Non-linear Tikhonov Regularization in Banach Spaces for Inverse Electromagnetic Scattering from Anisotropic Penetrable Non-Magnetic Media

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November 23, 2016

Abstract

We consider Tikhonov and sparsity-promoting regularization in Banach spaces for inverse electromagnetic scattering from penetrable linear inhomogeneous anisotropic non-magnetic media, which are free of sources. For that purpose we work with material parameters of an admissible set, equipped with the L^∞-topology. Further we use H^1-estimates for solutions of Maxwell's equations to analyze the dependence of scattered fields and their derivatives on the material parameter. Therewith we show convergence of a non-linear Tikhonov regularization against a minimum-norm solution to the inverse problem, and extend that method to a sparsity-promoting version.

1 Introduction

Although the field of inverse problems was treated with skepticism in the beginning, its significance was quickly realized during World War II, such that inverse scattering problems have become the most popular and well-studied amongst ill-posed problems (for a survey introduction see e.g. [11]). The first applications arises in the inventions of RADAR and SONAR, trying to determine the distance of an object by the use of acoustic and electromagnetic waves. However, whereas determining the location of a target is kind of straight-forward, the problem of identification is way more difficult to handle, since the solution of the modeled problem does not depend continuously on the measured data. Such problems are called to be ill-posed. Thus, it was not until Tikhonov and his Western pendant Miller established their mathematical theories, that prepared the ground for the invention of synthetic aperture radar (SAR), marking the first successful application in object identification using electromagnetic waves [10]. Subsequently theoretical and numerical improvements consolidated the position of inverse scattering in daily routine, such that nowadays the theory of inverse scattering is for example an inherent part of medical detection devices as in electrical impedance tomography (EIT) or ultrasonics.

However, detection and identification of objects using inverse scattering theory is relatively new in the field of non-destructive material testing, but has already proven itself as highly promising approach. Therein, testing the intactness of a machine-made component is about detection of non-obvious cracks, bumps or the like. Therefore one can legitimately assume that the object, that is to say the crack, is very small compared to its surrounding media, in fact the manufactured item. Mathematically the contrast of such a scatterer is described by few non-zero coefficients for a chosen basis and is commonly called sparse. Since a-priori informations are reasonably used, like in using

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ultrasound to image the human body, the idea is now to improve detection methods in material testing by regarding the assumption of sparse contrasts.

But, although methods involving sparsity constraints are widely accepted in image processing, they are barely used in the theory of inverse scattering. Thus inspired, [15] presented so called Tikhonov and soft-shrinkage regularization methods for non-linear inverse medium scattering problems with sparsity-promoting penalty terms. The therein shown analysis rely on the scattering of time-harmonic acoustic waves from inhomogeneous media described by scalar valued refractive indices. Their results were then extended to anisotropic penetrable media with matrix valued material parameter in [16]. Further herein, the formerly restriction to Hilbert spaces as image spaces of the contrast-to-measurement operator, seen in [15], was generalized to Banach spaces.

Whereas both previous works deal with acoustic waves, we now adapt the techniques of [16] to electromagnetic scattering for penetrable linear inhomogeneous non-magnetic anisotropic media (see aluminum-copper alloys as an example). Therefore we remind that in general the propagation of time-harmonic electromagnetic waves in three dimensions is governed by Maxwell's equations for the electric and magnetic fields E and H. Given a circular frequency $\omega > 0$ and a medium with electric permittivity ε, magnetic permeability μ, and conductivity σ, linear and time-harmonic electromagnetic waves are governed by the differential equations

$$\begin{aligned} \operatorname{curl} E - \mathrm{i}\omega\mu H &= 0\,, \\ \operatorname{curl} H + \mathrm{i}\omega\varepsilon E &= \sigma E \end{aligned} \quad \text{in } \mathbb{R}^3. \tag{1}$$

We assume the tangential components of E and H to be continuous on interfaces, where σ, ε and μ are discontinuous. (Instead, the normal components might jump across the material boundary.) Denoting the constant background permittivity and permeability by ε_0 and μ_0, we introduce the anisotropic relative permittivity ε_{r} and relative permeability μ_{r}

$$\varepsilon_{\mathrm{r}}(x) = \frac{\varepsilon(x)}{\varepsilon_0} + \mathrm{i}\frac{\sigma(x)}{\omega\varepsilon_0}, \quad \mu_{\mathrm{r}}(x) = \frac{\mu(x)}{\mu_0}.$$

In the following we assume that $\varepsilon \equiv \varepsilon_0$, $\mu \equiv \mu_0$, and $\sigma \equiv 0$ outside some bounded domain. Further the scattered fields satisfy the Silver-Müller radiation condition

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty, \text{ uniformly with respect to } \hat{x} := \frac{x}{|x|} \in \mathbb{S}^2.$$

As mentioned above, we intend to handle the important case of non-magnetic media, that is the magnetic permeability μ is constant and equal to the permeability μ_0 of vacuum such that $\mu_{\mathrm{r}} \equiv 1$. An example for that case can be seen during the solidification of an aluminum-copper alloy with a non-magnetic $\mathrm{Al_2CU}$ phase. Herein strong magnetic fields are used to control the crystal growth and the solid-liquid interface morphology to align the phase and therefore avoiding the embrittlement of the material. Hence we will work from now on with the magnetic field H only, which is divergence free in case of non-magnetic media. Thus the system (1) can be reduced to the second-order Maxwell system

$$\operatorname{curl}\left(\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl} H\right) - k^2 H = 0 \quad \text{in } \mathbb{R}^3 \tag{2}$$

for the positive wave number $k := \omega\sqrt{\varepsilon_0\mu_0} \in \mathbb{C} \setminus \{0\}$, such that $\operatorname{Re} k \geq 0$ and $\operatorname{Im} k \geq 0$. Accordingly, the electric field is determined by $E = \mathrm{i} \operatorname{curl} H/(\omega\varepsilon_0\varepsilon_{\mathrm{r}})$.

First of all in the ongoing part we generalize the scattering problem in more details. Section 3 is then devoted to the construction of a solution operator to the according scattering problem, mapping material parameters to scattered fields. Therefor we will establish H^1-regularity estimates, provided by [21], which are used to show continuity of the solution operator. Note that the chosen setting will yield convergence results for Tikhonov regularization, which seems to be impossible when one works in $H(\mathrm{curl})$. Following [16], we derive in Section 4 some results of differentiability for the solution operator, which we extend in Section 5 to a parameter-to-far field mapping, called the forward operator. Finally, Section 6 states sparsity promoting Tikhonov regularization results in wavelet bases and also for functions of bounded variation.

Notation: By $\mathbb{S}^2 = \{x \in \mathbb{R}^3,\ |x| = 1\}$ we denote the unit sphere in $\mathbb{R}^3$ and $B_R(x)$ is the ball of radius R about $x \in \mathbb{R}^3$. For any bounded Lipschitz domain $B \subset \mathbb{R}^3$ we denote the Sobolev space $W^{1,2}(B, \mathbb{C}^3) = H^1(B, \mathbb{C}^3)$. Hence, we define the Hilbert space $H(\mathrm{curl}, B) := \{v \in L^2(B, \mathbb{C}^3),\ \mathrm{curl}\, v \in L^2(B, \mathbb{C}^3)\}$, with inner product $(v, w)_{H(\mathrm{curl},B)} := (v, w)_{L^2(B)} + (\mathrm{curl}\, v, \mathrm{curl}\, w)_{L^2(B)}$. The closure of $C_0^\infty(B, \mathbb{C}^3)$ in the norm of $H(\mathrm{curl}, B)$ is named $H_0(\mathrm{curl}, B) = \{v \in H(\mathrm{curl}, B),\ \nu \times v = 0 \text{ on } \partial B\}$. Further,

$$H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3) := \left\{v\colon \mathbb{R}^3 \to \mathbb{C}^3,\ v|_B \in H(\mathrm{curl}, B) \text{ for all balls } B \subset \mathbb{R}^3\right\}$$

and $H_t^{-1/2}(\partial B) := \{v \in H^{-1/2}(\partial B, \mathbb{C}^3),\ v \cdot \nu = 0 \text{ a.e. on } \partial B\}$ in which ν denotes the unit outward normal to B. Therewith one defines the trace space of $H(\mathrm{curl}, B)$ with respect to the trace $v \mapsto \nu \times v$,

$$H^{-1/2}(\mathrm{Div}, \partial B) := \left\{v \in H_t^{-1/2}(\partial B),\ \nabla_{\partial B} \cdot v \in H^{-1/2}(\partial B)\right\},$$

where $\nabla_{\partial B}\cdot$ denotes the surface divergence. Its dual space is given by $H^{-1/2}(\mathrm{Curl}, \partial B) := \{v \in H_t^{-1/2}(\partial B),\ \nabla_{\partial B} \times v \in H^{-1/2}(\partial B)\}$, within use of the surface scalar curl $\nabla_{\partial B}\times$ (for details see, e.g,. [18, Section 3.4]). By abuse of notation, a duality pairing between the trace space of $H(\mathrm{curl}, B)$ and its dual (see, e.g., [18, Section 3.5.3], [6]) will for simplicity always be written as a boundary integral over ∂B. Analogously we have $H(\mathrm{div}, B) := \{v \in L^2(B, \mathbb{C}^3),\ \mathrm{div}\, v \in L^2(B, \mathbb{C}^3)\}$ with inner product $(v, w)_{H(\mathrm{div},B)} := (v, w)_{L^2(B)} + (\mathrm{div}\, v, \mathrm{div}\, w)_{L^2(B)}$. To improve readability, we use a generic constant C in our estimates, maybe changing its value from one occurrence to the other.

2 Scattering from non-magnetic media

We consider for now the time-harmonic Maxwell's equations to model scattering of an incident electromagnetic wave from a non-magnetic medium modeled by space-dependent relative electric permittivity ε_{r}. As the material parameter $\varepsilon_{\mathrm{r}} \in L^\infty(D, \mathrm{Sym}(3))$ take values in the complex-valued symmetric 3×3 matrices $\mathrm{Sym}(3) \subset \mathbb{C}^{3\times 3}$, its real part correlates physically to the electric permittivity, whereas the imaginary part is proportional to the electric conductivity σ. We assume that there exists a positive constant $\lambda > 0$ such that $\lambda|\xi|^2 \leq \mathrm{Re}(\overline{\xi}^\top \varepsilon_{\mathrm{r}}\ \xi)$ for all $\xi \in \mathbb{C}^3$ and for almost all x on the bounded Lipschitz domain $D \subset \mathbb{R}^3$ with connected complement $\mathbb{R}^3 \setminus \overline{D}$. Since in particular we have that also $\varepsilon_{\mathrm{r}}^{-1} \in L^\infty(D, \mathrm{Sym}(3))$, we suppose that the closure of D equals the support of $\mathrm{I}_3 - \varepsilon_{\mathrm{r}}^{-1}$ and, moreover, that the imaginary part of $\varepsilon_{\mathrm{r}}^{-1}$ is bounded from above, that is to say $\mathrm{Im}(\overline{\xi}^\top \varepsilon_{\mathrm{r}}^{-1} \xi) \leq 0$ for $\xi \in \mathbb{C}^3$. To generalize notation we thus abbreviate the material parameter as an element $\rho := \varepsilon_{\mathrm{r}}^{-1}$ of the bounded subset $\mathcal{P}$ of $L^\infty(D, \mathrm{Sym}(3))$, which is equipped with the L^∞-topology and defined for a $\lambda > 0$ as

$$\mathcal{P} = \left\{\rho \in L^\infty(D, \mathrm{Sym}(3)),\ \lambda|\xi|^2 \leq \mathrm{Re}(\overline{\xi}^\top \rho^{-1} \xi),\ \mathrm{Im}(\overline{\xi}^\top \rho\ \xi) \leq 0, \text{ a.e. in } D \text{ and for all } \xi \in \mathbb{C}^3\right\}.$$

Remember that we have already derived in the introduction that the total magnetic field solves

$$\mathrm{curl}\,(\rho\, \mathrm{curl}\, H) - k^2 H = 0 \quad \text{in } \mathbb{R}^3. \tag{3}$$

On interfaces where ρ^{-1} is discontinuous, the tangential components of the magnetic field H and of $\rho\, \mathrm{curl}\, H$ are continuous across the interface. In particular, if ρ^{-1} is discontinuous across ∂D, then

$$\nu \times [H]_{\partial D} = 0 \quad \text{and} \quad \nu \times [\rho\, \mathrm{curl}\, H]_{\partial D} = 0, \tag{4}$$

where $[\cdot]_{\partial D}$ denotes the jump of a function across ∂D. Assume that a time-harmonic incident plane wave

$$H^i(x,d;p) := p\,\mathrm{e}^{\mathrm{i}kx\cdot d}, \quad x\in\mathbb{R}^3, \quad \text{where } d\in\mathbb{S}^2,\ p\in\mathbb{C}^3,\ \text{and } p\cdot d=0,$$

with direction d and polarization p propagates through the inhomogeneity D. Due to the different material parameters inside D there arises a scattered wave H^s, solving

$$\operatorname{curl}(\rho\operatorname{curl}H^s) - k^2H^s = \operatorname{curl}\big((\mathrm{I}_3-\rho)\operatorname{curl}H^i\big) \quad \text{in } \mathbb{R}^3. \tag{5}$$

Since H^i solves $\operatorname{curl}^2 H^i - k^2H^i = 0$ in $\mathbb{R}^3$, the total field $H = H^i + H^s$ is still a solution to (3). Furthermore H^s is radiating, i.e. it satisfies the Silver-Müller radiation condition

$$\operatorname{curl}H^s(x)\times\hat{x} - \mathrm{i}kH^s(x) = \mathcal{O}\left(|x|^{-2}\right) \quad \text{as } |x|\to\infty, \text{ uniformly with respect to } \hat{x} := \frac{x}{|x|}\in\mathbb{S}^2, \tag{6}$$

and therefore has the asymptotic behavior

$$H^s(x) = \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}H^\infty(\hat{x},d;p) + \mathcal{O}\left(|x|^{-2}\right), \quad \text{as } |x|\to\infty,$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$. Here H^∞ is called the far field pattern of H^s, which (see, e.g. [11, Theorem 6.9]) is an analytic and tangential vector field on the unit sphere, i.e.,

$$H^\infty(\hat{x},d;p)\cdot\hat{x} = 0 \quad \text{for all } \hat{x}\in\mathbb{S}^2 \text{ and all } d\in\mathbb{S}^2 \text{ and } p\in\mathbb{C}^3 \text{ with } p\cdot d=0.$$

In particular, H^∞ belongs to the space of square-integrable tangential vector fields

$$L^2_t(\mathbb{S}^2) := \left\{g\in L^2(\mathbb{S}^2,\mathbb{C}^3),\ g(\hat{x})\cdot\hat{x} = 0 \text{ for a.e. } \hat{x}\in\mathbb{S}^2\right\} \subset L^2(\mathbb{S}^2,\mathbb{C}^3).$$

The far field patterns H^∞ define the far field operator $F:\ L^2_t(\mathbb{S}^2)\to L^2_t(\mathbb{S}^2)$ by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} H^\infty(\hat{x},d;g(d))\,\mathrm{d}S(d) \quad \text{for } \hat{x}\in\mathbb{S}^2, \tag{7}$$

which is linear since H^∞ depends linearly on p, i.e. $H^\infty(\hat{x},d;p) = \hat{H}^\infty(\hat{x},d)p$ for all $p\in\mathbb{C}^3$ with $p\cdot d = 0$ and $\hat{H}^\infty(\hat{x},d)\in\mathbb{C}^{3\times3}$. Due to reciprocity relations, H^∞ is moreover a smooth function in both variables $\hat{x}$ and d which implies that F is a compact operator on $L^2_t(\mathbb{S}^2)$. Note that Fg with $g\in L^2_t(\mathbb{S}^2)$, is the far field pattern of the magnetic field corresponding to an incident Herglotz wave function

$$v_g(x) = \int_{\mathbb{S}^2} H^i(x,d;g(d))\,\mathrm{d}S(d) = \int_{\mathbb{S}^2} \mathrm{e}^{\mathrm{i}kx\cdot d}g(d)\,\mathrm{d}S(d), \quad x\in\mathbb{R}^3, \quad \text{in } H(\operatorname{curl},B_R). \tag{8}$$

Regarding a generalization of source terms $f\in C^\infty(D,\mathbb{C}^3)$ on the right of (5), we seek weak radiating solutions $v\in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3)$ to

$$\operatorname{curl}(\rho\operatorname{curl}v) - k^2v = \operatorname{curl}((\mathrm{I}_3-\rho)\,f) \quad \text{in } \mathbb{R}^3, \tag{9}$$

$$\nu\times v|_- = \nu\times v|_+\,, \quad \nu\times\rho\operatorname{curl}v|_- - \nu\times\operatorname{curl}v|_+ = \nu\times(\mathrm{I}_3-\rho)\,f \quad \text{on } \partial D. \tag{10}$$

Note that we always implicitly use natural, homogeneous transmission conditions (10) on ∂D in the rest of this paper and that setting $f = \operatorname{curl}H^i$ yields the original problem (5). The weak radiating solution $v\in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3)$ thus needs to satisfy

$$\int_{\mathbb{R}^3}\left[\rho\operatorname{curl}v\cdot\operatorname{curl}\overline{\psi} - k^2v\cdot\overline{\psi}\right]\mathrm{d}x = \int_{\mathbb{R}^3}(\mathrm{I}_3-\rho)\,f\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x \tag{11}$$

for all $\psi\in H(\operatorname{curl},\mathbb{R}^3)$ with compact support.

Remark 1. (a) Choosing $\psi = \nabla\varphi$ to be a gradient field, the equation $\operatorname{curl}\nabla\varphi = 0$ implies that $\int_{\mathbb{R}^3} v \cdot \nabla\overline{\varphi}\,\mathrm{d}x = 0$ for all $\varphi \in H^1(\mathbb{R}^3)$ with compact support, i.e. $\operatorname{div} v = 0$ in $\mathbb{R}^3$. Thus, the solution v is divergence free.

(b) The Silver-Müller radiation condition is well-defined for any weak solution v to (11): Outside D the solution v solves $\operatorname{curl}^2 v - k^2 v = 0$ together with $\operatorname{div} v = 0$; thus, the identity $\Delta = \nabla \operatorname{div} - \operatorname{curl}^2$ implies that $\Delta v + k^2 v = 0$ and elliptic regularity results imply that v is a smooth function in $\mathbb{R}^3 \setminus \overline{D}$.

3 The solution operator

Now we transform the weak formulation (11) into a variational equation on a bounded domain. Therefore we denote by B_R a ball, containing the support $\overline{D}$ of $\mathrm{I}_3 - \rho$ in its interior and the tangential trace mapping $\gamma_t \colon H(\operatorname{curl}, B_R) \to H^{-1/2}(\operatorname{Div}, \partial B_R)$ by $\gamma_t(u) = \nu \times u|_{\partial B_R}$ for the outward unit normal vector $\nu = \nu(x)$ at $x \in \partial B_R$. Further the "dual" tangential trace $\gamma_T \colon H(\operatorname{curl}, B_R) \to H^{-1/2}(\operatorname{Curl}, \partial B_R)$ is given by $\gamma_T(u) = (\nu \times u)|_{\partial B_R} \times \nu$, see [18, Theorem 3.31] or [6] for a generalization to Lipschitz domains. If $v \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ solves (11), then v solves also

$$\int_{B_R} \left[\rho \operatorname{curl} v \cdot \operatorname{curl}\overline{\psi} - k^2 v \cdot \overline{\psi}\right] \mathrm{d}x + \int_{\partial B_R} \gamma_t(\operatorname{curl} v) \cdot \gamma_T(\overline{\psi})\ \mathrm{d}S = \int_D (\mathrm{I}_3 - \rho)\, f \cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x, \tag{12}$$

for all test functions $\psi \in H(\operatorname{curl}, B_R)$ with compact support included in B_R, since $\rho \equiv 1$ on ∂B_R. Regarding the transmission conditions (10) and the relation between the magnetic and electric fields, see below (2), we denote the exterior magnetic-to-electric Calderon operator by

$$\Lambda \colon H^{-1/2}(\operatorname{Div}, \partial B_R) \to H^{-1/2}(\operatorname{Div}, \partial B_R),$$

mapping $\varphi \in H^{-1/2}(\operatorname{Div}, \partial B_R)$ into $\left.\left(\nu \times \frac{\mathrm{i}}{\omega\varepsilon_0} \operatorname{curl} u\right)\right|_{\partial B_R}$, where u satisfies

$$\operatorname{curl}^2 u - k^2 u = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_R}, \quad \gamma_t(u) = \nu \times u = -\mathrm{i}\omega\varepsilon_0\varphi \text{ on } \partial B_R$$

and the Silver-Müller radiation condition (6). Therewith we can rewrite (12) as

$$\int_{B_R} \left[\rho \operatorname{curl} v \cdot \operatorname{curl}\overline{\psi} - k^2 v \cdot \overline{\psi}\right] \mathrm{d}x + \int_{\partial B_R} \Lambda(\gamma_t(v)) \cdot \gamma_T(\overline{\psi})\ \mathrm{d}S = \int_D (\mathrm{I}_3 - \rho)\, f \cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x \tag{13}$$

for all $\psi \in H(\operatorname{curl}, B_R)$. (We omit the trace operators γ_t and γ_T from now on if a tangential restriction to the boundary is obvious.)

Remark 2. If $v \in H(\operatorname{curl}, B_R)$ solves (13) then v can be extended into the exterior of B_R, for simplicity denoted by v again, such that the extension v solves (11).

We now define a sesquilinear form for $\rho \in \mathcal{P}$ and for all φ, $\psi \in H(\operatorname{curl}, B_R)$ by

$$a_\rho(\varphi, \psi) := \int_{B_R} \left[\rho \operatorname{curl}\varphi \cdot \operatorname{curl}\overline{\psi} - k^2 \varphi \cdot \overline{\psi}\right] \mathrm{d}x + \int_{\partial B_R} \Lambda(\nu \times \varphi) \cdot \overline{\psi}\,\mathrm{d}S,$$

and the solution operator $\mathrm{L} \colon \mathcal{P} \times H(\operatorname{curl}, B_R) \to H(\operatorname{curl}, B_R)$, which maps material parameters ρ and incident fields u^i to the solution of the variational problem

$$a_\rho(\mathrm{L}(\rho, u^i), \psi) = \int_D (\mathrm{I}_3 - \rho)\ \operatorname{curl} u^i \cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x \quad \text{for all } \psi \in H(\operatorname{curl}, B_R). \tag{14}$$

Thus, $\mathrm{L}(\rho, u^i) = v$ is still the weak solution to the variational formulation (13) for $f = \operatorname{curl} u^i$ and the radiating extension of v to $\mathbb{R}^3$ (see Remark 2) weakly solves

$$\operatorname{curl}(\rho \operatorname{curl} v) - k^2 v = \operatorname{curl}\left((\mathrm{I}_3 - \rho)\ \operatorname{curl} u^i\right) \quad \text{in } \mathbb{R}^3.$$

Using either this variational formulation in involving the exterior Calderon operator [18] or a volume integral approach [14], it is possible to show that the underlying problem (11) can be reduced to a Fredholm problem of index zero (see, e.g. [14, Lemma 2.4]), i.e. uniqueness implies existence of solution (see, e.g. [14, Theorem 2.5], [18, Theorem 10.2]):

Lemma 3. *The scattering problem* (9), (10), *and* (6) *satisfies the Fredholm alternative, i.e. there exists a unique radiating solution* $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ *of* (11) *for every* $f \in L^2(D, \mathbb{C}^3)$*, provided uniqueness holds for all* $\rho \in \mathcal{P}$*. If uniqueness holds, then there exists a constant* $C > 0$ *(depending on* B_R, k, ρ *only) such that*

$$\|v\|_{H(\mathrm{curl},B_R)} \leq C \|(\mathrm{I}_3 - \rho) f\|_{L^2(D,\mathbb{C}^3)} \tag{15}$$

for the right-hand side of (22)*. Further, the restriction* $v|_D$ *is the unique solution of* (22) *in* $H(\mathrm{curl}, B_R)$*.*

Assumption 4. *We assume in the following that for the connected, convex set* $\mathcal{P}$ *any solution to* (11) *for* $f \in L^2(D, \mathbb{C}^3)$ *is unique, such that existence and continuous dependence of this solution follow from uniqueness. For example, in the case of dielectric media (i.e.* $\sigma \equiv 0$*), this assumption is always satisfied if* $\rho \in \mathcal{P}$ *is globally Hölder continuous and differentiable, except at one point of Coulomb-type singularity, since, under this smoothness assumption, unique continuation results for Maxwell's equations are applicable, see [19],[23].*

Thus the solution operator $\mathrm{L}(\rho, \cdot)$ exists for all $\rho \in \mathcal{P}$, together with a constant $C = C(\mathcal{P})$ such that $\| \mathrm{L}(\rho, u^i)\|_{H(\mathrm{curl},B_R)} \leq C \|u^i\|_{H(\mathrm{curl},D)}$.

To handle derivatives of L in L^p-spaces, we use an H^1-estimate stated by [21]. Since we will work with a couple of solutions to equations with slightly different right-hand sides, we state the following result for a broader range of coefficients A on the right-hand side.

Theorem 5. *Let* $\rho \in \mathcal{P}$*,* $f \in L^2(D, \mathbb{C}^3)$ *and the support* $\overline{D}$ *of* $A \in L^\infty(B_R, \mathrm{Sym}(3))$ *be a subset of the ball* B_R*. If* v *in* $H(\mathrm{curl}, B_R)$ *is a weak solution of*

$$\mathrm{curl}\,(\rho\, \mathrm{curl}\, v) - k^2 v = \mathrm{curl}\,(A\, f) \quad \text{in } \mathbb{R}^3, \tag{16}$$

then $v \in H^1(B_R, \mathbb{C}^3)$ *and there holds that*

$$\|v\|_{H^1(B_R,\mathbb{C}^3)} \leq C\, \|Af\|_{L^2(B_R,\mathbb{C}^3)}, \tag{17}$$

for some constant C *depending on* B_R*,* k*, and* ρ *only.*

Proof. To apply results of [21], we have to ensure that the solution is part of an appropriate function space. Therefore we choose a cut-off function $\chi \in C_c^\infty(\mathbb{R}^3)$, such that $\chi \equiv 1$ in $B_R \supseteq \overline{D} = \mathrm{supp}\, A$ and vanishes outside of the convex domain $B_{2R} \supseteq B_R$. Then the function $w = \chi v$ satisfies $\nu \times w = 0$ on ∂B_{2R}, such that $w \in H_0(\mathrm{curl}, B_{2R})$. We further have $\mathrm{div}\, w \in L^2(B_{2R}, \mathbb{C}^3)$–as we will see in the next lines–and thus $w \in H_0(\mathrm{curl}, B_{2R}) \cap H(\mathrm{div}, B_{2R})$. Due to Theorem 4.2 of [21], the field w then satisfies

$$\|w\|_{H^1(B_{2R},\mathbb{C}^3)} \leq C \left(\|\mathrm{curl}\, w\|_{L^2(B_{2R},\mathbb{C}^3)} + \|\mathrm{div}\, w\|_{L^2(B_{2R},\mathbb{C}^3)} \right).$$

Considering that $w = \chi v$, we can rewrite the norms on the right-hand side by applying the product rules of the rotation, respectively of the divergence, as

$$\begin{aligned} \|\mathrm{curl}\, w\|_{L^2(B_{2R},\mathbb{C}^3)} &= \|\chi\, \mathrm{curl}\, v + \nabla\chi \times v\|_{L^2(B_{2R},\mathbb{C}^3)} < \infty, \\ \|\mathrm{div}\, w\|_{L^2(B_{2R},\mathbb{C}^3)} &= \|\chi\, \mathrm{div}\, v + \nabla\chi \cdot v\|_{L^2(B_{2R},\mathbb{C}^3)} < \infty. \end{aligned}$$

Further, for two vectors a and b, the identities

$$|a \times b| = |a||b| \sin \sphericalangle(a, b) \quad \text{and} \quad a \cdot b = |a||b| \cos \sphericalangle(a, b),$$

where the absolute values of sin and cos are bounded by one, provide some estimates for the cross product and dot product respectively. Therewith, after applying triangle-inequality, we gain that

$$\|\chi \operatorname{curl} v + \nabla\chi \times v\|_{L^2(B_{2R},\mathbb{C}^3)} \leq \|\chi\|_{L^\infty(B_{2R})} \|\operatorname{curl} v\|_{L^2(B_{2R},\mathbb{C}^3)} + \|\chi\|_{C^1(B_{2R})} \|v\|_{L^2(B_{2R},\mathbb{C}^3)},$$
$$\|\chi \operatorname{div} v + \nabla\chi \cdot v\|_{L^2(B_{2R},\mathbb{C}^3)} \leq \|\chi\|_{C^1(B_{2R})} \|v\|_{L^2(B_{2R},\mathbb{C}^3)} \quad \text{(respecting that } \operatorname{div} v = 0).$$

So far, we have shown that $\|w\|_{H^1(B_{2R},\mathbb{C}^3)} \leq C(\chi)\|v\|_{H(\operatorname{curl},B_{2R})}$. Using Lemma 3 and bearing in mind that $\operatorname{supp} A$ is strictly contained in B_R only, we have that

$$\|w\|_{H^1(B_{2R},\mathbb{C}^3)} \leq C(\chi,\rho)\|Af\|_{L^2(B_R,\mathbb{C}^3)}.$$

Finally, regarding that the H^1-norm of $w = \chi v$ over B_{2R} is bounded from below by the H^1-norm of v over B_R, finishes the proof. □

Corollary 6. *Let $\rho \in \mathcal{P}$ and $\operatorname{supp}(\mathrm{I}_3 - \rho) = \overline{D} \subset B_R \subset \mathbb{R}^3$. If v in $H(\operatorname{curl}, B_R)$ is a weak solution of (9), then $v \in H^1(B_R, \mathbb{C}^3)$ and there holds that*

$$\|v\|_{H^1(B_R,\mathbb{C}^3)} \leq C \,\|(\mathrm{I}_3 - \rho)\, f\|_{L^2(B_R,\mathbb{C}^3)}, \tag{18}$$

for some constant C depending on B_R, k, and ρ only.

Remark. Since the tangential trace of v is in $H^{-1/2}(\operatorname{Div}, \partial B_R)$, v is also an element of $H^{-1/2}(\partial B_R)$, which can be characterized as the completion of $L^2(\partial B_R)$ [17, p.98], and further satisfies $v \cdot \nu = 0$ a.e. on ∂B_R. We thus have that $\nu \times v \in L^2_t(\partial B_R)$, such that v is a function of

$$W_N = \left\{u \in H(\operatorname{curl}, B_R) \cap H(\operatorname{div}, B_R),\ \operatorname{div} u = 0 \text{ in } B_R,\ \nu \times v \in L^2_t(\partial B_R)\right\},$$

which is compactly embedded in $L^2(B_R, \mathbb{C}^3)$ [18, Corollary 3.49]. Therefore one could show Theorem 5 alternatively via Riesz-Fredholm theory.

At the end of this section we show that L is Lipschitz continuous:

Theorem 7. *Let Assumption 4 hold and $\rho' \in L^\infty(B_R, \mathrm{Sym}(3))$ be a small perturbation of $\rho \in \mathcal{P}$, such that $\rho + \rho' \in \mathcal{P}$, then*

$$\|\mathrm{L}(\rho + \rho', u^i) - \mathrm{L}(\rho, u^i)\|_{H^1(B_R,\mathbb{C}^3)} \leq C\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\operatorname{curl},B_R)},$$

where $C > 0$ depends on B_R, k and ρ, but is independent of ρ' and u^i.

Proof. For a fixed incident field u^i we set $v_{\rho+\rho'} = \mathrm{L}(\rho + \rho', u^i)$ and $v = \mathrm{L}(\rho, u^i)$ and denote the radiating extensions (see Remark 2) of these functions to $\mathbb{R}^3$ again by $v_{\rho+\rho'}$, v and the corresponding total fields by $u_{\rho+\rho'} = u^i + v_{\rho+\rho'}$ and $u = u^i + v$. The difference $v_{\rho+\rho'} - v = u_{\rho+\rho'} - u$ is the weak, radiating solution to

$$\operatorname{curl}\left(\rho \operatorname{curl}(u_{\rho+\rho'} - u)\right) - k^2(u_{\rho+\rho'} - u) = -\operatorname{curl}\left(\rho' \operatorname{curl} u_{\rho+\rho'}\right) \quad \text{in } \mathbb{R}^3.$$

Now applying Theorem 5, yields

$$\|u_{\rho+\rho'} - u\|_{H^1(B_R,\mathbb{C}^3)} \leq C(\rho)\,\|\rho' \operatorname{curl} u_{\rho+\rho'}\|_{L^2(B_R,\mathbb{C}^3)} \leq C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u_{\rho+\rho'}\|_{H(\operatorname{curl},B_R)}.$$

By triangle-inequality $\|u_{\rho+\rho'}\| \leq \|u^i\| + \|v_{\rho+\rho'}\|$ in $H(\operatorname{curl})$-norms, we get rid of the total field, where due to Assumption 4

$$\|v_{\rho+\rho'}\|_{H(\operatorname{curl},B_R)} \leq C(B_R,\rho)\|u^i\|_{H(\operatorname{curl},B_R)}.$$

Note that the underlying inequality originally gives an upper bound in D, but we simply increased the norm by enlarging the domain. This will be done implicitly during further estimates.

Putting this altogether yields the stated inequality. □

4 Differentiability of the solution operator

To have a glance at the differentiability of the solution operator, we fix the incident field and the parameter $\rho \in \mathcal{P}$ in this section, such that the solution operator $\mathrm{L}(\rho, \cdot)$ is bounded on $H(\mathrm{curl}, B_R)$. Further we introduce the function $v' \in H(\mathrm{curl}, B_R)$ by

$$a_\rho(v', \psi) = -\int_D \theta \,\mathrm{curl}\left(\mathrm{L}(\rho, u^i) + u^i\right) \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x \quad \text{for all } \psi \in H(\mathrm{curl}, B_R), \tag{19}$$

for which we will show in Theorem 10 that v' is indeed the derivative $\mathrm{L}'(\rho, u^i)[\theta]$ of L with respect to $\rho \in \mathcal{P}$ in direction $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$.

Lemma 8. *For every $\rho \in \mathcal{P}$ the mapping $\theta \mapsto \mathrm{L}'(\rho, u^i)[\theta]$ in $\mathcal{L}(L^\infty(B_R, \mathrm{Sym}(3)), H(\mathrm{curl}, B_R))$ has the following continuity property:*

$$\|\,\mathrm{L}'(\rho, u^i)[\theta]\|_{H^1(B_R,\mathbb{C}^3)} \le C\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\mathrm{curl},B_R)},$$

where $C > 0$ depends on B_R, k and ρ only.

Proof. In the following we denote by $u = \mathrm{L}(\rho, u^i) + u^i$ the total field, such that due to (19) applying the H^1-estimate of Theorem 5 gains

$$\begin{aligned}\|\,\mathrm{L}'(\rho, u^i)[\theta]\|_{H^1(B_R,\mathbb{C}^3)} &\le C(\rho)\,\|\theta\,\mathrm{curl}\,u\|_{L^2(B_R,\mathbb{C}^3)}\\ &\le C\,\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))}\left(\|u^i\|_{H(\mathrm{curl},B_R)} + \|\,\mathrm{L}(\rho, u^i)\|_{H(\mathrm{curl},B_R)}\right),\end{aligned}$$

where for the last step we estimated the L^2-norm by $H(\mathrm{curl})$-norm and separated the total field by triangle inequality. Herein Lemma 3 states that

$$\|\,\mathrm{L}(\rho, u^i)\|_{H(\mathrm{curl},B_R)} \le C\,\|\,\mathrm{I}_3 - \rho\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\,\mathrm{curl}\,u^i\|_{L^2(B_R,\mathbb{C}^3)} \le C\,\|u^i\|_{H(\mathrm{curl},B_R)}, \tag{20}$$

due to the boundedness of the L^∞-term. □

Theorem 9. *Under the assumptions of Theorem 7, the map $\rho \mapsto \mathrm{L}'(\rho, u^i)$ is locally Lipschitz continuous: There is a $C > 0$ independent of ρ' and u^i such that for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$ there holds*

$$\|\,\mathrm{L}'(\rho + \rho', u^i)[\theta] - \mathrm{L}'(\rho, u^i)[\theta]\|_{H^1(B_R,\mathbb{C}^3)} \le C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\mathrm{curl},B_R)}.$$

where $C > 0$ depends on B_R, k and ρ only.

Proof. For $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$, the functions $\mathrm{L}'(\rho + \rho', u^i)[\theta]$ and $\mathrm{L}'(\rho, u^i)[\theta]$ satisfy by (19) the variational formulations

$$\begin{aligned}a_{\rho+\rho'}(\mathrm{L}'(\rho + \rho', u^i)[\theta], \psi) &= -\int_D \theta\,\mathrm{curl}\,u_{\rho+\rho'} \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x,\\ a_\rho(\mathrm{L}'(\rho, u^i)[\theta], \psi) &= -\int_D \theta\,\mathrm{curl}\,u \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x,\end{aligned}$$

where the perturbed total field $u_{\rho+\rho'}$ consists of the perturbed scattered field $\mathrm{L}(\rho + \rho', u^i)$ and the incident field u^i, whereas $u = \mathrm{L}(\rho, u^i) + u^i$. Thus, $w := \mathrm{L}'(\rho + \rho', u^i)[\theta] - \mathrm{L}'(\rho, u^i)[\theta]$ satisfies

$$a_\rho(w, \psi) = -\int_D \theta\,\mathrm{curl}(\mathrm{L}(\rho + \rho', u^i) - \mathrm{L}(\rho, u^i)) \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x - \int_D \rho'\,\mathrm{curl}\,\mathrm{L}'(\rho + \rho', u^i)[\theta] \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x.$$

Therefore Theorem 5 states

$$\begin{aligned}\|w\|_{H^1(B_R,\mathbb{C}^3)} &\le C\Big(\|\theta \operatorname{curl}(\mathrm{L}(\rho+\rho',u^i)-\mathrm{L}(\rho,u^i))\|_{L^2(B_R,\mathbb{C}^3)} + \|\rho' \operatorname{curl}\mathrm{L}'(\rho+\rho',u^i)[\theta]\|_{L^2(B_R,\mathbb{C}^3)}\Big)\\ &\le C\Big(\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\mathrm{L}(\rho+\rho',u^i)-\mathrm{L}(\rho,u^i)\|_{H(\mathrm{curl},B_R)}\\ &\quad+\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\mathrm{L}'(\rho+\rho',u^i)[\theta]\|_{H(\mathrm{curl},B_R)}\Big).\end{aligned}$$

Here, the first $H(\mathrm{curl})$-norm is bounded by a constant times $\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\mathrm{curl},B_R)}$ due to Theorem 7 and the last one by a constant times $\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\mathrm{curl},B_R)}$ due to Lemma 8. □

Theorem 10. *Let Assumption 4 hold, then the solution operator* L *is differentiable in the sense that for every small perturbation $\rho' \in L^\infty(B_R,\mathrm{Sym}(3))$ of $\rho\in\mathcal{P}$ such that $\rho+\rho'\in\mathcal{P}$, it holds that*

$$\|\mathrm{L}(\rho+\rho',u^i)-\mathrm{L}(\rho,u^i)-\mathrm{L}'(\rho,u^i)[\rho']\|_{H^1(B_R,\mathbb{C}^3)} \le C\,\|\rho'\|^2_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\mathrm{curl},B_R)},$$

where $C>0$ depends on B_R, k and ρ only. Thus, if $\{\rho'_n\}_{n\in\mathbb{N}}\subset L^\infty(B_R,\mathrm{Sym}(3))$ such that $\rho+\rho'_n\in\mathcal{P}$ for all $n\in\mathbb{N}$ as well as $\|\rho'_n\|_{L^\infty(B_R,\mathrm{Sym}(3))}\to 0$ as $n\to\infty$, then

$$\frac{\|\mathrm{L}(\rho+\rho'_n,u^i)-\mathrm{L}(\rho,u^i)-\mathrm{L}'(\rho,u^i)[\rho'_n]\|_{H^1(B_R,\mathbb{C}^3)}}{\|\rho'_n\|_{L^\infty(B_R,\mathrm{Sym}(3))}}\to 0 \quad \text{as } n\to\infty.$$

Proof. For $w := \mathrm{L}(\rho+\rho',u^i)-\mathrm{L}(\rho,u^i)-\mathrm{L}'(\rho,u^i)[\rho']$ we first consider the variational formulations defining all three terms,

$$\begin{aligned} a_{\rho+\rho'}(\mathrm{L}(\rho+\rho',u^i),\psi) &= \int_D (\mathrm{I}_3-\rho-\rho')\operatorname{curl}u^i\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x,\\ a_\rho(\mathrm{L}(\rho,u^i),\psi) &= \int_D(\mathrm{I}_3-\rho)\operatorname{curl}u^i\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x,\\ a_\rho(\mathrm{L}'(\rho,u^i)[\rho'],\psi) &= -\int_D\rho'\operatorname{curl}(\mathrm{L}(\rho,u^i)+u^i)\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x,\end{aligned}$$

for all $\psi\in H(\mathrm{curl},B_R)$. Thus, for all $\psi\in H(\mathrm{curl},B_R)$ there holds

$$\begin{aligned} a_{\rho+\rho'}(w,\psi) &= a_{\rho+\rho'}(\mathrm{L}(\rho+\rho',u^i),\psi)-a_{\rho+\rho'}(\mathrm{L}(\rho,u^i)\psi)-a_{\rho+\rho'}(\mathrm{L}'(\rho,u^i)[\rho'],\psi)\\ &= a_{\rho+\rho'}(\mathrm{L}(\rho+\rho',u^i),\psi)-a_\rho(\mathrm{L}(\rho,u^i)\psi)-a_\rho(\mathrm{L}'(\rho,u^i)[\rho'],\psi)\\ &\quad-\int_{B_R}\rho'\operatorname{curl}\mathrm{L}(\rho,u^i)\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x-\int_{B_R}\rho'\operatorname{curl}\mathrm{L}'(\rho,u^i)[\rho']\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x\\ &= \int_D(\mathrm{I}_3-\rho-\rho')\operatorname{curl}u^i\cdot\operatorname{curl}\overline{\psi}\ \mathrm{d}x+\int_D\rho'\operatorname{curl}(\mathrm{L}(\rho,u^i)+u^i)\cdot\operatorname{curl}\overline{\psi}\ \mathrm{d}x\\ &\quad-\int_D(\mathrm{I}_3-\rho)\operatorname{curl}u^i\cdot\operatorname{curl}\overline{\psi}\ \mathrm{d}x-\int_D\rho'\operatorname{curl}\mathrm{L}(\rho,u^i)\cdot\operatorname{curl}\overline{\psi}\ \mathrm{d}x\\ &\quad-\int_D\rho'\operatorname{curl}\mathrm{L}'(\rho,u^i)[\rho']\cdot\operatorname{curl}\overline{\psi}\ \mathrm{d}x\\ &= -\int_D\rho'\operatorname{curl}\mathrm{L}'(\rho,u^i)[\rho']\cdot\operatorname{curl}\overline{\psi}\ \mathrm{d}x. \end{aligned}\tag{21}$$

Now the H^1-estimate of Theorem 5 implies that

$$\|w\|_{H^1(B_R,\mathbb{C}^3)}\le C(\rho)\,\|\rho'\operatorname{curl}\mathrm{L}'(\rho,u^i)[\rho']\|_{L^2(B_R,\mathbb{C}^3)}\le C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\mathrm{L}'(\rho,u^i)[\rho']\|_{H(\mathrm{curl},B_R)}.$$

Due to the equation (21), we gain by Lemma 3 that

$$\| \mathrm{L}'(\rho, u^i)[\rho'] \|_{H(\mathrm{curl}, B_R)} \le C \, \|\rho' \, \mathrm{curl}\, u\|_{L^2(B_R,\mathbb{C}^3)} \le C \, \|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))} \|u\|_{H(\mathrm{curl}, B_R)},$$

for the total field $u = \mathrm{L}(\rho, u^i) + u^i$. After separation into incident and scattered fields, again applying Lemma 3 like in (20) finally results in the stated estimate. □

5 The forward operator

In this section we define the so called forward operator which maps material parameters to their corresponding far field operators, such that the forward operator corresponds to the inverse scattering problem we are actually interested in.

Therefore we follow the volume integral approach mentioned in Section 3, by which one can show, see [14, Theorem 2.3], that the scattering problem (9), (10), and (6) is equivalent to an integro-differential equation defined via the radiating fundamental solution to the Helmholtz equation:

$$\Phi_k(x) = \frac{1}{4\pi|x|} \, \mathrm{e}^{\mathrm{i}k|x|}, \quad x \neq 0.$$

More precisely, $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ is a radiating solution to (11) iff v solves

$$v = \mathrm{curl} \int_{B_R} \Phi_k(\cdot - y)(\mathrm{I}_3 - \rho)(y) \, \mathrm{curl}\left[v(y) + u^i(y)\right] \mathrm{d}y \quad \text{in } H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3). \tag{22}$$

Analogously the radiating extension of the function $v' = \mathrm{L}'(\rho, u^i)[\theta]$ to $\mathbb{R}^3$ satisfies

$$v' = \mathrm{curl} \int_{B_R} \Phi_k(\cdot - y) \left[(\mathrm{I}_3 - \rho) \, \mathrm{curl}\, v'(y) - \theta \, \mathrm{curl}(\mathrm{L}(\rho, u^i) + u^i)(y)\right] \mathrm{d}y \tag{23}$$

in $H(\mathrm{curl}, B_R)$, because v' solves, by definition, the variational formulation (19). Now for a direction $\hat{x} \in \mathbb{S}^2$, the far field pattern of $v^\infty(\hat{x})$ hence equals

$$\begin{aligned} v^\infty(\hat{x}) &= \left(\mathrm{curl} \int_{B_R} \Phi_k(\cdot - y)(\mathrm{I}_3 - \rho)(y) \, \mathrm{curl}\left[v(y) + u^i(y)\right] \mathrm{d}y\right)^\infty (\hat{x}) \\ &= \int_{B_R} [\mathrm{curl}\, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}](\mathrm{I}_3 - \rho)(y) \, \mathrm{curl}\left[v(y) + u^i(y)\right] \mathrm{d}y \\ &= \mathrm{i}k\hat{x} \times \int_{B_R} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}(\mathrm{I}_3 - \rho)(y) \, \mathrm{curl}\left[v(y) + u^i(y)\right] \mathrm{d}y, \quad \hat{x} \in \mathbb{S}^2. \end{aligned} \tag{24}$$

This shows that the far field v^∞ is an analytic function, since the latter integral expression is analytic in $\hat{x}$. To keep notation simple, we introduce the integral operator

$$V \colon L^2(B_R, \mathbb{C}^3) \to H^2(B_R, \mathbb{C}^3), \quad Vf = \int_{B_R} \Phi_k(\cdot - y) f(y) \, \mathrm{d}y.$$

(See [11] for the mapping properties of V.) The scattered field restricted to B_R satisfies

$$v = \{\mathrm{I}_3 - \mathrm{curl}\, V\left((\mathrm{I}_3 - \rho) \, \mathrm{curl}(\cdot)\right)\}^{-1} \left[\mathrm{curl}\, V((\mathrm{I}_3 - \rho) \, \mathrm{curl}\, u^i)\right],$$

such that the total field $v + u^i$ equals $S_\rho u^i$, in particular

$$S_\rho(u^i) := [\mathrm{I}_3 - \mathrm{curl}\, V\left((\mathrm{I}_3 - \rho) \, \mathrm{curl}(\cdot)\right)]^{-1}(u^i) = v + u^i.$$

Thus we represent the far field pattern $v^\infty = \mathrm{L}(\rho, u^i)^\infty$, computed for direction $\hat{x} \in \mathbb{S}^2$ in (24), as

$$v^\infty(\hat{x}) = \mathrm{i}k \int_{B_R} \hat{x} \times (\mathrm{I}_3 - \rho)(y) \operatorname{curl}(S_\rho u^i)(y) \, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}y.$$

If we further introduce the integral operator

$$Z \colon L^r(B_R, \mathbb{C}^3) \to L^2_t(\mathbb{S}^2), \quad f \mapsto \mathrm{i}k \int_{B_R} \hat{x} \times f(y) \, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}y, \tag{25}$$

then there holds that

$$\mathrm{L}(\rho, u^i)^\infty = Z \circ [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho(u^i)].$$

Using smoothing properties of Z, which is a trace class operator (see [13]), the following lemma from [16, Lemma 5.1] shows that the composition on the right is well-defined and bounded, since $\mathrm{I}_3 - \rho \in L^\infty(B_R, \mathrm{Sym}(3))$ and $\operatorname{curl} S_\rho u^i \in L^2(B_R, \mathbb{C}^3)$.

Lemma 11. *Choose $m \in \mathbb{N}$, $1 < r < \infty$, and $f \in L^r(B_R, \mathbb{C}^3)$.*

(i) There is $C = C(m, r)$ such that $\|Zf\|_{C^m(\mathbb{S}^2)} \leq C(m, r)\|f\|_{L^r(B_R,\mathbb{C}^3)}$.

(ii) The operator Z is of trace class from $L^r(B_R, \mathbb{C}^3)$ into $L^2_t(\mathbb{S}^2)$.

Now we are able to introduce the forward operator, which maps material parameters to associated far field operators. As mentioned in Section 2, from now on we assume to have Herglotz wave functions v_g for $g \in L^2_t(\mathbb{S}^2)$, see (8), as incident fields. Thus note that $g \mapsto v_g|_{B_R}$ is a bounded mapping from $L^2_t(\mathbb{S}^2)$ into $H^1(B_R)$, see [11]. For the incident field v_g a far field operator $F = F_\rho \colon L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ defines by $Fg = (\mathrm{L}(\rho, v_g))^\infty$ for $g \in L^2_t(\mathbb{S}^2)$. We mention that F_ρ is compact, since the integral kernel $u^\infty = u^\infty_\rho \colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{C}$ of $\mathrm{F}(\rho)$ is analytic in both variables. Because of the summability of its singular values $s_j(F_\rho)$, i.e., $\|F_\rho\|_{\mathcal{S}_1} = \sum_{j\in\mathbb{N}} |s_j(F_\rho)| < \infty$, it even belongs to the set $\mathcal{S}_1$ of trace class operators on $L^2_t(\mathbb{S}^2)$. The embedding $\ell^p \subset \ell^q$ for $1 \leq p < q \leq \infty$ of the sequence spaces ℓ^p further implies that trace class operators belong to the qth Schatten class $\mathcal{S}_q$ for all $q \in [1, \infty)$, a Banach space of all compact operators on $L^2_t(\mathbb{S}^2)$ with q-summable singular values $s_j(F)$, equipped with the norm defined by

$$\|F\|^q_{\mathcal{S}_q} = \sum_{j\in\mathbb{N}} |s_j(F)|^q, \quad \text{for } q \geq 1.$$

Therewith the contrast-to-far field mapping defines as an operator from $\mathcal{P}$ into the qth Schatten class $\mathcal{S}_q$:

$$\mathrm{F}(\cdot)g \colon \mathcal{P} \to \mathcal{S}_q, \quad \mathrm{F}(\rho)g = Z \circ [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho(v_g)] \quad \text{for } g \in L^2_t(\mathbb{S}^2),\ q \geq 1, \tag{26}$$

Remark 12.

(i) Due to Lemma 11 with $r = 2$ and the continuity properties of the solution operator L, the composition $Z \circ [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho(v_g)]$ is well-defined in $L^2_t(\mathbb{S}^2)$.

(ii) As trace class operators form an ideal in the space of all bounded operators, and as $\mathrm{F}(\rho)g = Z((\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho(v_g))$ with a trace class operators Z, the forward operator is a trace class operator as well, and hence belongs to all spaces $\mathcal{S}_q$ for $q \geq 1$.

(iii) An alternative to the $\mathcal{S}_q$-norms are L^q-norms for integral operators on the sphere: Since $\mathrm{F}(\rho)g = \int_{\mathbb{S}^2} u^\infty(\cdot, \theta) g(\theta) \, \mathrm{d}S(\theta)$ is represented by the far field pattern $u^\infty(\cdot, \theta)$ of the scattered fields $u^s = \mathrm{L}(\rho, v_g)$, the L^q-Norm of u^∞ defines an operator norm for $\mathrm{F}(\rho)$ by $\|\mathrm{F}(\rho)\|_q := \|u^\infty\|_{L^q(\mathbb{S}^2\times\mathbb{S}^2)}$, $1 < q < \infty$. The contrast-to-far field map $\rho \mapsto \mathrm{F}(\rho)$ as defined in (26) is then continuous from

$L^{q'}(\mathbb{S}^2)$ into $L^q(\mathbb{S}^2)$ with $q' = q/(q-1)$, because $g \mapsto v_g|_D$ is continuous from $L^{q'}(\mathbb{S}^2)$ into $C^1(D)$ for all $q \in (1,\infty)$. For $q = 2$, it is well-known that $\|\cdot\|_{\mathcal{S}_2} = \|\cdot\|_2$. The advantage of the L^q-norms with respect to the implementation of inversion algorithms is that the computation of adjoints or subdifferentials is straightforward for these spaces. Since the subsequent theoretic results do not depend on the choice of the discrepancy norm, we continue to work with the Schatten norms $\|\cdot\|_{\mathcal{S}_q}$, noting that all results holds as well for the $\|\cdot\|_q$-norms.

Be aware that the far field of the radiating extension of $\mathrm{L}(\rho, v_g)$ depends boundedly and linearly on $\mathrm{L}(\rho, v_g)$. Thus, since $\mathrm{L}(\rho, v_g) = S_\rho(v_g) - v_g$, the derivative $\theta \mapsto \mathrm{F}'(\rho)[\theta] \in \mathcal{L}(L^\infty(B_R, \mathrm{Sym}(3)), \mathcal{S}_q)$ of F with respect to $\rho \in \mathcal{P}$ in direction $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$ equals, by the product rule in Banach spaces, see [24],

$$\mathrm{F}'(\rho)[\theta]g = Z \circ [(\mathrm{I}_3 - \rho)\,\mathrm{curl}(\mathrm{L}'(\rho, v_g)[\theta]) + \theta\,\mathrm{curl}(S_\rho(v_g))]. \tag{27}$$

Since the non-linear forward operator F is linked to the solution operator L, we are able to transfer the results of Theorem 7, 9, and 10 from L to F.

Corollary 13. *Let Assumption 4 hold and $\rho' \in L^\infty(B_R, \mathrm{Sym}(3))$ be a small perturbation of $\rho \in \mathcal{P}$, such that $\rho + \rho' \in \mathcal{P}$ and let $q \geq 1$.*

(i) There is a constant $C = C(\rho, B_R, k)$ such that

$$\|\mathrm{F}(\rho + \rho') - \mathrm{F}(\rho)\|_{\mathcal{S}_q} \leq C\|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))}. \tag{28}$$

(ii) The operator $\mathrm{F}'(\rho)$ is locally Lipschitz continuous with respect to $L^\infty(B_R, \mathrm{Sym}(3))$: There is $C = C(\rho, B_R, k)$ such that $\|\mathrm{F}'(\rho + \rho') - \mathrm{F}'(\rho)\|_{\mathcal{L}(L^\infty(B_R, \mathrm{Sym}(3)), \mathcal{S}_q)} \leq C\,\|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))}$.

(iii) The far field operator $\mathrm{F}(\rho)$ is differentiable in the sense that

$$\|\mathrm{F}(\rho + \rho') - \mathrm{F}(\rho) - \mathrm{F}'(\rho)[\rho']\|_{\mathcal{S}_q} \leq C\|\rho'\|^2_{L^\infty(B_R, \mathrm{Sym}(3))}$$

for a constant C depending on B_R, k and ρ. If $\{\rho'_n\}_{n\in\mathbb{N}} \subset L^\infty(B_R, \mathrm{Sym}(3))$ such that $\rho + \rho'_n \in \mathcal{P}$ for all $n \in \mathbb{N}$ as well as $\|\rho'_n\|_{L^\infty(B_R, \mathrm{Sym}(3))} \to 0$ as $n \to \infty$, then $\|\mathrm{F}(\rho + \rho') - \mathrm{F}(\rho) - \mathrm{F}'(\rho)[\rho']\|_{\mathcal{S}_q} / \|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))} \to 0$.

Proof. The basic ingredient of the proof is the smoothing property of the far field mapping Z defined in (25), which is a trace class operator from $L^2(B_R, \mathbb{C}^3)$ into $L^2_t(\mathbb{S}^2)$. Since the incident field u^i is chosen to be a Herglotz wave function v_g for some $g \in L^2(\mathbb{S}^2)$, we have

$$\begin{aligned}
\|\mathrm{F}(\rho + \rho') - \mathrm{F}(\rho)\|_{\mathcal{S}_q} &= \|g \mapsto Z\left[(\mathrm{I}_3 - (\rho + \rho'))\,\mathrm{curl}\,S_{\rho+\rho'}(v_g) - (\mathrm{I}_3 - \rho)\,\mathrm{curl}\,S_\rho(v_g)\right]\|_{\mathcal{S}_q} \\
&\leq C\,\|g \mapsto Z\left[(\mathrm{I}_3 - (\rho + \rho'))\,\mathrm{curl}\,S_{\rho+\rho'}(v_g) - (\mathrm{I}_3 - \rho)\,\mathrm{curl}\,S_\rho(v_g)\right]\|_{\mathcal{S}_1} \\
&\overset{(*)}{\leq} C\,\|g \mapsto \left[(\mathrm{I}_3 - (\rho + \rho'))\,\mathrm{curl}\,S_{\rho+\rho'}(v_g) - (\mathrm{I}_3 - \rho)\,\mathrm{curl}\,S_\rho(v_g)\right]\|_{\mathcal{L}(L^2_t(\mathbb{S}^2), L^2(B_R, \mathbb{C}^3))} \\
&\leq C \sup_{\|g\|_{L^2}=1} \Big[\|\rho'\,\mathrm{curl}\,S_{\rho+\rho'}(v_g)\|_{L^2(B_R, \mathbb{C}^3)} + \|(\mathrm{I}_3 - \rho)\,\mathrm{curl}[S_{\rho+\rho'}(v_g) - S_\rho(v_g)]\|_{L^2(B_R, \mathbb{C}^3)}\Big],
\end{aligned}$$

where inequality $(*)$ follows from Lemma 11. Now we obtain the bound

$$\begin{aligned}
\|\rho'\,\mathrm{curl}\,S_{\rho+\rho'}(v_g)\|_{L^2(B_R, \mathbb{C}^3)} &\leq \|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))}\|\,\mathrm{curl}\,S_{\rho+\rho'}(v_g)\|_{L^2(B_R, \mathbb{C}^3)} \\
&\leq \|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))}\|S_{\rho+\rho'}(v_g)\|_{H(\mathrm{curl}, B_R)},
\end{aligned}$$

where for the total field $\|S_{\rho+\rho'}(v_g)\|_{H(\mathrm{curl}, B_R)} \leq C\|v_g\|_{H(\mathrm{curl}, B_R)} \leq C\|g\|_{L^2(\mathbb{S}^2)} = C$ with a constant $C = C(\rho)$ independent of ρ', due to Assumption 4. The same technique yields

$$\|(\mathrm{I}_3 - \rho)\,\mathrm{curl}[S_{\rho+\rho'}(v_g) - S_\rho(v_g)]\|_{L^2(B_R, \mathbb{C}^3)} \leq \|\,\mathrm{I}_3 - \rho\|_{L^\infty(B_R, \mathrm{Sym}(3))}\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{H(\mathrm{curl}, B_R)}.$$

As $S_{\rho+\rho'}(v_g) - S_\rho(v_g) = \mathrm{L}(\rho+\rho', v_g) - \mathrm{L}(\rho, v_g)$, Theorem 7 further shows that

$$\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{H(\mathrm{curl},B_R)} \leq C\, \|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))} \|v_g\|_{H(\mathrm{curl},B_R)},$$

such that by plugging the last estimates together we deduce the statement. The bounds in (ii) and (iii) are shown analogously, using Theorems 9 and 10 instead of Theorem 7. □

6 Non-linear Tikhonov and sparsity regularization

We observe the stable approximation of ρ_{exa} from perturbed measurements of its far field operator $\mathrm{F}(\rho_{\text{exa}})$. This will be referred to as our inverse problem. In detail, we seek to approximate ρ by non-linear Tikhonov regularization for noisy measurements F^δ_{meas} with noise level $\delta > 0$ such that $\|\mathrm{F}(\rho_{\text{exa}}) - F^\delta_{\text{meas}}\|_{\mathcal{S}_q} \leq \delta$. Thus, for a convex regularization functional $\mathcal{R}$ we consider to minimize the Tikhonov functional

$$\mathcal{J}_{\alpha,\delta}(\rho) := \frac{1}{2}\|\mathrm{F}(\rho) - F^\delta_{\text{meas}}\|^2_{\mathcal{S}_q} + \alpha\mathcal{R}(\rho), \tag{29}$$

over some appropriate admissible parameter set included in $\mathcal{P}$. Note that under Assumption 4 the operator $\mathrm{F}(\rho)$ is well-defined.

Theorem 14 (Tikhonov regularization)**.** *If $\mathcal{D}(\mathrm{F})$ is a closed subset of a Banach space, equipped with the weak*-topology such that additionally $\mathcal{D}(\mathrm{F})$ is weak*-closed, and if the imagespace of F is also a Banach space for which any (norm-)bounded subset is weakly precompact, and if F is a (norm-norm)-continuous map, whose graph is (weak*,weak)-closed, then for any weak*-lower semicontinuous $\mathcal{R}$ with weak*-precompact level sets, such that $\mathcal{R}(\mathcal{D}(\mathrm{F})) \cap \mathbb{R} \neq \emptyset$, there exists a minimizer for the Tikhonov functional $\mathcal{J}_{\alpha,\delta}$, defined in* (29)*.*

If further $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that weak-converges to a solution $\rho^\dagger$ such that $\mathrm{F}(\rho^\dagger) = \mathrm{F}(\rho_{exa})$ holds in the imagespace of F and $\rho^\dagger$ minimizes $\mathcal{R}$.*

Proof. Combining Definition 5.2.1 and Theorems 5.2.2 to 5.2.4 of [20] yields a more general version of this theorem for a broader range of topologies. But due to [12, Chapter 5, Ex. 51], the weak*-topology is suitable. □

We thus apply this result to our setting by following the techniques of [20]. Therefore be aware that the domain of definition $\mathcal{P}$ of F, equipped with the weak*-topology, is a closed and bounded subset of the Banach space $L^\infty(B_R, \mathrm{Sym}(3))$. Alaoglu's theorem then states that closed balls are weak*-compact, in particular $\mathcal{P}$ is weak*-closed. Recalling (26), the forward operator F can be written as

$$\mathrm{F}(\cdot)g \colon \mathcal{P} \to \mathcal{S}_q, \quad \mathrm{F}(\rho)g = Z \circ [(\mathrm{I}_3 - \rho)\,\mathrm{curl}(\mathrm{L}(\rho, v_g) + v_g)] \in L^2_t(\mathbb{S}^2) \quad \text{for all } g \in L^2(\mathbb{S}^2),$$

for the Banach space $\mathcal{S}_q$ of qth Schatten-class operators, $q \geq 1$. Additionally, we like to quote the following statement from [20, Corollary 8.3.7], combining results for weak*-convergent sequences:

Lemma 15. *Let $\{f_n\} \in L^q(X)$ a weak*-convergent sequence, where (X, λ) is a finite measure space and $q \in (1, \infty]$. Then we find a subsequence $\{f_{n_k}\}$, converging in the L^r-norm for all $1 \leq r < q$.*

We now show the (weak*,weak)-closedness of the graph of F, i.e. F is sequentially closed from ($\mathcal{P}$,weak*) to $\mathcal{S}_q$ with its weak topology. Therefore one assumes to have a sequence of parameters $\{\rho_n\}_{n\in\mathbb{N}}$ from $\mathcal{P}$, such that $\rho_n \rightharpoonup^* \rho_0$ and $\mathrm{F}(\rho_n) \rightharpoonup F_{\rho_0}$ in $\mathcal{S}_q$ for $n \to \infty$. Then one has to show that this implies that $\mathrm{F}(\rho_0) = F_{\rho_0}$ (since $\rho_0 \in \mathcal{P}$ due to the weak*-closedness). Hence we define $v_n := \mathrm{L}(\rho_n, v_g)$, i.e. $\mathrm{F}(\rho_n)g = Z((\mathrm{I}_3 - \rho_n)\,\mathrm{curl}(v_n + v_g))$ for all $g \in L^2(\mathbb{S}^2)$.

Lemma 16. F *is sequentially closed from ($\mathcal{P}$,weak*) to $\mathcal{S}_q$ equipped with its weak topology.*

Proof. Since in Banach spaces weak*-convergent sequences are bounded (see e.g. [1]), the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is norm-bounded and due to Theorem 7 L is Lipschitz continuous. Hence the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in H^1-norm. (Alternatively, Theorem 5 states the same.) Thus there exists a subsequence $\{v_{n_m}\}_{m\in\mathbb{N}}$, weakly converging to a $v \in H^1(B_R, \mathbb{C}^3)$ by Alaoglu's theorem. Because of that, $\operatorname{curl} v_{n_m} \rightharpoonup \operatorname{curl} v$ in $L^2(B_R, \mathbb{C}^3)$ and due to the compact embedding of $H^1(B_R, \mathbb{C}^3)$ in $L^q(B_R, \mathbb{C}^3)$ for $1 \le q < 6$, we have also that $v_{n_m} \to v$ in $L^2(B_R, \mathbb{C}^3)$ for $m \to \infty$.

Further, by assumption, $\rho_n \rightharpoonup^* \rho_0$ in $L^\infty(B_R, \mathrm{Sym}(3))$, for the finite measure space (B_R, λ) (where λ denotes the Lebesgue-measure), such that Lemma 15 implies the existence of a subsequence $\{\rho_{n_m}\}_{m\in\mathbb{N}}$ which converges in $L^r(B_R, \mathrm{Sym}(3))$ for all $1 \le r < \infty$; hence $\rho_{n_m} \to \rho_0$ in $L^2(B_R, \mathrm{Sym}(3))$.

Therewith one can show that

$$\mathrm{i}k\hat{x} \times \int_{B_R} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\big[(\rho_0 - \rho_{n_m}) \operatorname{curl}(v_{n_m} - v)\big](y) \, \mathrm{d}y \to 0,$$

by rewriting the term as in (24):

$$\left| \int_{B_R} \big[\operatorname{curl} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\big] \big[(\rho_0 - \rho_{n_m}) \operatorname{curl}(v_{n_m} - v)\big](y) \, \mathrm{d}y \right| \\ \le \| \operatorname{curl} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\|_\infty \| \operatorname{curl}(v_{n_m} - v)\|_{L^2(B_R,\mathbb{C}^3)} \|\rho_0 - \rho_{n_m}\|_{L^2(B_R,\mathrm{Sym}(3))},$$

where the first norm is bounded as well as the second one, while the last norm vanishes as discussed above. Thus,

$$Z \circ [(\mathrm{I}_3 - \rho_{n_m}) \operatorname{curl}(v_{n_m} + v_g)] \to Z \circ [(\mathrm{I}_3 - \rho_0) \operatorname{curl}(v + v_g)] \quad \text{in } L^2_t(\mathbb{S}^2).$$

Since by assumption we have that $Z((\mathrm{I}_3 - \rho_n) \operatorname{curl}(v_n + v_g)) = \mathrm{F}(\rho_n)g \rightharpoonup F_{\rho_0}g$ in $L^2_t(\mathbb{S}^2)$ for all $g \in L^2(\mathbb{S}^2)$, we conclude that $F_{\rho_0}g = Z((\mathrm{I}_3 - \rho_0) \operatorname{curl}(v + v_g))$.

Now we show that $v = \mathrm{L}(\rho_0, v_g)$, because this implies $F_{\rho_0}g = Z((\mathrm{I}_3 - \rho_0) \operatorname{curl}(v + v_g)) = Z((\mathrm{I}_3 - \rho_0) \operatorname{curl}(\mathrm{L}(\rho_0, v_g) + v_g)) = \mathrm{F}(\rho_0)g$.
Therefor remember that, according to (14), v_{n_m} solves

$$a_{\rho_{n_m}}(v_{n_m}, \psi) = \int_D (\mathrm{I}_3 - \rho_{n_m}) \operatorname{curl} v_g \cdot \operatorname{curl} \overline{\psi} \, \mathrm{d}x \quad \text{for all } \psi \in C^1(\overline{B_R})$$

for

$$a_{\rho_{n_m}}(v_{n_m}, \psi) = \int_{B_R} [\rho_{n_m} \operatorname{curl} v_{n_m} \cdot \operatorname{curl} \overline{\psi} - k^2 v_{n_m} \overline{\psi}] \, \mathrm{d}x + \int_{\partial B_R} \Lambda(\nu \times v_{n_m}) \cdot \gamma_T(\overline{\psi}) \, \mathrm{d}S.$$

Note that the original equation was stated for test functions in $H(\mathrm{curl}, B_R)$, but since $C^1(\overline{B_R}, \mathbb{C}^3)$-functions are dense in $H(\mathrm{curl}, B_R)$, we switch to those test functions to profit from their boundedness in the maximum norm.

Now, instead of showing that $a_{\rho_{n_m}}(v_{n_m}, \psi) - a_{\rho_0}(v, \psi) \to 0$ we rewrite the difference into $a_{\rho_{n_m}}(v, \psi) - a_{\rho_0}(v, \psi) + a_{\rho_{n_m}}(v_{n_m} - v, \psi)$.

To show convergence of the difference of the first terms, note that both the boundary integrals and the integrals which do not contain any parameter ρ cancel themselves, such that we only have to have a glance at

$$\left| \int_{B_R} (\rho_{n_m} - \rho_0) \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} \, \mathrm{d}x \right| \le \|\rho_{n_m} - \rho_0\|_{L^2(B_R,\mathrm{Sym}(3))} \| \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi}\|_{L^2(B_R,\mathbb{C}^3)}.$$

As discussed above, we know that $\rho_{n_m} \to \rho_0$ in $L^2(B_R, \mathrm{Sym}(3))$. Since the other term is bounded, the integral tends to zero.

To show that

$$a_{\rho_{n_m}}(v_{n_m}-v,\psi) = \int_{B_R} [\rho_{n_m} \operatorname{curl}(v_{n_m}-v)\cdot\operatorname{curl}\overline{\psi} - k^2(v_{n_m}-v)\overline{\psi}]\ \mathrm{d}x + \int_{\partial B_R} \Lambda(\nu\times(v_{n_m}-v))\cdot\gamma_T(\overline{\psi})\ \mathrm{d}S$$

converges to zero, we first have a glance at the integral over B_R without material parameter. Recall that $v_{n_m} \to v$ in $L^2(B_R, \mathbb{C}^3)$ and thus

$$\left| k^2 \int_{B_R} (v_{n_m} - v)\overline{\psi}\ \mathrm{d}x \right| \le k^2 \|v_{n_m} - v\|_{L^2(B_R,\mathbb{C}^3)} \|\psi\|_{L^2(B_R,\mathbb{C}^3)} \to 0.$$

To show convergence of the integral containing the material parameter, recall that $\operatorname{curl} v_{n_m} \rightharpoonup \operatorname{curl} v$ in $L^2(B_R, \mathbb{C}^3)$ and by the same arguments as above we deduce again that $\rho_{n_m} \to \rho_0$ in $L^2(B_R, \mathrm{Sym}(3))$. Respecting the a.e. boundedness of ψ and $\operatorname{curl}\psi$, this implies

$$\begin{aligned}
&\left| \int_{B_R} \rho_{n_m} \operatorname{curl}(v_{n_m} - v)\cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x \right| \\
&\le \left| \int_{B_R} (\rho_{n_m} - \rho_0) \operatorname{curl}(v_{n_m} - v)\cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x \right| + \left| \int_{B_R} \rho_0 \operatorname{curl}(v_{n_m} - v)\cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x \right| \\
&\le \|\rho_{n_m} - \rho_0\|_{L^2(B_R,\mathrm{Sym}(3))} \|\operatorname{curl}(v_{n_m} - v)\cdot \operatorname{curl}\overline{\psi}\|_{L^2(B_R,\mathbb{C}^3)} + \left| \int_{B_R} \rho_0 \operatorname{curl}(v_{n_m} - v)\cdot \operatorname{curl}\overline{\psi}\ \mathrm{d}x \right|.
\end{aligned}$$

Since in Banach spaces also weak-convergent sequences are bounded (see e.g. [1]), $\operatorname{curl}(v_{n_m} - v)$ is bounded and since $\|\rho_{n_m} - \rho_0\|_{L^2(B_R,\mathrm{Sym}(3))} \to 0$, the first term converges. The last one converges since $\operatorname{curl}(v_{n_m} - v) \rightharpoonup 0$ in $L^2(B_R, \mathbb{C}^3)$.

At least, to see the convergence of the boundary integral, we have to be aware, that the solution is a smooth function on a neighborhood S of ∂B_R, not containing $\overline{D}$, such that $\overline{S}\cap\overline{D} = \emptyset$. To see this, we choose a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\mathrm{supp}(\chi) \subset S$, such that $\chi \equiv 1$ in a neighborhood of ∂B_R but vanishes elsewhere. Therefore $v|_S := \chi v$ is a smooth function outside D, see Remark 1(b), and thus for $j \ge 1$ one derives, using the integro-differential form (22) of the solution, the estimate

$$\begin{aligned}
\| v|_S \|_{C^j(\overline{S})} &\le C(S,j) \|(\mathrm{I}_3 - \rho)\,[\operatorname{curl} v + f]\|_{L^2(B_R,\mathbb{C}^3)} \\
&\le C(S,j) \|\,\mathrm{I}_3 - \rho\|_{L^\infty(B_R,\mathbb{C}^3)} \left(\|v\|_{H(\operatorname{curl},B_R)} + \|f\|_{L^2(B_R,\mathbb{C}^3)} \right),
\end{aligned}$$

where in fact $f = \operatorname{curl} v_g$. That shows $v_{n_m}|_S, v|_S \in C^\infty(\overline{S})$ for all $m \in \mathbb{N}$, implying $v_{n_m} \to v$ in $H(\operatorname{curl}, S)$ due to the density of $C^\infty(S)$ in $H(\operatorname{curl}, S)$, from where the tangential trace mapping γ_t and the exterior Calderon operator Λ maps $v_{n_m} - v$ into $H^{-1/2}(\mathrm{Div}, \partial B_R)$, the dual space of the range $H^{-1/2}(\mathrm{Curl}, \partial B_R)$ of the trace γ_T. Thus,

$$\int_{\partial B_R} \Lambda(\nu \times (v_{n_m} - v)) \cdot \gamma_T(\overline{\psi})\ \mathrm{d}S \to 0.$$

Hence, we have shown that

$$\begin{aligned}
&\int_{B_R} [\rho_{n_m} \operatorname{curl} v_{n_m} \cdot \operatorname{curl}\overline{\psi} - k^2 v_{n_m}\overline{\psi}]\ \mathrm{d}x + \int_{\partial B_R} \Lambda(\nu \times v_{n_m}) \cdot \gamma_T(\overline{\psi})\ \mathrm{d}S \\
&\qquad \to \int_{B_R} [\rho_0 \operatorname{curl} v \cdot \operatorname{curl}\overline{\psi} - k^2 v\overline{\psi}]\ \mathrm{d}x + \int_{\partial B_R} \Lambda(\nu \times v) \cdot \gamma_T(\overline{\psi})\ \mathrm{d}S,
\end{aligned}$$

i.e. $a_{\rho_{n_m}}(v_{n_m}, \psi) \to a_{\rho_0}(v, \psi)$ for all $\psi \in C^1(\overline{B}_R)$. Since due to Assumption 4 the problem is uniquely solvable, this implies that $v = \mathrm{L}(\rho_0, v_g)$. □

To gain sparse reconstruction techniques, we follow an approach with respect to basis functions of $\mathcal{D}(\mathrm{F})$. Note that, since B_R is of finite measure, we have $\mathcal{P} \subseteq L^\infty(B_R, \mathrm{Sym}(3)) \subseteq L^2(B_R, \mathrm{Sym}(3))$ with continuous embedding. We thus fix a biorthogonal wavelet Riesz basis $\{\psi_i\}_i$, $\{\widetilde{\psi}_i\}_i$, assuming that each ψ_i is also a function in $L^\infty(B_R, \mathrm{Sym}(3))$. Further, due to Hölder interpolation it holds that $(L^p(B_R, \mathrm{Sym}(3)) \cap \mathcal{P}) \subseteq L^2(B_R, \mathrm{Sym}(3))$ for $p \in (1, 2]$, such that we define our penalty term as some weighted ℓ^p-norm, i.e.

$$\mathcal{R}_p(\rho) := \frac{1}{p} \sum_{i \in \mathbb{N}} \omega_i |\langle \rho , \widetilde{\psi}_i \rangle|^p, \quad \rho \in \mathcal{P},\ p \in (1, 2], \tag{30}$$

with non-negative weights $(\omega_i)_i$, satisfying $\|\rho\|_{L^\infty} \leq \mathcal{R}_p(\rho)$. Note that such weights exist, since one can achieve a norm equivalence for an appropriate Besov space (see, e.g. [9, Theorem 3.7.7]), such that Sobolev/Besov embedding theorems yields an L^∞-embedding.

Theorem 17 (Sparsity regularization I). *For $p \in (1, 2]$, the Tikhonov functional $\mathcal{J}_{\alpha,\delta}$, defined in* (29)*, with $\mathcal{R} = \mathcal{R}_p$, defined in* (30)*, possesses a minimizer in $\mathcal{P} \cap L^p(B_R, \mathrm{Sym}(3))$.*

If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that converges $\mathcal{P}$-weakly to a $\mathcal{R}_p$-minimizing solution $\rho^\dagger \in \mathcal{P} \cap L^p(B_R, \mathrm{Sym}(3))$ of the equation $\mathrm{F}(\rho) = \mathrm{F}(\rho_{exa})$ in $\mathcal{S}_q$.

Recall that $\rho^\dagger$ is a $\mathcal{R}_p$-minimizing solution to $\mathrm{F}(\rho^\dagger) = \mathrm{F}(\rho_{\mathrm{exa}})$ if

$$\mathcal{R}_p(\rho^\dagger) = \min\{\mathcal{R}_p(\rho),\ \rho \in \mathcal{P} \cap L^p(B_R, \mathrm{Sym}(3)),\ \mathrm{F}(\rho) = F_{\mathrm{exa}}\}.$$

Proof. As carried out above, the choices of $\mathcal{P}$ and $\mathcal{S}_q$ satisfy the conditions for the Tikhonov regularization of Theorem 14, as well as the sequentially closedness of F, shown in Lemma 16. Further note that $|\langle \cdot , \widetilde{\psi}_i \rangle|^p$ is L^2-weakly lower semicontinuous and any $\mathcal{P}$-weak* convergent sequence is also L^2-weakly convergent due to continuous embedding of $\mathcal{P}$ into $L^2(B_R, \mathrm{Sym}(3))$. Since scalar multiplication does not impact lower semicontinuity properties as well as summation of lower semicontinuous functions (see, e.g. [5, Lemma 6.14]), the penalty term $\mathcal{R}_p$ is weak*-lower semicontinuous, for $p \in (1, 2]$. Finally, it has weak*-precompact sublevel sets, since $\mathcal{P}$ itself is weak*-compact by Alaoglu's theorem. □

To avoid Hölder continuous spaces we now give a second approach by adapting techniques of image processing, where the gradient is used to highlight edges of objects, whereas homogeneous regions stay as more connected areas. Traditionally this leads to Sobolev penalty terms $\|D^m w\|_{L^p(\Omega)}$ for all $w \in W^{m,p}(\Omega)$ with $p \in (1, \infty)$ and $m \geq 1$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. It can be shown that $p \to 1$ yields better reconstructions. Since the $W^{1,1}$-seminorm is not lower semicontinuous [5, Satz 6.101], the boundedness of a sequence does not imply the existence of a weak-convergent subsequence, such that the Sobolev penalty term for $p = 1$ can not be used directly.

In fact, this holds in general for $p = 1$, since $L^1(\Omega)$ without a σ-finite measure is not a reflexive dual space of $L^\infty(\Omega)$. Thus one generalizes the integral $\int_\Omega |Dw| \mathrm{d}x$ by regarding $L^1(\Omega)$ as a subset of the space of vector valued, finite Radon measures on Ω, that is

$$\mathfrak{M}(\Omega, \mathbb{R}^3) := \left\{\mu \colon \mathfrak{B}(\Omega) \to \mathbb{R}^3,\ \mu \text{ vector valued finite Radon measure}\right\},$$

equipped with the norm $\|\mu\|_{\mathfrak{M}(\Omega,\mathbb{R}^3)} = |\mu|(\Omega)$. Therefore $\mathfrak{M}(\Omega, \mathbb{R}^3)$ is a Banach space and isometrically isomorphic to the dual space $C_0(\Omega, \mathbb{R}^3)^*$ by Riesz-Markow's representation theorem. Note that the characterization as a dual space implies a weak*-convergence.

Due to that, one then defines the distributional gradient by the representation of such a vector-valued finite Radon measure:

Definition 18. For a domain $\Omega \subset \mathbb{R}^3$, $\mu \in \mathfrak{M}(\Omega, \mathbb{R}^3)$ is the distributional gradient of $w \in L^1_{loc}(\Omega)$, if for every $\psi \in C_0^\infty(\Omega, \mathbb{R}^3)$ it holds, that

$$\int_\Omega w \operatorname{div} \psi \, \mathrm{d}x = - \int_\Omega \psi \, \mathrm{d}\mu.$$

The norm of such a measure μ is called total variation of w and we write

$$TV_\Omega(w) := \begin{cases} \|Dw\|_{\mathfrak{M}(\Omega,\mathbb{R}^3)} & \text{if the measure } \mu =: Dw \text{ exists,} \\ \infty & \text{else.} \end{cases}$$

If the distributional derivative of w can be written as a finite Radon measure, we say the function w has bounded total variation. Therefore one often also writes

$$TV_\Omega(w) = \sup \left\{ \int_\Omega w \operatorname{div} \psi \, \mathrm{d}x, \ \psi \in C_0^\infty(\Omega), \ \|\psi\|_{L^\infty(\Omega)} \leq 1 \right\}. \tag{31}$$

The space of all functions with bounded total variation is thus defined by

$$BV(\Omega) := \left\{ w \in L^1(\Omega), \ TV_\Omega(w) < \infty \right\}.$$

Roughly speaking, $BV(\Omega)$ contains functions in $L^1(\Omega)$, whose distributional gradients are finite Radon measures, and is a Banach space equipped with the norm $\|w\|_{BV(\Omega)} := \|w\|_{L^1(\Omega)} + TV_\Omega(w)$, whereas $TV_\Omega(w)$ is the BV-seminorm (obviously $\|w\|_{W^{1,1}(\Omega)} = \|w\|_{BV(\Omega)}$ for $w \in W^{1,1}(\Omega)$). Note that compared to Sobolev spaces, the BV-space also contains piecewise smooth functions, such that by total variation as penalty, one can handle functions with discontinuities.

Since BV is the dual space of the separable space L^1 on which bounded sets are pre-compact [2, Theorem 3.23], [3, Remark B.7], a weak*-convergence can be defined which yields the general weak*-topology, i.e.

Definition 19. Let $w, w_n \in BV(\Omega)$, then $\{w_n\}_{n\in\mathbb{N}}$ is called weakly*-convergent to w in $BV(\Omega)$, if $w_n \to w$ in $L^1(\Omega)$ and $Dw_n \overset{\mathfrak{M}}{\rightharpoonup} Dw$ in Ω, i.e.

$$\lim_{n\to\infty} \int_\Omega \psi \, \mathrm{d}Dw_n = \int_\Omega \psi \, \mathrm{d}Dw \quad \text{for all } \psi \in C_0(\Omega).$$

We now restrict our set $\mathcal{P}$ and further operate on the set of material parameters, called

$$\mathcal{P}_{TV} := \left\{ \rho \in \mathcal{P}, \ TV_{(B_R, \mathrm{Sym}(3))}(\rho) < \infty \right\}.$$

Although, the authors of [22] suggest to use the BV semi-norm (31) as penalty, i.e. $\mathcal{R} = TV_\Omega$, [4] remarks that using only the BV semi-norm as penalty, does not guarantee the possibility to make the to-be-solved variational problem locally convex, such that the global minimum can be computed by local descent methods. Because of that, they add a multiple of the squared L^2-norm to gain sufficient compactness properties of the functional. Even though the full BV-norm as penalty would provide the same compactness properties, they state that numerical minimization can be handled easier by adding L^2-norm instead of L^1-norm. However, since [7] provide promising reconstructions for the full BV-norm (taking into account an additional term respecting some physical constraints), we suppose to use the penalty

$$\mathcal{R}_{BV}(\rho) := \|\rho\|_{BV(B_R, \mathrm{Sym}(3))} = \|\rho\|_{L^1(B_R, \mathrm{Sym}(3))} + TV_{(B_R, \mathrm{Sym}(3))}(\rho), \quad \rho \in \mathcal{P}. \tag{32}$$

Theorem 20 (Sparsity regularization II)**.** *The Tikhonov functional $\mathcal{J}_{\alpha,\delta}$, defined in* (29)*, with $\mathcal{R} = \mathcal{R}_{BV}$, defined in* (32)*, possesses a minimizer in $\mathcal{P}_{TV} \cap BV(B_R, \mathrm{Sym}(3))$.*

If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that converges $\mathcal{P}_{TV}$-weakly to an $\mathcal{R}_{BV}$-minimizing solution $\rho^\dagger \in \mathcal{P}_{TV} \cap BV(B_R, \mathrm{Sym}(3))$ of the equation $\mathrm{F}(\rho) = \mathrm{F}(\rho_{exa})$ in $\mathcal{S}_q$.

Proof. As in Theorem 17, $\mathcal{S}_q$ and especially $\mathcal{P}_{TV}$ satisfy the conditions for the Tikhonov regularization of Theorem 14, since $\mathcal{P}_{TV} \subset \mathcal{P}$. Further, Lemma 16, i.e. the sequentially closedness of F, also holds for $\mathcal{P}_{TV}$. Since the total variation TV is weak*-lower semicontinuous (see, e.g. [8, Proposition 3.7]) as well as the L^1-norm, the penalty $\mathcal{R}_{BV}$ is weak*-lower semicontinuous as well. Again, $\mathcal{R}$ has weak*-precompact sublevel sets, since $\mathcal{P}_{TV}$ is again weak*-compact by Alaoglu's theorem. ☐

A The adjoint of the forward operator's linearization

Most gradient-based schemes, which are used to solve the inverse scattering problem, i.e. stably solving the non-linear equation $\mathrm{F}(\rho) = F_{\mathrm{meas}}$ for some given data $F_{\mathrm{meas}} \in \mathcal{S}_q$, like iterated shrinkage algorithm, rely on the adjoint operator of the linearization F'. This is why we give an explicit and computable representation, following [16]. Therefore we fix $\rho \in \mathcal{P}$, consider $\mathrm{F}'(\rho)\colon L^\infty(B_R, \mathrm{Sym}(3)) \to \mathcal{S}_q$ and aim to determine $\mathrm{F}'(\rho)^*\colon \mathcal{S}_{q'} \to L^1(B_R, \mathrm{Sym}(3))$ such that

$$(\mathrm{F}'(\rho)[\theta], K)_{\mathcal{S}_2} \overset{!}{=} (\theta, \mathrm{F}'(\rho)^* K)_{L^2} \quad \text{for all } \theta \in L^\infty(B_R, \mathrm{Sym}(3)) \text{ and } K \in \mathcal{S}_{q'}. \tag{33}$$

Here, q' denotes the conjugate Lebesgue index to q, respectively, such that $1/q + 1/q' = 1$, and $(\cdot\,,\cdot)_{L^2(B_R,\mathrm{Sym}(3))}$ is the usual scalar product in $L^2(B_R, \mathrm{Sym}(3))$,

$$(A, B)_{L^2(B_R,\mathrm{Sym}(3))} = \int_{B_R} A : B \,\mathrm{d}x = \int_{B_R} \sum_{i,j=1}^{d} \overline{A}_{ij} B_{ij} \,\mathrm{d}x.$$

extended to the anti-linear dual product between $L^\infty(B_R, \mathrm{Sym}(3))$ and $L^1(B_R, \mathrm{Sym}(3))$. Further, $(\cdot,\cdot)_{\mathcal{S}_2}$ is the scalar product in the Hilbert space of Hilbert-Schmidt operators,

$$(\mathrm{F}, \mathrm{K})_{\mathcal{S}_2} = \sum_{j\in\mathbb{N}} s_j(\mathrm{F}) \overline{s_j(\mathrm{K})} = \sum_{j=1}^{\infty} (\mathrm{F} g_j,\, \mathrm{K}\, g_j)_{L^2_t(\mathbb{S}^2)}$$

for an arbitrary orthonormal basis $(g_j)_{j\in\mathbb{N}}$ of $L^2_t(\mathbb{S}^2)$. Consequently, (33) becomes

$$\sum_{j=1}^{\infty} (\mathrm{F}'(\rho)[\theta] g_j, \mathrm{K}\, g_j)_{L^2_t(\mathbb{S}^2)} \overset{!}{=} (\theta, \mathrm{F}'(\rho)^* \,\mathrm{K})_{L^2_t(B_R,\mathrm{Sym}(3))} \quad \text{for all } \theta \in L^\infty(B_R, \mathrm{Sym}(3)),\, \mathrm{K} \in \mathcal{S}_{q'}.$$

Thus, we consider at first a single L^2-scalar product and for fixed $\rho \in \mathcal{P}$ and $g \in L^2_t(\mathbb{S}^2)$ we seek for $A\colon L^2(\mathbb{S}^2) \to L^1(B_R, \mathrm{Sym}(3))$, such that

$$(\mathrm{F}'(\rho)[\theta] g, f)_{L^2_t(\mathbb{S}^2)} \overset{!}{=} (\theta, Af)_{L^2(B_R,\mathrm{Sym}(3))} \quad \text{for all } \theta \in L^\infty(B_R, \mathrm{Sym}(3)) \text{ and } f \in L^2_t(\mathbb{S}^2).$$

Recall from (23) that $\mathrm{L}'(\rho, v_g)[\theta] = v' \in H(\mathrm{curl}, B_R)$, a function whose radiating extension satisfies

$$v' = -S_\rho \left[\mathrm{curl}\, V(\theta \,\mathrm{curl}[\mathrm{L}(\rho, v_g) + v_g])\right] \;\text{ in } H(\mathrm{curl}, B_R),$$

where $S_\rho = [\mathrm{I}_3 - \mathrm{curl}\, V((\mathrm{I}_3 - \rho)\,\mathrm{curl})]^{-1}$. Since F' involves the far field of L', see (27), we note that

$$\begin{aligned} \mathrm{F}'(\rho)[\theta] g &= Z \circ \left[(\mathrm{I}_3 - \rho)\,\mathrm{curl}\, v' + \theta\,\mathrm{curl}\, S_\rho(v_g)\right] \\ &= Z \circ \left[\theta\,\mathrm{curl}\, S_\rho(v_g) - (\mathrm{I}_3 - \rho)\,\mathrm{curl}\, S_\rho\left[\mathrm{curl}\, V(\theta\,\mathrm{curl}\, S_\rho(v_g))\right]\right]. \end{aligned}$$

Consequently, we compute that

$$
\begin{aligned}
&(\mathrm{F}'(\rho)[\theta]g,\, f)_{L^2_t(\mathbb{S}^2)} = \big(\theta \operatorname{curl} S_\rho(v_g) - (\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho \left[\operatorname{curl} V(\theta \operatorname{curl} S_\rho(v_g))\right],\, Z^* f\big)_{L^2(B_R,\mathbb{C}^3)} \\
&= \big(\theta \operatorname{curl} S_\rho(v_g),\, Z^* f\big)_{L^2(B_R,\mathbb{C}^3)} - \big(\theta \operatorname{curl} S_\rho(v_g),\, [\operatorname{curl} V]^* \circ [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho]^* \circ Z^* f\big)_{L^2(B_R,\mathbb{C}^3)} \\
&= \big(\theta,\, [(\mathrm{I}_3 - [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho \circ \operatorname{curl} V]^*) \circ Z^* f] \otimes \operatorname{curl} \overline{S_\rho(v_g)}\big)_{L^2(B_R,\mathrm{Sym}(3))}
\end{aligned}
$$

where the last matrix-valued function is defined by $(a \otimes b)_{i,j} = a_i b_j$ for $1 \le i, j \le 3$.

Lemma 21. *For $\rho \in \mathcal{P}$ and $g \in L^2(\mathbb{S}^2)$, the adjoint of $\theta \mapsto \mathrm{F}'(\rho)[\theta](g)$ with respect to the L^2-inner product maps $L^2_t(\mathbb{S}^2)$ into $L^1(B_R, \mathrm{Sym}(3))$ and is represented by*

$$
g \mapsto \Big(\big[\,\mathrm{I}_3 - [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho \circ \operatorname{curl} V]^*\big] \circ Z^* g\Big) \otimes \operatorname{curl} \overline{S_\rho(v_g)}.
$$

For all orthonormal bases $\{g_j\}_{j\in\mathbb{N}}$ of $L^2_t(\mathbb{S}^2)$ and all $\mathrm{K} \in \mathcal{S}_{q'}$, the bounded operator $\mathrm{F}'(\rho)^* : \mathcal{S}_{q'} \to L^1(B_R, \mathrm{Sym}(3))$ is represented by

$$
\mathrm{F}'(\rho)^*(\mathrm{K}) = \sum_{j=1}^{\infty} \Big(\big[\,\mathrm{I}_3 - [(\mathrm{I}_3 - \rho) \operatorname{curl} S_\rho \circ \operatorname{curl} V]^*\big] \circ Z^*\big(\mathrm{K}\, g_j\big)\Big) \otimes \operatorname{curl} \overline{S_\rho[v_{g_j}]}.
$$

Acknowledgments

The author would like to thank the German Research Foundation (DFG), which supported the research leading to these results by grant Le 2499/2-1.

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