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A UNIFIED APPROACH TO THE APPROXIMATE SOLUTION OF PDE

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Abstract

This paper describes procedures for solving each of the three types of PDE (partial differential equations): hyperbolic, elliptic, and parabolic, by means of an *indefinite convolution procedure* combined with *Sinc approximation.* The PDE is first transformed to an equivalent integral equation, and then solved by means of "Sinc convolution". Whereas different numerical methods are used in practice for solving elliptic, parabolic, and hyperbolic PDE, the present paper uses essentially the same procedure for all three of these equations, over bounded or unbounded regions. The time complexity of computation to solve a d–dimensional problem on a sequential machine (i.e., the amount of time required to obtain a solution to within a uniform error of ε) under suitable assumptions of analyticity, allowing for possible singularities in the coefficients on the boundaries of the regions is of the order of $(\log(\varepsilon))^{2d+2}$. The method also lends itself readily to parallel computation, although we have not illistrated this feature in this paper. Several examples are presented, and time comparisons are made with efficient existing methods. We do not need to store the large matrices that current methods require, enabling us to achieve high accuracy, whereas this is not possible via current algorithms.

1 Introduction and Summary

Sinc methods offer a variety of approaches for solving PDE [14, 15, 5]. In the present article we present a unified approach to solve the integral equa-

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tion formulation of solutions to each of the three classes of PDE, via use of *indefinite convolutions* combined with *Sinc approximation*, a procedure described in [15] and [14], §4.6, and henceforth in this paper to be referred to as *Sinc convolution.* This *Sinc convolution* procedure is derived in [15] for approximating one–dimensional indefinite convolution integrals. It is readily extended to the approximation of definite convolution integrals, and to the approximation of multidimensional definite and indefinite integrals. The procedure has some remarkable features:

1. Only the Laplace transform (or its accurate approximation) of f is required to get an accurate approximation of convolution integrals of the form

$$\int_a^x f(x-t)\, g(t)\, dt\ , \qquad \int_x^b f(x-t)\, g(t)\, dt\,. \tag{1.1}$$

   The accurate approximation of such integrals, including e.g., *Abel–type* integrals, was hitherto difficult, especially in important cases of when $f(t)$ has an integrable singularity at $t=0$. Moreover, being able to accurately approximate each of the integrals in (1.1) enables us to accurately approximate definite integral convolutions of the form

$$\int_a^b f(x-t)\, g(t)\, dt\,. \tag{1.2}$$

2. Whereas Fourier transforms (FFT) can also be used to approximate the integrals (1.1), FFT can converge very slowly in cases of when $f(t)$ has an integrable singularity at $t=0$, or when (a,b) is finite or semi–infinite, and/or when g has isolated singularities on (a,b). For example, the *Sinc convolution* procedure offers a remarkably simple method of solving *Wiener–Hopf* integral equations.

3. The *Sinc convolution* procedure enables a surprising "separation of variables" procedure analogous to that used to solve multidimensional problems in e.g., PDE. This makes it possible to accurately approximate the multidimensional convolution integrals via use of one–dimensional matrix multiplications. It is applicable in all dimensions.

4. This separation of variables feature of the method enables solution of the PDE via parallel computation, although we have not used parallel computation in our illustrative examples.

5. A bi–product of using one–dimensional matrix multiplications is that we need not set up the analogous "big matrices" that are required for the solution of PDE via finite difference and finite element methods. This is one of the reasons why it is not possible to achieve high accuracy in solving PDE via these classical methods. Another reason is that the rate of convergence of the *Sinc convolution* technique is much more rapid. Indeed, these facts are illustrated in our comparison tests.

6. Finally, we add that the procedure can also be applied to convolution integrals over curvilinear regions in two or more dimensions, such as, e.g., in two dimensions, regions of the form

$$B = \{(x, y) : a_1 < x < b_1,\ a_2(x) < y < b_2(x)\} \tag{1.3}$$

and, of course, to translations and rotations of such regions. That is, it is applicable to the solution of most PDE problems over curvilinear regions arising in science and engineering.

The present paper is *not* a complete illustrations of solution of PDE via *Sinc* methods, i.e., we have not illustrated handling initial and/or boundary conditions. However, what we have omitted can, in fact be easily dealt with via Sinc methods, as has already been illustrated. That initial value ordinary differential equation problems can be easily dealt with via *Sinc* methods has been illustrated in the program package [7]. The present paper illustrates the construction of an approximate solution to a non–homogeneous PDE via use of Green's functions. Given a PDE along with initial and or boundary conditions, this same Green's function can be used to set up a boundary integral equation for determination of a solution of the homogeneous PDE, yielding a solution to the original PDE with the correct boundary conditions. That such boundary integral equations can in fact be efficiently and accurately solved via Sinc methods has already been illustrated in [14], §6.5, and [13, 3, 8, 4, 2, 6, 16, 17, 12]. Furthermore, the *Sinc convolution* procedure can also be used to solve integral equations, as we illustrate in this paper via a simply example, and as was illustrated in [11] on the solution of a five– dimensional convolution type integral equation.

We shall use Sinc terminology in the present paper. An excellent presentation of this terminology is given in [9]. For sake of completeness, we shall include the essence of this terminology in §3 of the present paper. We also present the *Sinc convolution* algorithm in this section.

In what follows, in §2 we illustrate the *Sinc convolution* procedure to obtain approximate solutions of several PDE, and we make time comparisons with other existing methods for solving such problems. In order to achieve accuracies that are possible via *Sinc convolution*, the matrices required by classical methods to achieve such accuracies in more than three and 4 dimensions very quickly reached the capacity of our computer, making it difficult to make complexity comparisons.

The Sinc terminology, including the Sinc convolution technique is given in §3, while §4 contains explicit derivations of the multidimensional Laplace transforms of standard Green's functions, based on a technique developed, essentially, in [11].

2 Applications

In this section we illustrate the application of the *Sinc convolution* procedures to obtain approximate solutions of elliptic, parabolic, and hyperbolic differential equations. We also illustrate the solution of an integral equation problem, as well as the solution of a PDE problem over a curvilinear region.

For all our numerical computations a two processor PC with Intel Pentium II (400 MHz), Linux operating system and 512MB main memory was used. The code for our algorithms was written in Matlab. To compare the results of the *Sinc convolution* computations with results obtained by *FEM*–methods, we used for the Poisson equation and Heat equation the program package KASKADE, developed at the Konrad-Zuse-Zentrum Berlin (ZIB), Germany (for a description of the used algorithm, program code and manual see ftp://elib.zib.de/pub/kaskade). For the two dimensional Wave equation the PDE–Toolbox of Matlab was used.

2.1 Sinc Convolution Solution of Poisson Problems

1. Our first illustration is that for the Poisson equation

$$\Delta\,\Psi(\bar r) = -g(\bar r),, \qquad \bar r \in V = \mathbb{R}^3\,, \tag{2.1}$$

Our *Sinc convolution* computations were based on the formula

$$\Psi(x,y,z) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \frac{g(\xi,\eta,\zeta)}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\xi\,d\eta\,d\zeta. \tag{2.2}$$

for $(x,y,z)\in V$. This multidimensional convolution integral can be readily split into 8 indefinite convolution integrals, such as, e.g.,

$$\Psi^{(1)}(x,y,z) = \int_{a_1}^{x}\int_{y}^{b_2}\int_{a_3}^{z} \frac{g(\xi,\eta,\zeta)}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\zeta\,d\eta\,d\xi. \tag{2.3}$$

To evaluate the 8 integrals of the form $\Psi^{(1)}$ at all of the Sinc points $\{(i\,h, j\,h, k\,h) : i=-N,\cdots,N, j=-N,\cdots,N, k=-N,\cdots,N\}$, we require 3 transformations $\varphi_i : (a_i,b_i) = \mathbb{R}\to\mathbb{R}, i=1,2,3$, but this is trivial, since each of the transformations is the same identity map. We thus determine $h_i = h$ and we form the matrices

$$\begin{aligned} A_i = A &= h\,I^{(-1)} = X_i\,S_i\,X_i^{-1},\ i=1,3 \\ A_2 = A^T &= h\,(I^{(-1)})^T = X_2\,S_2\,X_2^{-1}\,, \end{aligned} \tag{2.4}$$

where each $S_i = \mathrm{diag}[s^{(i)}_{-M_i},\ldots,s^{(i)}_{N_i}]$ is a diagonal matrix of eigenvalues of the matrix A_i, and X_i is the corresponding matrix of eigenvectors.

We then evaluate the array $[g_{ijk}] = [g(ih, jh, kh)]$, and we use the *Sinc convolution* algorithm (an explicit 3–dimensional version is given in [16]) to transform this array into an array $\left[\Psi^{(1)}_{ijk}\right]$, by means of the "Laplace transform" of the Greens function $1/(4\,\pi\,r)$ given in Lemma 4.3 of this paper. We then repeat this computation to get accurate approximations at the *Sinc* points for all of the remaining 7 *Sinc convolutions* $\Psi^{(\ell)}$, $\ell = 2, 3, \ldots, 8$. We then have

$$[\Psi(ih, jh, kh)] = \left[\sum_{\ell=1}^{8} \Psi^{(\ell)}(ih, jh, kh)\right] . \tag{2.5}$$

Using Sinc interpolation, we can then get an almost equally accurate approximation to the function Ψ at all points of V. It may moreover be shown, assuming that the function $g(\cdot, y, z)$, and, additionally, making similar assumptions about the functions $g(x, \cdot, z)$ and $g(x, y, \cdot)$) is analytic on (a_1, b_1), for all $(y, z) \in [a_2, b_2] \times [a_3, b_3]$ (which is, in fact so, for the case of this example) then the uniform error of approximation is of the order of $\exp(-c\,N^{1/2})$, with c a constant that is independent of N. In particular, for the case of the present problem we can take $c = \pi$.

Let us also give a more explicit picture of the (unstored) matrix that is involved in the above computation. Let us form a vector $\mathbf{g}$ from the array $[g_{ijk}]$, in which the subscripts appear in the order (call it lexicographic) dictated by the order of appearance of the subscripts in the Fortran do loop, "DO k $= -\mathrm{M}_3, \mathrm{N}_3$", followed by "DO j $= -\mathrm{M}_2, \mathrm{N}_2$", followed by "DO i $= -\mathrm{M}_1, \mathrm{N}_1$". We then also form the diagonal matrix $\hat{\mathbf{G}}$ in which the entries are the values $\hat{G}_{ijk} = \hat{G}(s^{(1)}_i, s^{(2)}_j, s^{(3)}_k)$, with the function $\hat{G}$ and the eigenvalues $s^{(i)}_j$ defined as above, and where we also list the values $\hat{G}_{ijk}$ in the same lexicographic order as for g_{ijk}. Then, similarly from the array $\Psi^{(1)}_{ijk}$, we can define a vector $\boldsymbol{\Psi}_1$ by listing the elements $\Psi^{(1)}_{ijk}$ in lexicographic order. It can then be shown that $\boldsymbol{\Psi}_1$ is defined by the matrix (Kronecker) product

$$\begin{aligned} \boldsymbol{\Psi}_1 &= \boldsymbol{\Phi}^{(1)}\,\mathbf{g} \\ \boldsymbol{\Phi}^{(1)} &= X_3 \otimes X_2 \otimes X_1\,\hat{\mathbf{G}}\,X_3^{-1} \otimes X_2^{-1} \otimes X_1^{-1}\,, \end{aligned} \tag{2.6}$$

and moreover, the analogous vector $\boldsymbol{\Psi}$ approximating the function Ψ defined in (2.2) above at the *Sinc* points is then given by

$$\boldsymbol{\Psi} = \left(\sum_{\ell=1}^{8} \boldsymbol{\Psi}^{(\ell)}\right) \mathbf{g}\,. \tag{2.7}$$

We emphasize that the matrices $\boldsymbol{\Psi}^{(\ell)}$ and $\boldsymbol{\Psi}$ need never be computed, since our algorithm involves performing a sequence of one–dimensional matrix multiplications. For example, with $N = 20$ we get at least 6 places of accuracy, and the size of the corresponding matrix $\boldsymbol{\Psi}$ is $41^3 \times 41^3$, or $68,921 \times 68,921$. Such a matrix, which is full, contains more than 4.75×10^9 elements. If such a matrix were to be obtained by a Galerkin scheme, with each entry requiring the evaluation of a three dimensional integral, and with each integral requiring 41^3 evaluation points, then more than 3.27×10^{14} function evaluations would be required, an ominous task indeed! On the other hand, our method accurately gives us *all* of these values for relatively little work.

For a test computation, we used as right hand side $g(\bar{r}) = \exp(-r^2)(6-4r^2)$. The exact solution is then given by $\Psi(\bar{r}) = \exp(-r^2)$, $r = |\bar{r}|$.

The computational domain for FEM (finite element method) with which we compared our results was $[-6,6]^3$ with zero boundary conditions. This restriction caused no problems because u and f are rapidly decreasing functions.

We computed the *Sinc* based solution with $m = 2N+1$ *Sinc* points. The corresponding values of h were computed by $h = \pi/\sqrt{N}$. The CPU time is listed in Table 1. The FEM solution was computed afterwards using more and more knots until the accuracy of the *Sinc* solutions was achieved. We compared the accuracy of the *Sinc* with the FEM solution based on the maximum difference of exact and approximate solution at the FEM knots. The table shows that to achieve one point of accuracy (i.e.10^{-1}) FEM is faster than our *Sinc* method. But even for two places of accuracy our *Sinc* procedure is three times as fast whereas three points cause problems with FEM methods concerning time and storage of the matrix. For *Sinc*, we were even able to compute an example with 161 *Sinc* points and an accuracy of 10^{-6} on the region $[-6,6]^3$.

2. Next, we illustrate the solution via *Sinc convolution* of an elliptic problem over a two dimensional curvilinear region.

The region is $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, with

$$\begin{aligned}\mathcal{B}_1 &= \left\{(x,y) : -\frac{3}{2} < x < 0,\ 0 < y < x^2 + \frac{\sqrt{3}}{2}\right\} \\ \mathcal{B}_2 &= \left\{(x,y) : 0 < x < \frac{3}{2},\ 0 < y < \sqrt{1 - \left(x - \frac{1}{2}\right)^2}\right\}.\end{aligned} \tag{2.8}$$

We are given functions g_1 and g_2 defined by the equations

FEM			SINC		
accuracy ($\|\cdot\|_\infty$)	# of knots	CPU time (sec)	accuracy ($\|\cdot\|_\infty$)	CPU time (sec)	N
0.2829 0.1437	158 247	0.85 1.79	0.171 0.171	5.3	10
0.0652 0.0244 0.0125	1003 4365 15373	6.16 26.25 98.71	0.0166	21.54	15
0.0125 0.0072 0.0062 0.0026	15373 28140 53706 143497	98.71 229.47 482.68 1164.63	0.0040	57	20
***	***	***	1.1672e-05	2977	60
***	***	***	1.0734e-06	11611	80

Table 1: FEM and SINC costs for solution of Laplace equation. *** indicates that FEM was not able to achieve a similar accuracy.

$$
\begin{aligned}
g_1(x,y) &= c_1\,(-x)^{1/7}\left(\frac{3}{2}-x\right)^{-6/7} y^{-1/4}\left(\frac{\sqrt{3}}{2}+x^2-y\right)^{-3/4} \\
g_2(x,y) &= c_2\,x^{\frac{1}{\sqrt{3}}-1}\,y^{-1/3}\left(1-\left(x-\frac{1}{2}\right)^2-y^2\right)^{-1/3},
\end{aligned}
\tag{2.9}
$$

where the constants c_1 and c_2 are selected so that

$$\int\!\!\int_{B_1} g_1(x,y)\,dx\,dy = -\int\!\!\int_{B_2} g_2(x,y)\,dx\,dy = 1\,,$$

i.e., $c_1 = -\sin(\pi/7)/(\sqrt(2)\,\pi^2)$, $c_2 = 1/\left((3/2)^{1/\sqrt{3}}\pi\right)$.

Let us use the notation $\bar{\rho} = (x,y)$, $\rho = \sqrt{x^2+y^2}$. The partial differential equation which we propose to solve is

$$
\begin{aligned}
&\nabla^2 U(\bar{\rho}) = g(\bar{\rho}) \quad \bar{\rho} \in \mathbb{R}^2 \\
&\lim_{\rho\to\infty} U(\bar{\rho}) = 0\,,
\end{aligned}
\tag{2.10}
$$

with $g = g_j$ in $\mathcal{B}_j$ $(j = 1,2)$, and with $g = 0$ on $\mathbb{R}^2 \setminus \{\mathcal{B}_1 \cup \mathcal{B}_2\}$, although we shall be interested in values of the solution only on $\mathcal{B}_1 \cup \mathcal{B}_2$.

(Notice that g is unbounded on the boundary of $\mathcal{B}_1 \cup \mathcal{B}_2$.) Evidently, the solution to this problem is given by

$$\begin{aligned} U(\bar\rho) \;&=\; \int\!\!\int_{B_1} \frac{1}{2\,\pi} \log\left\{\frac{1}{|\bar\rho - \bar\rho'|}\right\} g_1(\bar\rho')\, d\bar\rho' \\ &\quad + \int\!\!\int_{B_2} \frac{1}{2\,\pi} \log\left\{\frac{1}{|\bar\rho - \bar\rho'|}\right\} g_2(\bar\rho')\, d\bar\rho' \,, \end{aligned} \tag{2.11}$$

with $\bar\rho \in \mathcal{B}$.

To solve this problem, we split each integral over $\mathcal{B}_j$ into four indefinite convolution integrals, i.e.,

$$\int\!\!\int_{\mathcal{B}_j} = \int_{a_{j,1}}^{b_{j,1}} \int_{a_{j,2}(x)}^{b_{j,2}(x)} G(\bar\rho, \bar\rho')\, g_j(\bar\rho') dy'\, dx' = \sum_{i=1}^{4} Q_j^{(i)} \,,$$

with

$$Q_j^{(1)}(\bar\rho) = \int_{a_{j,1}}^{x} \int_{a_{j,2}(x)}^{y} \cdots dy'\, dx'$$

$$Q_j^{(2)}(\bar\rho) = \int_{a_{j,1}}^{x} \int_{y}^{b_{j,2}(x)} \cdots dy'\, dx'$$

$$Q_j^{(3)}(\bar\rho) = \int_{x}^{b_{j,1}} \int_{a_{j,2}(x)}^{y} \cdots dy'\, dx'$$

$$Q_j^{(4)}(\bar\rho) = \int_{x}^{b_{j,1}} \int_{y}^{b_{j,2}(x)} \cdots dy'\, dx' \,.$$

Inspection of the functions g_j shows that:

(a) $Q_1^{(i)}(x,y) \in \mathbf{Lip}_\alpha$ with respect to x, with $\alpha = \alpha_x^{(1)} = 6/7$ near $x = -3/2$, and $\alpha = \beta_x^{(1)} = 1/7$ near $x = 0$;

(b) $Q_1^{(i)}(x,y) \in \mathbf{Lip}_\alpha$ with respect to y, with $\alpha = \alpha_y^{(1)} = 3/4$ near $y = 0$ and with $\alpha = \beta_y^{(1)} = 1/4$ near $y = x^2 + \sqrt{3}/2$;

(c) $Q_2^{(i)}(x,y) \in \mathbf{Lip}_\alpha$ with respect to x, with $\alpha = \alpha_x^{(2)} = 1/\sqrt{3}$ near $x = 0$ and with $\alpha = \beta_y^{(2)} = 1$ near $x = 1$; and

(d) $Q_2^{(i)}(x,y) \in \mathbf{Lip}_\alpha$ with respect to y, with $\alpha = \alpha_y^{(2)} = 2/3$ near $y = 0$ and with $\alpha = \beta_y^{(2)} = 1/3$ near $y = \sqrt{1 - (x - 1/2)^2}$.

Let us (at this point, somewhat arbitrarily) select

$$h = \frac{1}{\sqrt{N_1}} \,. \tag{2.12}$$

Given some $\varepsilon > 0$, we select an integer N_1 so that

$$\exp\left(-\beta_x^{(1)}\, N_1\, h\right) = \exp\left(-\beta_x^{(1)}\, N_1^{1/2}\right) = \varepsilon\,.$$

We can then expect to achieve the same accuracy in all the variables by fixing M_j by means of the equations (see e.g., [14] §3.1)

$$\beta_x^{(1)}\, N_1^{(1)} = \alpha_x^{(1)}\, M_1^{(1)} = \alpha_y^{(1)}\, M_2^{(1)} = \beta_y^{(1)}\, N_2^{(1)}$$

$$= \beta_x^{(2)}\, N_1^{(2)} = \alpha_x^{(2)}\, M_1^{(2)} = \alpha_y^{(2)}\, M_2^{(2)} = \beta_y^{(2)}\, N_2^{(2)}\,.$$

We then need the matrices

$$A_j^{(i)} = h\, I_{m_j^{(i)}}^{(-1)}\, D_{m_j^{(i)}}$$

$$B_j^{(i)} = h\, \left(I_{m_j^{(i)}}^{(-1)}\right)^T\, D_{m_j^{(i)}}$$

with $i, j = 1, 2$, with h defined as above, with $m_j^{(i)} = M_j^{(i)} + N_j^{(i)} + 1$, and with

$$D_{m_j^{(i)}} = D\left(\frac{e^w}{1+e^w}\right), \quad w = k\, h_j\,, \ k = -M_j^{(i)}, \ldots, N_j^{(i)}\,.$$

We next approximate each of the integrals $Q_j^{(i)}$ via the above described *Sinc convolution algorithm*. To this end, we first opt to simplify the somewhat cumbersome notation that we have adopted above. In order to approximate $Q_1^{(1)}$, let us first set

$$p(x,y) = Q_1^{(1)}(x,y) = \int_{-3/2}^{x} \int_0^y \frac{1}{2\pi} \ln\left(\frac{1}{|\bar\rho - \bar\rho'|}\right) g_1((\bar\rho')\ d\bar\rho'\,.$$

We have to diagonalize the matrices $A_1^{(1)}$ and $B_1^{(1)}$, i.e., we set

$$A_1^{(1)} = X\, S_1\, X^{-1}; \quad X^{-1} = [x^{ij}]$$

$$B_1^{(1)} = Y\, S_2\, Y^{-1};$$
$$S_j = \operatorname{diag}\left[s_{M_j}^{(j)}, \ldots, s_{M_j}^{(j)}\right]\,.$$

With reference to *Algorithm 3.2*, we have

$$a_1^{(1)} = -\frac{3}{2}, \quad b_1^{(1)} = 0,$$

$$a_2^{(1)}(x) = 0, \quad b_2^{(1)}(x) = x^2 + \frac{\sqrt{3}}{2}\,.$$

The *Sinc points* which we shall require for $\mathcal{B}_1$ are

$$x_i^{(1)} = \frac{a_1^{(1)}}{1+e^{i\,h}}, \quad y_{i,j}^{(1)} = \frac{b_2\left(x_i^{(1)}\right)\, e^{j\,h}}{1+e^{j\,h}}$$

The algorithm for approximating the first integral on the right hand side of (2.11), with $(x,y) \in \mathcal{B}_1$ then is the following:

(a) Set up $[g_{i,j}] = \left[g_1\left(x_i^{(1)}, y_{i,j}^{(1)}\right)\right]$;

(b) Form $h_{i,\cdot} = Y^{-1}\, g_{i,\cdot}$;

(c) Use the "Laplace transform" $\hat{G}(u,v)$ given in (4.10);

(d) Form

$$r_{i,j}$$

$$= \sum_{k=-M_1^{(1)}}^{N_1^{(1)}} x^{ik}\, \hat{G}\left(\left[b_1 - a_1\right]\, s_i^{(1)}, \left[b_2\left(x_k^{(1)}\right) - b_2\left(x_k^{(1)}\right)\right]\, s_j^{(2)}\right)\, h_{k,j}\,;$$

(e) Form

$$q_{\cdot,j} = X\, r_{\cdot,j}\,; \quad p_{i,\cdot}^{(1)} = Y\, q_{i,\cdot}\,.$$

At this point we need to:

(a) Repeat the above steps to evaluate the 3 other indefinite convolutions over $\mathcal{B}_1$, to get the total *convolution contribution* $p_{i,j}$ to U from $\mathcal{B}_1$;

(b) Repeat the above steps for the second integral in (2.11);

(c) We then need to do a *Sinc quadrature* over $\mathcal{B}_2$, to determine the contribution $P_{ij}^{(1)}$ of the integral over $\mathcal{B}_2$ to the *Sinc points* in $\mathcal{B}_1$, and similarly, we also need to do a *Sinc quadrature* over $\mathcal{B}_1$ to determine the contribution of this convolution integral to the *Sinc points* in $\mathcal{B}_2$. These *Sinc quadratures* are possible since the Green's function $G(\bar{\rho}, \bar{\rho}')$ does not have any singularity on the region of integration. The contribution P_{ij} is then determines as follows:

$$
\begin{aligned}
&P_{i,j} \\
&= \int\int_{\mathcal{B}_2} G(\bar\rho_{i,j} - \bar\rho')\, g_2(\bar\rho')\, d\bar\rho' \\
&= \int_0^{3/2} \int_0^{\sqrt{1-(x^{(2)}-1/2)^2}} G\left(x_i^{(1)}, y_{i,j}^{(1)}\,; x^{(2)}, y^{(2)}\right) \cdot \\
&\quad \cdot g\left(x^{(2)}, y^{(2)}\right)\, dy^{(2)}\, dx^{(2)} \\
&= \int_0^{3/2} \int_0^1 G\left(x_i^{(1)}, y_{i,j}^{(1)}\,; x^{(2)}, y^{(2)} \sqrt{1-(x^{(2)}-1/2)^2}\right) \cdot \\
&\quad \cdot g\left(x^{(2)}, y^{(2)} \sqrt{1-(x^{(2)}-1/2)^2}\right) \cdot \\
&\quad \cdot \sqrt{1-(x^{(2)}-1/2)^2}\, dy^{(2)}\, dx^{(2)} \\
&\approx h^2 \sum_{k=-M_1^{(2)}}^{N_1^{(2)}} \sum_{\ell=-M_2^{(2)}}^{N_2^{(2)}} \frac{e^{(k+\ell)\,h}\, g_{k,\ell}}{\left(1+e^{k\,h}\right)^2 \left(1+e^{k\,h}\right)^2} \cdot \\
&\quad \cdot G\left(x_i^{(1)}, y_{i,j}^{(1)}; x_k^{(2)}, y_\ell^{(2)} \sqrt{1-\left(x_k^{(2)}-1/2\right)^2}\right) \cdot \\
&\quad \cdot g\left(x_k^{(2)}, y_\ell^{(2)} \sqrt{1-\left(x_k^{(2)}-1/2\right)^2}\right) \cdot \\
&\quad \cdot \sqrt{1-\left(x_k^{(2)}-1/2\right)^2}
\end{aligned}
$$

(d) This sum has to be done for all integers $(i,j) \in \left[-M_1^{(1)}, N_1^{(1)}\right] \times \left[-M_2^{(1)}, N_2^{(1)}\right]$, and each of these contributions $P_{i,j}$ then needs to be added to $p_{i,j}$. Then repeat, for approximating the integral over B_2.

2.2 Sinc Convolution Solution of a Heat Problem

Next, we consider the heat equation,

$$
\frac{\partial u(\bar r, t)}{\partial t} - \mu \Delta u(\bar r, t) = f(\bar r, t)\,,
$$

with $\mu = 1$, $\bar r = (x, y, z) \in \mathbb{R}^3$ and $r = |\bar r|$. For the numerical tests we chose as right sides the functions

$$
f_1(x, y, z, t) = e^{-\bar r^2 - 0.5 \cdot t}(1 + 5.5 \cdot t - 4 \cdot t \cdot r^2)
$$

Table 2: Results for example 1. *** indicates that FEM was not able to achieve this accuracy.

FEM			SINC		
accuracy ($\|\cdot\|_\infty$)	CPU time (sec)	# of time steps	accuracy ($\|\cdot\|_\infty$)	CPU time (sec)	N
0.0619	10.28 sec	10	0.0621	101 sec	8
0.0293	121.2 sec	10	0.0297	238 sec	10
0.0123	3517 sec	100	0.0132	680 sec	12
0.0055	12994 sec	100	0.0068	1614 sec	14
***	***	***	0.0041	2306 sec	15

Table 3: Results for example 2. *** indicates that FEM was not able to achieve this accuracy.

FEM			SINC		
accuracy ($\|\cdot\|_\infty$)	CPU time (sec)	# of time steps	accuracy ($\|\cdot\|_\infty$)	CPU (sec)	N
0.1139	14 sec	10	0.1458	80 sec	6
0.0435	125 sec	10	0.0394	438 sec	9
0.0198	5309 sec	200	0.0186	1228 sec	11
***	***	***	0.0012	8238 sec	20
***	***	***	0.0006	13592 sec	30

(Table 2) and

$$f_2(x,y,z,t) = e^{-\bar{r}^2-0.5\cdot t}\left(\frac{1}{2\sqrt{t}} + 5.5\sqrt{t} - 4\sqrt{t}\cdot r^2\right)$$

(Table 3); the corresponding solutions are

$$u_1(x,y,z,t) = t \cdot e^{-r^2-0.5\cdot t}$$

and

$$u_2(x,y,z,t) = \sqrt{t}\cdot e^{-r^2-0.5\cdot t} \quad .$$

The FEM solution was computed on a cubic area with center the origin, side length 12 and zero boundary conditions, the time interval was chosen as $t = [0,1]$. In the time variable t, a constant step size was used.

For both problems, the accuracy of the computations was compared on the mesh generated by the FEM method. The total number of Sinc points used in each direction as well as in the time t was $2N+1$. In order to get a higher accuracy for the FEM method, we had to choose smaller time steps in the second half of the computations. As indicated in the table, we failed in our attempt to get higher accuracy for FEM due to memory problems. On the other hand, we *were* able to achieve a higher accuracy via our *Sinc* procedure. We may note that FEM failed for example 2 even earlier than

Table 4: Results for the Wave equation. *** indicates that FEM was not able to achieve this accuracy.

FEM				SINC		
accuracy ($\|\cdot\|_\infty$)	CPU time (sec)	# of time steps	# of knots	accuracy ($\|\cdot\|_\infty$)	CPU (sec)	N
0.0614	7.52	10	181	0.066	1.3	8
0.0073	25.32	10	681	0.008	9.8	15
0.0021	100	20	2641	0.0023	25.83	20
0.0016	556.19	31	10401	0.0014	37.44	22
0.0015	3964	51	41218	0.0014	37.44	22
***	***	***	***	2e-06	717 sec	50
***	***	***	***	2e-07	1421e sec	60

for example 1 sue to a singularity of f_2 at $t = 0$, which resulted in our requirement of a finer mesh for FEM.

2.3 Wave Equation

For a numerical example for solving the 2d wave equation,

$$\frac{1}{c^2}\frac{\partial^2 u(\bar{r},t)}{\partial t^2} - \nabla^2 u(\bar{r},t) = f(\bar{r},t) \ ,$$

we took as right hand side the function

$$f(x,y,t) = e^{(-|r|^2-0.5\cdot t)}\left(\frac{3}{4\cdot\sqrt{t}} - \frac{3}{2}\sqrt{t} + \sqrt{t}^3\cdot\left(\frac{17}{4} - 4\cdot|r|^2\right)\right) \ .$$

The corresponding solution is

$$u(x,y,t) = t^{3/2}\cdot e^{(-|r|^2-0.5\cdot t)} \ .$$

As usual, $r = \sqrt{x^2+y^2}$, and c was set to 1. The results of both methods were compared with the exact solution (with maximum–norm) only in the knots of the FEM mesh. For the previous two examples, the program KSAKADE was used to produce the FEM solution. This time, we used the Matlab PDE–Toolbox. Matlab does not have an adaptive refinement of the mesh. To get different degrees of accuracy, the mesh for FEM was refined by hand; for $SINC$ a larger number of Sinc points was used (number of Sinc points=2N+1). The time was measured by using the Matlab command *cputime*.

It is obvious from Table 4 that the computing time for FEM increases rapidly with a finer mesh without a substantial improvement of the accuracy. Again, trying to use a finer mesh caused some memory problems for FEM. This is not the problem with *SINC*, a substantial improvement of the accuracy is still possible as is shown in the table.

2.4 Solving Burgers' Equation

[5]. Let $\mathbb{R}$ denote the real line, and let u_0 denote a given function defined on $\mathbb{R}$. We shall illustrate an integral equation procedure for solving the Burgers' equation problem

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) - \varepsilon \frac{\partial^2}{\partial x^2} u(x,t) &= -\frac{1}{2}\frac{\partial}{\partial x} u^2(x,t), \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{2.13}$$

We accomplish this by first transforming the problem (2.13) into the equivalent integral equation problem

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi\varepsilon t)^{1/2}} \int_{\mathbb{R}} \exp\left\{ -\frac{(x-\xi)^2}{4\varepsilon t} \right\} u_0(\xi)\, d\xi \\ &+ \pi \int_0^t \int_{\mathbb{R}} \frac{x-\xi}{\{4\pi\varepsilon(t-\tau)\}^{3/2}} \exp\left\{ -\frac{(x-\xi)^2}{4\varepsilon(t-\tau)} \right\} u^2(\xi,\tau)\, d\xi\, d\tau, \end{aligned} \tag{2.14}$$

which we discretize via the *Sinc* collocation procedure of the previous example, and then we solve the resulting discretized system via Neumann iteration.

We take

$$u_0(x) = a \exp\left\{ -b(x-c)^2 \right\}. \tag{2.15}$$

This choice of u_0 enables an explicit expression for the first term on the right-hand side of (2.14), so that we can now rewrite (2.14) in the form

$$\begin{aligned} u(x,t) &= v(x,t) \\ &+ \pi \int_0^t \left[\int_{-\infty}^x \frac{x-\xi}{\{4\pi\varepsilon(t-\tau)\}^{3/2}} \exp\left\{ -\frac{(x-\xi)^2}{4\varepsilon(t-\tau)} \right\} u^2(\xi,\tau)\, d\xi \right. \\ &\left. - \int_x^\infty \frac{\xi - x}{\{4\pi\varepsilon(t-\tau)\}^{3/2}} \exp\left\{ -\frac{(x-\xi)^2}{4\varepsilon(t-\tau)} \right\} u^2(\xi,\tau)\, d\xi \right] d\tau, \end{aligned} \tag{2.16}$$

where

$$v(x,t) = \frac{a}{\{1 + 4b\varepsilon t\}^{1/2}} \exp\left\{ -\frac{b(x-c)^2}{1 + 4b\varepsilon t} \right\}. \tag{2.17}$$

Due to this explicit form of the function $v(x,t)$, the form (2.15) for u_0 makes it possible to approximate an arbitrary continuous function u_0 defined on $\mathbb{R}$ by use of the function $F_3(\beta, h)$ defined in [14], §5.8.

We now proceed to discretize Equation (2.16) as outlined in Example 2.2. To this end we may note that it is possible to explicitly evaluate the "Laplace transform" of the convolution kernel in (2.16), i.e.,

$$\begin{aligned} F(s,\sigma) &= \int_0^\infty \int_0^\infty \exp\left\{-\frac{x}{s} - \frac{t}{\sigma}\right\} \frac{x}{\{4\pi\varepsilon t\}^{3/2}} \exp\left\{-\frac{x^2}{4\varepsilon t}\right\} dx\, dt \\ &= \frac{1}{4\varepsilon^{1/2}} \frac{s\,\sigma^{1/2}}{s+\varepsilon^{1/2}\sigma^{1/2}}. \end{aligned} \tag{2.18}$$

We now select $\varepsilon = 1/2$, $b = 1$, $c = 0$, $\phi_t(t) = \log(t)$, $\phi_x(x) = x$, $d_t = \pi/2$, $\alpha_t = \beta_t = 1/2$, $d_x = \pi/4$, $\alpha_x = \beta_x = 1$, and in this case it is convenient to take $M_t = N_t = M_x = N_x = N$ and $h = 2/\sqrt{N}$. We thus form matrices

$$\begin{aligned} &A_x = h_x I^{(-1)} = X_x S_x X_x^{-1}, \quad A_x' = h_x (I^{(-1)})^T = (X_x^{-1})^T S_x X_x^T, \\ &B_t = h_t I^{(-1)} D(1/\phi_t') = X_t S_t X_t^{-1}, \end{aligned} \tag{2.19}$$

where the superscript "T" denotes the transpose, and where S_x and S_t are diagonal matrices, and then proceed as in Example 2.2 above, and the notation of Equation (4.6) to reduce the integral equation problem (2.16) to the nonlinear matrix problem

$$[u_{ij}] = F(A_x, B_t, [u_{ij}^2]) - F(A_x', B_t, [u_{ij}^2]) + [v_{ij}], \tag{2.20}$$

with e.g., if we list as a single vector the columns of a rectangular matrix $[c_{i,j}]$, (denote it by $\mathrm{col}\{[c_{i,j}]\}$) then similarly list the columns of $F(A_x, B_t, [v_{i,j}]$, and then form a diagonal matrix $\mathbf{F}$ by listing the numbers $F(s_i, \sigma_j)$ in the same order, then

$$\mathrm{col}\{F(A_x, B_t, [c_{i,j}]\} = X_t \otimes X_x\, \mathbf{F}\, X_t^{-1} \otimes X_x^{-1}\, \mathrm{col}\{[c_{i,j}]\}\,, \tag{2.21}$$

and where the function v_{ij} may be evaluated *a priori*, via the formula $v_{ij} = v(ih_x, z_j)$, with $v(x,t)$ defined as in (2.17), and with $z_j = e^{j\,h_t}$.

The system (2.20) may be solved by Neumann iteration, for a (defined as in (2.15)) sufficiently small. Neumann iteration takes the form

$$[u_{ij}^{(k+1)}] = F(A_x, B_t, [(u_{ij}^{(k)})^2]) - F(A_x', B_t, [(u_{ij}^{(k)})^2]) + [v_{ij}], \tag{2.22}$$

for $k = 0, 1, 2, \ldots$, starting with $[u_{ij}^{(0)}] = [v_{ij}]$. For example, with $a = 1/2$, and using the map $\phi_t(t) = \log[\sinh(t)]$ we achieved convergence in 4 iterations, for all values of N (between 10 and 30) that we attempted. We can also solve the above equation via Neumann iteration for larger values of a, if we restrict the time t to a finite interval, $(0,T)$, via the map $\phi_t(t) = \log\{t/(T-t)\}$.

Let us now also consider the convergence of the iteration procedure (2.22). To this end, let us assume that we have determined the integers

$M_x = N_x = M_t = N_t = N$, as well as $h = 2/\sqrt{N}$ and the time interval T to enable achievement of a certain accuracy in the approximate solution to the problem (2.20). We wish to illustrate the existence of $T = T_0$, such that if the parameters are fixed in this manner, and the "time map" is selected by $w = \phi_t(t) = \log(t/(T-t))$, then (2.20) is a contraction map for all $T < T_0$. To this end, we note from (2.19) above, that A_x and A_x' are unchanged, whereas

$$B_t = h\,T\,I^{(-1)}\,D\left(\frac{e^w}{(1+e^w)^2}\right)\,,\quad w = k\,h_t\,,\quad k = -N,\ldots,N\,,$$

that is, the eigenvalues of the diagonal matrix S_t in (2.19) are proportional to T, whereas the eigenvector matrix X_t is independent of T. By (2.18) it thus follows that e.g.,

$$\begin{aligned}&\|\mathrm{col}\{F(A_x, B_t, [c_{i,j}]\}\| \\ &\le \|X_t\|\,\left\|X_t^{-1}\right\|\,\|X_x\|\,\|X_x^{-1}\|\,\|\mathbf{F}\|\,\|\mathrm{col}\{[c_{i,j}]\}\|\,,\end{aligned}$$

where, by (2.20), and the above expression for B_t,

$$\|\mathbf{F}\| = \mathcal{O}\left(T^{1/2}\right)\,,\quad T \to 0\,.$$

That is, the right hand side of (2.20) is a contraction map for all sufficiently small T.

Similar results obtain for the above cases of the wave and heat equations, as well as for the case of the electric field integral equation which is considered below, in the cases when the Green's function approach is used to reduce a PDE to an equivalent integral equation formulation.

2.5 Solving the Electric Field Integral Equation

[11].

Our final example involves the electric field integral equation, which takes the form

$$\mathbf{e}^{in}(\mathbf{r},t) = \mathbf{e}(\mathbf{r},t) - \int_V \int_0^t \left(\int_0^{t'} \gamma(\mathbf{r}',t'-\xi)\mathbf{e}(\mathbf{r}',\xi)d\xi\right) g(\mid \mathbf{r}-\mathbf{r}'\mid, t-t')dt'd^3\mathbf{r}', \tag{2.23}$$

where the time-domain Green's function is given in terms of $r = |\mathbf{r}|$, i.e.,

$$g(\mathbf{r},t) = \frac{1}{4\pi r}e^{-a\frac{r}{c}}\delta\left(t-\frac{r}{c}\right) + \frac{a\frac{r}{c}e^{-at}}{4\pi r\left(t^2-\frac{r^2}{c^2}\right)^{\frac{1}{2}}}I_1\left[a\left(t^2-\frac{r^2}{c^2}\right)^{\frac{1}{2}}\right]u\left(t-\frac{r}{c}\right), \tag{2.24}$$

where u is the *Heaviside function* and I_1 is the *modified Bessel function of the first kind* of order one, with $a = \frac{\sigma_0}{2\epsilon_0}$. The constant $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the velocity of the wave in V. Furthermore, $\gamma(\mathbf{r}, \mathbf{t})$ is the time-domain scattering potential, which is given as the product of two functions for $t > 0$, one with space variables and the other with time variable, i.e.

$$\gamma(\mathbf{r}, t) = \gamma_1(\mathbf{r})\gamma_2(\mathbf{r}, t), \tag{2.25}$$

where

$$\gamma_1(\mathbf{r}) = \mathbf{z}^{\frac{1}{2}} e^{-r^2} + j\mathbf{z}^{\frac{2}{3}} e^{-|\mathbf{r}-(0,0,3)|^2}, \tag{2.26}$$

and where the "Laplace transform" of $\gamma_2(\mathbf{r}, t)$ taken with respect to t is

$$\Gamma_2(\mathbf{r}, \tau) = \int_0^\infty \exp(-t/\tau)\, \gamma_2(\mathbf{r}, t)\, dt = \frac{(\sigma(\mathbf{r}) - \sigma_0)\tau + (\varepsilon(\mathbf{r}) - \varepsilon_0)}{\sigma_0 \tau + \varepsilon_0}, \tag{2.27}$$

We shall obtain an approximate solution to this integral equation for $\{(\mathbf{r}, t) \in V \times (0, T)\}$, where

$$V = \left\{ \mathbf{r} = (x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2, z > 0 \right\}. \tag{2.28}$$

The *Sinc-convolution* method of solution requires the "Laplace transform" $\mathcal{G}(u, v, w, \tau)$ of the kernel of the integral equation (2.23), which is the product not only the "Laplace transform" of $\gamma_2(\mathbf{r}, t)$ with respect to t, and the four dimensional "Laplace transform" of the time domain free space Green's function $g(x, y, z, t)$ with respect to all variables

$$\mathcal{G}(u, v, w, \tau) = \Gamma_2(\mathbf{r}, \tau)\, G(u, v, w, \tau). \tag{2.29}$$

These "Laplace transforms" may in fact be explicitly expressed in terms of the results given in §4 of this paper. Moreover, the resulting Sinc convolution layout can be solved via Neumann iteration, analogous to that of Burgers' equation above. Furthermore, this iteration scheme may be shown to converge provided the time interval $(0, T)$ is sufficiently small, via an argument similar to that used at the end of the Burgers equation example. However, we omit the lengthy details, which will be published elsewhere. It is, nevertheless interesting to compare the performance the Sinc convolution method with that of Ye's [18] finite difference solution method. For these comparisons, see Table 5. In this table, all entries except those with a "*" are actual computation times. The entries marked with a "*" are computed times based on the convergence rates obtained by Monk & Süli in [10].

3 Sinc Terminology

This appendix is a summary of the *Sinc* notation which we require for the presentation of the results of the paper. Most of the results are proved elsewhere, i.e., in [9, 14, 15]. The new results, such as the extension of *Sinc convolution* to curvilinear regions are presented with proofs. Our manner of description of the methods is in symbolic form. We include methods for collocation, function interpolation and approximation, for approximate

Table 5: The IBM RISC/560 workstation run-times Computation time required by Ye's Finite Difference and *Sinc-convolution* methods vs. desired precision. Computer run–time is shown as Days: Hours: Minutes: Seconds

Precision	Finite Difference Run-Time	Sinc-Convolution Run-Time
$\mathbf{10^{-1}}$	< 1 second	< 1 second
$\mathbf{10^{-2}}$	000:00:00:27	000:00:00:06
$\mathbf{10^{-3}}$	003:00:41:40*	000:00:02:26
$\mathbf{10^{-4}}$	> 82 years*	000:00:43:12
$\mathbf{10^{-5}}$	> 800,000 years*	000:06:42:20
$\mathbf{10^{-6}}$	> 8.2 billion years*	001:17:31:11

definite and indefinite integration, for the approximation of definite and indefinite convolutions, including multidimensional extensions of these for the approximate solution of partial differential and integral equations.

3.1 One Dimensional Sinc Spaces

Let $\mathcal{D}$ denote a simply–connected domain in the complex plane $\mathbb{C}$, let $1 \leq p \leq \infty$, and let $\mathbf{H}^p(\mathcal{D})$ denote the family of all functions f that are analytic in $\mathcal{D}$, such that

$$N_p(f, \mathcal{D}) \equiv \begin{cases} \left(\int_{\partial \mathcal{D}} |f(z)|^p |dz| \right)^{1/p} < \infty \ \text{ if } \ 1 \leq p < \infty, \\ \\ \sup\limits_{z \in \mathcal{D}} |f(z)| < \infty \ \text{ if } \ p = \infty. \end{cases} \tag{3.1}$$

In essence, we consider two spaces of functions $\mathbf{M}_{\alpha,\beta}(\Gamma)$ and $\mathbf{L}_{\alpha,\beta}(\Gamma)$ for purposes of *Sinc* approximation on an interval or contour. Consider first the case of a finite interval, (a, b). Perhaps the simplest concept of the space of functions $\mathbf{M}_{\alpha,\beta}(a, b)$, with $0 < \alpha \leq 1$, $0 < \beta \leq 1$, is that consisting of all functions that are analytic on the open interval (a, b), of class $\mathbf{Lip}_\alpha$ in a neighborhood of a, and of class $\mathbf{Lip}_\beta$ in a neighborhood of b. The corresponding space $\mathbf{L}_{\alpha,\beta}(a, b)$ consists of the set of all functions $f \in \mathbf{M}_{\alpha,\beta}(a, b)$ for which $f(a) = f(b) = 0$.

More generally, if (a, b) is a contour Γ, such as, e.g., the interval $(0, \infty)$, or the real line $\mathbb{R}$, (or even an analytic arc in the complex plane), the mapping ϕ is selected to be a conformal mapping of a domain $\mathcal{D}$ onto $\mathcal{D}_d$, with $\mathcal{D}_d$ defined as above, such that ϕ is also a one-to-one map of Γ onto $\mathbb{R}$. We define ρ by $\rho = e^{\phi}$. Note that $\rho(z)$ increases from 0 to ∞ as z traverses Γ from a to b.

Let α, β and d denote arbitrary, fixed positive numbers. We denote by $\mathbf{L}_{\alpha,\beta}(\Gamma)$ the family of all functions that are analytic and uniformly bounded in $\mathcal{D}$, such that

$$f(z) = \begin{cases} \mathcal{O}(|\rho(z)|^{\alpha}), & \text{uniformly as } z \to a \text{ from within } \overline{\mathcal{D}}, \\ \mathcal{O}(|\rho(z)|^{-\beta}), & \text{uniformly as } z \to b \text{ from within } \overline{\mathcal{D}}. \end{cases} \tag{3.2}$$

We next define the class of functions $\mathbf{M}_{\alpha,\beta}(\Gamma)$, but this time restricting α, β and d such that $\alpha \in (0,1]$, $\beta \in (0,1]$ and $d \in (0,\pi)$. This class consists of all those functions $g \in \mathbf{Hol}(\mathcal{D})$, that have finite limits at a and b, so that the function $\mathcal{L}f$ is well defined, where

$$\mathcal{L}g(z) = \frac{f(a) + \rho(z)\, f(b)}{1 + \rho(z)}, \quad \rho = e^{\phi}, \tag{3.3}$$

and such that if f is defined by

$$f = g - \mathcal{L}g \tag{3.4}$$

then $f \in \mathbf{L}_{\alpha,\beta}(\Gamma)$.

Note that if $0 < d < \pi$, then $\mathcal{L}(g)$ is uniformly bounded in $\overline{\mathcal{D}}$, the closure of $\mathcal{D}$, and moreover, $\mathcal{L}(g)(z) - f(a) = \mathcal{O}(|\rho(z)|)$ as $z \to a$, and $\mathcal{L}(g)(z) - f(b) = \mathcal{O}(1/|\rho(z)|)$ as $z \to b$, i.e., $\mathcal{L}(g) \in \mathbf{M}_{1,1}(\Gamma)$. Furthermore, $\mathbf{M}_{1,1}(\Gamma) \subseteq \mathbf{M}_{\alpha,\beta}(\Gamma)$ for any $\alpha \in (0,1]$, $\beta \in (0,1]$, and $d \in (0,\pi)$, and moreover for these restrictions on α, β, and d, the class $\mathbf{L}_{\alpha,\beta}(\Gamma)$ is contained in the class $\mathbf{M}_{\alpha,\beta}(\Gamma)$.

The spaces $\mathbf{L}_{\alpha,\beta}(\Gamma)$ and $\mathbf{M}_{\alpha,\beta}(\Gamma)$ are motivated by the premise that most scientists and engineers use calculus to model differential and integral equation problems, and under this premise the solution to these problems are (at least piecewise) analytic. The spaces $\mathbf{L}_{\alpha,\beta}(\Gamma)$ and $\mathbf{M}_{\alpha,\beta}(\Gamma)$ house nearly all solutions to such problems, including solutions with singularities at end points of (finite or infinite) intervals (or at boundaries of finite or infinite domains in more than one dimension). Although these spaces also house singularities, they are not as large as Sobolev spaces which assume the existence of only a finite number of derivatives in a solution, and consequently (see below) when *Sinc* methods are used to approximate solutions of differential or integral equations, they are usually more efficient than finite difference or finite element methods. In addition, *Sinc* methods are replete with interconnecting simple identities, including DFT (which is one of the *Sinc* methods, enabling the use of FFT), making it possible to use a *Sinc* approximation for nearly every type of operation arising in the solution of differential and integral equations.

Let us describe some specific spaces for one dimensional *Sinc* approximation.

Example 3.1: If $\Gamma = (0,1)$, and if $\mathcal{D}$ is the "eye–shaped" region, $\mathcal{D} = \{z \in \mathbb{C} : |\arg[z/(1-z)]| < d\}$, then $\phi(z) = \log[z/(1-z)]$, the relation

(3.3) reduces to $f = g - (1-x)g(0) - xg(1)$, and $\mathbf{L}_{\alpha,\beta}(\Gamma)$ is the class of all functions $f \in \mathbf{Hol}(\mathcal{D})$, such that for all $z \in \mathcal{D}$, $|f(z)| < c|z|^{\alpha}|1-z|^{\beta}$. In this case, if e.g. $\delta = \max\{\alpha, \beta\}$, and a function w is such that $w \in \mathbf{Hol}(\mathcal{D})$, and $w \in \mathbf{Lip}_{\delta}(\mathcal{D})$, then $w \in \mathbf{M}_{\alpha,\beta}(\Gamma)$. The *Sinc* points z_j are $z_j = e^{jh}/(1+e^{jh})$, and $1/\phi'(z_j) = e^{jh}/(1+e^{jh})^2$.

Example 3.2: If $\Gamma = (0, \infty)$, and if $\mathcal{D}$ is the "sector" $\mathcal{D} = \{z \in \mathbb{C} : |\arg(z)| < d\}$, then $\phi(z) = \log(z)$, the relation (3.3) reduces to $f(z) = g(z) - [g(0) + z\, g(\infty)]/(1+z)$, and the class $\mathbf{L}_{\alpha,\beta}(\Gamma)$ is the class of all functions $f \in \mathbf{Hol}(\mathcal{D})$ such that if $z \in \mathcal{D}$ and $|z| \le 1$ then $|f(z)| \le c|z|^{\alpha}$, while if $z \in \mathcal{D}$ and $|z| \ge 1$, then $|f(z)| \le c|z|^{-\beta}$. This map thus allows for algebraic decay at both $x = 0$ and $x = \infty$. The *Sinc* points z_j are defined by $z_j = e^{jh}$, and $1/\phi'(z_j) = e^{jh}$.

Example 3.3: If $\Gamma = (0, \infty)$, and if $\mathcal{D}$ is the "bullet–shaped" region $\mathcal{D} = \{z \in \mathbb{C} : |\arg(\sinh(z))| < d\}$, then $\phi(z) = \log(\sinh(z))$. The relation (3.3) then reduces to $f(z) = g(z) - [g(0) + \sinh(z)\, g(\infty)]/(1 + \sinh(z))$, and $\mathbf{L}_{\alpha,\beta}(\Gamma)$ is the class of all functions $f \in \mathbf{Hol}(\mathcal{D})$ such that if $z \in \mathcal{D}$ and $|z| \le 1$ then $|f(z)| \le c|z|^{\alpha}$, while if $z \in \mathcal{D}$ and $|z| \ge 1$, then $|f(z)| \le c\exp\{-\beta|z|\}$. This map thus allows for algebraic decay at $x = 0$ and exponential decay at $x = \infty$. The *Sinc* points z_j are defined by $z_j = \log[e^{jh} + (1 + e^{2jh})^{1/2}]$, and $1/\phi'(z_j) = (1 + e^{-2jh})^{-1/2}$.

Example 3.4: If $\Gamma = \mathbb{R}$, and if $\mathcal{D}$ is the above defined "strip", $\mathcal{D} = \mathcal{D}_d$, take $\phi(z) = z$. The relation (3.3) then reduces to $f(z) = g(z) - [g(-\infty) + e^{z}\, g(\infty)]/(1+e^{z})$. The class $\mathbf{L}_{\alpha,\beta}(\mathcal{D})$ is the class of all functions $f \in \mathbf{Hol}(\mathcal{D})$ such that if $z \in \mathcal{D}$ and $\Re z \le 0$ then $|f(z)| \le ce^{-\alpha|z|}$, while if $z \in \mathcal{D}$ and $\Re z \ge 0$, then $|f(z)| \le ce^{-\beta|z|}$. Thus this map allows for exponential decay at both $x = -\infty$ and $x = \infty$. The *Sinc* points z_j are defined by $z_j = jh$, and $1/\phi'(z_j) = 1$.

Example 3.5: If $\Gamma = \mathbb{R}$, and if $\mathcal{D}$ is the "hour glass–shaped" region, $\mathcal{D} = \{z \in \mathbb{C} : |\arg[z + (1+z^2)^{1/2}]| < d\}$, take $\phi(z) = \log[z + (1+z^2)^{1/2}]$. The relation (3.3) reduces to $f(z) = g(z) - [g(-\infty) + (z + (1+z^2)^{1/2})\, g(\infty)]/[1 + z + (1+z^2)^{1/2}]$, and the class $\mathbf{L}_{\alpha,\beta}(\Gamma)$ is the class of all functions $f \in \mathbf{Hol}(\mathcal{D})$ such that if $z \in \mathcal{D}$ and $\Re z \le 0$, then $|f(z)| \le c(1 + |z|)^{-\alpha}$, while if $z \in \mathcal{D}$ and $\Re z \ge 0$, then $|f(z)| \le c(1 + |z|)^{-\beta}$. This map thus allows for algebraic decay at both $x = -\infty$ and $x = \infty$. The *Sinc* points z_j are defined by $z_j = \sinh(jh)$, and $1/\phi'(z_j) = \cosh(jh)$.

Example 3.6: If $\Gamma = \mathbb{R}$, and if $\mathcal{D}$ is the "funnel–shaped" region, $\mathcal{D} = \{z \in \mathbb{C} : |\arg\{\sinh[z + (1+z^2)^{1/2}]\}| < d\}$, take $\phi(z) = \log\{\sinh[z + (1 + z^2)^{1/2}]\}$. The relation (3.3) then reduces to
$f(z) = g(z) - [g(-\infty) + \sinh(z + (1+z^2)^{1/2})\, g(\infty)]/[1 + \sinh(z + (1+z^2)^{1/2})]$, and $\mathbf{L}_{\alpha,\beta}(\Gamma)$ is the class of all functions $f \in \mathbf{Hol}(\mathcal{D})$ such that if $z \in \mathcal{D}$ and $\Re z \le 0$, then $|f(z)| \le c(1 + |z|)^{-\alpha}$, while if $z \in \mathcal{D}$ and $\Re z \ge 0$, then $|f(z)| \le ce^{-\beta|z|}$. This map thus allows for algebraic decay at $x = -\infty$ and exponential decay at $x = \infty$. The *Sinc* points z_j are defined by $z_j = (1/2)[t_j - 1/t_j]$, where $t_j = \log[e^{jh} + (1 + e^{2jh})^{1/2}]$, and $1/\phi'(z_j) =$

$(1/2)(1+1/t_j^2)(1+e^{-2jh})^{-1/2}$.

3.2 Multidimensional Sinc Spaces

Precise definitions of these spaces may be found in [14] §6.53, §6.6.2 & §7.3.2. In essence, a function f defined on e.g., a bounded region $V \in R^d$ belongs to a proper Sinc space if given a point $\mathbf{x} = (x_1, \ldots, x_d)$ in the closure of V, then corresponding to each $j = 1, \ldots, d$, let us fix all of the coordinates x_i, $i \neq j$, and call the resulting function $f_j(x_j)$. Denoting by $\Gamma_j =\in V$ the interval of longest length in the x_j direction that contains the point $\mathbf{x}$, we want f_j to be analytic in the interior of Γ_j, and to be of class $\mathbf{Lip}_\alpha$ on the closure of Γ_j. If all of these conditions are satisfied, then by taking $h = c_1/\sqrt{(}N)$, and performing Sinc approximation with N point evaluations in each dimension, i.e., for a total of N^d points, we are able to achieve an error of the order of $\exp\left(-c'\sqrt{N}\right)$.

3.3 One Variable Sinc Approximation

1. *Notation.*

   Sinc approximation in $\mathbf{M}_{\alpha,\beta}(\Gamma)$ is defined as follows. Let N denote a positive integer, and let integers M, and m, a diagonal matrix $D(u)$ and an operator V_m be defined as follows

$$\begin{aligned} N &= \text{positive integer} \\ M &= [\beta N/\alpha] \\ m &= M+N+1 \\ D(u) &= \text{diag}[u(z_{-M}), \ldots, u(z_N)] \\ V_m(u) &= (u(z_{-M}), \ldots, u(z_N))^T, \end{aligned} \tag{3.5}$$

   where $[\cdot]$ denotes the greatest integer function, where u is an arbitrary function defined on Γ, and where "T" denotes the transpose.

   We shall also define a norm by

$$\|f\| = \sup_{x\in\Gamma} |f(x)|,$$

   and throughout this section C will denote a generic constant, independent of N.

2. *Sinc Basis.*

   Letting $\mathbb{Z}$ denote the set of all integers, set

$$
\begin{aligned}
\operatorname{sinc}(z) &= \frac{\sin(\pi z)}{\pi z}, \\
h &= \left(\frac{\pi d}{\beta N}\right)^{1/2}, \\
z_j &= \phi^{-1}(jh), \quad j \in \mathbb{Z} \\
\gamma_j &= \operatorname{sinc}\{[\phi - jh]/h\}, \quad j = -M, \ldots, N, \\
w_j &= \gamma_j, \quad j = -M+1, \ldots, N-1, \\
w_{-M} &= \frac{1}{1+\rho} - \sum_{j=-M+1}^{N} \frac{1}{1+e^{jh}} \gamma_j, \\
w_N &= \frac{\rho}{1+\rho} - \sum_{j=-M}^{N-1} \frac{e^{jh}}{1+e^{jh}} \gamma_j, \\
\varepsilon_N &= N^{1/2} e^{-(\pi d \beta N)^{1/2}}.
\end{aligned}
\tag{3.6}
$$

We may thus define a row vector $\mathbf{w}$ of basis functions by

$$
\mathbf{w} = (w_{-M}, \ldots, w_N)
$$

with w_j defined as in (3.6) and for given vector $\mathbf{c} = (c_{-M}, \ldots, c_N)^T$, we have

$$
\mathbf{w}_m \mathbf{c} = \sum_{j=-M}^{N} c_j \, w_j \, . \tag{3.7}
$$

3. *Sinc Interpolation and Approximation.*

A proof of the following result may be found in [2, 3] (see e.g., [2, pp. 126–132].

Theorem 3.1 *If* $f \in \mathbf{M}_{\alpha,\beta}(\Gamma)$*, then*

$$
\| f - \mathbf{w}_m \, V_m \, f \| \leq C \varepsilon_N. \tag{3.8}
$$

The constants in the exponent in the definition of ε_N are the best constants for approximation in $\mathbf{M}_{\alpha,\beta}(\Gamma)$. Hence accurate *Sinc* approximation of f is based on our being able to make good estimates on α, β, and d. If these constants cannot be accurately estimated, e.g., if instead of as in (3.6) above, we define h by $h = \gamma / N^{1/2}$, with γ a constant independent of N, then the right-hand side of (3.8) is replaced by $C e^{-\delta N^{1/2}}$, where C and δ are some positive constants independent of N. Henceforth we shall take h as defined in (3.6).

Remark: We remark, that if $f \in \mathbf{L}_{\alpha,\beta}(\mathcal{D})$, then it is convenient to take $w_j = \operatorname{sinc}\{[\phi - jh]/h\}$, $j = -M, \ldots, N$, instead of as defined in (3.6), since the corresponding approximation of f as defined in (3.2) then also vanishes at the end points of Γ, just as f then vanishes at the end points of Γ.

4. *Sinc Collocation.*

   The following result, guarantees an accurate final approximation of f on Γ, provided that we know a good approximation to f at the *Sinc* points (for a proof, see [9], p. 132).

   Theorem 3.2 *Let $f \in \mathbf{M}_{\alpha,\beta}(\Gamma)$, and let the conditions of Theorem 3.1 be satisfied. Let $\mathbf{c} = (c_{-M}, \ldots, c_N)^T$ be a complex vector of order m, such that*

$$\left(\sum_{j=-M}^{N} |f(z_j) - c_j|^2 \right)^{1/2} < \delta, \tag{3.9}$$

   where δ is a positive number. If C and ε_N are defined as in (3.8), and if w_j is defined as in (3.6), then

$$\| f - \mathbf{w}_m \, \mathbf{c} \| < C \varepsilon_N + \delta. \tag{3.10}$$

5. *Sinc Quadrature.*

   We also record the standard *Sinc* quadrature formula, which belongs to the family of tools for solving differential and integral equations (see [14], §4.2).

   Theorem 3.3 *If $f/\phi' \in \mathbf{L}_{\alpha,\beta}(\Gamma)$, then*

$$\left| \int_a^b f(x)\, dx - h\, \{V_m(1/\phi')\}^T (V_m f) \right| \leq C \varepsilon_N. \tag{3.11}$$

6. *Sinc Indefinite Integration.*

   A detailed derivation and proof of Sinc indefinite integration is given in [14], §3.6 and 4.5.

   Let us next describe *Sinc* indefinite integration or convolution over an interval or a contour. At the outset, we define numbers σ_k and e_k, by

$$\begin{aligned} \sigma_k &= \int_0^k \operatorname{sinc}(x)\, dx, \quad k \in \mathbb{Z}, \\ e_k &= 1/2 + \sigma_k. \end{aligned} \tag{3.12}$$

   We use the notation of (3.6), and we define a Toeplitz matrix $I^{(-1)}$ of order m by $I^{(-1)} = [e_{i-j}]$, with e_{i-j} denoting the (i,j)th element of $I^{(-1)}$. We then define operators $\mathcal{J}$ and $\mathcal{J}'$, and matrices A_m and B_m by

$$\begin{array}{rclrcl}
(\mathcal{J} f)(x) & = & \int_a^x f(t)\,dt, & (\mathcal{J}^* f)(x) & = & \int_x^b f(t)\,dt, \\
A_m & = & h\,I^{(-1)} D(1/\phi'), & B_m & = & h\,(I^{(-1)})^T D(1/\phi'), \\
\mathcal{J}_m & = & \mathbf{w}_m\,A_m\,V_m, & \mathcal{J}_m^* & = & \mathbf{w}_m\,B_m\,V_m,
\end{array} \tag{3.13}$$

with $(I^{(-1)})^T$ denoting the transpose of $I^{(-1)}$. We thus obtain the following theorem [9],

Theorem 3.4 *If $f/\phi' \in \mathbf{L}_{\alpha,\beta}(\Gamma)$, then*

$$\begin{array}{rcl}
\|\mathcal{J} f - \mathcal{J}_m f\| & \le & C\varepsilon_N, \\
\|\mathcal{J}^* f - \mathcal{J}_m^* f\| & \le & C\varepsilon_N.
\end{array} \tag{3.14}$$

7. *Sinc Indefinite Convolution.*

Indefinite convolution integrals can also be effectively collocated via Sinc methods. To this end, we begin with the *model integrals*,

$$\begin{array}{rcl}
p(x) & = & \displaystyle\int_a^x f(x-t)\,g(t)\,dt, \\
q(x) & = & \displaystyle\int_x^b f(t-x)\,g(t)\,dt,
\end{array} \tag{3.15}$$

where $x \in \Gamma$. In presenting these convolution results, we shall assume that $\Gamma = (a,b) \subseteq \mathbb{R}$, unless otherwise indicated. Note also, that being able to collocate p and q enables us to collocate both definite convolutions

$$\begin{array}{c}
\displaystyle\int_a^b f(x-t)\,g(t)\,dt \\
\displaystyle\int_a^b f(|x-t|)\,g(t)\,dt.
\end{array} \tag{3.16}$$

Sinc collocation of p and q is possible under the following

Assumption 3.5 *We assume that the "Laplace transform",*

$$F(s) = \int_E f(t)\,e^{-t/s}\,dt \tag{3.17}$$

with E any subset of $\mathbb{R} = (-\infty,\infty)$ such that $E \supseteq (0, b-a)$, exists for all $s \in \Omega^+ \equiv \{s \in \mathbb{C} : \Re s > 0\}$.

In this notation, one gets the rather "esoteric results" [15],

$$p = F(\mathcal{J})\,g, \qquad q = F(\mathcal{J}^*)\,g\,. \tag{3.18}$$

However, the previous theorem suggests that the approximations $\mathcal{J}\,g \approx \mathcal{J}_m\,g$ and $\mathcal{J}^*\,g \approx \mathcal{J}_m^*\,g$ are accurate, at least for g in certain spaces of *functions, and this is indeed the case. In fact, with the above definition of $\mathcal{J}_m = \mathbf{w}_m A_m V_m$, upon diagonalization of A_m in the form $A_m = X_1 \Lambda X_1^{-1}$, with $\Lambda = \mathrm{diag}[\lambda_{-M_1}, \ldots, \lambda_{N_1}]$, and the esoteric forms (3.18) become computationally feasible, as follows:*

$$\begin{aligned} F(\mathcal{J})\, g &\approx F(\mathcal{J}_m)\, g \\ &= \mathbf{w}_m F(A_m)\, V_m\, g \\ &= \mathbf{w}_m X_1\, F(\Lambda)\, X_1^{-1}\, V_m\, g\,. \end{aligned} \tag{3.19}$$

As to convergence, let $P(r,x)$ be defined by

$$P(r,x) = \int_a^x f(r+x-t)\, g(t)\, dt. \tag{3.20}$$

We assume that

(i) $P(r,\cdot) \in \mathbf{M}_{\alpha,\beta}(\Gamma)$, uniformly for $r \in [0, b-a]$; and that

(ii) $P(\cdot, x)$ is of bounded variation on $(0, [b-a])$, uniformly for $x \in [a,b]$.

Under these assumptions, we have (see [14], §4.6, or [9] for a proof)

Theorem 3.6 *If the above assumptions are satisfied, and if A_m and B_m are defined as in (3.13), then [15]*

$$\begin{aligned} \|p - \mathbf{w}_m F(A_m) V_m g\| &\leq C\varepsilon_N, \\ \|q - \mathbf{w}_m F(B_m) V_m g\| &\leq C\varepsilon_N. \end{aligned} \tag{3.21}$$

Remark: We remark here that it may be shown [14] that every eigenvalue of the matrices $I^{(-1)}$ lies in $\overline{\Omega^+}$, where Ω^+ denotes the right half plane, and also, we have shown by direct computation, that every eigenvalue of the matrices $I^{(-1)}$ lies in Ω^+ for $m = 1, 2, \ldots, 513$. It thus follows, thus, at least for the case when (a,b) is a subinterval of $\mathbb{R}$, that the matrices $F(A_m)$ and $F(B_m)$ are well defined, and may be evaluated in the usual way, via diagonalization of A_m and B_m. (We have also tacitly assumed here that A_m and B_m can be diagonalized, which has so far always been the case for the problems that we have attempted.)

3.4 Sinc Approximation of Multidimensional Convolutions

In this section we illustrate the extension of one dimensional convolution to the approximation of multidimensional convolution integrals. The reader should note that this algorithm actually yields a separation of variables, enabling the approximation of multidimensional convolution integrals via a sequence of one–dimensional matrix multiplications. Thus, the "big matrix" that one requires for the solution of partial differential equations via classical finite difference and finite element techniques need never be stored, so that problems that require matrix sizes of e.g. $10^7 \times 10^7$ can readily be dealt with. At first we summarize the known results over rectangular regions, leaving out some details which may be found in [15], [14], §4.6. We then give a detailed derivation of indefinite convolution over curvilinear regions. The combination of these two algorithms enables us to solve most PDE problems stemming from applications whose solution can be expressed as convolution integrals over curvilinear regions.

1. *Convolutions over Rectangular Regions*

We briefly illustrate in what follows, an algorithm for evaluating a two dimensional convolution integral based on the *Sinc* convolution Theorem 3.8 above.

We illustrate here, the approximation of a convolution integral of the form

$$p(x,y) = \int_{a_2}^{y} \int_{x}^{b_1} f(x-\xi, \eta - y)\, g(\xi,\eta)\, d\xi d\eta, \tag{3.22}$$

where the approximation is sought over the region $\mathcal{B} = \prod_{i=1}^{2} \otimes (a_i, b_i)$, and with $(a_i, b_i) \subseteq \mathbb{R}$. We assume that the mappings $\phi_j : \mathcal{D}'_j \to \mathcal{D}_{d'}$ have been determined. We furthermore assume that positive integers N_j and M_j as well as positive numbers h_j $(j = 1, 2)$ have been selected, we set $m_j = M_j + N_j + 1$, and we define the *Sinc* points by $z_\ell^{(j)} = \phi^{-1}(\ell h_j)$, for $\ell = -M_j, \ldots, N_j; j = 1, 2$. Next, we determine matrices A_j, X_j, S_j and X_j^{-1}, such that

$$\begin{array}{rclcl} A_1 & = & h_1 \left(I_{m_1}^{(-1)}\right)^T D(1/\phi_1') & = & X_1 S_1 X_1^{-1}, \\ A_2 & = & h_2 I_{m_2}^{(-1)} D(1/\phi_2') & = & X_2 S_2 X_2^{-1}. \end{array} \tag{3.23}$$

In (3.23), $I_{m_j}^{(-1)}$ is defined as in (3.13) above, and the S_j are diagonal matrices,

$$S_j = \operatorname{diag}[s_{-M_j}^{(j)}, \ldots, s_{N_j}^{(j)}]. \tag{3.24}$$

We require the two dimensional "Laplace transform"

$$F(s^{(1)}, s^{(2)}) = \int_0^\infty \int_0^\infty f(x,y)\, e^{-x/s^{(1)} - y/s^{(2)}}\, dx\, dy, \tag{3.25}$$

which we assume to exist for all $s^{(j)} \in \Omega^+$, with Ω^+ denoting the right half plane. It can then be shown (see [9], or [14], §4.6) that the values $p_{i,j}$ which approximate $p(z_i^{(1)}, z_j^{(2)})$ can be computed via the following succinct algorithm. In this algorithm the we use the notation, e.g., $\mathbf{h}_{i,\cdot} = (h_{i,-M_2}, \ldots, h_{i,N_2})^T$. We again emphasize the obvious ease of adaptation of this algorithm to parallel computation.

Algorithm 3.1

(a) Form the arrays $z_i^{(j)}$, and $\frac{d}{dx}\phi^{(j)}(x)$ at $x = z_i^{(j)}$ for $j = 1, 2$, and $i = -M_j, \ldots, N_j$, and then form the block of numbers $[g_{i,j}] = [g(z_i^{(1)}, z_j^{(2)})]$.

(b) Determine A_j, S_j, X_j, and X_j^{-1} for $j = 1, 2$, as defined in (3.23).

(c) Form $\mathbf{h}_{\cdot,j} = X_1^{-1}\mathbf{g}_{\cdot,j}$, $j = -M_2, \ldots, N_2$;

(d) Form $\mathbf{k}_{i,\cdot} = X_2^{-1}\mathbf{h}_{i,\cdot}$, $i = -M_1, \ldots, N_1$;

(e) Form $r_{i,j} = F(s_i^{(1)}, s_j^{(2)})\, k_{i,j}$, $i = -M_1, \ldots, N_1$, $j = -M_2, \ldots, N_2$;

(f) Form $\mathbf{q}_{i,\cdot} = X_2 \mathbf{r}_{i,\cdot}$, $i = -M_1, \ldots, N_1$;

(g) Form $\mathbf{p}_{\cdot,j} = X_1 \mathbf{q}_{\cdot,j}$, $j = -M_2, \ldots, N_2$.

Remark: It is unnecessary to compute the matrices X_1^{-1} and X_2^{-1} in steps c and d of this algorithm, since the vectors $\mathbf{h}_{\cdot,j}$ and $\mathbf{k}_{i,\cdot}$ can be found via the LU factorization of the matrices X_1 and X_2.

Thus starting with the rectangular array $[g_{i,j}]$, Algorithm 3.1 transforms this into the rectangular array $[p_{i,j}]$.

Suppose, for example, that we form a vector $\mathbf{g}$ in which the subscripts appear in the order (call it lexicographic) dictated by the order of appearance of the subscripts in the Fortran do loop, "DO j = $-\mathrm{M}_2, \mathrm{N}_2$", followed by "DO i = $-\mathrm{M}_1, \mathrm{N}_1$". We then also form the diagonal matrix $\mathbf{F}$ in which the entries are the values $F_{ij} = F(s_i^{(1)}, s_j^{(2)})$, with the function F and the eigenvalues $s_j^{(i)}$ defined as above, and where we also list the values F_{ij} in the same lexicographic order as for g_{ij}. Then, similarly for the array $p_{i,j}$, we can define a vector $\mathbf{p}$ by listing the elements p_{ij} in lexicographic order. It can then be shown that if $\mathbf{p}_1$ is defined by the matrix (Kronecker) product

$$\begin{aligned} \mathbf{p} &= \mathbf{C}\,\mathbf{g} \\ \mathbf{C} &= X_2 \otimes X_1\ \mathbf{F}\ \otimes X_2^{-1} \otimes X_1^{-1} \end{aligned}$$

then the corresponding numbers p_{ij} are accurate approximations of the values $p(z_i^{(1)}, z_j^{(2)})$.

We emphasize here, due to the Kronecker product representation of the matrix $\mathbf{C}$ the numerical determination of the vector $\mathbf{p} = \mathbf{C}\,\mathbf{g}$ can be carried out in parallel, without storage of of the huge matrix $\mathbf{C}$ in this equation, which may be an asset, especially for problems in 3 or more dimensions.

Once the numbers $p_{i,j}$ have been computed, we can then use these numbers to approximate p on the region $\mathcal{B}$ via the use of a *Sinc* basis; upon setting $\rho^{(\ell)} = e^{\phi^{(\ell)}}$, we can define the functions

$$
\begin{aligned}
\gamma_i^{(\ell)} &= \operatorname{sinc}\{[\phi^{(\ell)} - ih]/h\}, \quad \ell = 1,2; \quad i = -M_\ell, \ldots, N_\ell, \\
w_i^{(\ell)} &= \gamma_i^{(\ell)}, \quad \ell = 1,2; \quad i = -M_\ell + 1, \ldots, N_\ell - 1, \\
w_{-M_\ell}^{(\ell)} &= \frac{1}{1+\rho^{(\ell)}} - \sum_{j=-M_\ell+1}^{N_\ell} \frac{1}{1+e^{jh_\ell}} \gamma_j^{(\ell)}, \\
w_{N_\ell}^{(\ell)} &= \frac{\rho^{(\ell)}}{1+\rho^{(\ell)}} - \sum_{j=-M_\ell}^{N_\ell-1} \frac{e^{jh_\ell}}{1+e^{jh_\ell}} \gamma_j^{(\ell)}.
\end{aligned}
\tag{3.26}
$$

We then get the approximation

$$
p(x,y) \approx \sum_{i=-M_1}^{N_1} \sum_{j=-M_2}^{N_2} p_{i,j}\, w_i^{(1)}(x) w_j^{(2)}(y). \tag{3.27}
$$

To get an idea of the complexity of the above procedure, we make the simplifying assumption that $M_j = N_j = N$, for $j = 1, 2$. We may readily deduce that if the above two dimensional "Laplace transform" F is either known explicitly, or if the evaluation of this transform can be reduced to the evaluation of a one–dimensional integral, then the complexity, i.e., the total amount of work required to achieve an error ε when carrying out the computations of the above algorithm (to approximate $p(x,y)$ at $(2N+1)^2$ points) on a sequential machine, is $\mathcal{O}([\log(\varepsilon)]^6)$.

The above algorithm extends readily to ν dimensions, in which case the complexity for evaluating a ν–dimensional convolution integral (at $(2N+1)^\nu$ points) by the above algorithm to within an error of ε is of the order of $[\log(\varepsilon)]^{2\nu+2}$.

The above results extend readily to "product regions" over more than one dimension.

2. *Convolutions over Curvilinear Regions*

   Curvilinear regions can also be dealt with relatively easily via *Sinc* methods. We now illustrate this, by deriving in detail a *Sinc convolution* algorithm over a two dimensional region, and then stating the algorithmic form of the three dimensional version (which is obvious at this point, once the derivation of the two dimensional algorithm has been carried out).

   Sinc Convolution over a Two Dimensional Curvilinear Region. Suppose that we are given a convolution integral over the region

$$
\mathcal{B} = \{(t,\tau) : a_1 < t < b_1,\ a_2(t) < \tau < b_2(t)\} \tag{3.28}
$$

where for purposes of this illustration a_1 and b_1 are finite and a_2 and b_2 are finite valued functions belonging to $\mathbf{M}_{\alpha,\beta}(\Gamma)$ can be readily dealt with by the following initial transformation of the differential equation:

$$\begin{aligned} t &= a_1 + (b_1 - a_1)\,\xi \\ \tau &= a_2(t) + (b_2(t) - a_2(t))\,\eta, \end{aligned} \tag{3.29}$$

which transforms the square $\{(x,y) : 0 \le \xi \le 1,\ 0 \le \eta \le 1\}$ onto the region $\mathcal{B}$. Here we have not excluded the possibility that $a_2(a_1) = b_2(a_1)$ or that $a_2(b_1) = b_2(b_1)$. The resulting differential equation problem over the square can now be easily dealt with, starting with a product–type approximation, using a double sum, based on (3.3) (see (4.8) below).

We next extend the above two dimensional *Sinc convolution* algorithm to such a region $\mathcal{B}$ defined in (3.28) above.

Consider the integral

$$r(x,y) = \int\int_{\mathcal{B}} f(x-t, y-\tau)\, g(t,\tau)\, dt\, d\tau\,, \tag{3.30}$$

where $\mathcal{B}$ is given in (3.28) above.

We decompose r into a sum of 4 integrals:

$$r = r_1 + r_2 + r_3 + r_4 \tag{3.31}$$

with

$$\begin{aligned} r_1 &= \int_{a_1}^{x} \int_{a_2(t)}^{y} f(x-t, y-\tau)\, g(t,\tau)\, d\tau\, dt\,, \\ r_2 &= \int_{x}^{b_1} \int_{a_2(t)}^{y} f(x-t, y-\tau)\, g(t,\tau)\, d\tau\, dt\,, \\ r_3 &= \int_{a_1}^{x} \int_{y}^{b_2(t)} f(x-t, y-\tau)\, g(t,\tau)\, d\tau\, dt\,, \\ r_4 &= \int_{x}^{b_1} \int_{y}^{b_2(t)} f(x-t, y-\tau)\, g(t,\tau)\, d\tau\, dt\,. \end{aligned} \tag{3.32}$$

Each of these integrals can be handled in exactly the same way. We thus illustrate here, an explicit procedure for approximating the first of the above integrals, i.e.,

$$p(x,y)) = \int_{a_1}^{x} \int_{a_2(t)}^{y} f(x-t, y-\tau)\, g(t,\tau)\, d\tau\, dt \tag{3.33}$$

over the region B defined above.

It is again essential to illustrate in detail the steps of the reduction. To this end, we shall require the use of the "Laplace transforms",

$$
\begin{aligned}
F(x,\sigma) &= \int_{E_2} \exp\left(-\frac{y}{\sigma}\right) f(x,y)\,dy, \quad E_2 \supset (0, \max_x (b_2(x) - a_2(x)) \,, \\
G(s,\sigma) &= \int_{E_1} \exp\left(-\frac{x}{s}\right) F(x,\sigma)\,dx\,. \quad E_1 \supset (0, \max_x (b_2 - a_2) \,,
\end{aligned}
\tag{3.34}
$$

where we assume that both integrals exist whenever both variables are on their respective open right half planes.

We first apply the *Sinc convolution* procedure to the inner integral. To this end, we set

$$
\begin{aligned}
&\tau = \varphi^{-1}(w) \equiv \frac{a_2(t) + b_2(t)\,e^w}{1+e^w} \qquad \Leftrightarrow \qquad w = \varphi(\tau) = \log\left(\frac{\tau - a_2(t)}{b_2(t) - \tau}\right) \\
&\frac{1}{\varphi'(\tau)} = [b_2(t) - a_2(t)]\,\frac{e^w}{(1+e^w)^2}, \qquad \tau_j = \tau_j(t) = \frac{a_2(t) + b_2(t)\,e^{jh_2}}{1+e^{jh_2}}\,.
\end{aligned}
\tag{3.35}
$$

In order to carry out indefinite convolution with respect to τ, we shall require the *indefinite integration matrix* $[b_2(t) - a_2(t)]\,\mathbf{A}_2$, with $\mathbf{A}_2 = h_2\,I^{(-1)} D\,(e^w (1+e^w)^{-2})$, and in which the variable w is evaluated at the points $j\,h_2$, $j = -M_2, \ldots, N_2$. We then set $m_2 = M - 2 + N_2 + 1$, $\mathbf{A_2} = \mathbf{X}_2\,\mathbf{S}_2\,\mathbf{X}_2^{-1}$, with $\mathbf{S}$ a diagonal matrix with entries $s_{-M_2}^{(2)}, \ldots, s_{N_2}^{(2)}$. We thus obtain

$$
p(x,y) \approx \int_{a_1}^{x} \mathbf{w}_2(y)\,\mathbf{X}_2\,F\left(x-t, [b_2(t) - a_2(t)]\ \mathbf{S}_2\right)\,\mathbf{X}_2^{-1}\,V_\tau\,g(t,\cdot)\,dt\,.
\tag{3.36}
$$

At this point $\mathbf{X}_2\,F\left(x-t, [b_2(t) - a_2(t)]\ \mathbf{S}_2\right)\,\mathbf{S}_2^{-1}\,V_y\,g(t,\cdot)$ is a vector of order m_2 which accurately approximates the vector

$$
V_y \int_{a_2(t)}^{y} f(x-t, y-\tau)\,d\tau
$$

at the points τ_j defined in (3.35) above.

We emphasize here that the operator V_y transforms $g(t,\tau)$ into a vector with entries obtained by replacing τ with the numbers τ_j defined in (3.35) above. That is, the points τ_j also depend upon t. The vector $\mathbf{w}_2(y)$ of basis functions interpolates at the points τ_j. Removal of the basis $\mathbf{w}$ in (3.36) therefore defines this vector $\mathbf{p}(x)$; setting $\mathbf{q}(x) = \mathbf{X}_2^{-1}\,\mathbf{p}(x)$, $\mathbf{h}(t) = \mathbf{X}_2^{-1}\,V_y\,g(t,\cdot)$, and then taking the j^{th} component, $q_j(x)$ of $\mathbf{q}(x)$ and $h_j(t)$ of $\mathbf{h}(t)$, we get

$$q_j(x) = \int_{a_1}^{x} F\left(x - t, [b_2(t) - a_2(t)]\ s_j^{(2)}\right)\ h_j(t)\, dt\,. \tag{3.37}$$

We now again apply Sinc convolution, in the variable t to this integral, to get in the notation of (5.13), and with $m_1 = M_1 + N_1 + 1$,

$$q_j \approx G\left(\mathcal{J}_{m_1}, [b_2(\cdot) - a_2(\cdot)]\ s_j^{(2)}\right)\ h_j\,, \tag{3.38}$$

where, corresponding to any function u,

$$\begin{gathered}
(\mathcal{J}_{m_1}\, u)\,(x) = \mathbf{w}_1(x)\,[b_1 - a_1]\ \mathbf{A}_1\, V_x\, u \\
\mathbf{A}_1 = h_1\, I^{(-1)}\, D\left(\frac{e^{jh_1}}{(1 + e^{jh_1})^2}\right)\,, \\
\mathbf{A}_1 = \mathbf{X}_1\, \mathbf{S}_1\, \mathbf{X}_1^{-1}, \\
\mathbf{X}_1 = \left[x_{i,j}^{(1)}\right], \qquad \mathbf{X}^{-1} = \left[x_{(1)}^{i,j}\right]\,.
\end{gathered} \tag{3.39}$$

If one of the variables $s^{(i)}$ in the function $G(s^{(1)}, s^{(2)})$ is fixed, then $G(s, \sigma)$ is assumed to be analytic with respect to the other variable, on the whole right half plane. Hence, denoting the right hand side of (3.37) by q_j^*, we have, formally, upon performing a power series expansion of G with respect to the first variable,

$$\begin{aligned}
q_j^* &= G\left(\mathcal{J}_{m_1}\,, [b_2(\cdot) - a_2(\cdot)]\ s_j^{(2)}\right)\ h_j(t) \\
&= \sum_{\ell \geq 0} \mathcal{J}_{m_1}^{\ell}\, G_\ell\left([b_2(\cdot) - a_2(\cdot)]\ s_j^{(2)}\right)\ h_j(\cdot) \\
&= \mathbf{w}_1\, \mathbf{X}_1 \sum_{\ell \geq 0} [b_1 - a_1]^{\ell}\, (S^{(1)})^{\ell}\, \mathbf{X}_1^{-1}\, V_x\, G_\ell\left([b_2(\cdot) - a_2(\cdot)]\ s_j^{(2)}\right)\ h_j(\cdot)\,.
\end{aligned} \tag{3.40}$$

Now, by applying the operator V_x to each side (or equivalently, dropping the vector $\mathbf{w}_1$), then multiplying each side of this equation by $\mathbf{X}_1^{-1}$, setting $\mathbf{X}_1^{-1}\, V_x\, q_j^* = \mathbf{r}_j$, and denoting $t_k = \left[a_1 + b_1\, e^{kh_1}\right]/(1 + e^{kh_1})$, we find that the i^{th} component of $\mathbf{r}_j$ is just

$$
\begin{aligned}
r_{i,j} &= \sum_{\ell \ge 0} [b_1 - a_1]^\ell \left(s_i^{(1)}\right)^\ell \sum_{k=-M_1}^{N_1} x_{(1)}^{i,k}\, G_\ell\left([b_2(\cdot) - a_2(\cdot)]\; \mu_j\right)\; h_j(\cdot) \\
&= \sum_{k=-M_1}^{N_1} x_{(1)}^{i,k} \sum_{\ell \ge 0} [b_1 - a_1]^\ell \left(s_i^{(1)}\right)^\ell G_\ell\left([b_2(t_k) - a_2(t_k)]\; s_j^{(2)}\right)\; h_j(t_k) \\
&= \sum_{k=-M_1}^{N_1} x_{(1)}^{i,k}\, G\left([b_1 - a_1]\; s_i^{(1)}\, , [b_2(t_k) - a_2(t_k)]\; s_j^{(2)}\right)\; h_j(t_k)\,.
\end{aligned}
\tag{3.41}
$$

Now, having gotten $r_{i,j}$, we get $q_{i,j}$ and $p_{i,j}$ from the equations

$$
q_{\cdot,j} = \mathbf{X}_1\, r_{\cdot,j} \qquad p_{i,\cdot} = \mathbf{X}_2\, q_{i,\cdot}\,. \tag{3.42}
$$

The final algorithm takes the form of Algorithm 3.2, which follows.

Algorithm 3.2

(a) Determine parameters M_ℓ and N_ℓ, set $m_\ell = M_\ell + N_\ell + 1$, $\ell = 1, 2$, and form the matrices

$$
\begin{aligned}
\mathbf{A}_\ell &= h_\ell\, I^{(-1)}\, D\left(\frac{e^w}{(1+e^w)^2}\right); \quad w = k\, h_\ell \\
&= \mathbf{X}_\ell\, \mathbf{S}_\ell\, \mathbf{X}_\ell^{-1} \\
\mathbf{X}_1^{-1} &= \left[x_{(1)}^{i,j}\right]\,, \quad \mathbf{S}_\ell = \mathrm{diag}\left[s_{-M_\ell}^{(\ell)}, \ldots, s_{N_\ell}^{(\ell)}\right], \quad \ell = 1, 2\,.
\end{aligned}
\tag{3.43}
$$

(b) Form the values $u_i = e^w/(1+e^w)$, with $w = i\, h_1$, $i = -M_1, \ldots, N_1$ as well as the values $v_j = e^w/(1+e^w)$, with $w = j\, h_2$, $j = -M_2, \ldots, N_2$. Then use these to get the Sinc points $t_i = (a_1 + (b_1 - a_1) u_i$, and $\tau_{i,j} = a_2(t_i) + (b_2(t_i) - a_2(t_i))\; v_j$.

(c) Form the $m_1 \times m_2$ array $[g_{i,j}]$ with

$$
g_{i,j} = g\left(t_i, \tau_{i,j}\right)\,.
$$

(d) Replace the array $[g_{i,j}]$ with $[h_{i,j}]$, where

$$
h_{i,\cdot} = \mathbf{X_2}^{-1}\, g_{i,\cdot}.
$$

(e) Obtain the "Laplace transform"

$$
G(s^{(1)}, s^{(2)}) = \int_{E_1} \int_{E_2} \exp\left\{-\frac{x}{s^{(1)}} - \frac{y}{s^{(2)}}\right\}\, f(x,y)\, dy\, dx,
$$

with the sets E_i defined as above.

(f) Replace the array $[h_{i,j}]$ with the array $[r_{i,j}]$, with

$$r_{i,j} = \sum_{k=-M_1}^{N_1} x_{(1)}^{i,k} \, G\left([b_1 - a_1] \, s_i^{(1)}, [b_2(t_k) - a_2(t_k)] \, s_j^{(2)} \right) \, h_{k,j} \, .$$

(g) Set

$$q_{\cdot,j} = \mathbf{X}_1 \, r_{\cdot,j} \, .$$

(h) Set

$$p_{i,\cdot} = \mathbf{X}_2 \, q_{i,\cdot} \, .$$

Remark: Having computed the array $\{p_{i,j}\}$, we can now approximate $p(x,y)$ at any point in $\mathcal{B}$ by means of the formula

$$p(x,y) \approx \mathbf{w}_1(x) \; [p_{i,j}] \, (\mathbf{w}_2(x,y))^T \, , \tag{3.44}$$

where $\mathbf{w}_i = (w_{M_i}^{(i)}, \ldots, w_{N_i}^{(i)})$, are defined as in (3.6), but with

$$\begin{aligned} \varphi_1(x) &= \log\left(\frac{x - a_1}{b_1 - x}\right) \\ \varphi_2(x,y) &= \log\left(\frac{y - a_2(x)}{b_2(x) - y}\right) \end{aligned} \tag{3.45}$$

Sinc Convolution over a Three Dimensional Curvilinear Region.

We now state an analogous algorithm for approximating indefinite convolutions over the region

$$\mathcal{B} = \{(x,y,z) : a_1 < x < b_1, \;\; a_2(x) < y < b_2(x), \;\; a_3(x,y) < z < b_3(x,y)\} \, . \tag{3.46}$$

We wish to approximate a convolution integral of the form

$$R(x,y,z) = \int\!\!\int\!\!\int_{\mathcal{B}} f(x-\xi, y-\eta, z-\zeta) \, g(\xi,\eta,\zeta) \, d\xi \, d\eta \, d\zeta \, . \tag{3.47}$$

This definite convolution integral can be split up into 8 indefinite convolution integrals, the approximation of each of which is similar. We now give an explicit algorithm for approximating one of these, namely,

$$p(x,y,z) = \int_{a_1}^{x} \int_{a_2(x)}^{y} \int_{a_3(x,y)}^{z} f(x-\xi, y-\eta, z-\zeta) \, g(\xi,\eta,\zeta) \, d\zeta \, d\eta \, d\xi \, . \tag{3.48}$$

Algorithm 3.3

(a) Determine parameters M_ℓ and N_ℓ, set $m_\ell = M_\ell + N_\ell + 1$, $\ell = 1, 2, 3$, and form the matrices

$$\begin{aligned}\mathbf{A}_\ell &= h_\ell\, I^{(-1)}\, D\left(\frac{e^w}{(1+e^w)^2}\right); \quad w = k\, h_\ell \\ &= \mathbf{X}_\ell\, \mathbf{S}_\ell\, \mathbf{X}_\ell^{-1} \\ \mathbf{X}_\ell^{-1} &= \left[x_{(\ell)}^{i,j}\right], \quad \mathbf{S}_\ell = \operatorname{diag}\left[s_{-M_\ell}^{(\ell)}, \ldots, s_{N_\ell}^{(\ell)}\right], \quad \ell = 1, 2, 3\,.\end{aligned} \tag{3.49}$$

(b) Form the values $u_i = e^\omega/(1+e^\omega)$, with $\omega = i\, h_1$, $i = -M_1, \ldots, N_1$ the values $v_j = e^\omega/(1+e^\omega)$, with $\omega = j\, h_2$, $j = -M_2, \ldots, N_2$, and $w_k = e^\omega/(1+e^\omega)$, with $\omega = k\, h_3$, $k = -M_3, \ldots, N_3$. Then use these to get the Sinc points $\xi_i = (a_1 + (b_1 - a_1)u_i$, $\eta_{i,j} = a_2(\xi_i) + [b_2(\xi_i) - a_2(\xi_i)]\, v_j$, and $\zeta_{i,j,k} = a_2(\xi_i, \eta_j) + [b_3(\xi_i, \eta_j) - a_3(\xi_i, \eta_j)]\, \zeta_k$.

(c) Form the $m_1 \times m_2 \times m_3$ array $[g_{i,j,k}]$ with

$$g_{i,j,k} = g\left(\xi_i, \eta_{i,j}, \zeta_{i,j,k}\right)\,.$$

(d) Replace the array $[g_{i,j,k}]$ with $[h_{i,j,k}]$, where

$$h_{i,j,\cdot} = \mathbf{X}_3^{-1}\, g_{i,j,\cdot}.$$

(e) It is convenient to obtain the "Laplace transform" by first taking

$$\begin{aligned} E_1 &\supset (0, [b_1 - a_1])\,, \\ E_2 &\supset \left(0, \sup\nolimits_{x \in [b_1 - a_1]}\right)\,, \\ E_3 &\supset \left(0, \sup\nolimits_{y:(x,y) \in \{(\xi,\eta): a_1 < \xi < b_1, a_2(\xi) < \eta < b_2(\xi)\}}\right)\,, \end{aligned}$$

and then setting

$$\begin{aligned} &G\left(s^{(1)}, s^{(2)}, s^{(3)}\right) \\ &= \int_{E_1}\int_{E_2}\int_{E_3} \exp\left\{-\frac{x}{s^{(1)}} - \frac{y}{s^{(2)}} - \frac{z}{s^{(3)}}\right\} f(x,y,z)\, dz\, dy\, dx\,. \end{aligned}$$

(f) Replace the array $[h_{i,j,k}]$ with the array $[r_{i,j,k}]$, with

$$\begin{aligned} \sigma_{i,j,k} = &\sum_{\ell=-M_1}^{N_1} x_{(1)}^{i,\ell} \sum_{m=-M_2}^{N_2} x_{(2)}^{j,m} \cdot \\ &\cdot G\left([b_1 - a_1]\, s_i^{(1)}, [b_2(\xi_\ell) - a_2(\xi_\ell)]\, s_j^{(2)}, [b_3(\xi_\ell, \eta_{l,m}) - a_3(\xi_\ell, \eta_{l,m})]\right) \cdot \\ &\cdot h_{\ell,m,k}\,. \end{aligned}$$

(g) Form

$$r_{\cdot,j,k} = X_1\, \sigma_{\cdot,j,k}$$

(h) Form

$$q_{i,\cdot,k} = \mathbf{X}_2\, r_{i,\cdot,k} \;.$$

(i) Form

$$p_{i,j,\cdot} = \mathbf{X}_3\, q_{i,j,\cdot} \;.$$

Remark: Having computed the array $\{p_{i,j,k}\}$, we can now approximate $p(x, y, z)$ at any point B by means of the formula

$$p(x,y,z) \approx \sum_{i=-M_1}^{N_1} \sum_{j=-M_2}^{N_2} \sum_{k=-M_3}^{N_3} p_{i,j,k} w_i^{(1)}(x)\, w_j^{(2)}(x,y)\, w_k^{(3)}(x,y,z)\,, \tag{3.50}$$

where $\mathbf{w}_i = (w_{M_i}^{(i)}, \ldots, w_{N_i}^{(i)})$, are defined as in (3.6), but with

$$\begin{aligned} \varphi_1(x) &= \log\left(\frac{x-a_1}{b_1-x}\right) \\ \varphi_2(x,y) &= \log\left(\frac{y-a_2(x)}{b_2(x)-y}\right) \\ \varphi_3(x,y,z) &= \log\left(\frac{z-a_3(x,y)}{b_3(x,y)-z}\right) \end{aligned} \tag{3.51}$$

4 Laplace Transforms of Green's Functions

In this section we carry the derivation of the "Laplace transforms" of standard Green's functions for solving Poisson problems in 2 and 3 space dimensions, wave equation problems in one, two and three space and one time dimension, Helmholtz equation problems in two and three space dimensions, and heat equation problems in one, two and three space and one time dimension. The procedure used to carry out these derivations is essentially that developed first in [11].

Let us state and prove a general lemma which is applicable for evaluation of the "Laplace transforms" of all three types of equations, elliptic, hyperbolic, and parabolic. The lemma involves the correct evaluation of the integral

$$Q(a) = \int_C \frac{dz}{z-a} \tag{4.1}$$

for given $a \in \mathbb{C}$, with $C = \{z \in \mathbb{C} : z = e^{i\theta}, \quad 0 < \theta < \pi/2\}$. As a convention, we let $\ln(a)$ denote the principal value of the logarithm, i.e., if $a = \xi + i\eta \in \mathbb{C}$, then $\ln(a) = \ln|a| + i\ \arg(a)$, with $\arg(a)$ taking its principal value in the range $-\pi < \arg(a) \leq \pi$.

Lemma 4.1 *Let $a = \xi + i\,\eta \in \mathbb{C}$, set $L(a) = \xi + \eta - 1$ and let A denote the region*

$$A = \{a = \xi + i\eta \in \mathbb{C}\} : |a| < 1,\ L(a) > 0\}. \tag{4.2}$$

- *If $a \in \mathbb{C} \setminus \overline{A}$, then*

$$Q(a) = \ln\left(\frac{i-a}{1-a}\right)\ ; \tag{4.3}$$

- *If $a = 1$ or $a = i$, then $Q(a)$ is infinite.*
- *If $a \in A$, then*

$$Q(a) = \ln\left|\frac{i-a}{1-a}\right| + i\left(2\pi - \arg\left(\frac{i-a}{1-a}\right)\right); \tag{4.4}$$

- *If $L(a) = 0$ with $|a| < 1$, then*

$$Q(a) = \ln\left|\frac{i-a}{1-a}\right| + i\,\pi; \tag{4.5}$$

- *If $L(a) > 0$, and $|a| = 1$, then*

$$Q(a) = \ln\left|\frac{i-a}{1-a}\right| + i\frac{\pi}{2}\,. \tag{4.6}$$

Proof. The proof is straight forward, the main difficulty stemming from the fact that if $a \in A$, then the principal value of $\ln\left(\frac{i-a}{1-a}\right)$ does not yield the correct value of $Q(a)$, since the principal value of $\ln\left(\frac{j-a}{1-a}\right)$ does not yield the correct value of $Q(a)$ as defined by the integral definition of $Q(a)$, since in that case the imaginary part of $Q(a)$ is larger than π. Perhaps the most difficult part of the proof is the verification of (4.6), which results from the fact that if the conditions on a are satisfied, then the integral (4.3) defining $Q(a)$ is a principal value integral along the arc C, and this value of $Q(a)$ is thus the average of the two limiting values of $Q(\zeta)$ as ζ approaches the point a on the circular arc from interior and from the exterior of A, which is easily seen to be $\pi/2$. ∎

4.1 Transforms of Green's Functions of Poisson Problems

1. *The Case for Planar Regions.* In this case, the Green's function $G(x, y)$ for which the expression

$$\Psi(x, y) = \int\int_B G(x - \xi, y - \eta)\, g(\xi, \eta)\, d\xi\, d\eta \tag{4.7}$$

defines a function u that solves the problem

$$\frac{\partial^2 \Psi(x, y)}{(\partial x)^2} + \frac{\partial^2 \Psi(x, y)}{(\partial y)^2} = -g(x, y), \quad (x, y) \in B \tag{4.8}$$

is given by the expression

$$G(x, y) = \frac{1}{2\pi} \log \left(\frac{1}{\sqrt{x^2 + y^2}} \right) . \tag{4.9}$$

Lemma 4.2 *If u and v are both on the open right half complex plane, then*

$$\begin{aligned} \hat{G}(u, v) &= \int_0^\infty \int_0^\infty \exp\left(-\frac{x}{u} - \frac{y}{v} \right) G(x, y)\, dx\, dy \\ &= \left(\frac{1}{u^2} + \frac{1}{v^2} \right)^{-1} \left(-\frac{1}{4} + \frac{1}{2\pi} \left(\frac{v}{u} (\gamma - \ln(v)) - \frac{u}{v} (\gamma - \ln(u)) \right) \right) \end{aligned} \tag{4.10}$$

Proof. Since

$$G_{xx}(x, y) + G_{yy}(x, y) = -\delta(x)\, \delta(y) \tag{4.11}$$

we have, using integration by parts,

$$\begin{aligned} &\int_0^\infty \exp\left(-\frac{x}{u} \right) G_{xx}(x, y)\, dx \\ &= G_x(x, y) \exp\left(-\frac{x}{u} \right) \Big|_{x=0}^{\infty} + \frac{1}{u} \int_0^\infty \exp\left(-\frac{x}{u} \right) G_x(x, y)\, dx \\ &= 0 + \frac{1}{u} \exp\left(-\frac{x}{u} \right) G(x, y) \Big|_{x=0}^{\infty} + \frac{1}{u^2} \int_0^\infty \exp\left(-\frac{x}{u} \right) G(x, y)\, dx \end{aligned} \tag{4.12}$$

since $G(0^+, y) = 0$. Next, multiplying both sides of this equation by $\exp(-y/v)$, integrating with respect to y over $(0, \infty)$, we get

$$
\begin{aligned}
-\frac{1}{u} \int_0^\infty & G(0,y) \exp\left(-\frac{y}{v}\right) dy \\
&= \frac{1}{2\pi u} \int_0^\infty \ln(y) \, \exp\left(\frac{y}{v}\right) \, dy \\
&= \frac{v}{2\pi u} \left(\Gamma'(1) + \ln(v)\right) \\
&= \frac{v}{2\pi u} \left(-\gamma + \ln(v)\right)
\end{aligned}
\tag{4.13}
$$

since $\Gamma'(1) = -\gamma = -0.577\ldots$.

Repeating the above steps, starting with G_{yy} instead of G_{xx}, and noting that the "Laplace transform" of $\delta(x)\,\delta(y)$ is $1/4$, we thus arrive at the statement of the lemma. ∎

2. *The "Laplace Transform" of* $G(x,y) = \left(x^2+y^2\right)^{-1/2}$.

It is sometimes convenient to know this result for boundary integral equations, as well as for two obtaining the transform of the Green's function in three dimensions. We want to evaluate the integral

$$
\hat{G}(u,v) = \int_0^\infty \int_0^\infty \exp\left(-\frac{x}{u} - \frac{y}{v}\right) G(x,y) \, dx \, dy \, . \tag{4.14}
$$

Setting

$$
\begin{aligned}
&\rho = \sqrt{x^2+y^2}\,, \quad x + i\,y = \rho\, z\,, \quad \left(z = e^{i\theta}\right) \\
&\lambda = \sqrt{\frac{1}{u^2} + \frac{1}{v^2}}, \quad \zeta = \sqrt{\frac{\frac{1}{u} + \frac{i}{v}}{\frac{1}{u} - \frac{i}{v}}}\,,
\end{aligned}
\tag{4.15}
$$

we have

$$
\begin{aligned}
\frac{x}{u} + \frac{y}{v} &= \frac{\rho\,\lambda}{2}\left(\frac{z}{\zeta} + \frac{\zeta}{z}\right) \\
u + i\,v &= \lambda\,\zeta\,.
\end{aligned}
\tag{4.16}
$$

Substituting these results into (4.14) above, we get, after integrating with respect to ρ,

$$
\hat{G}(u,v) = \frac{2}{\pi i} \int_C \frac{dz}{z\,\lambda\left(\frac{z}{\zeta} + \frac{\zeta}{z}\right)}\,, \tag{4.17}
$$

where C is defined as in (4.1) above. Hence, after rational simplification, we get

$$\hat{G}(u,v) = \frac{-1}{\lambda}\left\{Q(i\zeta) - Q(-i\zeta)\right\}. \tag{4.18}$$

where Q is defined as in (4.1) above.

3. *The* $3-d$ *Green's function* $(4\pi r)^{-1}$, *with* $r = \sqrt{x^2+y^2+z^2}$.

We shall here derive an explicit expression for

$$\hat{G}(u,v,w) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp\left\{-\frac{x}{u} - \frac{y}{v} - \frac{z}{w}\right\}}{4\pi\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz. \tag{4.19}$$

This result will enable us to obtain an accurate approximation for Ψ, with

$$\Psi(x,y,z) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \frac{g(\xi,\eta,\zeta)}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\xi\, d\eta\, d\zeta\,. \tag{4.20}$$

in $V = (a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)$. The function Ψ defined in (4.20) satisfies the equation $\Psi_{xx} + \Psi_{yy} + \Psi_{zz} = -g$ in V.

Lemma 4.3 *Let $\hat{G}(u,v,w)$ be defined as in (4.19) for arbitrary complex u, v, and w located on the right half complex plane. Then*

$$\begin{aligned}\hat{G}(u,v,w) &= \left(\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2}\right)^{-1}\cdot \\ &\quad \cdot\left\{-\frac{1}{8} + H(u,v,w) + H(v,w,u) + H(w,u,v)\right\},\end{aligned} \tag{4.21}$$

where, setting

$$\begin{aligned}\lambda &= \sqrt{\frac{1}{v^2}+\frac{1}{w^2}}, \\ \zeta &= \sqrt{\frac{\frac{1}{v}+\frac{i}{w}}{\frac{1}{v}-\frac{i}{w}}}\end{aligned} \tag{4.22}$$

we have from (4.14)-(4.18),

$$H(u,v,w) = -\frac{1}{8\pi u\lambda}\left\{Q(i\zeta) - Q(-i\zeta)\right\} \tag{4.23}$$

where $Q(a)$ is defined as in Lemma 4.1.

Proof. The *Green's function* $G(x,y,z) = (4\pi r)^{-1}$, with $r = \sqrt{x^2+y^2+z^2}$ satisfies the equation

$$\nabla^2 \left(\frac{1}{4\,\pi\, r} \right) = -\delta^3(\bar{r}) \tag{4.24}$$

where $\delta^3(\bar{r}) = \delta(x)\,\delta(y)\,\delta(z)$ denotes the three dimensional delta function. Thus, integrating the expression

$$J(y,z,u) \equiv \int_0^\infty G_{xx}(x,y,z)\, \exp(-x/u)\, dx \tag{4.25}$$

by parts, we have, noting that $G_x(0^+,y,z) = 0$,

$$\begin{aligned} &J(y,z,u) \\ &= G_x(x,y,z)\, \exp\left(-\frac{x}{u}\right)\Big|_{x=0}^{\infty} = \frac{1}{u}\int_0^\infty G_x(x,y,z)\, \exp(-x/u)\, dx \\ &= \frac{1}{u}\, G(x,y,z)\, \exp\left(-\frac{x}{u}\right)\Big|_{x=0}^{\infty} + \frac{1}{u^2}\int_0^\infty G(x,y,z)\, \exp\left(-\frac{x}{u}\right)\, dx \\ &= \frac{-1}{u}\frac{1}{4\pi\sqrt{y^2+z^2}} + \frac{1}{u^2}\int_0^\infty G(x,y,z)\, \exp\left(-\frac{x}{u}\right)\, dx. \end{aligned} \tag{4.26}$$

We now proceed as in (4.14)–(4.18) above, and then proceed similarly for G_{yy} and G_{zz} as we did for G_{xx} above, and then note that the three dimensional "Laplace transform" of $\delta^3(\bar{r})$ is $1/8$, to arrive at the statement of Lemma 4.3. ∎

4.2 Transforms of Green's Functions of Wave Problems.

We consider here the transforms of Green's functions for problems in d space and one time dimension, for $d = 1, 2, 3$. The Sinc convolution technique has a considerable advantage over such problems, since for small time intervals they enable uniformly (in space and time) accurate solution of the corresponding integral equations via use of successive aproximations, should therefore be useful for accurate and efficient solution of inverse problems stemming from the novel integral equation formulations that they offer.

1. *The* $d = 1$ *Case.* We want to evaluate the integral

$$\hat{G}(u,\tau) = \int_0^\infty G(x,t) \exp\left(-\frac{x}{u} - \frac{t}{\tau}\right)\, d\tau\, dx\,, \tag{4.27}$$

where the Green's function $G(x,t)$ is defined by the equations

$$
\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2 G(x,t)}{(\partial t)^2} - \frac{\partial^2 G(x,t)}{(\partial x)^2} = \delta(t)\,\delta(x) \quad x \in \mathbb{R}, \quad t \in (0,T)\,, \\
&\qquad G(x,0^+) = \left.\frac{\partial G(x,t)}{\partial t}\right|_{t=0^+} = 0\,, \quad x \in \mathbb{R}\,.
\end{aligned}
\tag{4.28}
$$

It is readily seen that the "Laplace transform" $\tilde{G}(x,\tau)$ of $G(x,t)$ then satisfies the differential equation

$$
\tilde{G}_{xx}(x,\tau) - \frac{1}{c^2\,\tau^2}\,\tilde{G}(x,\tau) = -\delta(x) \tag{4.29}
$$

and solving this, we find that

$$
\tilde{G}(x,\tau) = \frac{s\,\tau}{2}\exp\left(-\frac{|x|}{s\,\tau}\right)\,. \tag{4.30}
$$

Taking the "Laplace transform" of this equation with respect to the variable x, we now get

$$
\hat{G}(u,\tau) = \frac{c\,\tau\,u}{c\,\tau + u}\,. \tag{4.31}
$$

2. *The $d = 2$ Case.* We shall derive the "Laplace transform"

$$
\hat{G}(u,v,\tau) = \int_0^\infty\int_0^\infty G(x,y,t)\exp\left(-\frac{x}{u} - \frac{y}{v} - \frac{t}{\tau}\right)\,d\tau\,dx\,dy\,, \tag{4.32}
$$

where G is defined for $(x,y) \in \mathbb{R}^2$ and $t \in (0,T)$ by the equations

$$
\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2 G(x,y,t)}{(\partial t)^2} - \frac{\partial^2 G(x,y,t)}{(\partial x)^2} - \frac{\partial^2 G(x,y,t)}{(\partial y)^2} = \delta(t)\,\delta(x)\,\delta(y)\,, \\
&\qquad G(x,y,0^+) = \left.\frac{\partial G(x,y,t)}{\partial t}\right|_{t=0^+} = 0\,, \quad (x,y) \in \mathbb{R}^2\,.
\end{aligned}
\tag{4.33}
$$

Lemma 4.4 *Let $\hat{G}(u,v,\tau)$ be defined as in (4.32)–(4.33) for all $\Re u > 0$, $\Re v > 0$, and $\Re \tau > 0$. Then*

$$
\hat{G}(u,v,\tau) = \left(\frac{1}{c^2\tau^2} - \frac{1}{u^2} - \frac{1}{v^2}\right)^{-1}\left(\frac{1}{4} - \hat{H}(u,v,\tau) - \hat{H}(v,u,\tau)\right)\,, \tag{4.34}
$$

where

$$
\hat{H}(u,v,\tau) = \frac{1}{\pi\,u}\frac{1}{\sqrt{\frac{1}{c^2\tau^2} - \frac{1}{v^2}}}\arctan\sqrt{\frac{\frac{1}{c\tau} - \frac{1}{v}}{\frac{1}{c\tau} + \frac{1}{v}}} \tag{4.35}
$$

Proof. By first taking the "Laplace transform" of this equation with respect to t, and denoting the result by $\tilde{G}(x,y,\tau)$, we get the equation

$$\frac{\partial^2 \tilde{G}(x,y,\tau)}{(\partial x)^2} + \frac{\partial^2 \tilde{G}(x,y,\tau)}{(\partial y)^2} - \frac{1}{c^2\,\tau^2}\tilde{G}(x,y,\tau) = -\delta(x)\,\delta(y) \quad x,y \in \mathbb{R}^2\,, \tag{4.36}$$

whose solution is

$$\tilde{G}(x,y,\tau) = \frac{1}{2\pi}\,K_0\left(\frac{\sqrt{x^2+y^2}}{\tau\,c}\right)\,, \tag{4.37}$$

where K_0 denotes the Bessel function.

Let us now again take the "Laplace transform" of the differential equation, this time with respect to x and y. Using integration by parts, we find, e.g., that

$$\begin{aligned}
&\int_0^\infty \exp\left(-\frac{y}{v}\right)\left\{\int_0^\infty \exp\left(-\frac{x}{u}\right)\tilde{G}_{xx}(x,y,\tau)\,dx\right\}dy \\
&= \int_0^\infty \exp\left(-\frac{y}{v}\right)\left\{\tilde{G}_x(x,y,\tau)\exp\left(-\frac{x}{u}\right)\Big|_{x=0}^{\infty}\right. \\
&\quad\left. + \frac{1}{u}\int_0^\infty \exp\left(-\frac{x}{u}\right)\tilde{G}_x(x,y,\tau)\,dx\right\}dy \qquad (4.38) \\
&= \int_0^\infty \exp\left(-\frac{y}{v}\right)\left\{-\frac{1}{u}\,\tilde{G}(0^+,y,\tau)\right. \\
&\quad\left. + \frac{1}{u^2}\int_0^\infty \tilde{G}(x,y,\tau)\exp\left(-\frac{x}{u}\right)\right\}dy.
\end{aligned}$$

The last term on the right is just $u^{-2}\,\hat{G}(u,v,\tau)$. To evaluate the integral of the second last term, we substitute the representation (see [1], Eq. 9.6.23)

$$K_0\left(\frac{y}{c\tau}\right) = \int_1^\infty \exp\left(-\frac{y}{c\,\tau}\,\xi\right)\left(\xi^2-1\right)^{-1/2}\,d\xi \tag{4.39}$$

into this integral, and then interchange the order of integration, to get

$$\int_0^\infty\int_0^\infty \tilde{G}_{xx}(x,y,\tau)\,\exp\left(-\frac{x}{u}-\frac{y}{v}\right)dx\,dy = \frac{\hat{G}(u,v,\tau)}{u^2} + \hat{H}(u,v,\tau)\,, \tag{4.40}$$

where $\hat{H}(u,v,\tau)$ is given in (4.35) above. Furthermore, since the two dimensional "Laplace transform" of $\delta(x)\,\delta(y)$ is $1/4$, and proceeding similarly as above for G_{yy}, we arrive at (4.34). ∎

3. *Sinc Convolution Solution of a Wave Equation Problem* The Green's function $G(\bar{r},t) = G(x,y,z,t)$ for the wave equation satisfies the equations

$$\begin{gathered}
\frac{1}{c^2}\frac{\partial^2 G(\bar{r},t)}{\partial t^2} - \nabla^2 G(\bar{r},t) = \delta(t)\,\delta^3(\bar{r}, \quad \bar{r} \in \mathbb{R}^3,\ t \in (0,T) \\
G(\bar{r},0^+) = \frac{\partial G}{\partial t}(\bar{r},0^+) = 0,
\end{gathered} \tag{4.41}$$

The four dimensional "Laplace transform" of the function $G(\bar{r},t)$ is defined by

$$\begin{aligned}
&\hat{G}(u,v,w,\tau) \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty G(x,y,z,t) \exp\left\{-\frac{x}{u} - \frac{y}{v} - \frac{z}{w} - \frac{t}{\tau}\right\} dx\,dy\,dz\,dt.
\end{aligned} \tag{4.42}$$

Lemma 4.5 *Let $\hat{G}$ be defined for all $\Re u > 0$, $\Re v > 0$, $\Re w > 0$ and $\Re \tau > 0$ by (4.42). Then*

$$\begin{aligned}
&\hat{G}(u,v,w,\tau) \\
&\quad = \left(\frac{1}{c^2\tau^2} - \frac{1}{u^2} - \frac{1}{v^2} - \frac{1}{w^2}\right)^{-1} \cdot \\
&\quad \cdot \left\{\frac{1}{8} - \hat{H}(u,v,w,\tau) - \hat{H}(v,w,u,\tau) - \hat{H}(w,u,v,\tau)\right\}
\end{aligned} \tag{4.43}$$

where

$$\hat{H}(u,v,w,\tau) = \frac{i}{4\pi u\left(\frac{1}{c^2\tau^2} - \frac{1}{v^2} - \frac{1}{w^2}\right)}\ \{Q(z_1) - Q(z_2)\}, \tag{4.44}$$

where, with

$$\begin{aligned}
\zeta &= \sqrt{\frac{\frac{1}{v} + \frac{i}{w}}{\frac{1}{v} - \frac{i}{w}}}, \\
\lambda &= \sqrt{\frac{1}{v^2} + \frac{1}{w^2}},
\end{aligned} \tag{4.45}$$

we have

$$z_{1,2} = -\frac{\zeta}{\lambda}\left\{\frac{1}{c\tau} \pm \sqrt{\frac{1}{c^2\tau^2} - \lambda^2}\right\}. \tag{4.46}$$

and where $Q(\zeta)$ is defined as in Lemma 4.1.

Proof. Setting

$$\tilde{G}(x,y,z,\tau) = \int_0^\infty \exp\left(-\frac{t}{\tau}\right) G(x,y,z,t)\,dt\,, \tag{4.47}$$

it follows that

$$\nabla^2\tilde{G}(\bar{r},\tau) - \frac{1}{c^2\tau^2}\,\tilde{G}(\bar{r},\tau) = -\delta^3(\bar{r}), \tag{4.48}$$

an equation for which the solution is well known, i.e.,

$$\tilde{G}(\bar{r},\tau) = \frac{\exp\left\{-\frac{r}{c\tau}\right\}}{4\pi r}. \tag{4.49}$$

Let us evaluate the "Laplace transform" of $\tilde{G}_{xx}$ with respect to x as above. Since $\tilde{G}_x(0,y,z,\tau) = 0$ for $(y,z) \neq 0$, we have, upon integration by parts,

$$\begin{aligned}&\int_0^\infty e^{-x/u}\,\tilde{G}_{xx}(x,y,z\tau)\,dx\\ &\quad = -\frac{1}{u}\,\tilde{G}(0,y,z,\tau) + \frac{1}{u^2}\int_0^\infty e^{-x/u}\,\tilde{G}(x,y,z,\tau)\,dx.\end{aligned} \tag{4.50}$$

Hence, setting

$$\hat{H}(u,v,w,\tau) = \frac{1}{u}\int_0^\infty\int_0^\infty e^{-y/v-z/w}\,\hat{G}(0,y,z,\tau)\,dy\,dz\,, \tag{4.51}$$

we find, after converting to polar coordinates via use of the notation (4.45) above as well as $\rho = \sqrt{y^2+z^2}$, $y + i\,z = \rho\,\omega$, that

$$\hat{H}(u,v,w,\tau) = \frac{\zeta}{2\,\pi\,i\,u\,\lambda}\int_C \frac{d\omega}{\omega^2 + \frac{2\zeta\,z}{c\tau\lambda} + \zeta^2}, \tag{4.52}$$

where

$$C = \{z \in \mathbb{C} : z = e^{i\theta},\ 0 \le \theta \le \pi/2\}. \tag{4.53}$$

Upon denoting the roots of the quadratic in the denominator of the integrand by z_1 and z_2, we find that

$$z_{1,2} = -\frac{\zeta}{\lambda}\left(\frac{1}{c\,\tau} \pm \sqrt{\frac{1}{c^2\tau^2} - \lambda^2}\right), \tag{4.54}$$

which, after substitution into (4.52) enables us to arrive at (4.43)–(4.44). ∎

4. *Helmholtz Equations.* The Green's functions for these equations are simple replacements of the ones above, i.e., we have

$$\begin{aligned} &G(\bar\rho) = \tfrac{i}{4}\, H_0^{(1)}(k\,\rho) \\ &\nabla^2 G(\bar\rho) + k^2 G(\bar\rho) = -\delta^2(\bar\rho) \end{aligned} \tag{4.55}$$

in two dimensions, with $H_0^{(1)}$ denoting the Hankel function, and

$$\begin{aligned} &G(\bar r) = \frac{e^{i\,k\,r}}{4\,\pi\,r} \\ &\nabla^2 G(\bar r) + k^2 G(\bar r) = -\delta^3(\bar r) \end{aligned} \tag{4.56}$$

in three dimensions. The "Laplace transforms" of these can be readily obtained from the above by replacing $1/(c\tau)$ by $-ik$ in Lemmas 4.4 and 4.5 above.

4.3 Transforms of Green's Functions of Heat Problems.

We consider here, obtaining the d–dimensional "Laplace transforms" of the Free space Green's functions in $\mathbb{R}^d \times (0,\infty)$, where

$$G(r,t) = \frac{1}{(4\pi\varepsilon t)^{d/2}} \exp\left(-\frac{r^2}{4\varepsilon t}\right). \tag{4.57}$$

Suppose that we are interested in evaluating the integral

$$U(\bar r,t) \equiv \int_0^t \int_V G(|\bar r - \bar r'|, t - t')\, f(\bar r', t')\, d\bar r'\, dt', \tag{4.58}$$

with $V \subseteq \mathbb{R}^d$ and $t \in (0,T)$, for some (finite or infinite) value of T. In particular, let us assume the k^{th} variable in this expression, i.e., $x^k - \xi^k$ ranges over some subset $\mathcal{E}_k$ of $\mathbb{R}$, where $k = d+1$ corresponds to the time, $t - t'$. It is then convenient to recall that in the integral expression of the multidimensional "Laplace transform" it is necessary to only integrate over a subset E_k with respect to x^k so long as E_k contains $\mathcal{E}_k$. Thus, e.g., the particular "Laplace transform" obtained by integrating over $\mathbb{R}$ with respect to each space variable and over $(0,\infty)$ with respect to t covers all situations. It is then convenient to integrate first with respect to the space variables over $\mathbb{R}^d$ and then integrate with respect to the time variable t over $(0,\infty)$. That is, we take

$$\hat{G}(u^1, ..., u^d, \tau) = \int_0^\infty \int_{\mathbb{R}^d} \exp\left(-\frac{t}{\tau} - \sum_{k=1}^{d} \frac{x^k}{u^k}\right) G(r,t)\, d\bar{r}\, dt\,, \qquad (4.59)$$

where we use the notation $\bar{r} = (x, y, z)$, $\bar{r}' = (x', y', z')$, where $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$. The evaluations of the integrals are straight forward, and we omit the details. The results are given in the lemma which follows.

Lemma 4.6 *Let the d–dimensional Laplace transform $\hat{G}$ of the Green's function $G(r,t)$ be defined as in (4.59), for all $\Re u^k > 0$. Then, with $u^1 = u$, $u^2 = v$, and $u^3 = w$, we have*

$$\begin{aligned}
\hat{G}(u,\tau) &= \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon\tau} - \frac{1}{u^2}\right)^{-1}, \quad \text{if} \quad d = 1 \\
\hat{G}(u,v,\tau) &= \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon\tau} - \frac{1}{u^2} - \frac{1}{v^2}\right)^{-1}, \quad \text{if} \quad d = 2 \qquad (4.60) \\
\hat{G}(u,v,w,\tau) &= \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon\tau} - \frac{1}{u^2} - \frac{1}{v^2} - \frac{1}{w^2}\right)^{-1}, \quad \text{if} \quad d = 3\,.
\end{aligned}$$

It is possible to get alternate explicit expressions of $\hat{G}$ for the cases of $d = 1$ and $d = 3$, by first doing the time integration which is explicitly possible in these dimensions, thus reducing the problem to the case of the Helmholtz equation, which was considered above. However, this alternative does not appear to work for the case of $d = 2$.

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