{i=1}^{n} N_i,\ M := \sum_{i=1}^{n} M_i,\ x := (x_1^T, \ldots, x_n^T)^T,\ u := (u_1^T, \ldots, u_n^T)^T$ and $f = (f_1^T, \ldots, f_n^T)^T$, then the interconnected system of the form (2) can be written as one single system of the form (1), which we call the whole system.

The purpose of this paper is to analyse production networks, which can be written in the form (2), in view of stability. Therefore we introduce the following stability notion:

Definition 1. 1. System (1) is *locally input-to-state stable (LISS)* if there exist constants $\rho,\ \rho_u > 0,\ \gamma \in \mathscr{K}$, where $\mathscr{K}$ is the set of continuous functions with $\gamma(0) = 0$ and strictly increasing, and $\beta \in \mathscr{KL}$, where a function of class $\mathscr{KL}$ has to arguments: in the first argument it is a $\mathscr{K}$-function and in the second argument it is a continuous, strictly decreasing function with $\lim_{t\to\infty} \beta(\cdot,t) = 0$, such that for all initial values $|x_0| \leq \rho$ and all inputs $\|u\|_\infty \leq \rho_u$ the inequality

$$|x(t)| \leq \max\left\{\beta\left(|x_0|, t\right), \gamma\left(\|u\|_\infty\right)\right\}$$

is satisfied $\forall\ t \in \mathbb{R}_+$, where $|\cdot|$ denotes the Euclidean norm and $\|u\|_\infty := \operatorname*{ess\,sup}_{t\in[0,\infty)} |u(t)|$ is the essential supremum norm. γ is called (nonlinear) gain.

2. The i-th subsystem of (2) is called *LISS* if there exist constants $\rho_i,\ \rho_{ij},\ \rho_i^u > 0,\ \gamma_{ij},\ \gamma_i \in \mathscr{K}$ and $\beta_i \in \mathscr{KL}$ such that for all initial values $\left|x_i^0\right| \leq \rho_i$ and all inputs $\left\|x_j\right\|_\infty \leq \rho_{ij},\ \|u_i\|_\infty \leq \rho_i^u$ the inequality

$$|x_i(t)| \le \max\left\{\beta_i\left(\left|x_i^0\right|,t\right), \max_{j\neq i}\gamma_{ij}\left(\left\|x_j\right\|_\infty\right), \gamma_i\left(\|u_i\|_\infty\right)\right\}$$

is satisfied $\forall\, t \in \mathbb{R}_+$. γ_{ij} and γ_i are called (nonlinear) gains.

Note that, if $\rho, \rho_u = \infty$ then the system (1) is called *(global) ISS* and if $\rho_i, \rho_{ij}, \rho_i^u = \infty$ then the i-th subsystem of (2) is called (global) ISS. In particular LISS and ISS guarantee that the norm of the trajectories of each subsystem is bounded.

An important tool to verify LISS and ISS, respectively, of a system of the form (2) are Lyapunov functions.

Definition 2. We assume that for each subsystem of the interconnected system (1) there exists a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous, proper and positive definite. Then, for $i = 1,\ldots,n$ the function V_i is called a *LISS Lyapunov function of the i-th subsystem of* (2) if V_i satisfies the following two conditions:
There exist functions $\psi_{1i},\ \psi_{2i} \in \mathscr{K}_\infty$, where $\mathscr{K}_\infty$ is the subset of $\mathscr{K}$-functions that are unbounded, such that

$$\psi_{1i}\left(|x_i|\right) \le V_i(x_i) \le \psi_{2i}\left(|x_i|\right),\ \forall\, x_i \in \mathbb{R}^{N_i} \tag{3}$$

and there exist $\gamma_{ij},\ \gamma_i \in \mathscr{K}$, a positive definite function μ_i, which is continuous, $\mu_i(0) = 0$ and $\mu_i(r) > 0$, $\forall r \in \mathbb{R}$, and constants $\rho_i,\ \rho_{ij},\ \rho_i^u\ > 0$ such that

$$V_i(x_i) \ge \max\left\{\max_{j\neq i}\gamma_{ij}\left(V_j(x_j)\right), \gamma_i\left(|u_i|\right)\right\} \Rightarrow \nabla V_i(x_i)\cdot f_i(x,u) \le -\mu_i\left(V_i(x_i)\right) \tag{4}$$

for almost all $x_i \in \mathbb{R}^{N_i},\ \left|x_i^0\right| \le \rho_i,\ \left|x_j\right| \le \rho_{ij},\ u_i \in^{M_i},\ |u_i| \le \rho_i^u,\ \chi_{ii} = 0$, where ∇ denotes the gradient of the function V_i. Functions γ_{ij} are called *LISS Lyapunov gains*.

Note that, if $\rho_i, \rho_{ij}, \rho_i^u = \infty$ then the LISS Lyapunov function of the i-th subsystem becomes an ISS Lyapunov function of the i-th subsystem (see [11]). In general the LISS Lyapunov gains are different from the gains in Definition 1.

Condition (3) implies that V_i is proper, positive definite and radially unbounded. V_i can be interpreted as the energy of a system and the second condition (4) of a Lyapunov function means that if $V_i(x_i) \ge \max\left\{\max_{j\neq i}\gamma_{ij}\left(V_j(x_j)\right), \gamma_i\left(|u_i|\right)\right\}$ holds, then the energy decreases. If $V_i(x_i) < \max\left\{\max_{j\neq i}\gamma_{ij}\left(V_j(x_j)\right), \gamma_i\left(|u_i|\right)\right\}$ then the energy of the system is bounded by the expression on the left side of the previous inequality. Overall, the trajectory of a system is bounded.

Furthermore we define the *gain-matrix* $\Gamma := (\gamma_{ij})_{n\times n},\ i,j = 1,\ldots,n,\ \gamma_{ii} = 0$, which defines a map $\Gamma:\ \mathbb{R}_+^n\ \to\ \mathbb{R}_+^n$ by

$$\Gamma\left(s\right) := \left(\max_j \gamma_{1j}(s_j),\ldots,\max_j \gamma_{nj}(s_j)\right)^T,\ s \in \mathbb{R}_+^n. \tag{5}$$

Note that the matrix Γ describes in particular the interconnection structure of the network, moreover it contains the information about the mutual influence between the subsystems, which can be used to verify the (L)ISS property of networks.

Definition 3. Γ satisfies the *local small gain condition (LSGC) on* $[0,w^*]$, provided that

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\geq s,\ \forall s \in [0,w^*],\ s \neq 0. \tag{6}$$

Notation $\not\geq$ means that there is at least one component $i \in \{1,\ldots,n\}$ such that $\Gamma(s)_i < s_i$.

To check whether the interconnected system of the form (1) has the LISS property we use the scheme in Figure 1. To this end, one has to find a LISS Lyapunov function for each subsystem. If there exists a LISS Lyapunov function for each subsystem then it has the LISS property. Furthermore, if the LISS Lyapunov gains satisfy the local small-gain condition, then the whole system of the form (1) is LISS, which we recall in the following theorem (see [10]):

Theorem 1. *Consider the interconnected system* (2)*, where each subsystem has an LISS Lyapunov function* V_i*. If the corresponding gain-matrix* Γ *satisfies the local small-gain condition* (6)*, then there exist constants* $\rho, \rho_u > 0$*, such that the whole system of the form* (1) *is LISS.*

In [9] a similar ISS small-gain theorem for general networks was proved, where the small-gain condition is of the form

$$\Gamma(s) \not\geq s,\ \forall\, s \in \mathbb{R}^n_+ \setminus \{0\}.$$

2.2 Time-delay systems

In this section we introduce systems with time-delays that allow modelling of transportations times in logistic networks: material leaves one production location at time t and reaches the following location at time $t+\theta$, where $\theta > 0$ is the transportation time between these two production locations. Time-delay systems are described by continuous differential equations of the form

$$\dot{x}(t) = f(x^t, u), \tag{7}$$

where t, x, u and f are as in the previous section. Here the term $x^t := x(t+\tau),\ \tau \in [-\theta, 0],\ x^t \in C([-\theta,0];\mathbb{R}^N)$ represents the state, where $C([-\theta,0];\mathbb{R}^N)$ denotes the space of continuous functions defined on $[-\theta,0]$ equipped with the norm $\|x^t\|_{[-\theta,0]} := \sup_{t\in[-\theta,0]} |x(t)|$ and values in $\mathbb{R}^N$. θ can be interpreted as the maximal involved delay. We assume that the conditions for the existence and uniqueness of a solution of (7) are satisfied. Let the initial state be given by the function $\xi \in C([-\theta,0];\mathbb{R}^N)$.

The stability notions introduced in the previous section can be defined for time-delay systems as well:

Definition 4. System (7) is called LISS if there exist constants ρ, ρ_u and functions $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$, such that for every initial condition $\|\xi\|_{[-\theta,0]} \leq \rho$, every external input $\|u\|_\infty \leq \rho_u$ and for all $t \in \mathbb{R}_+$ it holds

$$|x(t)| \leq \max\{\beta(\|\xi\|_{[-\theta,0]}, t), \gamma(\|u\|_\infty)\},$$

where $\xi \in C([-\theta,0], \mathbb{R}^N)$.

If we consider n interconnected systems, we write each subsystem as

$$\dot{x}_i(t) = f_i(x_1^t, \ldots, x_n^t, u_i(t)), \tag{8}$$

where $x_j^t := x_j(t+\tau),\ \tau \in [-\theta,0]$ can be interpreted as internal input of the i-th subsystem, $i = 1,\ldots,n$. The initial functions are given by $\xi_i \in C([-\theta,0];\mathbb{R}^{N_i})$. Again, this network can be written in the form (7). The notion of LISS for interconnected time-delay systems is as follows:

Definition 5. The i-th subsystem of (8) is called LISS if there exist constants $\rho_i,\ \rho_{ij},\ \rho_i^u > 0$ and functions $\beta_i \in \mathscr{KL}$ and $\gamma_{ij}^d, \gamma_i^u \in \mathscr{K}$, $i,j = 1,\ldots,n,\ i \neq j$, such that for initial functions $\|\xi_i\|_{[-\theta,0]} \leq \rho_i$, for inputs $\|x_j\|_{[-\theta,\infty)} \leq \rho_{ij},\ \|u_i\|_\infty \leq \rho_i^u$ and for all $t \in \mathbb{R}_+$ it holds

$$|x_i(t)| \leq \max\{\beta_i(\|\xi_i\|_{[-\theta,0]}, t), \max_{j\neq i} \gamma_{ij}^d(\|x_j\|_{[-\theta,\infty)}), \gamma_i^u(\|u\|_\infty)\}, \tag{9}$$

where $\|x_j\|_{[-\theta,\infty)} := \sup_{t\in[-\theta,\infty)} |x_j(t)|$.

As in the delay-free case, Lyapunov functions are a useful tool to investigate stability of systems with time-delays, where one can use Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals (see [25], [16]). In this paper we only use Lyapunov-Razumikhin functions for the stability analysis. The existence of an ISS Lyapunov-Razumikhin function implies ISS for systems of the form (7). This was shown in [25] and can be transferred to LISS in a similar way. For the definition of LISS Lyapunov-Razumikhin functions we introduce the upper right-hand side derivative of a locally Lipschitz continuous function $V : \mathbb{R}^N \to \mathbb{R}_+$ along the solution $x(t)$, which is defined by

$$\mathrm{D}^+V(x(t)) = \limsup_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h}.$$

For interconnected time-delay systems the LISS Lyapunov-Razumikhin functions are defined in the following way:

Definition 6. We assume that for each subsystem of the interconnected system (8) there exists a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous, proper and positive definite. Then, for $i = 1,\ldots,n$ the function V_i is called an *LISS Lyapunov-Razumikhin function* for the i-th subsystem of (8) if there exist constants

$\rho_i,\ \rho_{ij},\ \rho_i^u > 0$ and functions $\beta_i \in \mathcal{KL}, \gamma_{ij}^d, \gamma_i^u \in \mathcal{K} \cup \{0\},\ \mu_i \in \mathcal{K},\ i,j = 1,\dots,n,$ such that

$$\psi_{1i}(|x_i|) \le V_i(x_i) \le \psi_{2i}(|x_i|),\ \forall x_i \in \mathbb{R}^{N_i}, \tag{10}$$

$$V_i(x_i) \ge \max\{\max_j \gamma_{ij}^d(||V_j^d(x_j)||), \gamma_i^u(|u|)\} \ \Rightarrow\ \mathrm{D}^+ V_i(x_i) \le -\mu_i(V_i(x_i)) \tag{11}$$

for all initial functions $\|\xi_i\|_{[-\theta,0]} \le \rho_i$, for all inputs $\left|x_j\right| \le \rho_{ij},\ |u_i| \le \rho_i^u$ and for all $t \in \mathbb{R}_+$, where $V_j^d(x_j(t)) := V_j(x_j(t+\tau)),\ \tau \in [\theta, 0]$ and $||V_j^d(x_j)|| := \max_{t-\theta \le s \le t} \left|V_j(x_j(s))\right|$.

Furthermore we define the gain-matrix for time-delay systems by $\overline{\Gamma} := (\gamma_{ij}^d)_{n\times n}$ and the map $\overline{\Gamma} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\overline{\Gamma}(s) := \left(\max_j \gamma_{1j}^d(s_j), \dots, \max_j \gamma_{nj}^d(s_j)\right)^T,\ s \in \mathbb{R}_+^n.$$

With help of the following theorem we can check, whether an interconnected system with time-delays is LISS.

Theorem 2. *Consider the interconnected system* (8)*, where each subsystem has a LISS Lyapunov-Razumikhin function* V_i*. If the corresponding gain-operator* $\overline{\Gamma}$ *satisfies the local small-gain condition from Definition 3, then there exist constants* $\rho, \rho_u > 0$*, such that the whole system of the form* (7) *is LISS.*

This follows from Theorem 1 in [8] with the corresponding changes according to the LISS property.

Remark 1. Another tool to check whether a system or a network has the ISS or LISS property is a Lyapunov-Krasovskii functional. We do not discuss and use this approach in this paper, but one can read the works [16] and [8] for further details.

Theorems 2 and 1 will be used in the following section for a stability analysis of production networks.

3 Modelling and stability analysis of production networks

In this section we model general production networks and perform a stability analysis, where the methods and tools presented in the previous section are used. We will derive a condition, which guarantees stability of a general network.

3.1 Description and Modelling of a general production network

We consider a production network, consisting of n market entities, which may be raw material suppliers (e.g., extracting or agricultural companies), producers, distributors and consumers, for example. Each entity is understood as a subsystem of the whole network. For simplicity we assume, that there is only one unified type of material, i.e., all primary products, used in the production network, can be measured as a number of units of this unified material.

The state of the i-th subsystem at time $t \in \mathbb{R}_+$ is the quantity of unprocessed material within the i-th subsystem at time t. It will be denoted by $x_i(t)$. The state of the whole network is denoted by $x(t) = (x_1(t),\ldots,x_n(t))^T$. A subsystem can get material from an external source, which is denoted by u_i, and from subsystems of the network (internal inputs).

Modelling without time-delays

At first we consider a production network without transportation times and use ordinary differential equations to model it. Let the i-th subsystem processes the raw material from its inventory with the rate $\tilde{f}_{ii}(t,x(t)) \geq 0$ and sends the produced goods (measured in units of unified material) to the j-th subsystem with the rate $\tilde{f}_{ji}(t,x(t))$. Thus, the total rate of the distribution from the i-th subsystem to other subsystems is $\sum_{j=1}^{n} \tilde{f}_{ji}(t,x(t))$ and the rest is sent to some customers not considered in the network.

For general functions $\tilde{f}_{ji}$ it is hard to derive stability conditions. Therefore we will investigate the special case $\tilde{f}_{ji}(t,x(t)) = c_{ji}(t,x(t))\tilde{f}_i(x_i(t)),\ c_{ji} \in \mathbb{R}_+$ and $\tilde{f}_{ii}(t,x(t)) = \tilde{c}_{ii}(t,x(t))\tilde{f}_i(x_i(t)),\ \tilde{c}_{ii} \in \mathbb{R}_+$, where $\tilde{f}_i(x_i(t)) \in \mathcal{K}$ is proportional to the processing rate of the system, $c_{ji}(t,x(t)),\ i \neq j$ are some positive distribution coefficients and $\tilde{c}_{ii}(t,x(t)) \geq 0$. We will denote $c_{ji}(t)$ for the sake of brevity. We interpret the constant distribution coefficients as central planning and on the other hand variable distribution coefficients can be used for some autonomous control method.

Under this assumptions the dynamics of the i-th subsystem is described by ordinary differential equations as in (2):

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(t)\tilde{f}_j(x_j(t)) + u_i(t) - \tilde{c}_{ii}(t)\tilde{f}_i(x_i(t)),\ i = 1,\ldots,n. \tag{12}$$

Denoting $c_{ii} := -\tilde{c}_{ii}$ we can rewrite the above equations as an interconnected system of the form (1) in a vector form

$$\dot{x}(t) = C(t)\tilde{f}(x(t)) + u(t), \tag{13}$$

where $\tilde{f}(x(t)) = (\tilde{f}_1(x_1(t)),...,\tilde{f}_n(x_n(t)))^T$, $u(t) = (u_1(t),\ldots,u_n(t))^T$ and $C(t) \in \mathbb{R}^{n\times n}$.

The given model will be used in the next subsection for a stability analysis of general production networks.

Modelling with time-delays

Now we model general production networks with transportation times using time-delay systems. The time needed for the transportation of material from the j-th to the i-th entity is denoted by $\tau_{ij} \in \mathbb{R}_+$. Then the dynamics of the i-th subsystem can be described by retarded differential equations similar to (12):

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(t)\tilde{f}_j(x_j(t-\tau_{ij})) + u_i(t) - \tilde{c}_{ii}(t)\tilde{f}_i(x_i(t)),\ i = 1,\ldots,n. \tag{14}$$

Here, the external input and the processing rate do not depend on any time-delay, but the internal inputs from other subsystems do, represented by the terms $c_{ij}(t)\tilde{f}_j(x_j(t-\tau_{ij}))$. This means, that the input of subsystem i at time t from subsystem j is the amount of material that was sent by the j-th subsystem at the time $t-\tau_{ij}$. The terms $c_{ij}(t)$ may also depend on $x_j(t-\tau_{ij})$, but we write $c_{ij}(t)$ for short.

In the next subsection we will perform a stability analysis for such systems, where we use the Lyapunov-Razumikhin approach.

3.2 Stability analysis

For the stability analysis we apply the framework shown in Figure 1. At first, we use the model based on ODEs. We choose an ISS Lyapunov function for each subsystem described in (12) and the corresponding Lyapunov gains. Then the conditions of a Lyapunov function and the small-gain condition are verified.

Stability analysis of production networks modeled without time-delays

At first, we consider the case $\tilde{f}_i \in \mathcal{K}_\infty,\ i = 1,\ldots,n$, in particular $\tilde{f}_i$ are unbounded. Later we will show how the same method can be applied with minimal modifications for bounded $\tilde{f}_i \in \mathcal{K} \setminus \mathcal{K}_\infty$. Note, that the conditions $\tilde{f}_i \in \mathcal{K}_\infty$, $c_{ii}(t) < 0$ and $c_{ij}(t) \geq 0,\ i \neq j$ imply, that if $x(0) \geq 0$ (that is $x_i(0) \geq 0\ \forall i = 1,\ldots,n$), then $x(t) \geq 0$ for all $t > 0$.

Let us check whether the function $V_i(x_i) = |x_i| = x_i$ is an ISS-Lyapunov function for the i-th entity. Obviously, $V_i(x_i)$ satisfies the condition (3). To prove, that the condition (4) holds, we choose the functions γ_{ij}, γ_i, μ_i (see Definition 2) as

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\tfrac{a_i}{a_j}\tfrac{1}{1+\delta_j}\tilde{f}_j(s)\right),\ \gamma_i(s) := \tilde{f}_i^{-1}\left(\tfrac{1}{r_i}s\right), \tag{15}$$

where δ_j, a_j, $j = 1,\ldots,n$ and r_i are positive reals. It follows

$$x_i \geq \gamma_{ij}(x_j)\ \Rightarrow \tilde{f}_j(x_j) \leq \tfrac{a_j}{a_i}(1+\delta_j)\tilde{f}_i(x_i),\ x_i \geq \gamma_i(|u_i|)\ \Rightarrow |u_i| \leq r_i\tilde{f}_i(x_i).$$

Using these inequalities and applying the following technical condition

$$\sum_{j=1, j\neq i}^{n} c_{ij}(t)\tfrac{a_j}{a_i}(1+\delta_j) + c_{ii}(t) + r_i \leq -h_i,\ h_i > 0, \tag{16}$$

we obtain

$$\begin{aligned}\frac{dV_i}{dt} &= \sum_{j=1}^{n} c_{ij}(t)\tilde{f}_j(x_j(t)) + u_i(t)\\ &\leq \left(\sum_{j=1, j\neq i}^{n} c_{ij}(t)\tfrac{a_j}{a_i}(1+\delta_j) + c_{ii}(t) + r_i\right)\tilde{f}_i\left(x_i(t)\right) \leq -\mu_i(V_i(x_i(t))),\end{aligned}$$

where $\mu_i(r) := h_i\tilde{f}_i(r)$ and thereby condition (4) is satisfied. Thus, under condition (16), $V_i(x_i) = |x_i|$ is an ISS Lyapunov function for the i-th entity.

To check whether the interconnected system (13) is ISS we need to verify the small-gain condition. It is known, that this condition is equivalent to the cycle condition (see [9]): for all $(k_1,...,k_p) \in \{1,...,n\}^p$, where $k_1 = k_p$, it holds

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ ... \circ \gamma_{k_{p-1}k_p}(s) < s. \tag{17}$$

Consider a composition $\gamma_{k_1k_2} \circ \gamma_{k_2k_3}$, then it holds

$$\begin{aligned}\gamma_{k_1k_2} \circ \gamma_{k_2k_3} =& \tilde{f}_{k_1}^{-1}\left(\tfrac{a_{k_1}}{a_{k_2}}\tfrac{1}{1+\delta_{k_3}}\tilde{f}_{k_2}\left(\tilde{f}_{k_2}^{-1}\left(\tfrac{a_{k_2}}{a_{k_3}}\tfrac{1}{1+\delta_{k_3}}\tilde{f}_{k_3}\left(s\right)\right)\right)\right) =\\ =& \tilde{f}_{k_1}^{-1}\left(\tfrac{a_{k_1}}{a_{k_3}}\tfrac{1}{\left(1+\delta_{k_3}\right)\left(1+\delta_{k_2}\right)}\tilde{f}_{k_3}\left(s\right)\right).\end{aligned}$$

In the same way we obtain the expression for the cycle condition in (17) (here we use, that $k_1 = k_p$):

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ ... \circ \gamma_{k_{p-1}k_p}(s) = \tilde{f}_{k_1}^{-1}\left(\tfrac{1}{\prod_{i=2}^{p}(1+\delta_{k_i})}\tilde{f}_{k_1}\left(s\right)\right) < s.$$

Thus, the small gain condition (17) holds true for all $\delta_i > 0$ and by Theorem 1 the whole system is ISS.

We assume that the c_{ij} are bounded, i.e., $\exists M > 0 : c_{ij}(t) \leq M$ for all $i, j = 1,\dots,n,\ i \neq j$, and the inequality (16) can be simplified:

$$\forall w_i > 0\ \exists \delta_j, j = 1,\dots,n : \sum_{j=1, j\neq i}^{n} c_{ij}(t)\tfrac{a_j}{a_i}\delta_j \leq M\left(\sum_{j=1, j\neq i}^{n}\tfrac{a_j}{a_i}\delta_j\right) < w_i.$$

Using these estimates, we can rewrite (16) by

$$\sum_{j=1, j\neq i}^{n} c_{ij}(t)a_j \leq -c_{ii}(t)a_i + \varepsilon_i,$$

where $\varepsilon_i = -a_i(r_i + h_i + w_i)$. In matrix notation, with $a = (a_1,\dots,a_n)^T$, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)^T$, it takes the form

$$C(t)a < \varepsilon. \tag{18}$$

We summarize our investigations in the following proposition.

Proposition 1. *Consider a network as in* (12) *and assume that the* c_{ij} *are bounded for all* $i,j=1,\dots,n,\ i\neq j$. *If* $\exists a\in\mathbb{R}^n,\ \varepsilon\in\mathbb{R}^n,\ a_i>0,\ \varepsilon_i<0,\ i=1,\dots,n$ *such that the condition* $C(t)a<\varepsilon$ *holds* $\forall t>0$, *then the whole network (13) is ISS.*

Remark 2. If the matrix C does not depend on t, then condition $Ca<\varepsilon$ is equivalent to $Ca<0$ (with $a,\ \varepsilon$ as in the proposition above). But if $C=C(t)$, then the existence of a positive vector $a,\ Ca<0$ is not enough to guarantee ISS of the system (13).

Remark 3. If C is a constant matrix and recalling (see [12], p. 301) that a matrix $C\in\mathbb{R}^{n\times n}$ is called diagonally dominant, if there are n numbers $a_i>0$, such that $c_{ii}a_i+\sum_{j\neq i}|c_{ij}|a_j<0$ for all $i=1,\dots,n$. Then, for $c_{ij}\geq 0$ this condition is equivalent to the existence of a positive vector a, such that $Ca<0$ holds. Consequently, for a case of time-independent matrices C diagonal dominance of C is a sufficient condition for ISS of a system of the form (13). Note, that every diagonally dominant matrix is Hurwitz (see, e.g., [12]), i.e., the real parts of all the eigenvalues are negative.

Now we consider $\tilde{f}_i\in\mathscr{K}\setminus\mathscr{K}_\infty$, i.e., function $\tilde{f}_i$ is monotonously increasing, but only up to a certain limit $\alpha_i:=\sup_{x_i}\{\tilde{f}_i(x_i)\}$. For such $\tilde{f}_i$ the global ISS property cannot be achieved, but we can establish the LISS property. We choose again the function $V_i=|x_i|=x_i$ as LISS Lyapunov function candidate for the i-th subsystem and the corresponding gains as follows

$$\gamma_{ij}(s):=\tilde{f}_i^{-1}\left(\frac{\alpha_i}{\alpha_j}\frac{1}{1+\delta}\tilde{f}_j(s)\right),\ \gamma_i(s):=\tilde{f}_i^{-1}\left(\frac{\alpha_i}{\|u_i\|_\infty r_i}s\right)$$

Note, that in contrast to the previous case, where the coefficients a_i involved in the gain functions were chosen arbitrarily, the α_j are taken from the boundedness assumptions on the functions $\tilde{f}_i$. The reason is to obtain a range of a function $\frac{\alpha_i}{\alpha_j}\tilde{f}_j(s)$ equal to the domain of definition of $\tilde{f}_i^{-1}$.

Applying the same methods as for $\tilde{f}_i\in\mathscr{K}_\infty$, we obtain the following proposition:

Proposition 2. *Consider a network as in* (12). *Define* $\|u\|_\infty:=(\|u_1\|_\infty,\dots,\|u_n\|_\infty)^T$. *Let* $\tilde{f}_j\in\mathscr{K}\setminus\mathscr{K}_\infty$, *and* $\alpha_j:=\sup_{x_j\in\mathbb{R}}\{\tilde{f}_j(x_j)\},\ j=1,\dots n,\ \alpha:=(\alpha_1,\dots,\alpha_n)^T$. *If* $\exists\varepsilon\in\mathbb{R}^n,\ \varepsilon_i<0,\ i=1,\dots,n$ *such that*

$$C(t)\alpha+\|u\|_\infty<\varepsilon, \tag{19}$$

then the whole network (13) is LISS.

Remark 4. The stability analysis for functions $\tilde{f}_i\in\mathscr{K}$ is skipped here, because some more technical details are necessary, that would increase the size of the paper drastically. The result is similar to Proposition 2.

Stability analysis of production networks with time-delays

Now we perform a stability analysis for general production networks with transportation times modeled by time-delay systems of the form (14), where we use the tools presented in the Section 2.2.

Consider the case $\tilde{f}_i \in \mathscr{K}_\infty,\ i = 1,\dots,n$, in particular $\tilde{f}_i$ are unbounded. We choose $V_i(x_i) = |x_i| = x_i$ as an ISS-Lyapunov-Razumikhin function candidate for the i-th entity. Obviously, $V_i(x_i)$ satisfies the condition (10). To prove, that the condition (11) holds true we choose the functions γ^d_{ij} and γ^u_i as γ_{ij}, γ_i in (15), where $\gamma^d_{ii} \equiv 0$ because there is no time-delay in the internal dynamic (see the term $\tilde{c}_{ii}(t)\tilde{f}_i(x_i(t))$ in the model). The difference to (15) is, that the time-delay is taken into account in the gains. From the condition (11) we have

$$V_i(x_i) \geq \max\{\max_j \gamma^d_{ij}(||V^d_j(x_j)||), \gamma^u_i(|u|)\},$$

where $V^d_j(x_j(t)) = V_j(x_j(t-\tau_{ij}))$ and $||V^d_j(x_j)|| = \max_{t-\tau_{ij}\leq s\leq t} \left|V_j(x_j(s))\right|$. This means $\gamma^d_{ij}(||V^d_j(x_j)||) \geq \gamma_{ij}(V_j(x_j))$ and furthermore for $\tau_{ij} > \tilde{\tau}_{ij} \ \Rightarrow\ \gamma^d_{ij}(|V_j(x_j(t-\tau_{ij}))|) \geq \gamma_{ij}(|V_j(x_j(t-\tilde{\tau}_{ij}))|)$.

From the definition of the gains we get by application of the Theorem 2 the following proposition by similar calculations as for the stability analysis based on ODEs.

Proposition 3. *Consider a network as in* (14).

1. *Assume that the c_{ij} are bounded for all $i, j = 1,\dots,n,\ i \neq j$. If $\exists a \in \mathbb{R}^n,\ \varepsilon \in \mathbb{R}^n,\ a_i > 0,\ \varepsilon_i < 0,\ i = 1,\dots,n$ such that the condition $C(t)a < \varepsilon$ holds $\forall t > 0$, then the whole network is ISS.*
2. *Define $\|u\|_\infty := (\|u_1\|_\infty,\dots,\|u_n\|_\infty)^T$. Let $\tilde{f}_j \in \mathscr{K}\setminus\mathscr{K}_\infty$, and $\alpha_j := sup_{x_j}\{\tilde{f}_j(x_j)\}$, $j = 1,\dots n,\ \alpha := (\alpha_1,\dots,\alpha_n)^T$. If $\exists \varepsilon \in \mathbb{R}^n,\ \varepsilon_i < 0,\ i = 1,\dots,n$ such that*

$$C(t)\alpha + \|u\|_\infty < \varepsilon, \tag{20}$$

   then the whole network is LISS.

This results are applied to a certain scenario of a production network in the following section.

4 Example of a certain scenario of a production network

System without time-delays

We consider a certain scenario of a production network without transportation times as in Figure 2. There, the numbers of the nodes are given in the centers of the corresponding circles. The first entity gets some raw material from an external supplier, denoted by u. At each entity the material will be processed with the rates $c_{ii}\tilde{f}_i = c_{ii}q_i\tilde{f},\ q_i \geq 0$ and immediately sent to the entities according to the network topology in Figure 2 with certain distribution coefficients c_{ij}. One half of the production of entity four will be sent to customers, not considered in the network. The distribution coefficients are given by

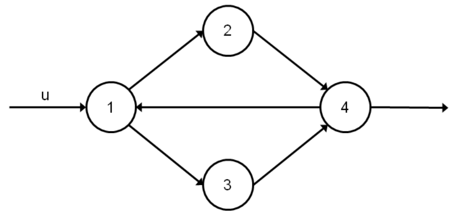

Fig. 2 Example of a scenario of a production network

$$C(t) = \begin{pmatrix} -2 & 0 & 0 & 0.5 \\ c_{21}(t) & -1.5 & 0 & 0 \\ c_{31}(t) & 0 & -2 & 0 \\ 0 & 1 & 1 & -2.5 \end{pmatrix}, \tag{21}$$

where we implement the queue length method by choosing

$$c_{21}(t) := \frac{\frac{c_{22}q_2}{x_2(t)+\varepsilon}}{\frac{c_{22}q_2}{x_2(t)+\varepsilon}+\frac{c_{33}q_3}{x_3(t)+\varepsilon}},\ c_{31}(t) := \frac{\frac{c_{33}q_3}{x_3(t)+\varepsilon}}{\frac{c_{22}q_2}{x_2(t)+\varepsilon}+\frac{c_{33}q_3}{x_3(t)+\varepsilon}}.$$

The term $\varepsilon > 0$ assures that the $c_{ij}(t)$ are well-defined and for simplicity one can choose $\varepsilon = 0$. Note, that $c_{21}(t) + c_{31}(t) = 1$.

To analyse whether the network has the ISS property we only have to check the condition (18), which can be easily verified with $a_i = 1,\ i = 1,\dots,4$. By Proposition 1 the whole network is ISS.

The gains are of the form

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\frac{1}{1+\delta_j}\tilde{f}_j(s)\right) = \tilde{f}^{-1}\left(\frac{q_i}{q_j}\frac{1}{1+\delta_j}\tilde{f}(s)\right),\ s \in \mathbb{R}_+,$$

where $\delta_j > 0$. For example, if we choose $\tilde{f}(s) = \sqrt{s}$ and $q_i = 1,\ i = 1,\dots,4$, then we have

$$\gamma_{ij}(s) = \frac{1}{(1+\delta_j)^2}s,\ s \in \mathbb{R}_+.$$

The differential equations that describe the systems behavior are of the form

$$\begin{aligned}
\dot{x}_1(t) &= u(t) + \tfrac{1}{2}\sqrt{x_4(t)} - 2\sqrt{x_1(t)},\\
\dot{x}_2(t) &= \frac{\frac{1.5}{x_2(t)}}{\frac{1.5}{x_2(t)}+\frac{2}{x_3(t)}}\sqrt{x_1(t)} - 1.5\sqrt{x_2(t)},\\
\dot{x}_3(t) &= \frac{\frac{2}{x_3(t)}}{\frac{1.5}{x_2(t)}+\frac{2}{x_3(t)}}\sqrt{x_1(t)} - 2\sqrt{x_3(t)},\\
\dot{x}_4(t) &= \sqrt{x_2(t)} + \sqrt{x_3(t)} - 2.5\sqrt{x_4(t)}.
\end{aligned}$$

Let the initial state be given by $x(0) = (2,5,4,3)^T$ and the input function be $u = 10 \cdot (sin(t)+1)$. Then we get the stable behavior, displayed in Figures 3 and 4, where a simulation is performed with Matlab.

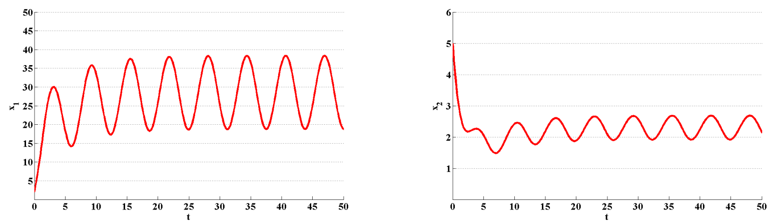

Fig. 3 Stable evolution of the amount of unprocessed parts within subsystems one and two

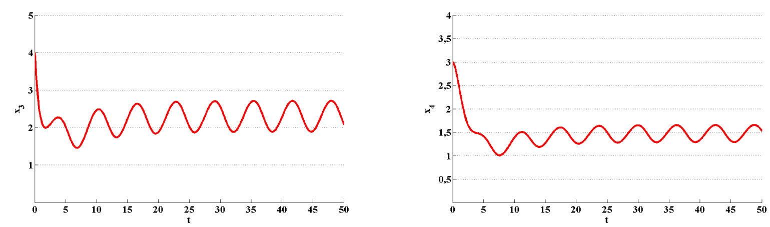

Fig. 4 Stable evolution of the amount of unprocessed parts within subsystems three and four

If the distribution coefficients are chosen as $c_{11} = -1$, $c_{22} = -1$, $c_{33} = -1$, $c_{44} = -1$, i.e., the condition (18) is not satisfied, then we get the following unstable behavior displayed in Figures 4 and 4. It means that the number of unprocessed parts within a subsystem increases up to infinity.

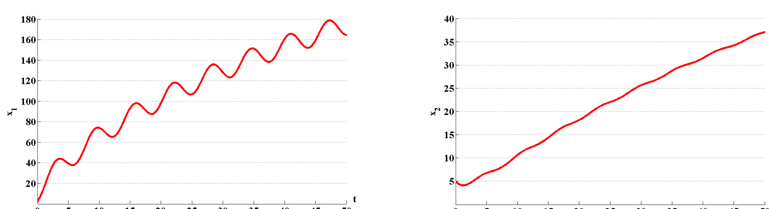

Fig. 5 Unstable evolution of the amount of unprocessed parts within subsystems one and two

System with time-delays

Now we consider the same scenario of a production network as in Figure 2, but with transportation times. The distribution coefficients c_{ij} for the stable situation are

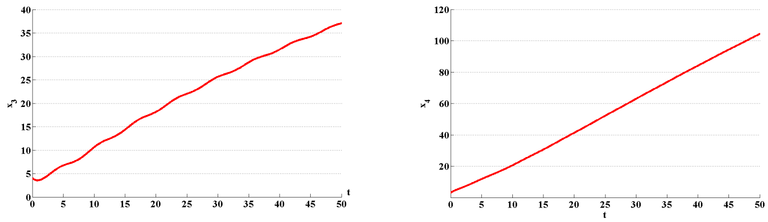

Fig. 6 Unstable evolution of the amount of unprocessed parts within subsystems three and four

given by (21) with c_{21} and c_{31} which represent the queue length method and take into account time-delays:

$$c_{21}(t) := \frac{\frac{c_{22}q_2}{x_2(t-\tau_{21})+\varepsilon}}{\frac{c_{22}q_2}{x_2(t-\tau_{21})+\varepsilon}+\frac{c_{33}q_3}{x_3(t-\tau_{21})+\varepsilon}},\ c_{31}(t) := \frac{\frac{c_{33}q_3}{x_3(t-\tau_{31})+\varepsilon}}{\frac{c_{22}q_2}{x_2(t-\tau_{31})+\varepsilon}+\frac{c_{33}q_3}{x_3(t-\tau_{31})+\varepsilon}}.$$

Now we choose $\tilde{f}_i(s) = q_i\sqrt{s}$ with $q_1 = 3,\ q_2 = 2,\ q_3 = 1.5,\ q_4 = 1.6$. The condition (18) is satisfied, which can be easily checked and therefore the network has the ISS property. The retarded differential equations of the system are of the form

$$\begin{aligned}
\dot{x}_1(t) &= u(t) + \tfrac{1.6}{2}\sqrt{x_4(t-\tau_{14})} - 6\sqrt{x_1(t)},\\
\dot{x}_2(t) &= \frac{\frac{3}{x_2(t-\tau_{21})}}{\frac{3}{x_2(t-\tau_{21})}+\frac{3}{x_3(t-\tau_{21})}} 3\sqrt{x_1(t-\tau_{21})} - 3\sqrt{x_2(t)},\\
\dot{x}_3(t) &= \frac{\frac{3}{x_3(t-\tau_{31})}}{\frac{3}{x_2(t-\tau_{31})}+\frac{3}{x_3(t-\tau_{31})}} 3\sqrt{x_1(t-\tau_{31})} - 3\sqrt{x_3(t)},\\
\dot{x}_4(t) &= 2\sqrt{x_2(t-\tau_{42})} + 1.5\sqrt{x_3(t-\tau_{43})} - 4\sqrt{x_4(t)}.
\end{aligned}$$

We choose $\tau_{ij} = 2$ and the initial function $x(s) \equiv (2,5,4,3)^T,\ s \in [-2,0]$. The input function is given by the constant function $u \equiv 20$ in contrast to the oscillating input used before. Then we get the stable behavior, displayed in Figure 7. Although we choose a constant input we observe an oscillating behavior of the number of unprocessed parts of the subsystems. The reason is the implemented queue length method in the terms $c_{21}(t)$ and $c_{31}(t)$: Here only the number of unprocessed parts at the time $t-\tau_{21}$ or $t-\tau_{31}$ is used for the calculation of the distribution coefficients $c_{i1}(t),\ i = 2,3$. The number of unprocessed parts, which has been sent during the time $(t-\tau_{i1}, 0]$ and has not yet been arrived at subsystem two or three, is not taken into account. Then, it happens that more parts are sent to a subsystem with larger queue than to the other subsystem until the distribution coefficients of both subsystems, depending on the number of unprocessed parts at time $t-\tau_{i1}$, are equal. After this point the proportionally higher number of sent parts arrive at the subsystem, which increases continuously the queue length and leads to a smaller distribution coefficient c_{i1} in contrast to the distribution coefficient of the other subsystem. Now

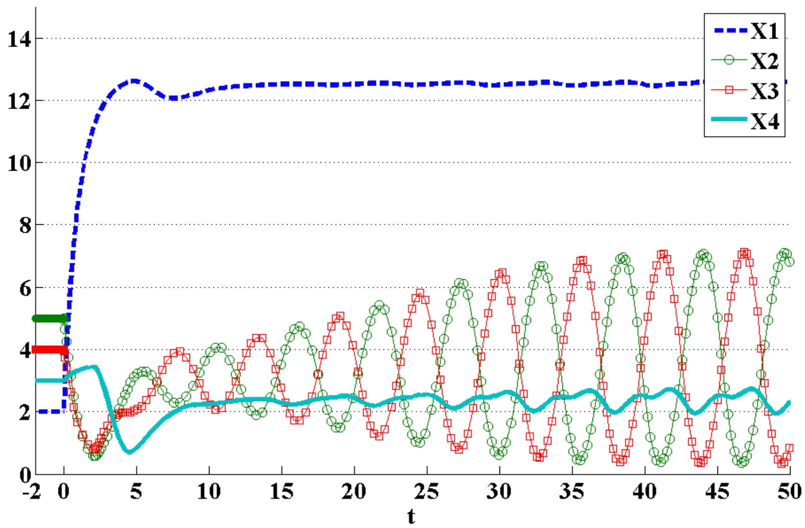

Fig. 7 Stable Evolution with the time-delays $\tau_{ij} = 2$

the procedure goes on in the opposite direction until the distribution coefficients are equal again. This cycle repeats and causes the observed oscillating behavior.

Now we increase the time-delays by choosing $\tau_{ij} = 4$ and the initial function $x(s) \equiv (2,5,4,3)^T,\ s \in [-4,0]$. Furthermore, we choose $\varepsilon = 0.001$ to assure that the distribution coefficients c_{1i} are well-defined. All other parameters are the same. Then we get the behavior of the number of unprocessed parts of the subsystems displayed in Figure 8. The increased time-delays $\tau_{ij} = 4$ cause higher amplitudes, i.e., larger maximal values of the number of unprocessed parts of a subsystem in contrast to the time-delays $\tau_{ij} = 2$ used in Figure 7. Furthermore, we observe as a result of this increased oscillations that for some time intervals the number of unprocessed parts of subsystem two and three equals or is close to zero, which means that the entities do not produce parts in these time intervals. In the conclusions we provide some ideas to avoid such negative outcomes.

5 Summary

5.1 Conclusions

We have modeled and investigated general production network in view of stability with and without transportation times. Two modelling methods were presented: modelling by differential equations with and without time-delays. They were used

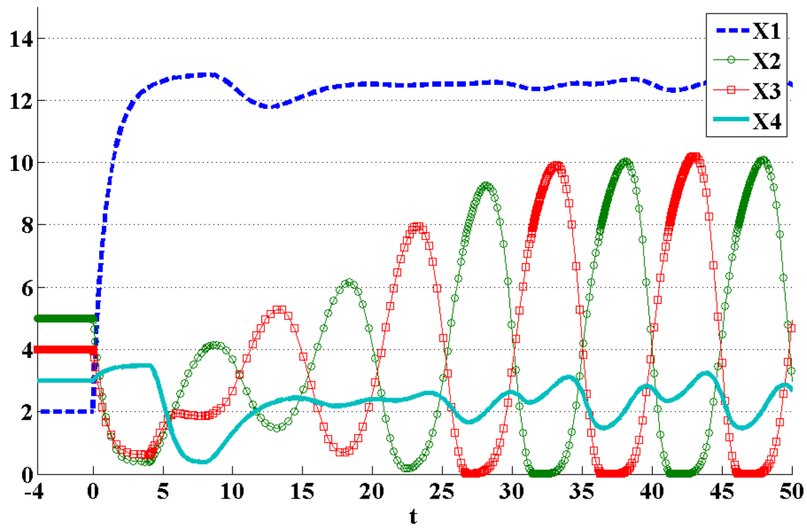

Fig. 8 Stable Evolution with the time-delays $\tau_{ij} = 4$

to model general production networks, where an autonomous control method, the queue length method, was implemented. Based on these models we have presented tools to perform a stability analysis using (L)ISS-Lyapunov or (L)ISS-Lyapunov-Razumikhin functions. By the application to our models we have performed a stability analysis for both approaches, where we have derived a condition which guarantee that a network possesses the (L)ISS property. This result was applied to a scenario of a production network with and without transportation times. Here we have found out that the maximum number of unprocessed parts of a subsystem with time-delays can be higher than of a subsystem without time-delays. Furthermore we have observed an oscillating behavior of the number of unprocessed parts of a subsystem with time-delays, which was caused by the modeled queue length method. The larger the time-delay is the higher is this oscillating behavior and could cause downtimes of the production.

5.2 Future work

The choices of the parameters c_{ij} for the modelling of the queue length method can be changed: the number of parts which are on the way to a subsystem, but not yet arrive there, can be taken into account. This means that full information access of the market entities of a network is necessary, which is not always available. This problem should be analysed. Another way of modelling the queue length method can be done by using switched systems [15]. For such modelling method the tools to perform a stability analysis for general networks have to be developed. One can extend the modelling of production networks by taking into account state jumps, e.g., loading and unloading processes, one can use hybrid or impulsive systems with and without time-delays [7]. Then the developed dwell-time condition plays a significant role and should be investigated in more detail.

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