Zentrum für Technomathematik
Fachbereich 3 – Mathematik und Informatik

On parameter identification for general linear elliptic problems of second order

Michael Wolff Michael Böhm

Report 18–01

Berichte aus der Technomathematik

Report 18–01 May 2018

On parameter identification for general linear elliptic problems of second order

M. Wolff, M. Böhm
Center for Industrial Mathematics, University Bremen, Germany
mwolff at math.uni-bremen.de

May 3, 2018

Contents

Abstract

This work deals with parameter identification for general linear elliptic problems of second order. The parameter dependence within the equation can be non-linear. Problems of such kind arise in many applications, in electrical impedance tomography and in elasticity, e.g. This paper aims to consider these special applied problems within a unique frame, focussing on two prototypical elliptic problems with mixed boundary conditions. At first we give a sound basis, providing a weak-formulation setting as well as main results concerning existence, uniqueness and continuity results of the solution operators. Moreover, from higher integrability of the gradient of weak solutions up to the boundary under mixed boundary conditions we deduce important properties of the solution operators (and of their Fréchet derivatives) with respect to an L^s topology for some $1 < s < \infty$. This is an important point of departure for a detailed study of arising inverse problems. Finally, we present and study cost functionals which compare calculated functions within the spatial domain where the elliptic problems are given.

Keywords: linear elliptic problems with parameters; regularity for mixed boundary-value problems; inverse-problem setting; application to linear non-homogeneous non-isotropic elasticity; special cost functionals

1 Introduction

Some important problems of parameter identification lead to inverse problems for linear elliptic boundary-value problems, for instance in elasticity as well as in electrical impedance tomography (for references see below in this introduction). Some of the approaches of parameter identification are characterized by the fact that the experiments yield *more* information than necessary for the calculation of the solution of the direct (forward) problem with known parameters. This additional information can be used to determine parameters not known or only inexactly known. In principle, with the aid of a *part* of experimental data one calculates the remaining data and compares them with their experimentally obtained counterpart. A minimization ("Fitting") of the deviation is the basis for finding parameters.

Depending on the experimental circumstances, in elasticity, e.g., displacements *and* normal stresses can be measured at (parts of) the surface. This renders two problems for *two alternative* calculations of the displacements inside the body. In Yun and Shang (2011), this idea has been brought into a special approach to inverse problems in continuum mechanics. In electrical impedance tomography, an analog approach for a single equation can be found in Knowles (1998) (further references are given below in this introduction). Note that in many cases, cost functionals are also based on comparison of boundary values (see subsection 4.2 for details and references).

Focussing on this idea, here, the parameter identification is based on a comparison of *two calculated* solutions *within* the spatial domain representing the body or sample in experiments or measurements.

The aim of this study is to give the approach of comparison of two calculated solutions within the domain a rigorous mathematical form in the framework of weak-solution theory for partial differential equations and to prove first results concerning the solutions operators and arising cost functionals.

In some simple cases, material parameters can be determined more or less directly from measured data. Young's modulus of a homogeneous isotropic material, e.g., provides an example. In more general situations with several parameters such a "Hand-Fitting" frequently fails. In the case of complex material behavior (elasto-plasticity with hardening, e.g.) *and of isotropy* a parameter identification can be performed with sufficient success based on uni- and bi-axial experiments with small (often cylindrical or hollow-cylindrical) samples in special testing machines (Gleeble®, e.g.). In Mahnken and Stein (1996a), a general algorithm for uni-axial experiments with visco-plastic materials has been developed. Special methods referred to cyclic plasticity and ratcheting in bi-axial experiments can be found in Bari and Hassan (2000), Abdel-Karim (2005), Taleb and Cailletaud (2010), Djimli et al. (2010), e.g.. In

Wolff et al. (2012), a semi-implicit algorithm for evaluation of uni-axial experimental data for creep and transformation-induced plasticity in steel has been developed. In Wolff et al. (2013), the underlying idea has been applied to multi-mechanism models with several kinematic variables for each mechanism in the case of plastic behavior. However, if the material behavior is anisotropic and/or non-homogeneous, major difficulties arise. There are experiments with specially manufactured samples which take the expectant anisotropy in the material into account (cf. Noman et al. (2010), Pietryga et al. (2012), e.g.). An alternative to these approaches consists in considering the samples as two- or three-dimensional work-pieces. Then, the behavior of these samples is modeled by partial differential equations including boundary conditions and corresponding two- or tree-dimensional simulations can be done.

In general, parameter identification means solving an inverse problem. While dealing with a forward problem (or direct problem), the determining system of equations (ordinary, partial or integral equations), the coefficients as well as boundary and initial conditions are (supposed to be) known, and a unique solution has to be determined. In applications, the coefficients (or coefficient functions) are mostly material and/or process parameters which are assumed to be given for the forward problems. When dealing with an inverse problem, the solution (in practical applications mostly only parts of them, at the boundary or inside of the body), boundary and initial conditions are known, but the equations are known only in their structure without knowledge of some or all coefficients (or coefficient functions) which have now to be determined. In doing so, more information is or needs to be known as it would be necessary for solving the direct problem. This additional information is the basis for solving inverse problems. Exemplarily we refer to Bard (1974), Banks and Kunisch (1989), Bui and Tanaka (1994), Mahnken and Stein (1996a). In Mahnken (2004), some practical problems have been discussed which might come up when identifying parameters. There may be an "over-parametrization" or data stemming from inappropriately planed experiments such that the phenomena to be investigated are not or only insufficiently active. Besides the determination of proper material parameters (or material functions) it is often, in related but different context, aimed at finding (optimal) process parameters. Heat-transfer coefficients provide an example. We refer to Alder et al. (2006), Hömberg and Weiss (2006), Kern (2011) as well as Liščić (2009), Liščić et al. (2011) and Frerichs et al. (2014) among others. Control problems (see Tröltzsch (2010), e.g.) in solid mechanics like in Herzog et al. (2012), Herzog et al. (2013) are mathematically closely related to problems in parameter identification.

When solving inverse problems, a manifold of mathematical challenges arises. This refers to existence and uniqueness of solutions of direct and inverse problems and with respect to the development of numerical algorithms for approximate computation. Usually, the underlying partial differential equations will be discretized in space and time and often solved approximately with the finite-element method. In the case of necessity, adaptive procedure will be applied. We refer to Tortorelli and Michaleris (1994), Vexler (2004), Meidner (2008) and to the references cited therein.

Concerning inverse problems in mechanics, we refer to the survey papers by Kajberg et al. (2004), Bonnet and Constantinescu (2005) as well as to Kohn and McKenney (1990), Gockenbach and Khan (2005), Grédiac and Pierron (2006), Avril and Pierron (2007), Feissel and Allix (2007), Pierron et al. (2007), Lecompte et al. (2007), Cooreman et al. (2007), Kajberg and Wikman (2007), Gockenbach and Khan (2007), Gockenbach and Khan (2008), Gockenbach et al. (2008), Andrieux and Baranger (2008), Pierron et al. (2010), Rossi and Pierron (2012), Banerjee et al. (2013), Le et al. (2015) e.g. In Suchocki et al. (2013), parameter identification in visco-elasticity is considered.

Concerning inverse problems in electrical impedance tomography, we refer to Kohn and Vogelius (1987), Kohn and McKenney (1990), Knowles (1998), Dahlke et al. (2010), Jin and Maaß (2012a), Jin et al. (2012), Jin and Maass (2012b), Pham (2015). For applications to geology see Lukaschewitsch et al. (2009), e.g.

As already pointed out; in order to solve the inverse problem requires more information than it would be necessary for solving the forward problem under assumed knowledge of all parameters. For problems in mechanics, this additional information could mean that both stress and displacement are known at parts of the boundary of the work-piece under consideration, or, alternatively, the displacement inside the body. For example, after solving the forward problem using the known stress at this boundary part, the calculated displacements at this boundary part can be compared with the measured ones. Hence, the occurring mathematical problem is a version of a Neumann-to-Dirichlet problem generally only for a port of the boundary. We refer to Isakov (2006), e.g. for general explanations and to Bonnet and Constantinescu (2005) in the context of linear elasticity.

So-called Cauchy problems of reconstruction of incomplete data at the surface are closely related to inverse problems for determining the elastic moduli. We refer to Andrieux et al. (2006), Andrieux and Baranger (2008).

In recent years, the technical preconditions for contactless optical measurement as well as the possibilities for an adequate evaluation of the occurring large data set have been considerably improved (see Avril et al. (2008), Pottier et al. (2011), Laurin et al. (2012) and the references cited therein). Thus, there are real chances to determine material parameters via comparison between calculated and (at boundary parts) measured quantities (displacements, e.g.) via fitting or other minimization approaches. Based on this idea, in Widany and Mahnken (2012), the parameter identification for an incompressible hyper-elastic material has been performed. In Mahnken and Dammann (2013), this approach has been applied to an originally isotropic polymer which develops a strain-induced anisotropy during the process. In special cases, for thin transparent samples, e.g., it is also possible to measure interior deformations and to use them for parameter identification ("full-field method", see Avril and Pierron (2007), Avril et al. (2008a), Pottier et al. (2011), e.g.).

If one encounters material behavior with a unique relation between stress and strain like in linear and non-linear elasticity in small and finite deformations, it is sufficient to deal with stationary (direct and inverse) problems. However, plastic or visco-plastic material behavior leads to non-stationary problems with major mathematical challenges as well as with major experimental effort.

In Yun and Shang (2011), a new approach has been proposed and applied to cyclic elasto-plasticity. In Shang and Yun (2012), a further experimental verification has been performed. The basis for parameter identification is the employment of twofold experimental information on a boundary *part* for two parallel simulations and to compare the results *inside* of the work-piece. Regardless of a formal similarity to full-field approaches there are *essential differences.* The authors call their method "Self-OPTIM" standing for "self-optimizing inverse analysis method". This is motivated by the fact that in the ongoing algorithm *the evaluation criterion consists in comparing of results of parallel simulations and not a permanent comparison with measured data.* The relevant data for the simulation are exclusively boundary data. These data only occur as boundary conditions for the simulations. In their context, the cost functional is implicit (see Shang and Yun (2012) for further explanations). Their approach to compare calculated quantities in the *interior* of the work-piece seems to be especially appropriate in the case of non-homogenous states and anisotropic material behavior. Numerical applications of this new approach can be found in Rahimi et al. (2012), Sadegh Zadeh and Montas (2014) and Weaver (2015).

It is the aim of this paper to *establish a sound mathematical foundation* of Yun and Shang's as well as Knowles' approach in the framework of weak-solution theory for partial differential equations. In difference to some contributions in electrical impedance tomography, we consider mixed-boundary conditions. Moreover, it is aimed to prove some results for the inverse problem. Furthermore, we discuss some relations to other approaches in use like Dirichlet-to-Neumann approaches and full-field methods.

Discretization methods (finite elements, e.g.) will be not considered here, though their great importance for numerical realization and practical application are beyond all question.

Outline of the remaining paper:

(i) In section 2, general parameter-dependent linear elliptic problems are considered. There are given the setting and the weak formulation of two closely related problems which play the role of *prototypical problems.* They cover some important applications both in linear elasticity as well as in electrical impedance tomography. We summarize widely well-known results concerning existence and uniqueness of weak solutions. Moreover, we establish some special continuity results of the solution operators, using global higher integrability of the gradient of weak solutions even under mixed boundary conditions. Based on this, Fréchet differentiability of the solution operators with respect to L^s topology for some $1 < s < \infty$ can be proved. This is an important point of departure for further investigations like of convergence behavior of minimizing sequences. The last item is not addressed in this study.

(ii) In section 3, we present some examples of boundary-value problems for linear elliptic equations and for systems of equations which can be dealt with in an analogical manner as the prototypical problems. The case of linear non-homogeneous non-isotropic elasticity is included, and, therefore, the general results presented are applicable.

(iii) In section 4, some results concerning the arising inverse problems are provided. Two variants of general cost functionals are dealt with which compare calculated solutions within the domain. Additionally, in subsection 4.2, our approach is related to other ones like Dirichlet-to-Neumann type mappings and full-field approaches. In subsection 4.4, some results about the existence of a global minimum of the cost functionals presented are proven.

(iv) An outlook concludes this study in section 5.

2 General parameter-dependent linear elliptic problems

In this section, it is the aim to provide two *prototypical* elliptic problems *for a single equation* dealt with. This is for convenience and for a better overview. As we will see in subsection 3, this is not an essential restriction, since some important elliptic problems (arising from elasticity or from electrical impedance tomography, e.g.) can be studied in a very analogous manner. We provide exact formulations and results on existence, uniqueness of weak solutions and on parameter dependence of the solution (or forward) operators. Furthermore, we prove some special continuity results of the solution operators, using higher integrability of the gradient of weak solutions.

2.1 Two prototypical problems

As already said in the introduction, in this work we study general linear elliptic problems of second order with a possibly non-linear dependence on a parameter as well as some *arising inverse problems* for determination of this parameter. To avoid the high technical effort for a comprehensive consideration of second-order elliptic problems stemming from general elliptic equations or from elliptic systems of equations, we mainly deal with *two prototypical mixed boundary-value problems* arising from a single linear elliptic equation without lower-order terms. However, as a rule, the results presented in the sequel can be transferred to more complex equations and systems of equations, including Lamé's equations of linear elasticity, performing only technical modifications. In subsection 3, we list up some of these cases and give comments.

Therefore, we consider the following two *alternative* mixed boundary-value problems for the same partial differential equation (PDE). For convenience, we formulate them in a classical setting. Later on, we deal with their weak formulations and give more precise definitions.

Let Ω be a domain with three boundary parts like in Fig. 1. For exact assumptions see (2.9) - (2.11).

Problem I: A function $u : \Omega \to \mathbb{R}$ is looked for fulfilling the following equation and boundary conditions.

$$-\mathrm{div}\,(b(\cdot,\kappa)\nabla u) = f \qquad \text{a.e. in } \Omega, \tag{2.1}$$

$$u = \phi \qquad \text{on } \Gamma_{DN}, \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{2.3}$$

$$b(\cdot,\kappa)\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma_N. \tag{2.4}$$

The function κ is a parameter function (defined on Ω), the functions f and ϕ are regarded as data, ν is the outer unit normal on the boundary parts Γ_{DN} and Γ_N. The parameter-dependent coefficient function b will be described later on.

Alternatively to problem I, we consider the corresponding problem II differing only in the boundary condition on Γ_{DN}:

Problem II: A function $v : \Omega \to \mathbb{R}$ is looked for fulfilling the following equation and boundary conditions (for further purpose we distinguish between u and v).

$$-\mathrm{div}\,(b(\cdot,\kappa)\nabla v) = f \qquad \text{a.e. in } \Omega, \tag{2.5}$$

$$b(\cdot,\kappa)\frac{\partial v}{\partial \nu} = \tau \qquad \text{on } \Gamma_{DN}, \tag{2.6}$$

$$v = 0 \qquad \text{on } \Gamma_D, \tag{2.7}$$

$$b(\cdot,\kappa)\frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \Gamma_N. \tag{2.8}$$

f (being the *same* as in (2.1)) and τ are regarded as data, ν is as before.

Problem I has Dirichlet conditions on Γ_D and on Γ_{DN} and a Neumann condition on Γ_N, while problem II has a Dirichlet conditions on Γ_D and Neumann conditions on Γ_{DN} and on Γ_N (see Fig. 1).

This setting is caused by the final purpose to deal with inverse problems. Considering a problem for an elliptic PDE like in (2.1), we assume that on Γ_D Dirichlet boundary conditions and on Γ_N Neumann boundary conditions are given. Moreover, as a *special issue*, on Γ_{DN} Dirichlet and Neumann conditions are *simultaneously* given corresponding to the *same* model situation (or to the same experiment or measurement). This leads to *two* alternative mixed boundary-value problems for the *same* equation, namely to the problems I and II. See remark 2.5 for non-homogeneous Dirichlet conditions on Γ_D and for Robin conditions on Γ_N and/or Γ_{DN}.

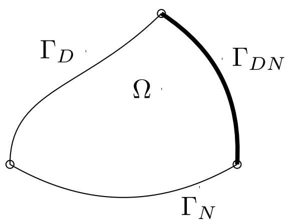

Figure 1: Scheme of the bounded domain Ω and its boundary parts. On Γ_{DN}, double information is available (see the explanation in the text).

Clearly, for models *exactly* describing deterministic scenarios these both problems must yield the same solution $u = v$, if the same experiment (or measurement) is performed. Due to almost always existing noise as well as deviations caused by numerical calculations, the calculated solutions generally differ. However, assuming that for each admissible parameter κ each problem has a unique solution $u = u_\kappa$ and $v = v_\kappa$, respectively, a comparison of u_κ and v_κ may be a point of departure for determining an "optimal" parameter κ.

In applications, the *double information* on Γ_{DN} coming from an experiment may be, for instance, displacement and normal stress in mechanics (see Yun and Shang (2011), Shang and Yun (2012), Shang et al. (2013)) or electrical current and electrical potential in electrical impedance tomography (see e.g., Jin and Maaß (2012a) for further explanation and references).

2.2 Assumptions concerning domain and coefficient function

Let a domain (= nonempty, open and connected set) Ω and its three boundary parts like in Fig. 1 have the following general properties.

(2.9)	$\Omega \subset \mathbb{R}^d$		bounded Lipschitz domain,
(2.10)	$\Gamma_D, \Gamma_{DN}, \Gamma_N \subset \partial\Omega,$	$\partial\Omega = \Gamma_D \cup \Gamma_{DN} \cup \Gamma_N$	mutually disjoint,
(2.11)	Γ_D, Γ_{DN}		closed with positive surface measure.

The space dimension d is generally a natural number, in applications often belonging to $\{1, 2, 3\}$. In more idealized situations, $\partial\Omega = \Gamma_{DN}$ can be assumed, avoiding mixed boundary conditions, see e.g., Jin and Maaß (2012a). We notice that Γ_D or Γ_N can be empty. For convenience, we let Γ_D have a positive surface measure (see (2.11)), ensuring uniqueness of weak solutions for both prototypical problems without an additional requirement like in pure Neumann boundary-value problems.

Let J be given by

$$J := [k_1, k_2] \subset \mathbb{R} \qquad \text{with } -\infty < k_1 < k_2 < \infty, \tag{2.12}$$

and let

$$b : \Omega \times J \to \mathbb{R} \tag{2.13}$$

be a Carathéodory function, i.e., for almost all $x \in \Omega$, b is continuous with respect to $s \in J$ and for all $s \in J$ it is Lebesgue measurable with respect to $x \in \Omega$ (see e.g., Showalter (1997)). Moreover, we assume

(2.14)	$\exists\, \beta_0 \ge 0, \beta_1 \ge 0$	f.a.a. $x \in \Omega$	$\forall\, s \in J$	:	$\lvert b(x,s)\rvert \le \beta_0 + \beta_1 \lvert s\rvert,$
(2.15)	$\exists\, \beta_2 > 0$	f.a.a. $x \in \Omega$	$\forall\, s \in J$	:	$b(x,s) \ge \beta_2,$
(2.16)	$\exists\, \beta_3 > 0$	f.a.a. $x \in \Omega$	$\forall\, s_1, s_2 \in J$	:	$\lvert b(x,s_1) - b(x,s_2)\rvert \le \beta_3 \lvert s_1 - s_2\rvert.$

We chose the *parameter space* and an *admissible set of parameters* in the following way. The sense of "admissible" will be clear in the sequel.

$$K := L^\infty(\Omega), \tag{2.17}$$

$$K_{ad} := \{\kappa \in K \,|\, \text{a.e. in } \Omega : \kappa(x) \in J\}. \tag{2.18}$$

Due to (2.12), K_{ad} is a closed, bounded and convex subset of K. This is the standard case, in particular for linear parameter dependence. Sometimes, a special behavior of b allows an unbounded interval J and a larger parameter space K than $L^\infty(\Omega)$. This may be useful in concrete applications.

For illustration, in the following remarks we consider some special cases of J, b, K and K_{ad}.

Remarks 2.1. (**Examples for** J, b, K **and** K_{ad}) The following examples are included in the general case considered above. To focus we drop a possible dependence on $x \in \Omega$. Obviously, to provide more general examples, all functions b listed here can be multiplied by a function $b_0 \in L^\infty(\Omega)$ bounded to below by a positive constant.

(i) (*Linear parameter dependence*) Let J be as in (2.12) with $0 < k_1$ and b be defined by

$$b(x,s) := s \qquad \text{f.a.a. } x \in \Omega \qquad \forall\, s \in J \tag{2.19}$$

Clearly, the choice (2.17), (2.18) for K and K_{ad} is suitable.
A single elliptic equation with this linear parameter dependence arises for instance in electrical impedance theory (see, e.g., Jin and Maaß (2012a), Jin and Maass (2012b), Pham (2015)) and in applications to geology (see, e.g., Lukaschewitsch et al. (2009))

(ii) A nonlinear example is given by J as in (i) and by

$$b(x,s) := s + \arctan(s) \qquad \text{f.a.a. } x \in \Omega \qquad \forall\, s \in J \tag{2.20}$$

and with K and K_{ad} as in (2.17), (2.18).

(iii) Let J be as in (2.12) with $k_1 = 0$ and b be defined by

$$b(x,s) := 1 + s + \arctan(s) \qquad \text{f.a.a. } x \in \Omega \qquad \forall\, s \in J \tag{2.21}$$

Again, the choice (2.17), (2.18) for K and K_{ad} is suitable.

(iv) Let be $J := \mathbb{R}$ and b as

$$b(x,s) := 1 + \frac{1}{\pi}\arctan(s) \qquad \text{f.a.a. } x \in \Omega \qquad \forall\, s \in J \tag{2.22}$$

Now, one can chose $K = K_{ad} := L^q(\Omega)$ with a fixed $1 \le q \le \infty$. In applications, $1 \le q < \infty$, in particular $q = 2$, can be useful.

(v) Let be $J := [k_1, \infty]$ with $k_1 > 0$ and b as

$$b(x,s) := \arctan(s) \qquad \text{f.a.a. } x \in \Omega \qquad \forall\, s \in J \tag{2.23}$$

Now, one can chose $K := L^q(\Omega)$ with a fixed $1 \le q \le \infty$ and K_{ad} as in (2.18).

2.3 Weak formulations

To define corresponding weak formulations of the problems I and II we introduce the basic Hilbert space V and the following test-function spaces arising from the given boundary conditions. Here, we only consider real-valued functions.

$$V : = W^{1,2}(\Omega), \tag{2.24}$$
$$V_0 : = \{\varphi \in V \,|\, \varphi = 0 \text{ on } \Gamma_D\}, \tag{2.25}$$
$$V_1 : = \{\varphi \in V \,|\, \varphi = 0 \text{ on } \Gamma_D \cup \Gamma_{DN}\}, \tag{2.26}$$

$W^{1,2}(\Omega)$ is the usual Sobolev space (of real-valued functions), see Showalter (1997), Adams and Fournier (2003), e.g. V_i^* are the corresponding dual spaces. Clearly, $V_1 \subsetneq V_0$. The dual pairing between $f \in V_i^*$ and $\varphi \in V_i$ is denoted by $\langle f\,,\,\varphi\rangle_{V_i^*,V_i}$.

The parameter space K and the set of admissible parameters $K_{ad} \subset K$ are defined in (2.17) and (2.18), respectively. In a standard way we define the form $a\,:\,K_{ad} \times V \times V \to \mathbb{R}$ being associated with the underlying linear elliptic problems (2.1) - (2.4) and (2.5) - (2.8):

$$a(\kappa,u,\varphi) := \int_\Omega b(x,\kappa)\nabla u \cdot \nabla\varphi \,\mathrm{d}x \qquad \forall\,(\kappa,u,\varphi) \in K_{ad} \times V \times V. \tag{2.27}$$

($\nabla u \cdot \nabla\varphi$ - scalar product in $\mathbb{R}^d$.) Clearly, there holds the following result.

Lemma 2.2. Let the assumptions (2.9) - (2.18) be given. Then there exist real numbers $\alpha_2 = \alpha_2(\beta_2) > 0$, $\alpha_1 = \alpha_1(\beta_0, \beta_1, k_1, k_2) > 0$ and $\alpha_3 = \alpha_3(\beta_3) > 0$ such that the form a defined in (2.27) fulfils

$$\forall \kappa \in K_{ad} \quad : \quad a(\kappa, \cdot, \cdot) : V \times V \to \mathbb{R} \quad \text{bilinear,} \tag{2.28}$$

$$\forall (\kappa, u, v) \in K_{ad} \times V \times V \quad : \quad |a(\kappa, u, v)| \leq \alpha_1 \|u\|_V \|v\|_V, \tag{2.29}$$

$$\forall (\kappa, u) \in K_{ad} \times V_0 \quad : \quad a(\kappa, u, u) \geq \alpha_2 \|u\|_V^2, \tag{2.30}$$

$$\forall (\kappa_1, u, v), (\kappa_2, u, v) \in K_{ad} \times V \times V \quad : \quad |a(\kappa_1, u, v) - a(\kappa_2, u, v)| \leq \leq \alpha_3 \|u\|_V \|v\|_V \|\kappa_1 - \kappa_2\|_K. \tag{2.31}$$

Thus, for $\kappa \in K_{ad}$ the family of bilinear forms $a(\kappa, \cdot, \cdot)$ is uniformly continuous and coercive. Moreover, on bounded sets of u and v they are uniformly Lipschitz continuous with respect to the parameter κ. Therefore, the admissibility of $K_{ad} \subset K$ means, that for all $\kappa \in K_{ad}$ the bilinear form $a(\kappa, \cdot, \cdot)$ has the properties (2.28) - (2.31).

Sometimes, in more general situations like in (3.1), the assumption of symmetry in the sense of

$$\forall (\kappa, u, v) \in K_{ad} \times V \times V \quad : \quad a(\kappa, u, v) = a(\kappa, v, u). \tag{2.32}$$

is helpful. Clearly, in the case of the prototypical problems, the arising form a in (2.27) follows automatically. However, besides in the special cases like in linear elasticity, the assumption (2.32) is *not* needed for existence and uniqueness results.

Remark 2.3. (**General linear elliptic problems**) In more general cases of linear elliptic problems with parameters (see subsection 3 for examples), the parameter space K is usually assumed to be a Banach space of functions defined on Ω. Then, a set $K_{ad} \subset K$ is called admissible, if the corresponding bilinear form a has the properties (2.28) - (2.31).

Sometimes, one needs the continuous embedding of V into $W^{\frac{1}{2},2}(\Gamma_{DN})$ in the sense of trace (see Showalter (1997), Adams and Fournier (2003), e.g.). Thus, there holds for the trace $u_{|\Gamma_{DN}}$ of u

$$\exists\, c_1 > 0 \quad \forall\, u \in V \quad : \quad \|u_{|\Gamma_{DN}}\|_{W^{\frac{1}{2},2}(\Gamma_{DN})} \leq c_1 \|u\|_V. \tag{2.33}$$

We assume for the Dirichlet-boundary data as well as for the Neumann-boundary data τ on Γ_{DN}:

$$\phi \in W^{\frac{1}{2},2}(\Gamma_{DN}) \tag{2.34}$$

$$\tau \in (W^{\frac{1}{2},2}(\Gamma_{DN}))^*. \tag{2.35}$$

To homogenize the problem I ("generalized Dirichlet problem") we assume, that the function ϕ has an extension $\widetilde{\phi} \in V_0$ with the estimate

$$\|\widetilde{\phi}\|_V \leq c_2 \|\phi\|_{W^{\frac{1}{2},2}(\Gamma_{DN})}, \tag{2.36}$$

where $c_2 > 0$ is independent of ϕ (cf. Kufner et al. (1977), Adams and Fournier (2003), e.g.). Clearly, this continuation property generally yields further restrictions to the boundary $\partial\Omega$, in particular to the intersections $\Gamma_{DN} \cap \Gamma_D$ and $\Gamma_{DN} \cap \overline{\Gamma_N}$.

Now we give weak formulations for both problems I and II described in (2.1) - (2.4) and (2.5) - (2.8), respectively. To avoid to much repetitions we assume (2.9) - (2.18) for the rest of this subsection.

Weak formulation of problem I (non-homogenized form) ("generalized Dirichlet problem")
Let $\kappa \in K_{ad}$, (2.34) and

$$f \in V_1^* \tag{2.37}$$

be given. A function $u \in V_0$ is called a (weak) solution to problem I (2.1) - (2.4), if it fulfils

$$u = \phi \qquad \text{on } \Gamma_{DN} \tag{2.38}$$

$$a(\kappa, u, \varphi) = \langle f\,,\, \varphi \rangle_{V_1^*, V_1} \qquad \forall\, \varphi \in V_1. \tag{2.39}$$

Note that the function u generally does *not* belong to the test-function space V_1.

Weak formulation of problem I (homogenized form) ("generalized Dirichlet problem")
Let $\kappa \in K_{ad}$, (2.34), (2.36) and (2.37) be given. A function $u \in V_0$ is called a (weak) solution to problem I (2.1) - (2.4), if there holds

$$u = w + \widetilde{\phi}, \tag{2.40}$$

while $w \in V_1$ satisfies

$$a(\kappa, w, \varphi) = -a(\kappa, \widetilde{\phi}, \varphi) + \langle f \, , \, \varphi\rangle_{V_1^*, V_1} \qquad \forall\, \varphi \in V_1. \tag{2.41}$$

Weak formulation of problem II ("generalized Neumann problem")
Let $\kappa \in K_{ad}$, (2.35) and

$$f \in V_0^* \tag{2.42}$$

be given. A function $v \in V_0$ is called a (weak) solution to problem II (2.5) - (2.8), if there holds

$$a(\kappa, v, \varphi) = \langle \tau \, , \, \varphi\rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} + \langle f \, , \, \varphi\rangle_{V_0^*, V_0} \qquad \forall\, \varphi \in V_0. \tag{2.43}$$

Obviously, the weak formulations above can be brought into equivalent abstract operator equations.

Lemma 2.4. **(i)** Let $\kappa \in K_{ad}$, (2.34) and (2.37) be given. Then the weak formulation (2.38), (2.39) of problem I can be equivalently expressed by

$$A_I(\kappa, u) = f \qquad \text{in } V_1^*, \tag{2.44}$$

with the operator $A_I \,:\, K \times \{u \in V_1 \,|\, u = \phi \text{ on } \Gamma_{DN}\} \to V_1^*$ being continuous in u, Lipschitz continuous in κ and defined by

$$\langle A_I(\kappa, u) \, , \, \varphi\rangle_{V_1^*, V_1} := a(\kappa, u, \varphi) \qquad \forall\, (\kappa, u, \varphi) \in K \times \{u \in V_1 \,|\, u = \phi \text{ on } \Gamma_{DN}\} \times V_1. \tag{2.45}$$

(ii) Let $\kappa \in K_{ad}$, (2.34), (2.36), (2.37) and (2.40) be given. Then the weak formulation (2.41) of the homogenized problem I can be equivalently expressed by

$$A_{I,hom}(\kappa, w) = F_{I,hom}(\kappa, f, \widetilde{\phi}) \qquad \text{in } V_1^*, \tag{2.46}$$

with F_I defined by

$$\langle F_{I,hom}(\kappa, f, \widetilde{\phi}) \, , \, \varphi\rangle_{V_1^*, V_1} := -a(\kappa, \widetilde{\phi}, \varphi) + \langle f \, , \, \varphi\rangle_{V_1^*, V_1} \qquad \forall\, \varphi \in V_1, \tag{2.47}$$

and with the operator $A_{I,hom} \,:\, K \times V_1 \to V_1^*$ being linear and continuous in u, Lipschitz continuous in κ and defined by

$$\langle A_{I,hom}(\kappa, w) \, , \, \varphi\rangle_{V_1^*, V_1} := a(\kappa, w, \varphi) \qquad \forall\, (\kappa, w, \varphi) \in K \times V_1 \times V_1. \tag{2.48}$$

(iii) Let $\kappa \in K_{ad}$, (2.35) and (2.42) be given. Then the weak formulation (2.43) of problem II can be equivalently expressed by

$$A_{II}(\kappa, v) = F_{II}(f, \tau) \qquad \text{in } V_0^*, \tag{2.49}$$

with F_{II} defined by

$$\langle F_{II}(f, \tau) \, , \, \varphi\rangle_{V_0^*, V_0} := \langle \tau \, , \, \varphi\rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} + \langle f \, , \, \varphi\rangle_{V_0^*, V_0} \qquad \forall\, \varphi \in V_0, \tag{2.50}$$

and with the operator $A_{II} \,:\, K \times V_0 \to V_0^*$ being linear and continuous in u, Lipschitz continuous in κ and defined by

$$\langle A_{II}(\kappa, v) \, , \, \varphi\rangle_{V_0^*, V_0} := a(\kappa, v, \varphi) \qquad \forall\, (\kappa, w, \varphi) \in K \times V_0 \times V_0. \tag{2.51}$$

Note that in the first case without homogenization the corresponding operator A_I is not linear in u, since its domain is not linear. Thus, to overcome this inconvenience one usually prefers the homogenized version. In this case, the new right-hand side explicitly depends on κ (see (2.41)). After establishing the existence of the "auxiliary" weak solution w, one can go back to the equivalent weak formulation (2.38), (2.39).

Moreover, if the weak solution, the function b, the parameter function κ and the data exhibit better regularity, the weak formulations yield the differential equations (2.1) and (2.5) as well as the Neumann boundary conditions (2.4) and (2.6), (2.8), respectively, in a standard way. Generally, these equations and conditions can only be interpreted as relations between distributions, see subsection 2.6.

Remarks 2.5. (**More general boundary conditions**) Further boundary conditions are possible and can be treated within the same frame with some modifications.

(i) (Non-homogeneous Dirichlet conditions on Γ_D) Due to applications, on Γ_D may be given the same non-homogeneous Dirichlet condition for problem I and II. Generally, this can be dealt with like in the weak formulation (2.38), (2.39). Assuming a sufficient regularity of the boundary function, after a standard homogenization there arises a new right-hand side in the equation generally depending on κ like in (2.41).

(ii) (Robin conditions on Γ_{DN} and Γ_N) On Γ_{DN} (for problem II) and/or Γ_N (for both problems) may be given a Robin boundary condition like

$$b(\cdot,\kappa)\frac{\partial u}{\partial \nu} + d(\cdot,\kappa)u = \tau \qquad \text{on } \Gamma_{DN} \tag{2.52}$$

corresponding to equation (2.1). d is a suitable Carathéodory function generally depending on κ (cf. (2.12) - (2.16)). Clearly, now the elements of K must have a trace on Γ_{DN} and Γ_N. Therefore, K must be narrower than L^∞. This case can be handled in the standard way leading to an extended form $\hat{a}$ which reads for problem II

$$\hat{a}(\kappa,u,v) := \int_\Omega b(x,\kappa)\nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_{\Gamma_{DN}} d(x,\kappa)uv\,\mathrm{d}_\Gamma x. \tag{2.53}$$

Thus, d must fulfil further assumptions to ensure the properties (2.29) - (2.31) of $\hat{a}$. A non-linear Robin condition (with $d(\cdot,\kappa,u)$ instead of $d(\cdot,\kappa)u$) leads to a *non-linear problem* which requires other tools for proving existence and uniqueness results. This will not be considered here.

2.4 Existence, Uniqueness, well-posedness for linear elliptic problems

The well-posedness of problems I and II, resp., is standard. Lax-Milgrams's theorem and the assumptions (2.28) - (2.31) are the essential ingredients (cf. Zeidler (1990a), Showalter (1997), e.g.). Again, to avoid to much repetitions, we assume (2.9) - (2.18) as common assumptions for both problems I and II. Moreover, we keep lemma 2.2 in mind. From now on we regard Problem I as given in its weak formulation (2.38), (2.39) and analogously, problem II as given by (2.43). Clearly, in this context, solution means weak solution.

Theorem 2.6. (**Existence and uniqueness**)

(i) (**Problem I**) Let (2.34), (2.36) and (2.37) be given. Than for each $\kappa \in K_{ad}$ there exists a unique solution $u \in V_0$ to problem I fulfilling the estimate

$$\|u\|_V \le \frac{c_2(\alpha_2+\alpha_1)}{\alpha_2}\|\phi\|_{W^{\frac{1}{2},2}(\Gamma_{DN})} + \frac{1}{\alpha_2}\|f\|_{V_1^*} \le c_3\big(\|\phi\|_{W^{\frac{1}{2},2}(\Gamma_{DN})} + \|f\|_{V_1^*}\big) \tag{2.54}$$

with some constant $c_3 = c_3(\alpha_1,\alpha_2,c_2) > 0$.

(ii) (**Problem II**) Let (2.35) and (2.42) be given. Than for each $\kappa \in K_{ad}$ there exists a unique solution $v \in V_0$ of problem II fulfilling the estimate

$$\|v\|_V \le \frac{c_1}{\alpha_2}\|\tau\|_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*} + \frac{1}{\alpha_2}\|f\|_{V_0^*} \le c_4\big(\|\tau\|_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*} + \|f\|_{V_0^*}\big) \tag{2.55}$$

with some constant $c_4 = c_4(\alpha_1,\alpha_2,c_1) > 0$.

Solution operators: We define the solution operators (often called forward operators, too) of both elliptic problems. Under the assumptions of theorem 2.6 the solution operator of problem I, L_I is defined by

$$u = L_I(f,\phi,\kappa) \tag{2.56}$$

with

$$L_I\,:\,V_1^*\times W^{\frac{1}{2},2}(\Gamma_{DN})\times K_{ad}\,\to\,V_0\subset V\;(=W^{1,2}(\Omega)). \tag{2.57}$$

Analogously, the solution operator of problem II, L_{II} is given by

$$v = L_{II}(f,\tau,\kappa) \tag{2.58}$$

where

$$L_{II} : V_0^* \times (W^{\frac{1}{2},2}(\Gamma_{DN}))^* \times K_{ad} \to V_0 \subset V. \tag{2.59}$$

To underline the dependence of the solutions u and v of the problems I and II, respectively, on the parameter κ, we also will use the notations $u_\kappa = L_I(f, \phi, \kappa)$ and $v_\kappa = L_{II}(f, \tau, \kappa)$. Besides, this makes sense since the remaining values f, ϕ, τ are often kept fixed, when looking for an optimal parameter. In the context of inverse problems the solution operators are usually called *forward operators*. We will return to this matter in section 4.

The preceding theorem easily leads to the following one by standard arguments. Here, the assumed Lipschitz continuity of the form a with respect to $\kappa \in K_{ad}$ (see (2.16), or more general (2.31)) is used for the first time.

Theorem 2.7. (**Well-posedness**)

(i) (**Problem I**) Let $\kappa_i \in K_{ad}$, $f_i \in V_1^*$, and $\phi_i \in W^{\frac{1}{2},2}(\Gamma_{DN})$ with the continuation property (2.36) as well as (2.31) be given ($i = 1, 2$). Then the corresponding unique weak solutions of problem I (2.38), (2.39), $u_1, u_2 \in V_0$, fulfil the estimate

$$\begin{aligned} \|u_1 - u_2\|_V \le c_5 \Big\{ &\|f_1 - f_2\|_{V_1^*} + \|\phi_1 - \phi_2\|_{W^{\frac{1}{2},2}(\Gamma_{DN})} + \\ &+ \big(\|\phi_2\|_{W^{\frac{1}{2},2}(\Gamma_{DN})} + \|f_2\|_{V_1^*}\big) \|\kappa_1 - \kappa_2\|_K \Big\} \end{aligned} \tag{2.60}$$

with some constant $c_5 = c_5(\alpha_1, \alpha_2, \alpha_3, c_2) > 0$.

(ii) (**Problem II**) Let $\kappa_i \in K_{ad}$, $f_i \in V_0^*$ and $\tau_i \in (W^{\frac{1}{2},2}(\Gamma_1))^*$ and (2.31) be given ($i = 1, 2$), and let $v_1, v_2 \in V_0$ be the corresponding unique weak solutions of problem II (2.43). Then there holds the estimate

$$\begin{aligned} \|v_1 - v_2\|_V \le c_6 \Big\{ &\|\tau_1 - \tau_2\|_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*} + \|f_1 - f_2\|_{V_1^*} + \\ &+ \big(\|\tau_2\|_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*} + \|f_2\|_{V_0^*}\big) \|\kappa_1 - \kappa_2\|_K \Big\} \end{aligned} \tag{2.61}$$

with some constant $c_6 = c_6(\alpha_1, \alpha_2, \alpha_3, c_1) > 0$.

Often only the parameter $\kappa \in K$ varies, while the remaining data is kept fixed. Thus, the preceding estimates considerably simplify.

Corollary 2.8. Let $\kappa_1, \kappa_2 \in K_{ad}$. Then there hold the following estimates for the solutions $u_i = L_I(f, \phi, \kappa_i)$ and $v_i = L_{II}(f, \tau, \kappa_i)$ ($i = 1, 2$) of problems I and II, respectively:

$$\|u_1 - u_2\|_V \le \frac{c_7}{\alpha_2} \min\big\{\|u_1\|_V, \|u_2\|_V\big\} \|\kappa_1 - \kappa_2\|_K, \tag{2.62}$$

$$\|v_1 - v_2\|_V \le \frac{c_7}{\alpha_2} \min\big\{\|v_1\|_V, \|v_2\|_V\big\} \|\kappa_1 - \kappa_2\|_K \tag{2.63}$$

with some constant $c_7 = c_7(\alpha_1, \alpha_2, \alpha_3, c_2) > 0$ and $c_7 = c_7(\alpha_0, \alpha_1, \alpha_3, c_1) > 0$, respectively.

Clearly, the minima on the right-hand sites of (2.62) and (2.63) can be estimates using (2.54) and (2.55), respectively. The following continuity statements for the solution operators can be derived from theorem 2.7.

Theorem 2.9. (**Continuity assertions for the solutions operators**) Under the assumptions of theorem 2.7, the following assertions hold:

(i) (**Problem I**)

$$L_I \quad : \quad V_1^* \times W^{\frac{1}{2},2}(\Gamma_{DN}) \times K_{ad} \to V_0 \subset V \tag{2.64}$$

Lipschitz continuous on bounded subsets of $V_1^* \times W^{\frac{1}{2},2}(\Gamma_{DN}) \times K_{ad}$.

(ii) (**Problem II**)

$$L_{II} \quad : \quad V_0^* \times (W^{\frac{1}{2},2}(\Gamma_{DN}))^* \times K_{ad} \to V_0 \subset V \tag{2.65}$$

Lipschitz continuous on bounded subsets of $V_0^* \times (W^{\frac{1}{2},2}(\Gamma_{DN}))^* \times K_{ad}$.

2.5 Higher integrability of the gradient of weak solutions

2.5.1 General remarks

A global higher integrability of the gradients of weak solutions holds for a wide class of generally quasi-linear second-order elliptic systems of PDE under moderate assumptions (see e.g., Bensoussan and Frehse (2013), Giaquinta (2016) for general setting of such systems). Higher integrability means that the (components of the) gradient belong to a *better* Lebesgue space than it is necessary for a correct definition of the corresponding weak solution.

In particular, in the linear cases under consideration (2.1) - (2.4) and (2.5) - (2.8), the gradient ∇u "originally" belongs to $(L^2(\Omega))^d$ by definition of the weak solution. If the data exhibit a better integrability than required for a correct definition of the weak solution, then ∇u belongs to $(L^p(\Omega))^d$ for some $p > 2$. Later on there will be clear formulations for our setting.

This "global higher integrability of the gradient" has been intensively investigated during the last decades, at first locally, after that globally, for elliptic and parabolic systems. We refer to Gröger (1989), Gröger and Rehberg (1989), Naumann and Wolff (1991), Naumann and Wolff (1995) and Haller-Dintelmann and Rehberg (2008) for general elliptic (and parabolic) problems like in subsections 3.1 and 3.2 as well as to Dahlberg et al. (1988), Shi and Wright (1994), Brown and Mitrea (2009), Ott and Brown (2013) and Herzog et al. (2011) with respect to the equations of elasticity provided in subsection 3.3. In Wolff (1996), a corresponding global result concerning the stationary Stokes problem with mixed boundary condition was proved. Moreover, we refer to Fiaschi et al. (2013), where an analogous higher integrability of the gradient of minimizers to a wide class of functionals was studied.

Although the results "global higher integrability of the gradient under mixed boundary condition" are very similar in Naumann and Wolff (1991) and Gröger (1989), Gröger and Rehberg (1989), the techniques of the proofs are entirely different. As a consequence, more restrictive assumptions on the coefficient functions are needed in Gröger (1989), Gröger and Rehberg (1989). More historical remarks can be found in Fiaschi et al. (2013). Regarding our special setting, more comments will be given in remark 2.11 (i).

As we will see below, in connection with inverse problems for elliptic problems this global higher integrability of the gradient allows to obtain special results.

2.5.2 Results for the prototypical problems I and II

We continue to focus on our prototypical problems I and II, given by (2.1) - (2.4) and (2.5) - (2.8), respectively. There will be comments concerning more general elliptic problems in remarks.

Thus, we consider the weak formulation for the single elliptic equation in (2.1) and (2.5) leading to the corresponding form a defined in (2.27). For convenience, we repeat it her:

$$a(\kappa, u, \varphi) := \int_\Omega b(x,\kappa)\nabla u \cdot \nabla\varphi \,\mathrm{d}x \qquad \forall\,(\kappa, u, \varphi) \in K_{ad} \times V \times V \tag{2.66}$$

with $V := W^{1,2}(\Omega)$, V_0, V_1 as in (2.25), (2.26) as well as with Ω, Γ_D, Γ_{DN}, Γ_N, $J \subset \mathbb{R}$, K, K_{ad} and $b : \Omega \times J \to \mathbb{R}$ in accordance with (2.9) - (2.18).

Clearly, the case of linear parameter dependence (2.19) is included into this setting.

It can be proved that under somewhat better assumptions on the data ϕ, f and τ, f, respectively, as well as on the boundary parts Γ_D, Γ_{DN} and Γ_N the weak solutions of problem I and II belong to $W^{1,p}(\Omega)$ for some $p > 2$. Additional notations are needed.

$$V^p := W^{1,p}(\Omega), \tag{2.67}$$

$$V_0^p := \{\varphi \in V^p \,|\, \varphi = 0 \text{ on } \Gamma_D\}, \tag{2.68}$$

$$V_1^p := \{\varphi \in V^p \,|\, \varphi = 0 \text{ on } \Gamma_D \cup \Gamma_{DN}\}, \tag{2.69}$$

$W^{1,p}(\Omega)$ is the usual Sobolev space for $1 \le p \le \infty$. By $(V^p)^*$ and $(V_i^p)^*$ $i = 0, 1$ we denote the dual spaces of V^p and V_i^p, respectively. It is $V_i = V_i^2$ (see (2.24) - (2.26)).

$$\langle\cdot\,,\,\cdot\rangle_{(V_0^p)^*, V_0^p} : (V_0^p)^* \times V_0^p \to \mathbb{R}$$

is the dual pairing. $p' := p/_{p-1}$ - conjugate exponent. Additionally to (2.9) - (2.11), the parts Γ_D, Γ_{DN} and Γ_N of the boundary $\partial\Omega$ should smash each other in a not too bad way. We assume

$$\Omega \cup \Gamma_N, \quad \Omega \cup (\Gamma_N \cup \Gamma_{DN}) \qquad \text{regular in the sense of Gröger,} \tag{2.70}$$

(cf. Gröger (1989), Gröger and Rehberg (1989), Haller-Dintelmann and Rehberg (2008) and Herzog et al. (2011) for details. In Naumann and Wolff (1991), the analogous requirement is, that $\Gamma_N \cap \Gamma_D$ and $\Gamma_N \cap \Gamma_{DN}$ are locally the graph of a Lipschitz function of $d-2$ variables.)

Theorem 2.10. (**Higher integrability of the gradient**) Let (2.9) - (2.18) and (2.67) - (2.70) be given. Let the form a be given by (2.66). Then there exists a $\overline{p} > 2$ such that for all $p \in [2, \overline{p}]$ the following assertions hold.

(i) (**Problem I**) Let $f \in (V_1^{p'})^*$, $\phi \in W^{1-\frac{1}{p},p}(\Gamma_{DN})$ with an analogous to (2.36) continuation property in V^p be given. Then for each $\kappa \in K_{ad}$ there exists a unique weak solution $u \in V_0^p$ of problem I (2.39), (2.40) with the estimate

$$\|u\|_{V^p} \leq c_8 \big(\|\phi\|_{W^{1-\frac{1}{p},p}(\Gamma_{DN})} + \|f\|_{(V_1^{p'})^*} \big). \tag{2.71}$$

The constant $c_8 > 0$ is independent of u, ϕ and f and universal for all $\kappa \in K_{ad}$ and for all $p \in [2, \overline{p}]$.

(ii) (**Problem II**) If $\boldsymbol{f} \in (V_0^{p'})^*$, $\tau \in (W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^*$, then for each $\kappa \in K_{ad}$ there exists a unique weak solution $v \in V_0^p$ of problem II (2.43) with the estimate

$$\|v\|_{V^p} \leq c_9 \big(\|\tau\|_{(W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^*} + \|f\|_{(V_0^{p'})^*} \big). \tag{2.72}$$

The constant $c_9 > 0$ is independent of v, τ, f and universal for all $\kappa \in K_{ad}$ and for all $p \in [2, \overline{p}]$.

For simplicity, we include the case $p = 2$ which is in deed covered by theorem 2.6 (with estimates without the one inside the brackets).

Remarks 2.11. **(i)** Theorem 2.10 follows from the main result in Naumann and Wolff (1991) [section 1, estimate (1.13_1)] due to the fact that the elliptic equation is linear.

(ii) As pointed out in subsection 2.5.1, higher integrability of the gradients of weak solutions can be proved for a wide class of quasi-linear elliptic systems of PDE. However, in the general case, the estimates corresponding to (2.71) and (2.71) are more complex. Moreover, the proof works with *assumed* weak solutions without ensuring their existence and a-priori estimates. Thus, on the right-hand sides of the resulting more general estimates than (2.71) and (2.72), there arise a further additive constant and the L^2 norm of the gradient of the (assumed!) weak solution, see e.g. Naumann and Wolff (1991).

(iii) The results in Gröger (1989), Gröger and Rehberg (1989) also cover theorem 2.10.

(iv) Theorem 2.10 only yields the existence of a $\overline{p} > 2$ without any further information. Hence, in the case of two spatial dimensions $d = 2$, the embedding theorem gives a continuous embedding $V^{\overline{p}} \subset C^{0,\lambda}(\overline{\Omega})$ for $0 < \lambda \leq 1 - 2/\overline{p}$, which is *compact* for $0 < \lambda < 1 - 2/\overline{p}$.

2.5.3 Special continuity results with respect to the parameter

From theorem 2.10 we get that under the corresponding assumption the solution operators L_I and L_{II} defined in (2.56) and (2.58), respectively, are also mappings in the following constellation for all $p \in [2, \overline{p}]$.

$$L_I : (V_1^{p'})^* \times W^{1-\frac{1}{p},p}(\Gamma_{DN}) \times K_{ad} \to V_0^p \subset V_0, \tag{2.73}$$

$$L_{II} : (V_0^{p'})^* \times (W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^* \times K_{ad} \to V_0^p \subset V_0. \tag{2.74}$$

For studying inverse problems, the (partial) continuity properties of the solution operators from the parameter space into the solution space are of great interest. Theorem 2.9 yields partial Lipschitz continuity from $K = L^\infty(\Omega)$ into $V_1 \subset W^{1,2}(\Omega)$ and $V_0 \subset W^{1,2}(\Omega)$, respectively. However, the space $L^\infty(\Omega)$ is *not* well-suited in further investigations. The higher-integrability results can close this gap in our situation. Here, we modify the arguments used in Jin and Maaß (2012a) and Jin and Maass (2012b). These authors deal with pure boundary conditions (i.e. not with mixed ones), and they use local (inner) higher integrability of the gradient, since in their setting the parameter function looked for has a compact support.

Based on theorem 2.10 and remark 2.11 (i), one can prove more special continuity properties of the solution operators.

Theorem 2.12. (**Continuity properties**) Let the assumptions of theorem 2.10 concerning problems I and II be given.

(i) (**Lipschitz continuity in the sense** $L^s \to V$) For any $p \in [2, \overline{p}]$, $s \in [\frac{2p}{p-2}, \infty]$ and $\kappa_0 \in K_{ad}$, $\kappa \in K$ with $\kappa_0 + \kappa \in K_{ad}$ the solution operators are Lipschitz continuous in the following sense.

$$\begin{aligned} (2.75) \qquad \|u_{\kappa_0+\kappa} - u_{\kappa_0}\|_V &= \|L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0)\|_V \leq \\ &\leq c_{10}\,\|\kappa\|_{L^s(\Omega)}\big(\|\phi\|_{W^{1-\frac{1}{p},p}(\Gamma_{DN})} + \|f\|_{(V_1^{p'})^*}\big), \\ (2.76) \qquad \|v_{\kappa_0+\kappa} - v_{\kappa_0}\|_V &= \|L_{II}(f,\tau,\kappa_0+\kappa) - L_{II}(f,\tau,\kappa_0)\|_V \leq \\ &\leq c_{11}\,\|\kappa\|_{L^s(\Omega)}\big(\|\tau\|_{(W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^*} + \|f\|_{(V_0^{p'})^*}\big). \end{aligned}$$

The constants c_{10} and $c_{11} > 0$ depend of s, Ω and c_8 and c_9, respectively.

(ii) (**Uniform continuity in the sense** $L^s \to V^p$) For any $2 < p < \min\{\overline{p}, 4\}$, $s \in [\frac{2p}{p-2}, \infty]$ and $\kappa_0 \in K_{ad}$, $\kappa \in K$ with $\kappa_0 + \kappa \in K_{ad}$ the solution operators are uniformly continuous in the following sense.

$$(2.77) \qquad \lim_{\|\kappa\|_{L^s(\Omega)}\to 0} \|u_{\kappa_0+\kappa} - u_{\kappa_0}\|_{V^p} = 0, \qquad \lim_{\|\kappa\|_{L^s(\Omega)}\to 0} \|v_{\kappa_0+\kappa} - v_{\kappa_0}\|_{V^p} = 0.$$

Proof. **(i)** Using the fact, that $u_{\kappa_0+\kappa} \in V_0^p \subset V_0$ and $u_{\kappa_0} \in V_0^p \subset V_0$ are weak solutions corresponding to $\kappa_0 + \kappa$ and κ_0, respectively, for the *same* data ϕ and f of problem I (2.38), (2.39), one easily obtains:

$$(2.78) \qquad a(\kappa_0, u_{\kappa_0+\kappa} - u_{\kappa_0}, \varphi) = a(\kappa_0, u_{\kappa_0+\kappa}, \varphi) - a(\kappa_0+\kappa, u_{\kappa_0+\kappa}, \varphi) \qquad \forall\, \varphi \in V_1.$$

Clearly, $u_{\kappa_0+\kappa} - u_{\kappa_0} \in V_1$ is an admissible test-function in (2.78). Using the structure of a given by (2.66), the Lipschitz continuity condition (2.16), Hölder's inequality with the exponents $2p/p-2$, p and 2 as well es the estimate (2.71), one obtains the estimate (2.75), at first for $s = 2p/p-2$, and after that for all $s \geq 2p/p-2$. The estimate (2.76) can be proved analogously.
(ii) The limiting relations in (2.77) can be proved in the following way. The relation (2.78) means that $u_{\kappa_0+\kappa} - u_{\kappa_0}$ is a weak solution of problem I with vanishing Dirichlet data on Γ_{DN} and with a right side of the form

$$(2.79) \qquad \langle F\,,\,\varphi\rangle := a(\kappa_0, u_{\kappa_0+\kappa}, \varphi) - a(\kappa_0+\kappa, u_{\kappa_0+\kappa}, \varphi) \qquad \forall\, \varphi \in V_1.$$

We show that F belongs to $(V_1^{p'})^*$ and estimate its norm for $2 < p < \min\{\overline{p}, 4\}$. Let δ be a real number with

$$(2.80) \qquad 0 < \delta < \min\{\overline{p} - p, \frac{p^2-2p}{4-p}\}.$$

Now, after using the Lipschitz continuity of b (2.16), it is admissible to apply Hölder's inequality with the exponents $p(p+\delta)/\delta$, $p+\delta$ and $p' = p/(p-1)$ to (each summand of) the right-hand side of (2.79). Using (2.71), we get

$$\begin{aligned} (2.81) \qquad |\langle F\,,\,\varphi\rangle| \leq & \,\overline{c}\,\|\kappa\|_{L^{\frac{p(p+\delta)}{\delta}}(\Omega)}\|u_{\kappa_0+\kappa}\|_{V^{(p+\delta)}}\,\|\varphi\|_{V^{p'}} \leq \\ & \leq \overline{c}c_8\,\|\kappa\|_{L^{\frac{p(p+\delta)}{\delta}}(\Omega)}\big(\|\phi\|_{W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma_{DN})} + \|f\|_{(V_1^{\overline{p}'})^*}\big)\,\|\varphi\|_{V^{p'}} \qquad \forall\, \varphi \in V_1^{p'}. \end{aligned}$$

Due to

$$(2.82) \qquad \frac{p(p+\delta)}{\delta} > \overline{s} := \frac{2p}{p-2}$$

and the boundedness of K_{ad} in $L^\infty(\Omega)$ (see (2.17), (2.18)) one gets

$$\begin{aligned} (2.83) \qquad \|\kappa\|_{L^{\frac{p(p+\delta)}{\delta}}(\Omega)} &= \Big(\int_\Omega |\kappa(x)|^{\frac{p(p+\delta)}{\delta}}\,\mathrm{d}x\Big)^{\frac{\delta}{p(p+\delta)}} = \\ &= \Big(\int_\Omega |\kappa(x)|^{\frac{p(p+\delta)}{\delta}-\overline{s}}\,\mathrm{d}x\Big)^{\frac{\delta}{p(p+\delta)}}\Big(\int_\Omega |\kappa(x)|^{\overline{s}}\,\mathrm{d}x\Big)^{\frac{\delta}{p(p+\delta)}} \leq k_2^{1-\frac{\delta\overline{s}}{p(p+\delta)}}\,\|\kappa\|_{L^{\overline{s}}(\Omega)}^{\frac{\overline{s}\delta}{p(p+\delta)}} \end{aligned}$$

with the constant k_2 (see (2.12), (2.17), (2.18)).

Hence, $F \in (V_1^{p'})^*$, and theorem 2.10 (i) is applicable to the auxiliary problem defined by (2.78). Moreover, the norm of F tends to zero in $(V_1^{p'})^*$, if the norm of κ tends to zero in $L^{\overline{s}}(\Omega)$. Therefore, we obtain the first assertion in (2.77), at first for $\overline{s} = 2p/p-2$, and, after that for all $s \in [\frac{2p}{p-2}, \infty]$. In an analogous manner one proves the second assertion in (2.77). $\square$

The proof of the second part of the last theorem yields a further Lipschitz continuity result, however with the price of a larger exponent. We need it in the sequel.

Corollary 2.13. (**Lipschitz continuity in the sense** $L^{\overline{s}} \to V^p$) For any $p \in [2, \overline{p}]$, $\overline{s} \in [\frac{p(p+\delta)}{\delta}, \infty]$ with $0 < \delta < \min\{\overline{p} - p, \frac{p^2-2p}{4-p}\}$ and for $\kappa_0 \in K_{ad}$, $\kappa \in K$ with $\kappa_0 + \kappa \in K_{ad}$ the solution operators are Lipschitz continuous in the following sense.

$$\begin{aligned} \|u_{\kappa_0+\kappa} - u_{\kappa_0}\|_{V^p} &= \|L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0)\|_{V^p} \leq \\ &\leq \overline{c}c_8\, \|\kappa\|_{L^{\overline{s}}(\Omega)} \big(\|\phi\|_{W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma_{DN})} + \|f\|_{(V_1^{\overline{p}'})^*}\big), \end{aligned} \tag{2.84}$$

$$\begin{aligned} \|v_{\kappa_0+\kappa} - v_{\kappa_0}\|_{V^p} &= \|L_{II}(f,\tau,\kappa_0+\kappa) - L_{II}(f,\tau,\kappa_0)\|_{V^p} \leq \\ &\leq \overline{c}c_9\, \|\kappa\|_{L^{\overline{s}}(\Omega)} \big(\|\tau\|_{(W^{1-\frac{1}{\overline{p}'},\overline{p}'}(\Gamma_{DN}))^*} + \|f\|_{(V_0^{\overline{p}'})^*}\big). \end{aligned} \tag{2.85}$$

The constant $\overline{c}$ does not depend of κ.

Remark 2.14. The case $s = \infty$ is included in theorem 2.12, whereat the assertions in part (i) directly follow from theorem 2.9.

2.6 Generalized co-normal derivatives of weak solutions

A general difficulty of the setting under consideration is the fact, that the weak formulations of problem I and II exhibit *different* test-function spaces. This drawback can be partially overcome with the use of a generalized co-normal derivative on Γ_{DN} (in the sense of distributions) of the solution of problem I, u_κ.

Theorem 2.15. (**Generalized co-normal derivative of** u_κ **on** Γ_{DN})

(i) Under the assumptions of theorem 2.6 for problem I, let f additionally fulfil (2.42), and, let $u_\kappa = L_I(f,\phi,\kappa) \in V_0$ be the weak solution of problem I. The relation

$$\begin{aligned} &\langle \partial_n(L_I(f,\phi,\kappa))\,, \varphi\rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*,(W^{\frac{1}{2},2}(\Gamma_{DN}))^m} := \\ &\qquad = a(\kappa, L_I(f,\phi,\kappa), \varphi) - \langle f\,,\, \varphi\rangle_{V_0^*,V_0} \qquad \forall\, \varphi \in V_0. \end{aligned} \tag{2.86}$$

defines an element $\partial_n(L_I(f,\phi,\kappa)$ $(= \partial_n(u_\kappa)$ in short) in $(W^{1/2,2}(\Gamma_{DN}))^*$.

(ii) Under the assumptions of theorem 2.10 for problem I the generalized co-normal derivative $\partial_n(L_I(f,\phi,\kappa)$ belongs to $(W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^*$ for all $p \in [2, \overline{p}]$ with $p' = p/_{p-1}$.

Proof. For fixed f, ϕ, κ the right-hand site of (2.86) is a linear continuous functional on the space of test functions used in (2.86). Due to (2.39), this functional only depends on the trace of φ on Γ_{DN}, and it becomes zero, if this trace vanishes. Therefore, $\partial_n(f,\phi,\kappa)$ defines indeed a distribution in $W^{1/2,2}(\Gamma_{DN})^*$. Finally, under the assumptions of theorem 2.10, it belongs to $(W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^*$. □

Remarks 2.16. **(i)** Due to (2.9), the outer unit normal vector n on Γ_{DN} exists almost everywhere. Therefore, we use the index n within the notation of $\partial_n(u_\kappa)$.

(ii) For the form a given in (2.66), and for $u \in W^{2,2}(\Omega)$, $\kappa \in C^{0,1}(\overline{\Omega})$ and b Lipschitz continuous on $\overline{\Omega} \times J$, the co-normal derivative is a function and given by $\partial_n(u_\kappa) = b(\cdot,\kappa)(\nabla u)_{|\Gamma_{DN}} \cdot \boldsymbol{n}$.

(iii) If the solutions of problem I and II coincide, one has $\partial_n(u_\kappa) = \tau$.

To overcame the above mentioned drawback of different test-function spaces $V_1 \subsetneq V_0$, we reformulate problem I. Under the assumptions of theorem 2.6 for problem I, and (2.42), a relation with test functions from V_0 follows:

$$a(\kappa, u_\kappa, \varphi) = \langle f\,,\, \varphi\rangle_{V_0^*,V_0} + \langle \partial_n(u_\kappa)\,,\, \varphi\rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*,W^{\frac{1}{2},2}(\Gamma_{DN})} \qquad \forall\, \varphi \in V_0. \tag{2.87}$$

Clearly, in the case of higher integrability of the gradient, this relation remains valid for test-function from $V_0^{p'}$.

Note that $u_\kappa - v_\kappa \in V_0$ for the weak solutions u_κ and v_κ of problem I and II, respectively. Thus one gets easily the following assertions.

Lemma 2.17. **(i)** Under the assumptions of theorem 2.6, problems I and II, there hold

$$a(\kappa, u_\kappa - v_\kappa, \varphi) = \langle \partial_n(u_\kappa) - \tau \,,\, \varphi \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} \qquad \forall \varphi \in V_0, \tag{2.88}$$

$$\|u_\kappa - v_\kappa\|_V \le \frac{c_1}{\alpha_0} \, \|\partial_n(u_\kappa) - \tau\|_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*}. \tag{2.89}$$

(ii) Under the assumptions of theorem 2.10 for problems I and II, there holds for all $p \in [2, \overline{p}]$ with $p' = {}^{p}/_{p-1}$

$$\|u_\kappa - v_\kappa\|_{V^p} \le c_9 \, \|\partial_n(u_\kappa) - \tau\|_{(W^{1-\frac{1}{p'},p'}(\Gamma_{DN}))^*}. \tag{2.90}$$

Using the co-normal derivative, one can prove some generalized reciprocity relations. For simplicity we assume $f \equiv 0$. Otherwise, assuming $f \in V_0^*$, one obtains more complex expressions. For convenience, we formulate the following assertions only with respect to spaces based on $V = W^{1,2}(\Omega)$. Clearly, an extension to the case of higher integrability is straightforward.

Theorem 2.18. Let the assumptions of theorem 2.6 for problems I and II and $f \equiv 0$ be given. Then the weak solutions $u_\kappa = L_I(0, \phi, \kappa) \in V_0$ and $v_{\overline{\kappa}} = L_{II}(0, \tau, \overline{\kappa}) \in V_0$ fulfill:
(i) (Reciprocity relation)

$$\begin{aligned} a(\kappa, u_\kappa, v_{\overline{\kappa}}) - a(\overline{\kappa}, u_\kappa, v_{\overline{\kappa}}) = {} & \langle \partial_n(u_\kappa) \,,\, v_{\overline{\kappa}} \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} + \\ & - \langle \tau \,,\, \phi \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})}. \end{aligned} \tag{2.91}$$

(ii) (Generalized Betti theorem)

$$\langle \partial_n(u_\kappa) \,,\, v_\kappa \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} = \langle \tau \,,\, \phi \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})}. \tag{2.92}$$

(iii) If the b depends linearly on κ, then (2.91) simplifies to

$$\begin{aligned} a(\kappa - \overline{\kappa}, u_\kappa, v_{\overline{\kappa}}) = {} & \langle \partial_n(u_\kappa) \,,\, v_{\overline{\kappa}} \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} + \\ & - \langle \tau \,,\, \phi \rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})}. \end{aligned} \tag{2.93}$$

Note that for general bilinear form a one needs the symmetry relation (2.32).

Based on an reciprocity relation in the case of $\partial\Omega = \Gamma_{DN}$ (full-displacement and full-traction setting), an approach for parameter identification ("reciprocity-gap method") in the case of linear elasticity can be developed. We refer to Bonnet and Constantinescu (2005) for details and further references. The formula (2.92) is a generalization of the classical Betti theorem (see e.g. Salençon (2001)) for mixed boundary conditions.

2.7 Fréchet differentiability of the solution operators

Now we establish the continuous Fréchet differentiability of the solution operators L_I and L_{II} under additional assumptions on the coefficient function b. In the sequel we need only the first and second derivatives. However, in the case of linear parameter dependence (or for infinitely differentiable b), the infinite differentiability can be proved in the same manner. We refer to Gockenbach and Khan (2007) for a slightly different setting.

2.7.1 Preparations

In difference to Gockenbach and Khan (2007), we also consider the solution operators L_I and L_{II} as mappings defined on K_{ad} as a subset of $L^s(\Omega)$ for a finite s under assumptions which ensure higher integrability of the gradient of the weak solutions. For this reason, the concept of Fréchet differentiability will be extended to points $\kappa_0 \in K_{ad}$ being limit points of K_{ad} with respect to the L^s norm as well as to increments of the operator for arguments belonging to K_{ad}. In doing so, we introduce a further subset (besides K_{ad}) of $K = L^\infty(\Omega)$, see (2.17), (2.18). For all $\kappa_0 \in K_{ad}$ we define:

$$L_{\kappa_0} := \operatorname{span}\{\kappa \in K \mid \kappa_0 + \kappa \in K_{ad}\}, \tag{2.94}$$

whereby span$\{X\}$ denotes the linear hull of a subset X of a given linear space. As a result, the Fréchet derivative in a limit point $\kappa_0 \in K_{ad}$ is a linear operator defined on L_{κ_0}. We note some assertions concerning the properties of K_{ad} as a subset of $L^s(\Omega)$ with $1 \le s < \infty$.

Remarks 2.19. Let be $1 \le s < \infty$.

(i) Due to (2.9), there holds $K_{ad} \subset L^s(\Omega)$.

(ii) Due to the continuous embedding $L^\infty(\Omega) \subset L^s(\Omega)$, each limit point with respect to the L^∞ norm is also a limit point with respect to the L^s norm.

(iii) The set $K_{ad} \subset L^\infty(\Omega)$ does *not* contain inner points (and hence open balls) with respect to the L^s-topology.

(iv) In *general* situations, K_{ad} is not closed in the L^s-topology, see the example in remark 2.1 (iii) with $K := K_{ad} := L^\infty(\Omega)$. Thus, in this general case, the admissible set K_{ad} is often restricted, assuming

$$(2.95) \qquad \kappa \in K_{ad} \quad \Rightarrow \quad \kappa \quad \text{limit point of } K_{ad} \quad \text{w.r.t. the } L^\infty \text{ norm.}$$

(v) Under (2.12), (2.17) and (2.18) there hold additionally

(a) The set K_{ad} is closed, bounded and convex both in L^∞- and L^s-topology.

(b) Each element of K_{ad} is a limit point both in L^∞- and L^s-topology.

In the case considered here, the Fréchet derivative can be easily extended to an operator defined on K, using the special structure. Concerning general differential calculus in Banach spaces, we refer to Showalter (1997) and Amann and Escher (2006), e.g.

2.7.2 Fréchet differentiability in the general case

It is natural, that the Fréchet differentiability of the solution operators requires differentiability of the coefficient function b. Thus, besides (2.12) - (2.16) we assume

$$(2.96) \qquad \text{F.a.a. } x \in \Omega \quad : \quad b(x,\cdot) \, : \, J \to \mathbb{R} \quad \text{differentiable,}$$

and

$$(2.97) \qquad \frac{\partial b}{\partial s}(\cdot,\cdot) \, : \, J \times \Omega \to \mathbb{R} \quad \text{Carathéodory function,}$$

$$(2.98) \qquad \exists\, m_1 > 0 \quad \text{f.a.a. } x \in \Omega \quad \forall\, s \in J \quad : \quad |\frac{\partial b}{\partial s}(x,s)| \le m_1,$$

$$(2.99) \qquad \exists\, m_2 > 0 \quad \text{f.a.a. } x \in \Omega \quad \forall\, s_1, s_2 \in J \quad : \quad |\frac{\partial b}{\partial s}(x,s_1) - \frac{\partial b}{\partial s}(x,s_2)| \le m_2|s_1 - s_2|,$$

as well as

$$(2.100) \qquad \text{f.a.a. } x \in \Omega \quad \forall\, s_0 \in J \quad \forall\, s \in \mathbb{R} \quad \text{with } s_0 + s \in J \; :$$
$$: \quad b(x, s_0 + s) - b(x, s_0) = \frac{\partial b}{\partial s}(x, s_0)s + O(x, s_0, s)s,$$
$$\text{f.a.a. } x \in \Omega \quad \forall\, s_0 \in J \quad : \quad \lim_{s \to s_0} |O(x, s_0, s)| = 0,$$
$$\exists\, m_3 > 0 \quad \text{f.a.a. } x \in \Omega \quad \forall\, s_0 \in J \quad \forall\, s \in \mathbb{R} \quad \text{with } s_0 + s \in J \; : \; |O(x, s_0, s)| \le m_3.$$

In other words, the function b must be partially differentiable with respect to the parameter, and the partial derivative must be Lipschitz continuous and some boundedness and uniformity conditions are required. Again, we modify arguments used in Jin and Maaß (2012a) and Jin and Maass (2012b) for our setting.

Theorem 2.20. Let the assumptions of theorem 2.10 and (2.96) - (2.100) be given. Moreover, let be $2 < p < \min\{\overline{p}, 4\}$ and $s \in [2p/(p-2), \infty]$.

(i) Let $f \in (V_1^{\overline{p}'})^*$, $\phi \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma_{DN})$ with an analogous to (2.36) continuation property in $V^{\overline{p}}$ be given and let L_I be the solution operator introduced in (2.56), (2.57).

(a) The mapping $L_I(f, \phi, \cdot) \, : \, K_{ad} \subset L^s(\Omega) \to V_0$ is Fréchet differentiable on K_{ad}. The Fréchet differential $\partial L_I/\partial\kappa(\kappa_0)\kappa$ belongs to $V_1^{\overline{p}}$ and is given for all $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$ by

$$(2.101) \qquad a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, \varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x, \kappa_0(x))\kappa(x)\nabla L_I(f, \phi, \kappa_0) \cdot \nabla\varphi \, \mathrm{d}x \quad \forall\, \varphi \in V_1,$$

or, equivalently by

$$(2.102) \qquad \frac{\partial L_I}{\partial \kappa}(f, \phi, \kappa_0)\kappa = L_I(F^I_{\kappa_0}(\kappa), 0, \kappa_0) \qquad \text{for all } \kappa_0 \in K_{ad}, \kappa \in K.$$

The quantity $F^I_{\kappa_0}(\kappa) := F^I_{f,\phi,\kappa_0}(\kappa) \in (V_1^{p'})^*$ is defined by

$$\langle F^I_{\kappa_0}(\kappa)\,,\,\varphi\rangle_{(V_1^{p'})^*,V_1^{p'}} := -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla L_I(f,\phi,\kappa_0)\cdot\nabla\varphi\,\mathrm{d}x \quad \forall \boldsymbol{\varphi} \in V_1^{p'}. \tag{2.103}$$

(b) The Fréchet differential ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)\kappa$ is uniformly continuous with respect to κ_0 as a mapping from $L^s(\Omega)$ into V_1.
(c) The Fréchet derivative ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)$ is uniformly continuous with respect to κ_0 as a mapping from $L^s(\Omega)$ into $L(L^\infty(\Omega),V_1)$, i.e. ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)$ is uniformly continuous with respect the operator norm.
(d) For all $\overline{s} \geq \max\{s, 2r, \overline{r}\}$ with $r := {}^{2\overline{p}}/_{\overline{p}-2}$ and $\overline{r} := {}^{\overline{p}p}/_{\overline{p}-p}$ the Fréchet differential ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)\kappa$ is uniformly continuous with respect to κ_0 as a mapping from $L^{\overline{s}}(\Omega)$ into V_1 and the Fréchet derivative ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)$ is uniformly continuous with respect to κ_0 as a mapping from $L^{\overline{s}}(\Omega)$ into $L(L^{\overline{s}}(\Omega),V_1)$, i.e. ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)$ is uniformly continuous with respect to the operator norm.
(e) For all $\overline{s} \geq \max\{s, 2r, \overline{r}, {}^{p(p+\delta)}/_{\delta}\}$ with $r := {}^{2\overline{p}}/_{\overline{p}-2}$, $\overline{r} := {}^{\overline{p}p}/_{\overline{p}-p}$ and $0 < \delta < \min\{\overline{p}-p, {}^{p^2-2p}/_{4-p}\}$ the Fréchet derivative ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)$ is Lipschitz continuous with respect to κ_0 as a mapping from $L^{\overline{s}}(\Omega)$ into $L(L^{\overline{s}}(\Omega),V_1)$, i.e. ${}^{\partial L_I}/_{\partial\kappa}(f,\phi,\kappa_0)$ is Lipschitz continuous with respect to the operator norm.

(ii) Let $\boldsymbol{f} \in (V_0^{\overline{p}'})^*$, $\tau \in (W^{1-\frac{1}{\overline{p}'},\overline{p}'}(\Gamma_{DN}))^*$ be given and let L_{II} be the solution operator introduced in (2.58), (2.59).
(a) The mapping $L_{II}(f,\tau,\cdot) : K_{ad} \subset L^s(\Omega) \to V_0$ is Fréchet differentiable on K_{ad}. The Fréchet differential ${}^{\partial L_{II}}/_{\partial\kappa}(\kappa_0)\kappa$ belongs to $V_1^{\overline{p}}$ and is given for all $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$ by

$$a(\kappa_0, \frac{\partial L_{II}}{\partial\kappa}(\kappa_0)\kappa, \varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla L_{II}(f,\tau,\kappa_0)\cdot\nabla\varphi\,\mathrm{d}x \quad \forall\,\varphi \in V_0. \tag{2.104}$$

or, equivalently by

$$\frac{\partial L_{II}}{\partial\kappa}(f,\tau,\kappa_0)\kappa = L_{II}(F^{II}_{\kappa_0}(\kappa),0,\kappa_0) \qquad \text{for all } \kappa_0 \in K_{ad}, \kappa \in K. \tag{2.105}$$

The quantity $F^{II}_{\kappa_0}(\kappa) = F^{II}_{f,\tau,\kappa_0}(\kappa) \in (V_0^{p'})^*$ is defined by

$$\langle F^{II}_{\kappa_0}(\kappa)\,,\,\varphi\rangle_{(V_0^{p'})^*,V_0^{p'}} := -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla L_{II}(f,\tau,\kappa_0)\cdot\nabla\varphi\,\mathrm{d}x \quad \forall\,\varphi \in V_0^{p'}. \tag{2.106}$$

(b) The Fréchet differential ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)\kappa$ is uniformly continuous with respect to κ_0 as a mapping from $L^s(\Omega)$ into V_0.
(c) The Fréchet derivative ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)$ is uniformly continuous with respect to κ_0 as a mapping from $L^s(\Omega)$ into $L(L^\infty(\Omega),V_0)$, i.e. uniform continuity in the operator norm.
(d) For all $\overline{s} \geq \max\{s, 2r, \overline{r}\}$ with $r := {}^{2\overline{p}}/_{\overline{p}-2}$ and $\overline{r} := {}^{\overline{p}p}/_{\overline{p}-p}$ the Fréchet differential ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)\kappa$ is uniformly continuous with respect to κ_0 as a mapping from $L^{\overline{s}}(\Omega)$ into V_0 and the Fréchet derivative ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)$ is uniformly continuous with respect to κ_0 as a mapping from $L^{\overline{s}}(\Omega)$ into $L(L^{\overline{s}}(\Omega),V_0)$, i.e. ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)$ is uniformly continuous with respect to the operator norm.
(e) For all $\overline{s} \geq \max\{s, 2r, \overline{r}, {}^{p(p+\delta)}/_{\delta}\}$ with $r := {}^{2\overline{p}}/_{\overline{p}-2}$, $\overline{r} := {}^{\overline{p}p}/_{\overline{p}-p}$ and $0 < \delta < \min\{\overline{p}-p, {}^{p^2-2p}/_{4-p}\}$ the Fréchet derivative ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)$ is Lipschitz continuous with respect to κ_0 as a mapping from $L^{\overline{s}}(\Omega)$ into $L(L^{\overline{s}}(\Omega),V_0)$, i.e. ${}^{\partial L_{II}}/_{\partial\kappa}(f,\tau,\kappa_0)$ is Lipschitz continuous with respect to the operator norm.

Proof. We only deal with the solution operator L_I. The proof for L_{II} only differs slightly. Moreover, for convenience, we often write u_κ instead of $L_I(f,\phi,\kappa)$.
(a) For $\kappa_0 \in K_{ad}$ and $\kappa_0+\kappa \in K_{ad}$ t he relations (2.38), (2.39) yield

$$a(\kappa_0+\kappa, L_I(f,\phi,\kappa_0+\kappa),\varphi) = \langle f\,,\,\varphi\rangle_{V_1^*,V_1} \qquad \forall\,\varphi \in V_1, \tag{2.107}$$

$$a(\kappa_0, L_I(f,\phi,\kappa_0),\varphi) = \langle f\,,\,\varphi\rangle_{V_1^*,V_1} \qquad \forall\,\varphi \in V_1, \tag{2.108}$$

as well as

$$L_I(f,\phi,\kappa_0+\kappa) = L_I(,\phi,\kappa_0) = \phi \qquad \text{on } \Gamma_{DN}. \tag{2.109}$$

Using (2.107) and (2.108), one obtains without difficulties for all $\varphi \in V_1$

$$
\begin{aligned}
(2.110)\quad a(\kappa_0, L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0),\varphi) &= -\big(a(\kappa_0+\kappa, L_I(f,\phi,\kappa_0),\varphi) - a(\kappa_0, L_I(f,\phi,\kappa_0),\varphi)\big)+ \\
&\quad - a(\kappa_0+\kappa, L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0),\varphi)+ \\
&\quad + a(\kappa_0, L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0),\varphi) =: A_1 + A_2 + A_3 \qquad \forall\,\varphi \in V_1.
\end{aligned}
$$

At first we consider A_1. Due to the differentiability assumptions on b we get for all $\varphi \in V_1$ (We also write u_κ instead of $L_I(f,\phi,\kappa)$, sometimes the dependence on x is dropped.)

$$
\begin{aligned}
(2.111)\qquad A_1 &= -\int_\Omega (b(x,\kappa_0+\kappa) - b(x,\kappa_0))\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x = \\
&= -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega O(x,\kappa_0(x),\kappa(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x.
\end{aligned}
$$

The first integral on the right-hand side is a linear and continuous functional with respect to $\varphi \in V_1$ for fixed $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$. Thus, applying the Lax-Milgram theorem, there exists a $\psi \in V_1$ with

$$
(2.112)\qquad a(\kappa_0,\psi,\varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x \qquad \forall\,\varphi\in V_1.
$$

Moreover, ψ depends linearly and continuously on $\kappa \in L_{\kappa_0}$. Thus, there exists a linear and continuous operator from L_{κ_0} into V_1 yielding ψ. We denote this operator by $\frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)$ and obtain

$$
(2.113)\qquad a(\kappa_0, \frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)\kappa,\varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x \qquad \forall\,\varphi\in V_1.
$$

Therefore the element $\frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)\kappa \in V_1$ is a weak solution to problem I with vanishing Dirichlet data on Γ_{DN} and with a right-hand side given in (2.113). Due to the assumptions ensuring higher integrability of the gradient, $\frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)\kappa$ belongs to V_1^p and there holds

$$
(2.114)\quad \|\frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)\kappa\|_{V^p} \le c\|\kappa\|_{L^\infty(\Omega)}\|L_I(f,\phi,\kappa_0)\|_{V^p} \le c\|\kappa\|_{L^\infty(\Omega)}\big(\|\phi\|_{W^{1-\frac{1}{p},p}(\Gamma_{DN})} + \|f\|_{(V_1^{p'})^*}\big).
$$

Moreover, we have a suggestion for the Fréchet derivative. From (2.110), (2.111) and (2.113) one deduces

$$
\begin{aligned}
(2.115)\qquad & a(\kappa_0, L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0) - \frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)\kappa,\varphi) = \\
&\qquad = -\int_\Omega O(x,\kappa_0(x),\kappa(x))|\kappa(x)|\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x + A_2 + A_3 \qquad \forall\,\varphi\in V_1.
\end{aligned}
$$

The second argument of the form a on the left-hand side is an admissible test-function in V_1. For convenience, we denote it by $\hat\varphi$. After inserting it into (2.115), on the left-hand side the quare of the V-norm of $\hat\varphi$ arises. The first term on the right-hand side can be estimated in following way, using Hölder's inequality.

$$
\begin{aligned}
(2.116)\qquad & \Big|\int_\Omega O(x,\kappa_0(x),\kappa(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\hat\varphi\,\mathrm{d}x\Big| \le \\
&\qquad \le c\|\kappa\|_{L^s(\Omega)}\,\|\hat\varphi\|_V\Big(\int_\Omega |O(x,\kappa_0(x),\kappa(x))|^p\,\|\nabla u_{\kappa_0}\|_{\mathbb{R}^d}^p\,\mathrm{d}x\Big)^{\frac{1}{p}}.
\end{aligned}
$$

The sum $A_2 + A_3$ will be estimated together, using Lipschitz continuity of b and Hölder's inequality.

$$
\begin{aligned}
(2.117)\qquad |A_2 + A_3| &= \Big|\int_\Omega \big(b(x,\kappa_0(x)+\kappa(x)) - b(x,\kappa_0(x))\big)\nabla(u_{\kappa_0+\kappa} - u_{\kappa_0})\cdot\nabla\hat\varphi\,\mathrm{d}x\Big| \le \\
&\le c\|\kappa\|_{L^s(\Omega)}\,\|\hat\varphi\|_V\|L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0)\|_{V^p}.
\end{aligned}
$$

Thus, from (2.115) we obtain the following estimate.

$$
\begin{aligned}
(2.118)\qquad & \|L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0) - \frac{\partial L_I}{\partial\kappa}(f,\phi,\kappa_0)\kappa\|_V \le \\
&\qquad \le c\,\|\kappa\|_{L^s(\Omega)}\|L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0)\|_{V^p} + \\
&\qquad + c\,\|\kappa\|_{L^s(\Omega)}\Big(\int_\Omega |O(x,\kappa_0(x),\kappa(x))|^p\,\|\nabla u_{\kappa_0}\|_{\mathbb{R}^d}^p\,\mathrm{d}x\Big)^{\frac{1}{p}}.
\end{aligned}
$$

After dividing this inequality by $\|\kappa\|_{L^s(\Omega)}$, the right-hand side tends to zero when κ does so (with respect to the L^s norm). This is based on the uniform continuity of the solution operator (see theorem 2.12 (ii)) for the first summand. For the second summand the reasoning uses Lebesgue's convergence theorem, the convergence principle ("arbitrary subsequence of an arbitrary subsequence") and the result that an L^s-convergent sequence consists a an almost everywhere convergent subsequence which has an L^s-majorant (see e.g., Zeidler (1990b) [Appendix, (36)]).

Thus, the Fréchet differential of L_I exists and one has for all $\varphi \in V_1$

$$a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa, \varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x =: \langle F^I_{\kappa_0}(\kappa)\,,\,\varphi\rangle_{V_1^*,V_1}. \tag{2.119}$$

Or, in other words, the Fréchet differential $\partial L_I/\partial\kappa(f,\phi,\kappa_0)\kappa \in V_1^{\overline{p}}$ is a weak solution of problem I for the data $F^I_{\kappa_0}(\kappa) \in (V_1^{p'})^*$, $0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma_{DN})$ and $\kappa_0 \in K_{ad}$, $\kappa \in L_{\kappa_0}$. Thus, (2.101) and (2.102) follow.

(b,c) The (uniform) continuity of the Fréchet differential $\partial L_I/\partial\kappa(f,\phi,\kappa_0)\kappa$ with respect to κ_0 (as a mapping from $L^s(\Omega)$ into V_1) can be established in the following way. Using the relation (2.119) for $\kappa_0 \in K_{ad}$ and $\overline{\kappa}_0 \in K_{ad}$, one gets after some re-arrangements for all $\varphi \in V_1$:

$$\begin{aligned} a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa - \frac{\partial L_I}{\partial \kappa}(f,\phi,\overline{\kappa}_0)\kappa, \varphi) &= -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla(u_{\kappa_0}-u_{\overline{\kappa}_0})\cdot\nabla\varphi\,\mathrm{d}x + \\ &-\int_\Omega \Big(\frac{\partial b}{\partial s}(x,\kappa_0(x)) - \frac{\partial b}{\partial s}(x,\overline{\kappa}_0(x))\Big)\kappa(x)\nabla u_{\overline{\kappa}_0}\cdot\nabla\varphi\,\mathrm{d}x + \\ &- a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(f,\phi,\overline{\kappa}_0)\kappa,\varphi) + a(\overline{\kappa}_0, \frac{\partial L_I}{\partial \kappa}(f,\phi,\overline{\kappa}_0)\kappa,\varphi) =: B_1 + B_2 + B_3. \end{aligned} \tag{2.120}$$

Using the assumptions concerning b (2.16), (2.97) - (2.99), theorem 2.10 (i) and (2.114), one gets the following estimates.

$$|B_1| \le c\,\|\kappa\|_{L^s(\Omega)}\,\|u_{\kappa_0}-u_{\overline{\kappa}_0}\|_{V^p}\|\varphi\|_V, \tag{2.121}$$

$$|B_2| \le c\,\|\kappa\|_{L^\infty(\Omega)}\,\|\kappa_0-\overline{\kappa}_0\|_{L^s(\Omega)}\,\|u_{\overline{\kappa}_0}\|_{V^p}\,\|\varphi\|_V, \tag{2.122}$$

$$|B_3| \le c\,\|\kappa\|_{L^\infty(\Omega)}\,\|\kappa_0-\overline{\kappa}_0\|_{L^s(\Omega)}\,\|u_{\overline{\kappa}_0}\|_{V^p}\,\|\varphi\|_V. \tag{2.123}$$

After inserting the admissible test-function $\varphi := \partial L_I/\partial\kappa(f,\phi,\kappa_0)\kappa - \partial L_I/\partial\kappa(f,\phi,\overline{\kappa}_0)\kappa$ into (2.120), there holds

$$\begin{aligned} \|\frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa - \frac{\partial L_I}{\partial \kappa}(f,\phi,\overline{\kappa}_0)\kappa\|_V &\le \\ &\le c\Big(\|\kappa\|_{L^s(\Omega)}\,\|u_{\kappa_0}-u_{\overline{\kappa}_0}\|_{V^p} + \|\kappa\|_{L^\infty(\Omega)}\,\|\kappa_0-\overline{\kappa}_0\|_{L^s(\Omega)}\,\|u_{\overline{\kappa}_0}\|_{V^p}\Big). \end{aligned} \tag{2.124}$$

Thus, based on theorem 2.12 (ii), this last estimate yields the uniform continuity of the Fréchet differential $\partial L_I/\partial\kappa(f,\phi,\kappa_0)\kappa$ with respect to κ_0. In other words, there holds:

$$\|\frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa - \frac{\partial L_I}{\partial \kappa}(f,\phi,\overline{\kappa}_0)\kappa\|_V \to 0, \qquad \text{if} \qquad \|\kappa_0-\overline{\kappa}_0\|_{L^s(\Omega)} \to 0. \tag{2.125}$$

(d) Due to $2 \le p < \overline{p}$ and theorem 2.10 (i), one also obtains from (2.113) (besides (2.114))

$$\|\frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa\|_{V^p} \le c\|\kappa\|_{L^{\overline{r}}(\Omega)}\|L_I(f,\phi,\kappa_0)\|_{V^{\overline{p}}} \le c\|\kappa\|_{L^{\overline{r}}(\Omega)}\big(\|\phi\|_{W^{1-\frac{1}{\overline{p}},\overline{p}}(\Gamma_{DN})} + \|f\|_{(V_1^{(\overline{p})'})^*}\big) \tag{2.126}$$

with $\overline{r} := \overline{p}p/\overline{p}-p$ and $(\overline{p})'$ being the dual exponent to $\overline{p}$. Using the fact, that $u_{\kappa_0} = L_I(f,\phi,\kappa_0)$ also belongs to $V^{\overline{p}}$, instead of (2.122) and (2.123) one obtains the alternative estimates

$$|B_2| \le c\,\|\kappa\|_{L^{2r}(\Omega)}\,\|\kappa_0-\overline{\kappa}_0\|_{L^{2r}(\Omega)}\,\|u_{\overline{\kappa}_0}\|_{V^{\overline{p}}}\,\|\varphi\|_V, \tag{2.127}$$

$$|B_3| \le c\,\|\kappa\|_{L^{\overline{r}}(\Omega)}\,\|\kappa_0-\overline{\kappa}_0\|_{L^s(\Omega)}\,\|u_{\overline{\kappa}_0}\|_{V^{\overline{p}}}\,\|\varphi\|_V \tag{2.128}$$

with $r := 2\overline{p}/\overline{p}-2$. Choosing $\overline{s} \ge \max\{s, 2r, \overline{r}\}$ and using (2.120), (2.121), (2.127) and (2.128), one gets the asserted continuity results.

(e) Again, we only deal with the solution operator L_I. From the relation (2.101) we obtain for all $\kappa_0 \in K_{ad}$, $\kappa \in L_{\kappa_0}$, $\xi \in L^{\overline{s}}(\Omega)$ as well as for all $\varphi \in V_1$

$$a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\xi, \varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\xi(x)\nabla L_I(f,\phi,\kappa_0)\cdot\nabla\varphi\,\mathrm{d}x, \tag{2.129}$$

$$a(\kappa_0+\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa)\xi, \varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x)+\kappa(x))\xi(x)\nabla L_I(f,\phi,\kappa_0+\kappa)\cdot\nabla\varphi\,\mathrm{d}x. \tag{2.130}$$

Using the definition of the form a in (2.27) and subtracting (2.129) from (2.130) one gets

$$(2.131)\quad a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa)\xi - \frac{\partial L_I}{\partial \kappa}(\kappa_0)\xi, \varphi) = -\int_\Omega \Big(b(\kappa_0+\kappa) - b(\kappa_0)\Big)\nabla\big(\frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa)\xi\big)\cdot\nabla\varphi\,\mathrm{d}x +$$
$$-\int_\Omega \Big(\frac{\partial b}{\partial s}(x,\kappa_0(x)+\kappa(x)) - \frac{\partial b}{\partial s}(x,\kappa_0(x))\Big)\xi(x)\nabla L_I(f,\phi,\kappa_0+\kappa)\cdot\nabla\varphi\,\mathrm{d}x, +$$
$$-\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\xi(x)\Big(\nabla L_I(f,\phi,\kappa_0+\kappa) - \nabla L_I(f,\phi,\kappa_0)\Big)\cdot\nabla\varphi\,\mathrm{d}x =: J_1(\varphi)+J_2(\varphi)+J_3(\varphi).$$

Due to part (a) of the proof, there holds

$$(2.132)\qquad w := \frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa)\xi - \frac{\partial L_I}{\partial \kappa}(\kappa_0)\xi \in V_1,$$

thus, w is an admissible test function in (2.131). Inserting it yields

$$(2.133)\qquad \|\frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa)\xi - \frac{\partial L_I}{\partial \kappa}(\kappa_0)\xi\|_V^2 \le c\Big(|J_1(w)|+|J_2(w)|+|J_3(w)|\Big).$$

Based on the previous results, the terms $|J_i(w)|$ $(i=1,2,3)$ can be estimated in the following way.

Using the Lipschitz continuity of b and the uniform continuity of the Fréchet derivative of the solution operator from part (d), there holds the estimate

$$(2.134)\qquad |J_1(w)| \le \int_\Omega \big|b(\kappa_0+\kappa)-b(\kappa_0)\big|\,\Big|\nabla\big(\frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa)\xi\big)\cdot\nabla\varphi\Big|\,\mathrm{d}x \le c\|\kappa\|_{L^{\bar{s}}}\|\xi\|_{L^{\bar{s}}}\|w\|_V.$$

Essentially from the Lipschitz continuity of the derivative of b (see (2.99)) the subsequent estimate follows

$$(2.135)\qquad \begin{aligned}|J_2(w)| &\le \int_\Omega \Big|\frac{\partial b}{\partial s}(x,\kappa_0(x)+\kappa(x)) - \frac{\partial b}{\partial s}(x,\kappa_0(x))\Big||\xi(x)|\Big|\nabla L_I(f,\phi,\kappa_0+\kappa)\cdot\nabla\varphi\Big|\,\mathrm{d}x\\ &\le c\|\kappa\|_{L^{\bar{s}}}\|\xi\|_{L^{\bar{s}}}\|w\|_V.\end{aligned}$$

To prove the estimate for $J_3(w)$, one needs the special Lipschitz continuity of the solution operator (see corollary 2.13):

$$(2.136)\qquad \begin{aligned}|J_3(w)| &\le \int_\Omega \Big|\frac{\partial b}{\partial s}(x,\kappa_0(x))\Big||\xi(x)|\Big|\Big(\nabla L_I(f,\phi,\kappa_0+\kappa) - \nabla L_I(f,\phi,\kappa_0)\Big)\cdot\nabla\varphi\Big|\,\mathrm{d}x\\ &\le c\|\kappa\|_{L^{\bar{s}}}\|\xi\|_{L^{\bar{s}}}\|w\|_V.\end{aligned}$$

From the last three estimates we finally get the asserted Lipschitz continuity of the Fréchet derivative of the solution operator:

$$(2.137)\qquad \Big\|\frac{\partial L_I}{\partial \kappa}(\kappa_0+\kappa) - \frac{\partial L_I}{\partial \kappa}(\kappa_0)\Big\|_V \le L\|\kappa\|_{L^{\bar{s}}},$$

with a Lipschitz constant L not depending on κ_0. □

The Lipschitz continuity of the Fréchet derivatives of the solution operators is *essential* for further investigation concerning convergence rates of regularization methods, e.g. This is not the topic in this study, we refer to Jin and Maaß (2012a), Hofmann et al. (2007) and for the references cited therein.

Based on part (e) of the previous theorem, there holds following estimate.

Corollary 2.21. Under the assumptions of theorem 2.20, part (e), there hold for all $\kappa_0 \in K_{ad}$, $\kappa \in L^\infty(\Omega)$ with $\kappa_0+\kappa \in K_{ad}$

$$(2.138)\qquad \|L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0) - \frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa\|_V \le \frac{L}{2}\|\kappa\|^2_{L^{\bar{s}}_I(\Omega)},$$

$$(2.139)\qquad \|L_{II}(f,\tau,\kappa_0+\kappa) - L_{II}(f,\tau,\kappa_0) - \frac{\partial L_{II}}{\partial \kappa}(f,\tau,\kappa_0)\kappa\|_V \le \frac{L}{2}\|\kappa\|^2_{L^{\bar{s}}_{II}(\Omega)},$$

where L is a common Lipschitz constant of the Fréchet derivatives.

Proof. Starting with the obvious relation

$$L_I(f,\phi,\kappa_0+\kappa) - L_I(f,\phi,\kappa_0) - \frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa = \\ = \int_0^1 \Big(\frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0+t\kappa)\kappa - \frac{\partial L_I}{\partial \kappa}(f,\phi,\kappa_0)\kappa\Big)\,\mathrm{d}t, \tag{2.140}$$

and using the Lipschitz continuity of the Fréchet derivative, one gets (2.138), and analogously (2.139). □

Remarks 2.22. (i) The extended version of (Fréchet) differentiability also includes differentiability in boundary points.

(ii) The Fréchet differentials (respectively the Fréchet derivatives) of the solution operators can be indicated as "sensitivities" like it takes place sometimes for the derivative of the displacement field respective to the parameter. In particular, this concerns numerical investigations (cf. Tortorelli and Michaleris (1994). Widany and Mahnken (2012), e.g.).

Based on arguments from the proofs of theorems 2.12 and 2.20 as well as on the representations of the Fréchet differentials by (2.101) and (2.104), respectively, some further assertions can be derived. We collect them into the following corollary. As a rule, the given data f, ϕ and τ are regarded as fixed, and, therefore will be omitted in some formulas.

Corollary 2.23. Let the assumptions of theorem 2.20 be given.

(i) For all $2 \le p < \min\{\overline{p}, 4\}$ and $s \in [2p/(p-2), \infty]$ the Fréchet derivatives $\partial L_I/\partial\kappa(\kappa_0) : L_{\kappa_0} \subset L^s(\Omega) \to V_1$ and $\partial L_{II}/\partial\kappa(\kappa_0) : L_{\kappa_0} \subset L^s(\Omega) \to V_0$ can be extended to linear bounded operators on $L^s(\Omega)$. And, there hold the estimates and relations

$$\|\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa\|_V \le c\,\|\kappa\|_{L^s(\Omega)}\,\|u_{\kappa_0}\|_{V^p}, \qquad \|\frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa\|_V \le c\,\|\kappa\|_{L^s(\Omega)}\,\|v_{\kappa_0}\|_{V^p}, \tag{2.141}$$

$$\lim_{\|\kappa\|_{L^s(\Omega)}\to 0} \|\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa\|_{V^p} = 0, \qquad \lim_{\|\kappa\|_{L^s(\Omega)}\to 0} \|\frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa\|_{V^p} = 0. \tag{2.142}$$

(ii) The Fréchet differential $\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa$ fulfills for all $\varphi \in V_0$

$$a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, \varphi) = \\ = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla u_{\kappa_0}\cdot\nabla\varphi\,\mathrm{d}x + \langle \partial_{\boldsymbol{n}}(\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa)\,,\,\varphi\rangle_{(W^{\frac12,2}(\Gamma_{DN}))^*, W^{\frac12,2}(\Gamma_{DN})}. \tag{2.143}$$

(iii) The difference of the Fréchet differentials fulfils

$$a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa - \frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa, \varphi) = -\int_\Omega \frac{\partial b}{\partial s}(x,\kappa_0(x))\kappa(x)\nabla\big(u_{\kappa_0} - v_{\kappa_0}\big)\cdot\nabla\varphi\,\mathrm{d}x + \\ + \langle \partial_{\boldsymbol{n}}(\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa)\,,\,\varphi\rangle_{(W^{\frac12,2}(\Gamma_{DN}))^*, W^{\frac12,2}(\Gamma_{DN})} \qquad \forall\,\varphi \in V_0. \tag{2.144}$$

2.7.3 Fréchet differentiability in the case of linear parameter dependence

Obviously, the case of linear parameter dependence (within the equation!), i.e., $b(\cdot, s) := s$ (see (2.19)), is covered by theorem 2.20. Additionally, the proof simplifies and the relation (2.101) reduces to a more convenient one:

$$a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, \varphi) = -a(\kappa, u_{\kappa_0}, \varphi) \quad \forall\,\varphi \in V_1, \tag{2.145}$$

The same concerns the relation (2.104). Moreover, this special case leads to results which may be useful for further investigations.

Lemma 2.24. Let the assumptions of theorem 2.10 (both cases) as well as (2.95) be given.

(i) If there holds for some $\kappa_0 \in K_{ad}$ and for all $\kappa \in L_{\kappa_0}$

$$\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa = 0 \qquad \Big(\frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa = 0\Big), \tag{2.146}$$

then the following assertions are valid.

$$L_I(f,\phi,\kappa_0) = L_I(f,\phi,\kappa_1) \qquad \Big(L_{II}(f,\phi,\kappa_0) = L_{II}(f,\phi,\kappa_1)\Big) \qquad \forall\,\kappa_0,\kappa_1 \in K_{ad}, \tag{2.147}$$

$$\frac{\partial L_I}{\partial \kappa}(\kappa_0) = 0 \qquad \Big(\frac{\partial L_{II}}{\partial \kappa}(\kappa_0) = 0\Big) \qquad \forall\,\kappa_0 \in K_{ad}. \tag{2.148}$$

(ii) If (2.146) holds only for some $\kappa_0 \in K_{ad}$ and for some $\kappa_1 \in L_{\kappa_0}$, then we get only

$$L_I(f,\phi,\kappa_0) = L_I(f,\phi,\kappa_0 + t\kappa_1) \qquad \Big(L_{II}(f,\phi,\kappa_0) = L_{II}(f,\phi,\kappa_0 + t\kappa_1)\Big) \tag{2.149}$$
$$\forall\, t \in \mathbb{R} \text{ with } \kappa_0 + t\kappa_1 \in K_{ad}.$$

(iii) If $\overset{\circ}{K}_{ad} \neq \emptyset$ and $f \neq 0$, then

$$\frac{\partial L_I}{\partial \kappa}(\kappa_0) \neq 0 \qquad \frac{\partial L_{II}}{\partial \kappa}(\kappa_0) \neq 0, \qquad \forall\,\kappa_0 \in \overset{\circ}{K}_{ad}\,. \tag{2.150}$$

Proof. **(i)** Let $\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa = 0$ for some $\kappa_0 \in K_{ad}$ and for all $\kappa \in L_{\kappa_0}$. Due to (2.145) and the fact, that u_{κ_0} is the (unique) solution of problem I corresponding to κ_0, we get easily

$$\forall\,\kappa \in L_{\kappa_0} \quad : \quad a(\kappa_0 + \kappa, u_{\kappa_0}, \varphi) = \langle f\,,\,\varphi\rangle_{V_1^*,V_1} \qquad \forall\,\varphi \in V_1. \tag{2.151}$$

Obviously, for $\kappa_1 \in K_{ad}$ we have $\kappa_1 - \kappa_0 \in K$ and, therefore, $\kappa_0 + (\kappa_1 - \kappa_0) = \kappa_1 \in K_{ad}$. Thus, (2.151) yields (2.147) and (2.148), at first for L_I, and, analogously, for L_{II}.
(ii) This assertion easily follows from the proof of (i).
(iii) Let be $\kappa_0 \in \overset{\circ}{K}_{ad}$ with $\frac{\partial L_I}{\partial \kappa}(\kappa_0) = 0$. Then there exists a real interval $J \subset \mathbb{R}$ with $t\kappa_0 \in K_{ad}$ for all $t \in J$ (clearly, $0 \notin J$ and $1 \in J$). Since $f \neq 0$, there exists a $\overline{\varphi} \in V_1$ with $\langle f\,,\,\varphi\rangle_{V_1^*,V_1} \neq 0$. Moreover, taking (2.145) and (2.147) into account, we get

$$\forall\, t \in J \quad : \quad t\, a(\kappa_0, u_{\kappa_0}, \overline{\varphi}) = \langle f\,,\,\overline{\varphi}\rangle_{V_1^*,V_1} \neq 0. \tag{2.152}$$

The arbitrariness of $t \in J$ leads to a contradiction. To prove the assertion for problem II, one chooses a test function $\overline{\varphi} \in V_0$ vanishing on Γ_{DN} with $\langle f\,,\,\varphi\rangle_{V_0^*,V_0} \neq 0$. □

2.7.4 Second-order Fréchet differentiability in the case of linear parameter dependence

In a similar manner as theorem 2.20 one can proof that the solution operators are also twice continuously Fréchet differentiable, if the coefficient function b fulfils additional conditions, and that the second Fréchet differentials are represented as weak solutions of problems of type I and II, respectively, in a similar manner as the first differentials. However, due to the high technical effort, we only deal with the case of linear parameter dependence, i.e., we assume

$$\forall\, s \in J \quad : \quad b(\cdot, s) := s. \tag{2.153}$$

Theorem 2.25. Let the assumptions of theorem 2.20 (cases I and II, respectively) as well as (2.153) be given.
(i) The mapping $L_I(f,\phi,\cdot) : K_{ad} \subset L^s(\Omega) \to V_0$ is twice continuously Fréchet differentiable on K_{ad} for all $2 \le p < \min\{\overline{p}, 4\}$ and $s \in [2p/(p-2), \infty]$. The second Fréchet differential $\partial^2 L_I/\partial\kappa^2(\kappa_0)(\kappa,\xi)$ belongs to $V_1^{\overline{p}}$ and is given for all $\kappa_0 \in K_{ad}$ and $\kappa, \xi \in L_{\kappa_0}$ by

$$a(\kappa_0, \frac{\partial^2 L_I}{\partial \kappa^2}(\kappa_0)(\kappa,\xi), \varphi) = -a(\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\xi, \varphi) - a(\xi, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, \varphi) \qquad \forall\,\varphi \in V_1 \tag{2.154}$$

or, equivalently by

$$\frac{\partial^2 L_I}{\partial \kappa^2}(f,\phi,\kappa_0)(\kappa,\xi) = L_I(G^I_{\kappa_0}(\kappa,\xi), 0, \kappa_0), \tag{2.155}$$

while $G^I_{\kappa_0}(\kappa,\xi) = G^I_{f,\phi,\kappa_0}(\kappa,\xi) \in V_1^*$ continuously depends on κ_0 and is defined by

$$\langle G^I_{\kappa_0}(\kappa,\xi)\,,\,\boldsymbol{\varphi}\rangle_{V_1^* V_1} := -a(\kappa, L_I(F^I_{\kappa_0}(\xi), 0, \kappa_0), \varphi) - a(\xi, L_I(F^I_{\kappa_0}(\kappa), 0, \kappa_0), \varphi)\ \forall\,\varphi \in V_1. \tag{2.156}$$

(ii) The mapping $L_{II}(f,\tau,\cdot) : K_{ad} \subset L^s(\Omega) \to V_0$ is twice continuously Fréchet differentiable on K_{ad} for all $2 \le p < \min\{\overline{p}, 4\}$ and $s \in [2p/(p-2), \infty]$. The second Fréchet differential $\partial^2 L_{II}/\partial\kappa^2(\kappa_0)(\kappa,\xi)$ belongs to $V_0^{\overline{p}}$ and is given for all $\kappa_0 \in K_{ad}$ and $\kappa, \xi \in L_{\kappa_0}$ by

$$a(\kappa_0, \frac{\partial^2 L_{II}}{\partial\kappa^2}(\kappa_0)(\kappa,\xi), \varphi) = -a(\kappa, \frac{\partial L_{II}}{\partial\kappa}(\kappa_0)\xi, \varphi) - a(\xi, \frac{\partial L_{II}}{\partial\kappa}(\kappa_0)\kappa, \varphi) \qquad \forall\, \varphi \in V_0, \tag{2.157}$$

or, equivalently by

$$\frac{\partial^2 L_{II}}{\partial\kappa^2}(f,\tau,\kappa_0)(\kappa,\xi) = L_{II}(G^{II}_{\kappa_0}(\kappa,\xi), 0, \kappa_0), \tag{2.158}$$

while $G^{II}_{\kappa_0}(\kappa,\xi) = G^{II}_{f,\tau,\kappa_0}(\kappa,\xi) \in V_0^*$ continuously depends on κ_0 and is defined by

$$\langle G^{II}_{\kappa_0}(\kappa,\xi)\,,\, \boldsymbol{\varphi}\rangle_{V_0^* V_0} := -a(\kappa, L_{II}(F^{II}_{\kappa_0}(\xi), 0, \kappa_0), \varphi) - a(\xi, L_{II}(F^{II}_{\kappa_0}(\kappa), 0, \kappa_0), \varphi) \;\; \forall\, \varphi \in V_0. \tag{2.159}$$

Again, some further assertions can be proved.

Corollary 2.26. Let the assumptions of theorem 2.20 and (2.153) be given.

(i) For all $2 \le p < \min\{\overline{p}, 4\}$ and $s \in [2p/(p-2), \infty]$ the second Fréchet derivatives $\partial^2 L_I/\partial\kappa^2(\kappa_0)$: $(L_{\kappa_0} \times L_{\kappa_0}) \subset (L^s(\Omega) \times L^s(\Omega)) \to V_1$ and $\partial L_{II}/\partial\kappa(\kappa_0)$: $(L_{\kappa_0} \times L_{\kappa_0}) \subset (L^s(\Omega) \times L^s(\Omega)) \to V_0$ can be extended to bounded operators on $L^s(\Omega) \times L^s(\Omega)$. Moreover, these extensions are bounded on $L^\infty(\Omega) \times L^\infty(\Omega)$, and there hold the estimates and relations (for $i \in \{I, II\}$)

$$\|\frac{\partial^2 L_i}{\partial\kappa^2}(\kappa_0)(\kappa,\xi)\|_V \le c\left(\|\kappa\|_{L^s(\Omega)}\, \|\xi\|_{(L^\infty(\Omega))^k} + \|\kappa\|_{L^\infty(\Omega)}\, \|\xi\|_{(L^s(\Omega))^k}\right). \tag{2.160}$$

$$\lim_{\|\kappa\|_{L^s(\Omega)} \to 0} \|\frac{\partial^2 L_i}{\partial\kappa^2}(\kappa_0)(\kappa,\xi)\|_{V^p} = 0, \qquad \lim_{\|\xi\|_{L^s(\Omega)} \to 0} \|\frac{\partial^2 L_i}{\partial\kappa^2}(\kappa_0)(\kappa,\xi)\|_{V^p} = 0. \tag{2.161}$$

(ii) The second Fréchet differential $\partial^2 L_I/\partial\kappa^2(\kappa_0)(\kappa,\xi)$ fulfills

$$\begin{aligned} a(\kappa_0, \frac{\partial^2 L_I}{\partial\kappa^2}(\kappa_0)(\kappa,\xi), \varphi) &= -a(\kappa, \frac{\partial L_I}{\partial\kappa}(\kappa_0)\xi, \varphi) - a(\xi, \frac{\partial L_I}{\partial\kappa}(\kappa_0)\kappa, \varphi) + \\ &+ \Big\langle \partial_{\boldsymbol{n}}(\frac{\partial^2 L_I}{\partial\kappa^2}(\kappa_0)(\kappa,\xi))\,,\,\varphi \Big\rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} \qquad \forall\, \varphi \in V_0. \end{aligned} \tag{2.162}$$

(iii) There holds for all $\kappa_0 \in K_{ad}$, $\kappa, \xi \in L_{\kappa_0}$:

$$\begin{aligned} a(\kappa_0, &\frac{\partial^2 L_I}{\partial\kappa^2}(\kappa_0)(\kappa,\xi) - \frac{\partial^2 L_{II}}{\partial\kappa^2}(\kappa_0)(\kappa,\xi), \varphi) = \\ &= -\, a(\kappa, \frac{\partial L_I}{\partial\kappa}(\kappa_0)\xi - \frac{\partial L_{II}}{\partial\kappa}(\kappa_0)\xi, \varphi) - a(\xi, \frac{\partial L_I}{\partial\kappa}(\kappa_0)\kappa - \frac{\partial L_{II}}{\partial\kappa}(\kappa_0)\kappa, \varphi) + \\ &+ \Big\langle \partial_{\boldsymbol{n}}(\frac{\partial^2 L_I}{\partial\kappa^2}(\kappa_0)(\kappa,\xi))\,,\,\varphi \Big\rangle_{(W^{\frac{1}{2},2}(\Gamma_{DN}))^*, W^{\frac{1}{2},2}(\Gamma_{DN})} \qquad \forall\, \varphi \in V_1. \end{aligned} \tag{2.163}$$

3 Examples of linear elliptic problems with parameters

For a better readability the investigations above have been focused on the two prototypical mixed boundary-value problems I (2.1) - (2.4) and II (2.5) - (2.8). The solutions looked for are scalar functions and the underlying PDE (2.1) is simple. However, the results presented in subsection 2.4, and, with some care in subsections 2.5 and 2.7, can be extended to many important cases with more complex equations or with systems of equations. As a rule, the proofs of these extended assertions have the same structure. Generally, the parameter space K is a normed space of (vector) functions defined on Ω, in some cases $(L^\infty(\Omega))^k$ with some $k \in \mathbb{N}$, sometimes $(L^q(\Omega))^k$ with some $1 \le q < \infty$.

We provide some examples of linear elliptic problems with parameters, generalizing problems I and II. To avoid repetitions we assume for all cases considered in the sequel boundary conditions like in (2.2) - (2.4) for the problem I and in (2.6) - (2.8) for the problem II. Besides, the domain Ω and its boundary parts are assumed to fulfil (2.9) - (2.11).

3.1 General linear elliptic equations of second order

This more complex case is given by

$$(3.1)\qquad a(\kappa,u,\varphi) := \sum_{i,j=1}^{d} \int_\Omega a_{ij}(x,\kappa)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x + \sum_{i=1}^{d}\int_\Omega a_i(x,\kappa)u\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x +$$
$$+\sum_{i=1}^{d}\int_\Omega b_i(x,\kappa)\frac{\partial u}{\partial x_i}v\,\mathrm{d}x + \int_\Omega b(x,\kappa)uv\,\mathrm{d}x \qquad \forall\,(\kappa,u,\varphi)\in K\times V\times V$$

with V, V_0, V_1 as in (2.24) - (2.26). Generally, each coefficient function a_{ij}, a_i, b_i and b can depend on its own parameter. This issue can be handled by a vector function. Hence, the general approach is $K := (L^\infty(\Omega))^k$ for some $k \in \mathbb{N}$. In special situations, $K := (L^q(\Omega))^k$ for some $1 \le q < \infty$ may be possible, see remarks 2.1 (v), (iv).

To fulfill the properties (2.29), (2.30), (2.31), we let the functions a_{ij}, a_i, b_i and b be special Carathéodory functions, i.e., they are Lebesgue measurable for $\kappa \in K$ and for almost all $x \in \Omega$ they are Lipschitz continuous. Moreover, to ensure the existence of the integrals in (3.1) we require for all $\kappa \in K_{ad}$ and for all indices $i,j \in \{1,\dots,d\}$

$$(3.2)\qquad a_{ij}(\cdot,\kappa)\in L^\infty(\Omega),$$
$$(3.3)\qquad a_i(\cdot,\kappa), b_i(\kappa,\cdot)\in L^d(\Omega),\qquad b(\kappa,\cdot)\in L^{\frac{d}{2}}(\Omega),\qquad \text{if } d>2,$$
$$(3.4)\qquad a_i(\cdot,\kappa), b_i(\cdot,\kappa)\in L^{2+\delta}(\Omega),\qquad b\in L^{1+\delta}(\Omega),\qquad \text{if } d=2,\qquad \text{with a fixed } \delta>0.$$

Finally, the set K_{ad} must be chosen so that the *coercivity property* (2.30) is additionally fulfilled. Clearly, this is a restriction to K_{ad} and to the functions a_{ij}, a_i, b_i and b. The corresponding classical formulation of problem I reads as

$$(3.5)\qquad -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(\cdot,\kappa)\frac{\partial u}{\partial x_j} + a_i(\cdot,\kappa)u\right) + \sum_{i=1}^{n} b_i(\cdot,\kappa)\frac{\partial u}{\partial x_i} + b(\cdot,\kappa)u = f \quad \text{in} \quad \Omega \quad \text{in } V_1^*,$$
$$(3.6)\qquad u=\phi \qquad \text{on } \Gamma_{DN},$$
$$(3.7)\qquad u=0 \qquad \text{on } \Gamma_D,$$
$$(3.8)\qquad \sum_{i,j=1}^{n}\left(a_{ij}(\cdot,\kappa)\frac{\partial u}{\partial x_j} + a_i(\cdot,\kappa)u\right)\nu_i = 0 \qquad \text{on } \Gamma_N.$$

Remarks 3.1. (i) (**Existence results**) Due to the above assumptions, ensuring in particular the property (2.30) for the (whole) form a given by (3.1), the existence and uniqueness results of subsection 2.4 are ensured.

(ii) (**Global higher integrability of the gradient**) Since the elliptic equation is linear and the existence of a unique weak solution is ensured, the results of section 2.5 continue to hold.

(iii) (**Only lower-order terms depend on** κ) Special cases arise if some groups of coefficient functions, say a_{ij}, do *not* depend on κ. Under this assumption, the study of the arising inverse problem of identification of the parameter κ may be considerably easier. For instance, one can consider the following equation.

$$(3.9)\qquad -\mathrm{div}\,(b(x)\nabla u) + c(x,\kappa)u = f \quad \text{in} \quad \Omega \quad \text{in } V_1^*,$$

with suitable functions $b : \Omega \to \mathbb{R}$ and $c : \Omega \times K \to \mathbb{R}$.

(iv) (**Linear dependence on** κ) In this case, one has $a_{ij}(x,\kappa(x)) = a_{ij0}(x)\kappa(x)$ and corresponding representations for the remaining coefficients a_i, b_i and b. Thus, for convenience one can define new parameter functions, setting $\kappa_{ij}(x) := a_{ij0}(x)\kappa(x)$ etc. Clearly, these new functions have to fulfil (3.2) - (3.4). To ensure admissibility, the components of K_{ad} must form bounded sets in the corresponding spaces given in (3.2) - (3.4). Moreover, they must ensure a suited uniform ellipticity condition.

3.2 Linear elliptic systems of second order

Linear elliptic systems depending on a parameter can be dealt with in an analogous manner as in the case of a single elliptic equation. However, the complexity can considerably increase, in particular, in practical applications and numerical calculations. For convenience we only write down a system without lower order terms for $u \in V$ looked for and $V := (W^{1,2}(\Omega))^m$ analogous as in (2.24).

$$-\sum_{s=1}^{m}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}^{rs}(x,\kappa)\frac{\partial u^s}{\partial x_j}\right) = f^r(x) \quad \text{in } \Omega. \quad r = 1,\dots,m \tag{3.10}$$

The corresponding test-function spaces V_0 and V_1 can be chosen as in (2.25), (2.26), if all components u_i of u have the same boundary parts for Dirichlet and Neumann conditions, respectively. However, generally, there can be individual boundary parts $\Gamma_{D_i}, \Gamma_{DN_i}, \Gamma_{N_i} \subset \partial\Omega$ for $i \in \{1,\dots,m\}$, leading to suitable test-function spaces V_0 and V_1. A parameter space K and an admissible set K_{ad} can be chosen in dependence of the behavior of the functions a_{ij}^{rs} like in the case (ii) in subsection 3.1.

3.3 Stationary linear anisotropic non-homogeneous elasticity

The stationary equations of linear anisotropic non-homogeneous elasticity lead to a special linear elliptic system of PDE with linear parameter dependence. As show the formulations in (3.12) - (3.19), there is a great similarity to the prototypical problems in subsection 2.1. In paragraph 3.3.4, see corollary 3.3, we list up and comment the corresponding existence and regularity results.

For convenience we sketch main things in short and refer to the broad literature for linear elasticity, see, e.g., Ciarlet (1988), Zeidler (1997) for mathematical aspects, and Haupt (2002), Bertram and Glüge (2015) for mechanical aspects.

3.3.1 Preparations

Let $\Omega \subset \mathbb{R}^d$ and its boundary parts as in (2.9) - (2.11), however, the case $d \in \{2,3\}$ is of practical relevance. Now, the spaces V, V_0 and V_1 corresponding to these ones defined (2.24) - (2.26) are given as vectorial variants, i.e., we set

$$V := (W^{1,2}(\Omega))^d, \quad V_0 := \{\varphi \in V \,|\, \varphi = 0 \text{ on } \Gamma_D\}, \quad V_1 := \{\varphi \in V \,|\, \varphi = 0 \text{ on } \Gamma_D \cup \Gamma DN\}. \tag{3.11}$$

For convenience we do not want to use new symbols. From the mechanical point of view, the closure $\overline{\Omega}$ is assumed to be the reference configuration of an elastic (stress-free) body. In the framework of small deformations the boundary-value problems for determining the displacement vector $u \in V_0$ (corresponding to the prototypical problems (2.1) - (2.4) and (2.5) - (2.8)) read as

Problem I ("displacement-driven") Find u with

$$-\mathrm{div}\,(\boldsymbol{\sigma}) = f \quad \text{in } \Omega, \tag{3.12}$$
$$u = 0, \quad \text{on } \Gamma_D, \tag{3.13}$$
$$\boldsymbol{\sigma}\nu = 0 \quad \text{on } \Gamma_N, \tag{3.14}$$
$$u = \phi \quad \text{on } \Gamma_{DN}. \tag{3.15}$$

Problem II ("traction-driven") Find u with

$$-\mathrm{div}\,(\boldsymbol{\sigma}) = f \quad \text{in } \Omega, \tag{3.16}$$
$$u = 0, \quad \text{on } \Gamma_D, \tag{3.17}$$
$$\boldsymbol{\sigma}\nu = 0 \quad \text{on } \Gamma_N, \tag{3.18}$$
$$\boldsymbol{\sigma}\nu = \tau \quad \text{on } \Gamma_{DN}. \tag{3.19}$$

For simplicity we take homogenous boundary conditions on Γ_D and Γ_N. This is not an essential restriction.

The notation is standard and in accordance with the general setting in section 2.3: $\boldsymbol{\sigma}$ - (Cauchy) stress tensor, ϕ - displacement vector on Γ_{DN}, τ - normal stress on Γ_{DN}, f - volume densities of external forces, ν - outer unit normal vector on $\partial\Omega$. Sometimes, instead of f one writes $\varrho\,\overline{f}$ with ϱ being the mass density in the reference configuration as well as with the mass density of external forces $\overline{f}$ (gravitational acceleration, e.g.).

3.3.2 Linear elastic behavior

In linear elasticity, the stress tensor, the linearized Cauchy-Green strain tensor $\boldsymbol{\varepsilon}$ and the displacement vector u are connected in a characteristic manner.

$$\boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(u) := \frac{1}{2}\big(\nabla u + (\nabla u)^T\big). \tag{3.20}$$

$\mathbb{E}$ is the fourth-order elasticity tensor (stiffness tensor). In the 3-dimensional case, being in the focus here, its 81 components are generally position-dependent, and they suffice the following symmetry conditions (cf. Haupt (2002), Bertram and Glüge (2015), e.g.).

$$\forall\, i,j,k,l \in \{1,2,3\} \qquad \text{f.a.a } x \in \Omega \qquad : \qquad \mathbb{E}_{ijkl}(x) = \mathbb{E}_{jikl}(x) = \mathbb{E}_{klji}(x). \tag{3.21}$$

Due to these symmetry conditions only maximally $n_{ai} = 21$ entries are independent. (ai for “anisotropic”.) There are seven mechanically relevant values of n_{ai} standing for eight symmetry groups (see Bertram and Glüge (2015), chapter 4.1 for details)

$$n_{ai} \in \{2, 3, 5, 6, 9, 13, 21\}. \tag{3.22}$$

Moreover, for a fixed $x \in \Omega$ the elasticity tensor $\mathbb{E}$ linearly depends on these n_{ai} parameters generally depending on x. Thus, we write in the sequel

$$\mathbb{E} = \mathbb{E}(\kappa), \qquad \mathbb{E}(x) = \mathbb{E}(\kappa)(x) \qquad \text{f.a.a. } x \in \Omega. \tag{3.23}$$

Another consequence of (3.21) is, that the elasticity tensor $\mathbb{E}$ can be expressed as a symmetric 6×6 matrix by the Voigt representation. Writing the symmetric second-order tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ as vectors with six components, the material law (3.20) can be brought into a form more suitable for applications in anisotropic elasticity (see Bertram and Glüge (2015), paragraph 2.1.15 for details). However, for our mathematical investigations we do not need this representation.

It is well-known, that in the isotropic case $n_{ai} = 2$ the relation (3.20) is reduced to

$$\boldsymbol{\sigma} = \mathbb{E}(\kappa)\boldsymbol{\varepsilon} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}. \tag{3.24}$$

Hence, we have $\kappa := (\mu, \lambda)$. The Lamé coefficients $\mu > 0$ and $\lambda \geq 0$ are functions of x, if the material is nonhomogeneous. Inserting (3.24) into the equation (3.20), one obtains the Lamé equations of stationary linear isotropic elasticity.

3.3.3 Admissible set of material parameter functions

In accordance with (3.22) and (3.23) we define the set K of material parameter functions for the nonhomogeneous linear elasticity with anisotropy by

$$K := (L^\infty(\Omega))^{n_{ai}}. \tag{3.25}$$

In order to underline the parameter dependence we write $\mathbb{E}(\kappa)$ for $\kappa \in K$. For fixed $\kappa \in K$ the values $\mathbb{E}(\kappa)(x)$ are defined point-wise almost everywhere on Ω and the tensor function $\mathbb{E}$ is supposed to fulfil

$$\mathbb{E} \in L((L^\infty(\Omega))^{n_{ai}}, (L^\infty(\Omega))^{81}). \tag{3.26}$$

Thus, there holds the estimate

$$\exists\, c_E > 0\ \forall\, \kappa \in K \quad : \quad \|\mathbb{E}(\kappa)\|_{(L^\infty(\Omega))^{81}} \leq c_E \|\kappa\|_{(L^\infty(\Omega))^{n_{ai}}}. \tag{3.27}$$

Any nonempty set

$$K_{ad} \subset K \tag{3.28}$$

is called a set of *admissible parameter functions* κ, if

$$\exists\, M = M(K) > 0 \quad : \quad \|\kappa\|_{(L^\infty(\Omega))^{n_{ai}}} \leq M \qquad \forall\, \kappa \in K, \tag{3.29}$$

$$\exists\, c_0 = c_0(K) > 0 \quad : \quad \mathbb{E}(\kappa)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \geq c_0\, \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \qquad \forall\, \kappa \in K, \quad \forall\, \boldsymbol{\varepsilon} \in \mathrm{Sym}(3). \tag{3.30}$$

Thus, $K_{ad} \subset K$ is bounded and closed. Moreover, (3.30) means that the set of $\mathbb{E}(\kappa)$ with $\kappa \in K_{ad}$ is uniformly positive definite.

Remark 3.2. Obviously, the equations of stationary linear elasticity depend linearly on the parameter $\kappa \in K_{ad}$. However, it is thinkable, that κ on its part depends on a further parameter like temperature or damage. In this case, one gets a more general setting like in subsection 2.1.

3.3.4 Weak-formulation setting and main results

The trilinear form a (cf. (2.28)) corresponding to the elasticity equation is given by

$$a(\kappa, u, \varphi) := \int_\Omega \mathbb{E}(\kappa)\, \varepsilon(u) : \varepsilon(\varphi)\, \mathrm{d}x \qquad \forall\, (\kappa, u, \varphi) \in K \times V \times V, \tag{3.31}$$

Due to the assumption on $\mathbb{E}$ and K_{ad}, (3.20) - (3.23), (3.25), (3.26), (3.28)- (3.30) as well as based on Poincaré's and Korn's inequality, the properties (2.29), (2.30), (2.31) and (2.32) can be easily verified.

Thus, the stationary equations for the linear elasticity lead to an elliptic system in the sense that the arising form a is coercive. Moreover, there is a great similarity to the prototypical problems with *linear* parameter dependence, see subsection 2.1. Thus, the main results presented above concerning weak solutions remain valid. We collect them in short.

Corollary 3.3. (**Results concerning weak solutions in stationary linear elasticity**)
Let be given the assumptions (2.9) - (2.11) for the domain Ω and its parts Γ_D, Γ_{DN} and Γ_N as well (3.11). Let the data ϕ, τ and f fulfil the vectorial variants of (2.34) - (2.36) as well as (2.37) and (2.42), respectively. Moreover, we assume (3.20) - (3.23), (3.25), (3.26), (3.28) - (3.30) for the elasticity tensor. Than there hold:

(i) (**Existence, uniqueness and well-posedness**) All results presented in section 2.4) hold with slight technical modifications concerning the arising norms and constants.

(ii) (**Higher integrability of the gradient**) All results presented in section 2.5) hold under (2.70) and analogous assumption for the data with slight technical modifications. Again, the scalar spaces in (2.67) - (1.3.3) are replaced by their vectorial counterparts.
The result in Herzog et al. (2011) covers the extension of theorem 2.10 to linear elasticity. In Shi and Wright (1994), an alternative proof for linear elasticity is presented. Note that the Lamé equations of linear elasticity are a special elliptic system. Thus, particular difficulties can arise when proving "analogous" assertions near the boundary (cf. Shi and Wright (1994), e.g.). Based on theorem 2.10, the results of theorem 2.12 follow easily for weak solutions of the Lamé equations.

(iii) (**Generalized co-normal derivative**) In the case of linear elasticity, the co-normal derivative is formally given by $\mathbb{E}(\kappa)\varepsilon(u)_{|\Gamma_{DN}} \cdot n$ and expresses the (distributional) normal stress on Γ_{DN}. It is only a function, if $u \in (W^{2,2}(\Omega))^m$ $(m \in \{2, 3\})$ and $\kappa \in C^{0,1}(\overline{\Omega})$.

(iv) (**Fréchet differentiability of solution operators**) Finally, the results in section 2.7 remain valid. Additionally, due to the linear parameter dependence, the assertions of theorem 2.20 and their proofs undergo some simplifications.

3.3.5 Special case - isotropic elasticity

In the case of isotropy, the general linear relation (3.20) is reduced to (3.24). The position dependence of the Lamé coefficients leads to the setting $\kappa = (\mu, \lambda) \in K_{max} := (L^\infty(\Omega))^2$. There are the following relations with Young's modulus E, Poisson's ratio ν and the compression (bulk) modulus k:

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad k = \lambda + \frac{2}{3}\mu, \tag{3.32}$$

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \nu}, \qquad \nu = \frac{\lambda}{2(\mu + \lambda)}, \qquad k = \frac{E}{3(1-2\nu)}. \tag{3.33}$$

Sometimes, it is useful to write the material law (3.24) in an equivalent form, dividing the stress into a deviatoric and a spherical part:

$$\boldsymbol{\sigma} = \mathbb{E}(\kappa)\boldsymbol{\varepsilon} = 2\mu\, \boldsymbol{\varepsilon}^* + k\, \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}. \tag{3.34}$$

($\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon} - {}^1\!/_3\, \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}$ - deviator of $\boldsymbol{\varepsilon}$, $\mathrm{tr}(\boldsymbol{\varepsilon})$ - trace of $\boldsymbol{\varepsilon}$). In Constantinescu (1995), this item was discussed in relation to the identification problem. Here, we do not need this.

A suitable subset $K_{ad} \subset K$ can be easily obtained in the isotropic case. One can choose the following (physically plausible) bounds for the Young modulus E and the Poisson ratio ν which have to be determined by preliminary considerations or tests.

$$0 < E_0 \leq E(x) \leq E_1 < \infty, \qquad 0 \leq \nu(x) \leq \nu_1 < {}^1\!/_2 \qquad \text{for almost all } x \in \Omega. \tag{3.35}$$

Due to (3.32), bounds for μ and λ follow.

$$\frac{E_0}{3} \le \mu(x) \le \frac{E_1}{2}, \qquad 0 \le \lambda(x) \le \frac{\nu_1 E_1}{1-2\nu_1} \qquad \text{for almost all } x \in \Omega. \tag{3.36}$$

The isochoric case $\nu = 1/2$ (this corresponds to $\lambda = \infty$) will be excluded. In this case divergence-free displacement fields have to be considered. We refer to Widany and Mahnken (2012) for an application to rubber-like material. The preceding considerations suggests the introduction of the bounded closed and convex subset $K_{ad} \subset K$:

$$K_{ad} = \{(\mu,\lambda) \in (L^\infty(\Omega))^2 \,|\, \mu,\lambda \text{ fulfil } (3.36)\}. \tag{3.37}$$

Obviously, for this set some $M > 0$ and $c_0 > 0$ exist such that the conditions (3.29) and (3.30) are fulfilled.

3.3.6 Special case - elastic polycrystals and composite materials

In addition to the general case of anisotropy and non-homogeneity, the following extension covers composite materials or polycrystals, e.g. We assume that the body can be decomposed into a number of subbodies Ω_i:

$$\overline{\Omega} = \bigcup_{l=1}^{N} \overline{\Omega}_l, \qquad \Omega_l \text{ - Lipschitz domain}, \qquad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j. \tag{3.38}$$

Moreover, we assume that the body behaves linear elastic in each Ω_l. Thus, for all l there exists an elasticity tensor $\mathbb{E}_l$ with corresponding symmetry properties (3.21) as well as a subset $K_{ad,l} \subset (L^\infty(\Omega_l))^{n^l_{ai}}$ with properties analogous to (3.28) - (3.30). This leads to the following generalization of $\mathbb{E}$:

$$\mathbb{E} \,:\, \Omega \times (L^\infty(\Omega_1))^{n^1_{ai}} \times \cdots \times (L^\infty(\Omega_N))^{n^N_{ai}} \to (L^\infty(\Omega))^{81}, \tag{3.39}$$

$$\mathbb{E}(x,\kappa) := \sum_{l=1}^{N} \chi_l(x)\, \mathbb{E}_l(\kappa_l). \tag{3.40}$$

χ_l - characteristic function of Ω_l, $\kappa = \{\kappa_1,\ldots,\kappa_N\}$ with $\kappa_l \in (L^\infty(\Omega_1))^{n^l_{ai}}$. Thus:

$$\text{f.f.a } x \in \Omega \quad : \quad \mathbb{E}(x,\cdot) \in \mathrm{Hom}\big((L^\infty(\Omega_1))^{n^1_{ai}} \times \cdots \times (L^\infty(\Omega_N))^{n^N_{ai}}; (L^\infty(\Omega))^{81}\big). \tag{3.41}$$

Moreover, a corresponding parameter set for $\mathbb{E}$ is given via

$$K_{ad} := K_{ad,1} \times \cdots \times K_{ad,N}, \tag{3.42}$$

and conditions like (3.28) - (3.30) are fulfilled. In polycrystals made of the same material, the tensors $\mathbb{E}_l$ arise from each other by orthogonal rotations. In composite materials these tensors have generally different grades of anisotropy.

3.4 Non-local linear elliptic problems

Many results concerning usual elliptic problems also remain valid in non-local situations. As an example, we consider the following non-local equation with the boundary conditions (2.2) - (2.4) for problem I.

$$-\mathrm{div}\,(b(x,\kappa)\nabla u) + \int_\Omega c(x,y)u(y)\,\mathrm{d}y = f \quad \text{in} \quad \Omega. \tag{3.43}$$

Clearly, this equation yields the corresponding form a being linear in u and φ:

$$a(\kappa,u,\varphi) := \int_\Omega b(x,\kappa)\nabla u \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega\int_\Omega c(x,y)u(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \qquad \forall\,(\kappa,u,\varphi) \in K \times V \times V \tag{3.44}$$

with V, V_0 and b, K and K_{ad} as for the prototypical problems, see subsection 2.2. The function $c \in L^2(\Omega\times\Omega)$ is assumed to have the property

$$\int_\Omega\int_\Omega c(x,y)u(y)u(x)\,\mathrm{d}y\,\mathrm{d}x \ge 0 \qquad \forall\, u \in L^2(\Omega). \tag{3.45}$$

Clearly, the form a has the properties (2.29) - (2.32).

4 Some inverse problems arising from linear elliptic equations

Under the assumptions of theorem 2.6 both prototypical problems I (2.38), (2.39) and II (2.43) have uniquely determined weak solutions $u = u_\kappa \in V_0$ and $v = v_\kappa \in V_0$ for $\kappa \in K_{ad}$, assuming the data as given. If the underlying situation is described *exactly* by some κ (e.g., if the elastic behavior is correctly described by some elasticity tensor $\mathbb{E}(\kappa)$ in the case of linear elasticity, see subsection 3.3.2), and if the boundary data (2.34) and (2.35) belong to the *same* experiment, and if $f \in V_0^*$, then both solutions u_κ and v_κ coincide in theory. However, in *real* situations both solutions generally will be different. Based on these discrepancies, cost functionals can be constructed in order to an optimal parameter κ.

Here, our interest is to consider cost functionals in integral form ("continuous" functionals) which take both calculated solutions *within* the domain Ω into account. In some application-oriented contributions, only discrete cost functionals are dealt with, often based on finite-element schemes for the underlying problems.

We continue to consider the *prototypical situation* described in subsections 2.1 - 2.3. An essential item is the special structure of the arising form a in (2.27). Thus, the subsequent investigations and results cover linear second-order elliptic problems (for single equations and for systems of equations) without lower-order terms and with possibly non-linear parameter dependence as described in subsection 2.2. Hence, the case of linear elasticity is included (see subsection 3.3.2). With some care, some results can be modified for general elliptic equations like in (3.1).

For convenience and for a better overview, if possible, we only formulate the results for our prototypical situation with the scalar equation in (2.1). If necessary, remarks concerning linear elasticity will be given.

At first, in subsection 4.1, we introduce cost functionals which are capable to compare the solutions of the prototypical problems within the domain Ω. Furthermore, in subsection 4.2, we relate the cost functionals presented before with some other cost functionals in use. Finally, in subsection 4.3, we prove some results concerning existence of minimizers. In subsection 4.4, we present some special results connected with Fréchet differentiability. Finally, in subsection 4.5, for some completeness, we provide general necessary and sufficient conditions for the existence of local minimizers for differentiable functional on Banach spaces.

4.1 Cost functionals extending the approaches by Knowles as well as Yun and Shang

The following first cost functional $J_\alpha^{(1)}$ extends an approach by Knowles (1998) (see also Jin et al. (2012)) for investigation of an inverse problem in electrical impedance tomography. There, the main essence consists in comparison of the solutions of problems corresponding to our problems I and II above - but with *non-mixed* boundary conditions - *within* the domain Ω (i.e., within the body in applications). This Idea was applied by Yun and Shang (2011) in its *discrete* version and with *mixed* boundary conditions to mechanical problems (see remarks 4.1 (i), (ii) for further comments and references). We continue to consider linear elliptic problems with possibly non-linear parameter dependence defined in subsections 2.1 - 2.3, which are essentially given by the form a in (2.27). Under the assumptions of theorem 2.6 (for both cases, with $f \in V_0^*$) we define two cost functional $J_\alpha^{(i)}$ $(i = 1, 2)$ by

$$\begin{aligned}
(4.1) \qquad J_\alpha^{(1)}(\kappa, \phi, \tau, f) :&= a(\kappa, u_\kappa - v_\kappa, u_\kappa - v_\kappa) + \alpha\, \Phi^{(1)}(\kappa) = \\
&= a(\kappa, L_I(f, \phi, \kappa) - L_{II}(f, \tau, \kappa), L_I(f, \phi, \kappa) - L_{II}(f, \tau, \kappa)) + \alpha\, \Phi^{(1)}(\kappa) \\
(4.2) \qquad J_\alpha^{(2)}(\kappa, \phi, \tau, f) :&= \|u_\kappa - v_\kappa\|_W^r + \alpha\, \Phi^{(2)}(\kappa) = \\
&= \|L_I(f, \phi, \kappa) - L_{II}(f, \tau, \kappa)\|_W^r + \alpha\, \Phi^{(2)}(\kappa)
\end{aligned}$$

with a regularization parameter α and with stabilizing functionals $\Phi^{(i)}$ $(i = 1, 2)$ fulfilling

$$(4.3) \qquad \alpha \geq 0,$$

$$(4.4) \qquad \Phi^{(i)} : K_{ad} \to [0, \infty], \qquad \text{proper.}$$

as well as with an exponent r and with a function space W fulfilling

$$(4.5) \qquad 1 \leq r < \infty,$$

$$(4.6) \qquad V_0^p \subseteq W, \qquad \text{continuously embedded.}$$

(See (2.68) for V_0^p). Clearly, due to (4.4), the *effective domains* of $\Phi^{(i)}$ are not empty. For convenience, they are assumed to be subsets of K_{ad}, i.e., there holds

$$\emptyset \neq D(\Phi^{(i)}) = \{\kappa \in K_{ad} \,|\, \Phi^{(i)}(\kappa) < \infty\}. \tag{4.7}$$

These approaches for $J_\alpha^{(i)}$, in particular the second one, are inspired by the so-called Tikhonov approach, see Hofmann et al. (2007) and Jin and Maaß (2012a), e.g. Usually, the Tikhonov approach contains only one solution (forward) operator. The functionals $\Phi^{(i)}$ are called stabilizing functional or, sometimes, penalty or regularization term.

Moreover, the approach in (4.1) compares the solutions u_κ and v_κ, using the weak-formulation structure in the shape of the form a in (2.27) and (3.31), respectively: a is bilinear and coercive for a *fixed* $\kappa \in K_{ad}$. In the particular case of linear elasticity, the functional $J_0^{(1)}$ (i.e. for $\alpha = 0$) can be interpreted as the stored elastic energy related to the difference of the displacements u_κ and v_κ.

In many cases, the quantities ϕ, τ and f are kept fixed, while the parameter κ is assumed to vary. Thus, if there will be not confusion, we will write $J_\alpha^{(i)}(\kappa)$ $(i = 1, 2)$ instead of $J_\alpha^{(i)}(\kappa, \phi, \tau, f)$. Bearing this in mind, mostly we consider $J_\alpha^{(i)}$ as (generally numeric) functions depending only on κ, i.e.

$$J_\alpha^{(i)} \,:\, K_{ad} \to \mathbb{R} \cup \{\infty\}. \tag{4.8}$$

Investigating the functionals $J_\alpha^{(i)}$ $(i = 1, 2)$ in the inverse-problem context, important questions arise. Some of them are:

(i) *Question of identifiability*: Are there different κ_1 and κ_2 leading to equal solutions $u_{\kappa_1} = u_{\kappa_2}$ and $v_{\kappa_1} = v_{\kappa_2}$, respectively, for fixed data ϕ, f_1 and τ, f_2? Clearly, for a positive answer the functionals $J_\alpha^{(i)}$ $(i = 1, 2)$ may have *different* minimizers, especially for $\alpha = 0$. Even in the case of linear parameter dependence this question is not trivial. In subsection 4.2.3, we return to this question in short. For further discussion, partial results and references we refer to Kohn and Vogelius (1987), Bonnet and Constantinescu (2005), Imanuvilov et al. (2012).

(ii) *Question of existence of minimizers*: Clearly, a minimizer can be interpreted as a best fit in applications. Due to the infinite dimension of the parameter space K, the admissible parameter set $K_{ad} \subset K$ is only bounded and closed, and the continuity of $J_\alpha^{(i)}$ $(i = 1, 2)$ generally does *not* ensure the existence of a minimizer. One needs additional restrictions ensuring some kinds of compactness. Moreover, it is of great interest to know, if an existing minimizer is unique. In subsection 4.4, we return to this question. We exemplarily refer to Hofmann et al. (2007) and Jin and Maass (2012b).

(iii) *Question of approximation*: If there is a minimizer, in which way it can be approximated? What happens if for all $\alpha_n > 0$ a minimizer exists and if α_n tends to zero? Clearly, these questions are of great importance in practical application. It is not in the focus here. Again, we exemplarily refer to Daubechies et al. (2004) and Jin and Maass (2012b).

We end this subsection with some comments.

Remarks 4.1. (i) To our best knowledge, for the first time, an approach like in (4.1) with $\alpha = 0$ appeared in Knowles (1998) in connection with an inverse problem in electrical impedance tomography. The underlying mathematical problem consists of a scalar elliptic partial differential equation like in (2.1) with linear parameter dependence as in (2.19) completed with a pure Dirichlet and, alternatively, with a pure Neumann boundary condition.

(ii) In Yun and Shang (2011), the idea of comparison of *two calculated* solutions to problems in mechanics of solids was applied, in particular to elasto-plastic behavior. The *discrete* cost functionals introduced in Yun and Shang (2011), Shang and Yun (2012) (the "implicit objective function") compares both calculated solutions using their discrete values in the Gauß points of a finite-element mesh. Further modifications and applications of this approach can be found in Shang et al. (2013), Rahimi et al. (2012), Weaver (2015) as well as in Yun and Shang (2016).

(iii) Note that our proposal (4.1) (with $\alpha = 0$) is *not* a continuous counterpart to Yun and Shang's discrete approaches. We use the idea of comparison of *two calculated* solutions *inside* the domain Ω, following Knowles (1998) with respect to its mathematical realization.

(iv) As usual in inverse-problem theory, we consider the functionals in (4.1) and (4.2) with regularization terms $\Phi^{(i)}$. At this stage, $\alpha = 0$ is allowed, especially for the functional in (4.1). Furthermore, the regularizing functionals $\Phi^{(i)}$ $(i = 1, 2)$ are general. Widely spread are approaches with norms in suitable function spaces. An example may be given by

$$\Phi^{(i)}(\kappa) := \|\kappa - \kappa^*\|_{L^s(\Omega)} \tag{4.9}$$

with some estimated value $\kappa^* \in K_{ad}$ and some $1 \leq s \leq \infty$. Besides the classic Tichonov regularization built up on an L^2 norm (cf. Engl et al. (1996), e.g.), approaches in general Banach spaces play an important role. We refer to Scherzer (2009) and Schuster et al. (2012).

(v) However, for identification of material parameters in mechanics a regularization term is frequently avoided. Corresponding explanations can be found in Mahnken and Stein (1996b) and Thielecke (1998), e.g. A problem is that the choice of a regularization term as well as the size of the parameter α are a-priori not clear. And, moreover, the results depend on this choice.

(vi) In Constantinescu (1995), Geymonat and Pagano (2003) and Avril and Pierron (2007), cost functionals in integral form were considered, related to problems in mechanics, in particular in linear elasticity.

(vii) One encounters frequently discrete approaches for cost functionals in application-oriented papers. See Avril and Pierron (2007), Cooreman et al. (2007), Lecompte et al. (2007), Kajberg and Wikman (2007), e.g. for applications in mechanics.

4.2 Relations between our approaches and some other ones

Now it is the aim to relate the approaches presented in (4.1) and (4.2) with some others. At first, we compare them with so-called Dirichlet-to-Neumann as well as Neumann-to-Dirichlet mappings playing an important role in inverse-problem investigations. Often these mappings are considered for full boundary conditions. Here, due to our mixed-boundary setting, we consider them related only to a part of the boundary. For references and applications we refer to Kohn and Vogelius (1987), Constantinescu (1995), Bonnet and Constantinescu (2005), Isakov (2006), Lukaschewitsch et al. (2009), Jin et al. (2011), Jin and Maaß (2012a), Imanuvilov et al. (2012), Widany and Mahnken (2012), Mahnken and Dammann (2013) e.g.

Moreover, we discuss the differences to full-field approaches. For trace operators we refer to Amann (1993), Showalter (1997), e.g. Let the assumptions of theorem 2.6 be fulfilled in this subsection. For a better overview, we do not use formulations based on higher integrability of the gradients of weak solutions. Clearly, this can be done in the case of necessity.

4.2.1 Partial Neumann-to-Dirichlet mapping

Due to theorem 2.6 (ii), for given $\kappa \in K_{ad}$, f and τ there exists a unique $v_\kappa \in V_0 \subset W^{1,2}(\Omega)$. Thus, it makes sense to define a partial Neumann-to-Dirichlet mapping (abbreviated as pNtD mapping) as the trace of v_κ on the boundary part Γ_{DN}:

$$G_{pNtD}(f, \tau, \kappa) := L_{II}(f, \tau, \kappa)_{|\Gamma_{DN}} = v_{\kappa|\Gamma_{DN}}. \tag{4.10}$$

($v_{\kappa|\Gamma_{DN}}$ - trace of v_κ on Γ_{DN}.) Obviously, based on theorem 2.9 (see (2.65)) and the well-known trace theorem (cf. Showalter (1997), e.g.) one gets

$$\begin{aligned} G_{pNtD} \,&: V_1^* \times (W^{\frac{1}{2},2}(\Gamma_{DN}))^* \times K_{ad} \to W^{\frac{1}{2},2}(\Gamma_{DN}) \\ &\text{Lipschitz continuous on bounded subsets of } V_1^* \times (W^{\frac{1}{2},2}(\Gamma_{DN}))^* \times K_{ad}. \end{aligned} \tag{4.11}$$

Since this mapping only refers to a part of the boundary, it is called partial NtD mapping.

The pNtD mapping may be the basis for the inverse problem of determining an optimal parameter $\kappa \in K_{ad}$. Assume that (besides given f) both boundary data on Γ_{DN}, τ and ϕ are known. Thus, one can seek a $\kappa_0 \in K_{ad}$ such that the deviation between $G_{pNtD}(f, \tau, \kappa)$ and ϕ becomes a minimum:

$$\begin{aligned} J_{pNtD}(f, \tau, \phi, \kappa_0) &= \min_{\kappa \in K_{ad}} \Big\{ J_{pNtD}(f, \tau, \phi, \kappa) \Big\} \\ \text{with} \quad J_{pNtD}(f, \tau, \phi, \kappa) &:= \|G_{pNtD}(f, \tau, \kappa) - \phi\|_{DN} + \alpha\, \Phi(\kappa) \end{aligned} \tag{4.12}$$

with a regularization parameter $\alpha \geq 0$ and with a functional $\Phi : K_{ad} \to [0, \infty]$. The expression $\| \cdot \|_{DN}$ stands for a positive definite functional defined on $W^{1/2,2}(\Gamma_{DN})$, for instance, for squares of appropriate norms.

4.2.2 Partial Dirichlet-to-Neumann mapping

This mapping is defined in analogy to the pNtD mapping above. Due to theorem 2.6 (i), for given $\kappa \in K_{ad}$, f and ϕ there exists a unique $u_\kappa \in V_0 \subset W^{1,2}(\Omega)$. If f additionally fulfills (2.42), then the normal stress on Γ_{DN} corresponding to u_κ is defined as a distribution, based on theorem 2.15. Thus, under the given weak assumptions a pDtN mapping can be defined:

$$G_{pDtN}(f,\phi,\kappa) := \partial_{\boldsymbol{n}}(L_I((f,\phi,\kappa)). \tag{4.13}$$

Obviously, based on theorems 2.9 (see (2.64)) and 2.15 one gets

$$\begin{aligned} G_{pDtN} \,:& V_1^* \times W^{\frac{1}{2},2}(\Gamma_{DN}) \times K_{ad} \;\to\; (W^{\frac{1}{2},2}(\Gamma_{DN}))^* \\ & \text{Lipschitz continuous on bounded subsets of } V_1^* \times W^{\frac{1}{2},2}(\Gamma_{DN}) \times K_{ad}. \end{aligned} \tag{4.14}$$

Clearly, applying the pDtN mapping to the solution of problem II returns the given normal stress τ on Γ_{DN}. Contrary to the former pNtD mapping, the handling of the pDtN mapping is more difficult. Only in the case of better regularity, the pDtN mapping yields a function in $L^2(\Gamma_{DN})$.

Again, the pDtN mapping may be the basis for the inverse problem of determining a parameter $\kappa \in K_{ad}$. Assume that (besides given f) both boundary data on Γ_{DN}, ϕ and τ are known. Thus, one can seek a $\kappa_0 \in K_{ad}$ such that the deviation between $G_{pDtN}(f,\phi,\kappa)$ and τ becomes a minimum:

$$\begin{aligned} J_{pDtN}(f,\tau,\phi,\kappa_0) &= \min_{\kappa \in K_{ad}} \Big\{ J_{pDtN}(f,\tau,\phi,\kappa) \Big\} \\ &\text{with} \quad J_{pDtN}(f,\tau,\phi,\kappa) := \|G_{pDtN}(f,\phi,\kappa) - \tau\|_{DN} + \alpha\, \Phi(\kappa) \end{aligned} \tag{4.15}$$

with a regularization parameter $\alpha \geq 0$ and with a functional $\Phi : K_{ad} \to [0,\infty]$. The expression $\|\cdot\|_{DN}$ stands for a positive definite functional defined on $(W^{1/2,2}(\Gamma_{DN}))^*$.

4.2.3 Relations of our approaches to pNtD and pDtN mappings in the case $\alpha = 0$

Now we consider the approaches in (4.1), (4.2), (4.12) and (4.15) *without* regularization terms, or in other words, with $\alpha = 0$. We assume that minimizers $\kappa_0^{(i)} \in K_{ad}$ $(i = 1,2)$ of $J_0^{(i)}$ exist:

$$J_0^{(i)}(f,\tau,\phi,\kappa_0^{(i)}) = \min_{\kappa \in K_{ad}} \Big\{ J_0^{(i)}(f,\tau,\phi,\kappa) \Big\}. \tag{4.16}$$

Based on theorem 2.6, some straightforward assertions follow.

Lemma 4.2. Let the assumptions of theorem 2.6 be given. For all $\kappa_0 \in K_{ad}$ and $\kappa_0^{(i)} \in K_{ad}$, respectively, the following implications hold.

$$J_{pDtN}(f,\tau,\phi,\kappa_0) = 0 \quad \Leftrightarrow \quad J_{pNtD}(f,\tau,\phi,\kappa_0) = 0. \tag{4.17}$$

$$J_{pDtN}(f,\tau,\phi,\kappa_0^{(i)}) = 0 \quad \Leftrightarrow \quad J_0^{(i)}(f,\tau,\phi,\kappa_0^{(i)}) = 0, \tag{4.18}$$

$$J_0^{(i)}(f,\tau,\phi,\kappa_0^{(i)}) = 0 \quad \Leftrightarrow \quad u_{\kappa_0^{(i)}} = v_{\kappa_0^{(i)}}. \tag{4.19}$$

Thus, if some $\kappa_0 \in K_{ad}$ is a null of one of these three functionals, then it is a null of the remaining ones, and the corresponding solutions u_{κ_0} and v_{κ_0} coincide. Generally, one *cannot* expect that such a null exists for arbitrarily given data (and a given set K_{ad}).

The question whether a (possible) null $\kappa_0 \in K_{ad}$ is unique is *still open*, in the case of linear parameter dependence and in excess of the general case considered here. For example, choosing the function b in (2.1) like $b(\kappa) := 2 + \sin(\kappa)$, the minimizers of $\Phi_0^{(i)}$ *cannot* be unique. For the case $\Gamma_{DN} = \partial\Omega$, in Bonnet and Constantinescu (2005), the authors present arguments letting guess a negative answer in the case of non-homogeneous anisotropic linear elasticity. However, under some restrictions, uniqueness was proved for isotropic elasticity by Imanuvilov et al. (2012) in the case of mixed boundary conditions.

4.2.4 Remarks on full-field approaches

In full-field approaches, the "full" solution of the direct problem is assumed to be given by measurements (and interpolation in practical applications). For an overview we refer to Avril and Pierron (2007) and the literature cited therein. The main idea means in the setting here, that a function $z \in V_0$ is regarded as the known solution of problem I or II (in Gockenbach and Khan (2007), the z is regarded as a solution of problem I). This choice depends on the availability of the boundary data on Γ_{DN}. However, there is no essential difference between these two cases. The task is to find a $\kappa \in K_{ad}$, that the corresponding solution $u_\kappa = L_I(f, \phi, \kappa)$ (or $v_\kappa = L_I(f, \tau, \kappa)$) gives the best approximation "in Ω" of $z \in V_0$ with $z = \phi$ on Γ_{DN} (or of $z \in V_0$). To formulate the inverse problem, several functionals are in use. We present in short three examples.

Output least-square approach: Referring to Gockenbach et al. (2008), Khan and Rouhani (2007) and to the references therein, we define the functional (if Dirichlet data on Γ_{DN} is available)

$$J_{ols}(f, \phi, \kappa, z) := \|L_I(f, \phi, \kappa) - z\|_\Omega. \tag{4.20}$$

Again, $\|\cdot\|_\Omega$ is a positive definite functional on V, e.g., a norm in V. The main drawback is that this functional is not convex.

Equation-error approach: Referring to Gockenbach et al. (2008) and to the references therein again, we define the functional (if Neumann data on Γ_{DN} is available)

$$J_{eea}(f, \tau, \kappa, z) := \|A_{II}(\kappa, z) - F_{II}(f, \tau)\|_\Omega, \tag{4.21}$$

with A_{II} and F_{II} given by (2.51) and (2.50), respectively. Now, $\|\cdot\|_\Omega$ is a positive definite functional on V_0^*, i.e., on a space of distributions. This functional is convex (in special situations), in Gockenbach et al. (2008), this approach was analyzed and applied to the 2d problem of an elastic membrane for determining the Lamé coefficients. We also refer to Geymonat and Pagano (2003) for the two-dimensional case.

Modified output least-square approach: If Dirichlet data on Γ_{DN} and a "measured" function $z_0 \in V_0$ are available, we define the functional

$$J^I_{mols}(f, \phi, \kappa, z) := a(\kappa, L_I(f, \phi, \kappa) - z, L_I(f, \phi, \kappa) - z). \tag{4.22}$$

This functional is convex. We refer to Khan and Rouhani (2007), Gockenbach and Khan (2007), Gockenbach and Khan (2008) for further references and detailed investigations for the case of an underlying scalar equation.

Analogously, assuming given Neumann data on Γ_{DN} and a "measured" function $w \in V_0$, we define the functional

$$J^{II}_{mols}(f, \tau, \kappa, w) := a(\kappa, L_{II}(f, \tau, \kappa) - w, L_{II}(f, \tau, \kappa) - w). \tag{4.23}$$

Comparing with full-field methods, in the approaches (4.1) and (4.2) only a second *calculated* solution is available but *not* a measured one. And, this calculated solution, say $v_\kappa = L_{II}(f, \tau, \kappa)$, clearly depends on the parameter κ. Although our approach in (4.1) (with $\alpha = 0$) arises formally from this one in (4.22), changing z by v_κ, one obtains an *essentially different* problem. As we will see in subsection **??**, the convexity of the functional J in (4.1) is not ensured. Thus, the methods applied in Khan and Rouhani (2007), Gockenbach and Khan (2007) cannot be simply transferred to our setting.

4.2.5 Remarks on approaches with constraints

The cost functional in (4.1) is formulated without constrains (side conditions). Again, we consider the case $\alpha = 0$. In an equivalent way, a formulation with side conditions and Lagrange multipliers can be chosen (cf. Ito and Kunisch (2008), e.g.). In doing so, in the functional $J_0^{(1)}$ in (4.1) instead of the solutions u_κ and v_κ there are admissible arbitrary $u, v \in V_0$ fulfilling certain suitable conditions. Thus, in our case we get the new functional Φ_{con} defined by

$$J_{con}(\kappa, u, v) := a(\kappa, u - v, u - v) \tag{4.24}$$

as well as the constraints

$$u = L_I(f, \phi, \kappa), \qquad v = L_{II}(f, \tau, \kappa). \tag{4.25}$$

Clearly, up to now one has an equivalence between the formulations without and with constraints. However, the side conditions in (4.25) contain explicitly the solution operators. This drawback can be avoided. We show this for the linear-elasticity problem (see subsection 3.3). Thus, we consider the functional

$$\widehat{J}_{con}(\kappa, \boldsymbol{u}, \boldsymbol{\sigma}) := \int_\Omega \big(\mathbb{E}(\kappa)\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\sigma}\big) : \big(\mathbb{E}(\kappa)\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\sigma}\big)\,\mathrm{d}x \tag{4.26}$$

with the constraints

$$\boldsymbol{u} \in V_{adm} := \{u \in V_0 \,|\, \boldsymbol{u} = \boldsymbol{\phi} \text{ on } \Gamma_{DN}\}, \tag{4.27}$$

$$\boldsymbol{\sigma} \in \Sigma_{adm} := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^6 \,|\, \boldsymbol{\sigma} = \boldsymbol{\sigma}^T\} \qquad \text{fulfilling} \tag{4.28}$$

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,\mathrm{d}x = \langle \boldsymbol{\tau} \,,\, \boldsymbol{\varphi}\rangle_{((W^{\frac{1}{2},2}(\Gamma_{DN}))^3)^*(W^{\frac{1}{2},2}(\Gamma_{DN}))^3} + \langle \boldsymbol{f} \,,\, \boldsymbol{\varphi}\rangle_{V_0^* V_0} \quad \forall \boldsymbol{\varphi} \in V_0. \tag{4.29}$$

Now, due to the mixed boundary conditions, an equivalence between the unconstraint form (4.1) and the constraint form (4.26) - (4.28) is only easily seen if the functionals have *zero* minimums. Contrary to the case $\Gamma_{DN} = \partial\Omega$ (full displacement vs. full traction, e.g. see Constantinescu (1995)), now, the field $\boldsymbol{u}$ is not restricted on Γ_N and the field $\boldsymbol{\sigma}$ is not restricted on Γ_D.

4.3 Existence of minimizers for $J_\alpha^{(i)}$

Now, we will present some first results concerning existence of minimizers for the cost functionals introduced in subsection 4.1. In the sequel, for convenience, we formulate detailed results for the prototypical problems given in (2.1) - (2.4) and (2.5) - (2.8), respectively, in the framework of weak solutions. As already pointed out (see corollary 3.3, too), there hold analogous results for *stationary linear elasticity.* Thus, we give only comments at some places, if needed.

4.3.1 Continuity of the functionals $J_\alpha^{(i)}$

Based on the continuity results concerning the solution operators in theorem 2.12 as well as on Lemma 2.17, one easily proves first basic results. For convenience, we deal ad once with the case of higher integrability of the gradients.

Theorem 4.3. (**Continuity of the functionals**) Let the assumptions of theorem 2.10 (cases I and II, respectively) as well as $f \in (V_0^{p'})^*$ be given. Moreover, let be given (4.7) and

$$\Phi^{(i)} \,:\, D(\Phi^{(i)}) \to \mathbb{R} \qquad\qquad \text{continuous w.r.t. } L^\infty \text{ topology.} \tag{4.30}$$

Then there hold for all $p \in [2, \overline{p}]$, $s \in [\frac{2p}{p-2}, \infty]$, for all $\alpha \geq 0$ and for $i = 1, 2$

$$\forall\, \kappa \in K_{ad} \quad : \quad J_\alpha^{(i)}(\kappa) \geq 0, \tag{4.31}$$

$$\forall\, \kappa \in D(\Phi^{(i)}) \subset K_{ad} \quad : \quad J_\alpha^{(i)}(\kappa) \in \mathbb{R}, \tag{4.32}$$

$$J_\alpha^{(i)} \,:\, D(\Phi^{(i)}) \subset K_{ad} \subset L^s(\Omega) \to \mathbb{R} \qquad \text{continuous w.r.t. } L^s \text{ topology,} \tag{4.33}$$

$$J_0^{(1)}(\kappa) = \big\langle \partial_{\boldsymbol{n}}(L_I(f, \phi, \kappa)) - \tau \,,\, \phi - v_{\kappa|\Gamma_{DN}} \big\rangle_{((W^{\frac{1}{p},p}(\Gamma_{DN}))^m)^*(W^{\frac{1}{p},p}(\Gamma_{DN}))^m}. \tag{4.34}$$

Proof. **(i)** The structure of $J_\alpha^{(i)}$ $(i = 1, 2)$ defined in (4.1) and (4.2) as well as the (4.3) and (4.4) immediately lead to (4.31) and (4.32).
(ii) Clearly, the continuity of the solution operators L_I and L_{II} from $L^s(\Omega)$ to V^p (see theroe 2.12 (ii)) and the structure of $J_\alpha^{(i)}$ $(i = 1, 2)$ as well as (4.7) and (4.30) yield the asserted continuity.
(iii) The relation (4.34) follows from (2.88). □

4.3.2 Existence of minimizers - I

Now, we will investigate the functionals $J_\alpha^{(i)}$ $(i = 1, 2)$ on the existence of minimizers. Due to (4.31), the functionals obviously have an *infimum* over all non-emty subsets of K_{ad} (including K_{ad}). However, the existence of a *minimum* (and therefore of a minimizer) is generally *not* ensured, even on closed and bounded subsets of K_{ad}. Note, that due to (2.12), (2.17) and (2.18) the admissible parameter set K_{ad} for our prototypical problems is bounded, closed and convex in $L^\infty(\Omega)$. For linear elasticity, due to (3.25), (3.28) - (3.30) the admissible parameter set K_{ad} is bounded and closed in $(L^\infty(\Omega)^{n_{ai}}$. However, in both cases, the set K_{ad} is *not* compact. Besides, in the case of elasticity, the convexity of K_{ad} has to be *additionally* assumed, if it is needed (see also subsection 3.3.5 concerning the isotropic case).

Summing up these findings, it is useful to restrict the domain of definition of the solutions operators L_I and L_{II} (see (2.56) - (2.59)). Thus, let be L_I and L_{II} restricted to a common new domain

$$\emptyset \neq D(L) \subseteq K_{ad}. \tag{4.35}$$

In other situations, it can be helpful to restrict the co-domain of the solutions operators, see Jin and Maass (2012b) (Chapt. 6).

Obviously, Weierstrass' theorem ensures the existence of a minimizer, if additionally compactness of $D(L)$ is assumed. More precisely, there hold the following assertion.

Theorem 4.4. (**Existence of minimizers under compactness assumption**) Under the assumptions of theorem 4.3 let be additionally given:

$$\emptyset \neq D := D(\Phi^{(i)}) \cap D(L) \qquad \text{compact.} \tag{4.36}$$

Then the functionals $J_\alpha^{(i)}$ $(i = 1, 2)$ have a minimizer $\kappa^\dagger \in D\,(\subset K_{ad})$, i.e.

$$J_\alpha^{(i)}(\kappa^\dagger) = \inf_{\kappa \in D} \{J_\alpha^{(i)}(\kappa)\}, \tag{4.37}$$

as well as a minimizing sequence (κ_m) in D, being convergent in $L^s(\Omega)$ for $1 \leq s < \infty$ to $\kappa^\dagger$. And, additionally, the corresponding sequences of images $L_I(\kappa_m)$ and $L_{II}(\kappa_m)$ converge to $L_I(\kappa^\dagger)$ and $L_{II}(\kappa^\dagger)$, respectively, in V^p for $p \in [2, \overline{p}]$.

There are some possibilities to chose a compact set $D(L)$, depending on concrete applications.

Examples 4.5. (**Suitable compact sets**)

(i) (**Piece-wise constant functions**) We consider a representation of the (bounded) domain Ω like in (3.39) in connection with polycrystalls, i.e., we assume

$$\overline{\Omega} = \bigcup_{l=1}^{N} \overline{\Omega}_l, \qquad \Omega_l \ \text{- Lipschitz domain,} \qquad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j. \tag{4.38}$$

Furthermore, we define the subset of (all most) piece-wise constant functions of K_{ad}:

$$D_{const}(L) := \{w \in K_{ad} \,|\, \forall\, j \in \{1, \ldots, N\} \,:\, w_{|\Omega_j} = \text{const.}\}. \tag{4.39}$$

Obviously, for any $N \in \mathbb{N}$ $D_{const}(L)$ is a compact subset of K_{ad}, and, if $D(\Phi^{(i)}) \cap D_{const}(L) \neq \emptyset$, theorem 4.4 is applicable.

(ii) (**Special triangulation of** Ω) If the domain Ω is polygonal, then the subdomains Ω_j in (4.38) can be chosen as simplexes. Additionally, in numerics, in particular in finite-elements theory, only an admissible decomposition (or conform triangulation) is allowed, see Knabner and Angermann (2000), e.g. Again, the set of (all most) piece-wise constant parameter functions $D_{const}(L)$ can be chosen as in (4.39).
Sometimes, for further investigations one considers a sequence of decreasing to zero regularization parameters α_m. Then it is possible to chose a corresponding sequence of nested conform triangulations $(\Omega_j^{(m)})$ $(j \in \{1, \ldots, N(m)\})$ with maximal diameters $d^{(m)} := \alpha_m$, tending also to zero.
Instead of piece-wise constant functions, one can take piece-wise linear or quadratic functions ("finite elements").

4.3.3 Existence of minimizers - II

Existence results can be also achieved under alternative assumptions in the framework of weak-sequential continuity and closedness, respectively. We present some general results, adopting arguments in Hofmann et al. (2007) and Jin and Maass (2012b) for analogous settings for slightly different functionals.

A first inspection shows that there are some differences between cost functionals $J_\alpha^{(1)}$ and $J_\alpha^{(2)}$, requiring different assumptions. Thus, we present the results consecutively.

Theorem 4.6. (**Existence of a minimizer for** $J_\alpha^{(2)}$) Let the assumptions of theorem 2.10 (cases I and II, respectively), $f \in (V_0^{p'})^*$ and (2.12), (2.17) and (2.18) for $K_{ad} \subset K$ be given. Moreover, let $\Phi^{(2)}$ fulfil (4.3), (4.4), and let (4.5), (4.6) be given. Moreover, we assume

$$\Phi^{(2)} \,:\, D(\Phi^{(2)}) \subset K_{ad} \to \mathbb{R} \qquad \text{weakly}^* \text{ sequentially lower semi-continuous,} \tag{4.40}$$

$$L_I, L_I \,:\, D(L) \subset K_{ad} \to V \qquad \text{weakly}^* \text{ sequentially closed,} \tag{4.41}$$

$$D := D(\Phi^{(2)}) \cap D(L) \neq \emptyset. \tag{4.42}$$

Then there exist a minimizer $\kappa^\dagger \in D\,(\subset K_{ad})$, i.e.

$$J_\alpha^{(2)}(\kappa^\dagger) = \inf_{\kappa\in D}\{J_\alpha^{(2)}(\kappa)\}, \tag{4.43}$$

as well as a minimizing sequence (κ_m) in D, being weakly* convergent in $L^\infty(\Omega)$ and weakly convergent in $L^s(\Omega)$ for $1 \le s < \infty$ to $\kappa^\dagger$.
And, additionally, the corresponding sequences of images $L_I(\kappa_m)$ and $L_{II}(\kappa_m)$ weakly converge to $L_I(\kappa^\dagger)$ and $L_{II}(\kappa^\dagger)$, respectively, in V^p for $p \in [2, \overline{p}]$.

Proof. **(i)** (**Existence of a minimizing sequence**) Due to (4.42) there exists an $\hat\kappa \in D(\subset K_{ad})$ with $J_\alpha^{(2)}(\hat\kappa) < +\infty$. Moreover, due to $J_\alpha^{(2)}(\kappa) \ge 0$ for all $\kappa \in D$ we get

$$0 \le d_0 := \inf_{\kappa\in D}\{J_\alpha^{(2)}(\kappa)\} \le J_\alpha^{(2)}(\hat\kappa) < +\infty. \tag{4.44}$$

Thus, there exists a minimizing sequence (κ_m) in D with

$$0 \le d_0 = \lim_{m\to\infty} J_\alpha^{(2)}(\kappa_m). \tag{4.45}$$

(ii) (**Choice of weakly convergent subsequences**) Due to (2.12), (2.17) and (2.18) the subset $K_{ad} \subset K$ is bounded, closed and convex. Hence, there exists a weakly*-convergent subsequence in $K_{ad} \subset L^\infty(\Omega)$, without any loss of generality also named by (κ_m), i.e., there exists $\tilde\kappa \in K_{ad}$ with

$$\tilde\kappa = \lim_{m\to\infty} \kappa_m \qquad\qquad \text{(weakly}^* \text{ convergence in } L^\infty(\Omega)). \tag{4.46}$$

Clearly, this subsequence is bounded $L^\infty(\Omega)$ (more precisely in K_{ad}) as well as in $L^s(\Omega)$ for any $1 \le s < \infty$. Due to (4.40) and (4.4) one has $\Phi^{(2)}(\tilde\kappa) \le \liminf_{m\to\infty} \Phi^{(2)}(\kappa_m) < +\infty$, therefore follows $\tilde\kappa \in D(\Phi^{(2)})$. Due to theorem 2.10, the sets of images of the solutions operators, $\{L_I(\kappa_m)\}$ and $\{L_{II}(\kappa_m)\}$, are bounded in $V^p(= W^{1,p}(\Omega))$. Therefore, one can choose a further subsequence of (κ_m), again named as before, such that the corresponding subsequences $(L_I(\kappa_m))$ and $(L_{II}(\kappa_m))$ are weakly convergent in V^p. Thus, there exist elements $\tilde u, \tilde v \in V^p$ being weak limits, i.e., there hold

$$\tilde u = \lim_{m\to\infty} L_I(\kappa_m), \qquad \tilde v = \lim_{m\to\infty} L_{II}(\kappa_m) \qquad \text{(weak convergence in)} V^p. \tag{4.47}$$

Clearly, this subsequence (κ_m) fulfills (4.46), too. Hence, the assumption (4.41) on weak sequential closedness yields $\tilde\kappa \in D$ as well as $\tilde u = L_I(\tilde\kappa)$ and $\tilde v = L_{II}(\tilde\kappa)$.
(iii) (**Limit process and existence of a minimizer**) The assumptions (4.40) together with (4.47) and the well-known weak sequential lower semi-continuity of a norm (see Edwards (1965), Yosida (1965), e.g.) yield

$$\begin{aligned} d_0 \le J_\alpha^{(2)}(\tilde\kappa) &= \|L_I(\tilde\kappa) - L_{II}(\tilde\kappa)\|^p + \alpha\Phi^{(2)}(\tilde\kappa) \le \\ &\le \liminf_{m\to\infty} \|L_I(\kappa_m) - L_{II}(\kappa_m)\|^p + \alpha \liminf_{m\to\infty} \Phi^{(2)}(\kappa_m) \le \liminf_{m\to\infty} J_\alpha^{(2)}(\kappa_m) = d_0. \end{aligned} \tag{4.48}$$

Thus $\kappa^\dagger := \tilde\kappa$ is a minimizer of $J_\alpha^{(2)}$, and (κ_m) is weakly* convergent in $L^\infty(\Omega)$ to $\kappa^\dagger$. Finally, a sequence being weakly* convergent in $L^\infty(\Omega)$ is also weakly convergent in $L^s(\Omega)$ for any $1 \le s < \infty$ due to the boundedness of Ω. □

Corollary 4.7. (**Existence of a minimizer in the case of linear elasticity**) An inspection of the proof shows, that theorem 4.6 remains valid under analogous assumptions concerning the stationary linear elasticity, if additionally the convexity of K_{ad} is assumed (see also corollary 3.3).

A typical difficulty consists in proving weakly* sequentially closedness of the solution operators. Note that their (strong) closedness or continuity does *not* imply the corresponding weak properties.

Moreover, a simple transfer of theorem 4.6 to the cost functional $J_\alpha^{(1)}$ is *not* possible, due to the special structure of the form a in (2.27), even in the case of linear parameter dependence like in (2.19). The form $a(\kappa, u, u)$ is only a norm on V for a *fixed* $\kappa \in K_{ad}$. Thus, under analogous assumptions for the cost functional $J_\alpha^{(1)}$, either the convergence of (κ_m) or the convergence of $L_I(\kappa_m)$ and $L_{II}(\kappa_m)$ must be better. Thus, theorem 4.6 becomes valid for the functional $J_\alpha^{(1)}$, if the solutions operators exhibits a type of reinforced continuity which maps weakly* sequentially convergent sequences in $L^\infty(\Omega)$ into convergent ones in V^p. Of course, such a property is difficult to prove.

A way out consists in choosing a *compact* subset $D \subsetneq K_{ad}$ like in the examples 4.5. In this case, the (strong) continuity of the solution operators ensures their weak* sequential closedness (and, moreover, their reinforced continuity), and theorem 4.6 becomes valid for both functionals $J_\alpha^{(i)}$ $(i = 1, 2)$.

4.4 Some results concerning the functionals $J_\alpha^{(i)}$ in a special case

We present some results connected with the Fréchet differentiability of the special cost functional $J_0^{(1)}$. For convenience, we *only* deal with the case of linear parameter dependence like in (2.19).

As before, the prototypical problems given in (2.1) - (2.4) and (2.5) - (2.8), respectively, are in the focus. Analogous results for stationary linear elasticity will be only commented at some places.

4.4.1 Fréchet differentiability of the cost functional $J_0^{(1)}$

We want to investigate the Fréchet differentiability of the special cost functional $J_0^{(1)}$ (see (4.1)). For convenience, we only deal with the case of linear parameter dependence like in (2.19). Clearly, under additional conditions on the function b, the general case can be investigated (cf. subsection 2.7.2).

Furthermore, in the case of linear parameter dependence, the subsequent results can be transferred in full analogy to the case of linear elasticity.

Due to the trilinearity of the form a and the differentiability results for the solution operators L_I and L_{II} in subsection 2.7, the Fréchet differentiability of the cost functional $J_0^{(1)}$ can be easily proved. As above, the dependence of $J_0^{(1)}$ on the fixed given data f, ϕ and τ is not explicitly mentioned.

Theorem 4.8. Let the assumptions of theorem 2.20 (cases I and II, respectively) as well as $f \in (V_0^{p'})^*$ be given. Let the form a be given as in (2.27) with b in accordance with (2.19). Then the cost functional $J_0^{(1)} : K_{ad} \subset L^s(\Omega) \to \mathbb{R}$ defined in (4.1) is twice continuously Fréchet differentiable on K_{ad} for all $2 \leq p < \min\{\overline{p}, 4\}$ and $s \in [{}^{2p}/{}_{(p-2)}, \infty]$. Moreover, there hold:

(i) The Fréchet differential is given for all $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$ by

$$(4.49)\quad \begin{aligned} \frac{\partial J_0^{(1)}}{\partial \kappa}(\kappa_0)\kappa &= a(\kappa, u_{\kappa_0} - v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + 2a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa - \frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa, u_{\kappa_0} - v_{\kappa_0}) = \\ &= -a(\kappa, u_{\kappa_0} - v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + 2\Big\langle \partial_{\boldsymbol{n}}(\frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa)\, ,\, \phi - v_{\kappa_0}\Big\rangle_{((W^{\frac{1}{p},p}(\Gamma_{DN}))^m)^*(W^{\frac{1}{p},p}(\Gamma_{DN}))^m}, \end{aligned}$$

$$(4.50)\quad \frac{\partial J_0^{(1)}}{\partial \kappa}(\kappa_0)\kappa = a(\kappa, u_{\kappa_0}, u_{\kappa_0}) - a(\kappa, v_{\kappa_0}, v_{\kappa_0}).$$

(ii) The (special) second Fréchet differential is given for all $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$ by

$$(4.51)\quad \begin{aligned} \frac{\partial^2 J_0^{(1)}}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa) =& a(\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa - \frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa - \frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa) + \\ &+ 2\Big\langle \partial_{\boldsymbol{n}}(\frac{\partial^2 L_I}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa))\, ,\, \phi - v_{\kappa_0}\Big\rangle_{((W^{\frac{1}{p},p}(\Gamma_{DN}))^m)^*(W^{\frac{1}{p},p}(\Gamma_{DN}))^m}, \end{aligned}$$

$$(4.52)\quad \frac{\partial^2 J_0^{(1)}}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa) = 2a(\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, u_{\kappa_0}) - 2a(\kappa, \frac{\partial L_{II}}{\partial \kappa}(\kappa_0)\kappa, v_{\kappa_0}).$$

Proof. The assertions (4.49) and (4.51) follow after simple calculations, repeating arguments from the proofs of theorems 2.20 and 2.25 using the trilinearity and symmetry of a. The relation (4.50) (after that in an easy manner (4.52)) can be proved, using some tricks and re-arrangements.

$$(4.53)\quad \begin{aligned} J_0^{(1)}(\kappa_0+\kappa) - J_0^{(1)}(\kappa_0) &= a(\kappa_0+\kappa, u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa}, u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa}) - a(\kappa_0, u_{\kappa_0} - v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) = \\ &= a(\kappa_0+\kappa, u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa}, u_{\kappa_0+\kappa} - u_{\kappa_0} - v_{\kappa_0+\kappa} + v_{\kappa_0}) + \\ &\quad + a(\kappa_0+\kappa, u_{\kappa_0+\kappa} - u_{\kappa_0} - v_{\kappa_0+\kappa} + v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + a(\kappa, u_{\kappa_0} - v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) = \\ &\quad = a(\kappa_0+\kappa, u_{\kappa_0+\kappa} - u_{\kappa_0}, u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa} + u_{\kappa_0} - v_{\kappa_0}) + \\ &\quad - a(\kappa_0+\kappa, v_{\kappa_0+\kappa} - v_{\kappa_0}, u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa} + u_{\kappa_0} - v_{\kappa_0}) + a(\kappa, u_{\kappa_0} - v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}). \end{aligned}$$

Exploiting the fact that $u_{\kappa_0+\kappa} - u_{\kappa_0} \in V_1$ is an admissible test function for the Dirichlet problem and that $u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa} + u_{\kappa_0} - v_{\kappa_0} \in V_0$ is an admissible test function for the Neumann problem, one obtains:

$$(4.54)\quad \begin{aligned} J_0^{(1)}(\kappa_0+\kappa) - J_0^{(1)}(\kappa_0) =& a(\kappa, u_{\kappa_0+\kappa} - u_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + a(\kappa, u_{\kappa_0} - v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + \\ &+ a(\kappa, v_{\kappa_0}, u_{\kappa_0+\kappa} - v_{\kappa_0+\kappa} + u_{\kappa_0} - v_{\kappa_0}) = \\ &= a(\kappa, u_{\kappa_0} + v_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + 2a(\kappa, u_{\kappa_0+\kappa} - u_{\kappa_0}, u_{\kappa_0} - v_{\kappa_0}) + \\ &+ a(\kappa, u_{\kappa_0} - v_{\kappa_0}, v_{\kappa_0+\kappa} - v_{\kappa_0}) + a(\kappa, u_{\kappa_0}, u_{\kappa_0+\kappa} - u_{\kappa_0} - v_{\kappa_0+\kappa} + v_{\kappa_0}). \end{aligned}$$

Due to (2.75), the last three terms are $o(\kappa)$ with respect the L^s norm. The first term is linear in κ and represents the Fréchet differential. Hence, one gets (4.11), and, after that (4.13). $\square$

Remarks 4.9. (i) Using the relations (2.102) and (2.105) as well as (2.155) and (2.158) for the first and second differentials of the solution operators L_I and L_{II}, the first two Fréchet differentials of the cost functional $J_0^{(1)}$ can be written, using merely the solution operators and not their Fréchet differentials.

(ii) Higher Fréchet differentiability can be proved in an analogous manner, yielding more complex formulas.

4.4.2 Comparison of $J_0^{(1)}$ with the full-field approach

We continue to deal with the cost functional $J_0^{(1)}$ defined in (4.1). In order to demonstrate the difference between Knowles' approach and the *seemingly similar* full-filled approaches in the case of the modified output least-square approach we come back to the cost functionals in (4.22) and (4.23) (see subsection 4.2.4):

$$\Phi^I_{mols}(f,\phi,\kappa,z) := a(\kappa, L_I(f,\phi,\kappa)-z, L_I(f,\phi,\kappa)-z) = a(\kappa, u_\kappa - z, u_\kappa - z), \tag{4.55}$$

$$\Phi^{II}_{mols}(f,\tau,\kappa,w) := a(\kappa, L_{II}(f,\tau,\kappa)-w, L_{II}(f,\tau,\kappa)-w) = a(\kappa, v_\kappa - w, v_\kappa - w), \tag{4.56}$$

with $z \in V_0$ and $z = \phi$ on Γ_{DN} as well as $w \in V_0$ being a "measured" solutions of problem I and II, respectively, for the same fixed data f, ϕ and f, τ, respectively. Therefore, $u_\kappa - z \in V_1$ and $v_\kappa - w \in V_0$ are admissible test functions for problem I and II, respectively. Similarly as in subsection 4.4, we can prove Fréchet differentiability of Φ^I_{mols} and Φ^{II}_{mols}, with respect to the L^s norm in the case of a better z and w, respectively. The following results generalize corresponding ones in Gockenbach and Khan (2007) for a scalar case corresponding to problem I with the parameter space $\mathrm{L}^\infty(\Omega)$. Moreover, we obtain more convenient formulas as in the case of the cost functional $J_0^{(1)}$ in (4.1) (with $\alpha = 0$).

At first we formulate a corresponding result to theorem 4.3. The data f, ϕ and f, τ are regarded as fixed. Again, it is evident, that the subsequent assertions can be simply transferred to linear elasticity.

Theorem 4.10. Let the assumptions of theorem 2.10 (cases I and II) as well as $z, w \in V_0$ with $z = \phi$ on Γ_{DN} and $\nabla z, \nabla w \in (L^{\overline{p}}(\Omega))^d$ be given. Then there hold for all $p \in [2, \overline{p}]$, $s \in [\frac{2p}{p-2}, \infty]$

$$\forall\, \kappa \in K_{ad} \quad : \quad \Phi^I_{mols}(\kappa) \geq 0, \qquad \Phi^I_{mols}(\kappa) \geq 0, \tag{4.57}$$

$$\Phi^I_{mols}, \Phi^{II}_{mols} \,:\, K_{ad} \subset L^s(\Omega) \to \mathbb{R} \qquad \text{continuous w.r.t } L^s \text{ topology}, \tag{4.58}$$

$$\Phi^I_{mols}(\kappa) = \langle f\,,\, u_\kappa - z\rangle_{(V_1^{p'})^*, V_1^{p'}} - a(\kappa, z, u_\kappa - z), \tag{4.59}$$

$$\Phi^{II}_{mols}(\kappa) = \langle \tau\,,\, v_\kappa - w\rangle_{W^{\frac{1}{p},p}(\Gamma_{DN})^* W^{\frac{1}{p},p}(\Gamma_{DN})} + \langle f\,,\, v_\kappa - w\rangle_{(V_0^{p'})^*, V_0^{p'}} - a(\kappa, w, v_\kappa - w). \tag{4.60}$$

The following theorem holds in full analogy for both functionals Φ^I_{mols} and Φ^{II}_{mols}. Thus, we do not repeat the assertions for Φ^{II}_{mols}.

Theorem 4.11. Let the assumptions of theorem 2.20 (case I) be given. Let be $z \in V_0$ with $z = \phi$ on Γ_{DN} and $\nabla z \in (L^{\overline{p}}(\Omega))^d$. Then the cost functional $\Phi_{mols} \,:\, K_{ad} \subset L^s(\Omega) \to \mathbb{R}$ defined in (4.55) is twice continuously Fréchet differentiable on K_{ad} for all $2 \leq p < \min\{\overline{p}, 4\}$ and $s \in [2p/(p-2), \infty]$.

(i) The Fréchet differential is given for all $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$ by

$$\frac{\partial \Phi_{mols}}{\partial \kappa}(\kappa_0)\kappa = -a(\kappa, u_{\kappa_0} + z, u_{\kappa_0} - z). \tag{4.61}$$

(ii) The (special) second Fréchet differential is given for all $\kappa_0 \in K_{ad}$ and $\kappa \in L_{\kappa_0}$ by

$$\frac{\partial^2 \Phi_{mols}}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa) = 2a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa) \geq \alpha_0 \Big\| \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa \Big\|_V^2. \tag{4.62}$$

(iii) If $K_{ad} \subset L^\infty(\Omega)$ is convex, then $\Phi_{mols} \,:\, K_{ad} \subset L^\infty(\Omega) \to \mathbb{R}$ is a convex functional.

Proof. Here, we only derive the formulas. The continuity assertions with respect to the arising Banach spaces are very similar to the corresponding assertions in theorem 4.8.
(I) Taking $u_\kappa - z \in V_1$ into account, differentiating of Φ_{mols} and using the relation (2.101) yields

$$\begin{aligned}(4.63)\qquad \frac{\partial \Phi_{mols}}{\partial \kappa}(\kappa_0)(\kappa) &= a(\kappa, u_{\kappa_0} - z, u_{\kappa_0} - z) + 2a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, u_{\kappa_0} - z) = \\ &= a(\kappa, u_{\kappa_0} - z, u_{\kappa_0} - z) - 2a(\kappa, u_{\kappa_0}, u_{\kappa_0} - z) = -a(\kappa, u_{\kappa_0} + z, u_{\kappa_0} - z).\end{aligned}$$

(II) Starting with the last relation and differentiating again, one obtains

$$\begin{aligned}(4.64)\qquad \frac{\partial^2 \Phi_{mols}}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa) &= -a(\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, u_{\kappa_0} - z) - a(\kappa, u_{\kappa_0} + z, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa) = \\ &= -2(\kappa, u_{\kappa_0}, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa) = 2a(\kappa_0, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa, \frac{\partial L_I}{\partial \kappa}(\kappa_0)\kappa).\end{aligned}$$

Finally, the assumption (2.30) gives the estimate in (4.62)
(III) If $K_{ad} \subset L^\infty(\Omega)$ is convex, the convexity of Φ_{mols} follows from (4.62) by standard arguments. □

The structure of the $\frac{\partial^2 J_0^{(1)}}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa)$ in (4.51) *essentially differs* from $\frac{\partial^2 \Phi^I_{mols}}{\partial \kappa^2}(\kappa_0)(\kappa,\kappa)$ in (4.62). Thus, the functional Φ^I_{mols} is convex (as a functional on $K_{ad} \subset L^\infty(\Omega)$ for a convex K_{ad}).

However, it remains unclear whether from the representation (4.51) the convexity of $J_0^{(1)}$ can be derived. And, as a consequence, the methods and results connected with the modified output least-square approach *cannot* be simply adopted to the Knowles' approach which compares *two calculated* solutions within the domain Ω.

Based on lemma 2.24 we obtain the following corollary. Again, only the functional Φ^I_{mols} will be considered.

Corollary 4.12. Under the assumptions of theorem 2.20 (cases I), $z \in V_0$ with $z = \phi$ on Γ_{DN}, (2.32) and $f \neq 0$ there holds

$$(4.65)\qquad \forall\, \kappa_0 \in \overset{\circ}{K}_{ad} \quad : \quad \frac{\partial^2 \Phi_{mols}}{\partial \kappa^2}(\kappa_0) \neq 0.$$

In particular, there holds

$$(4.66)\qquad \forall\, \kappa_0 \in \overset{\circ}{K}_{ad} \quad : \quad \frac{\partial^2 \Phi_{mols}}{\partial \kappa^2}(\kappa_0)(\kappa_0,\kappa_0) \neq 0.$$

Finally, due to (4.62) and (2.150), in inner points of K_{ad}, the cost functional Φ_{mols} is strictly convex.

4.5 Necessary and sufficient conditions for a local minimum of cost functionals in terms of Fréchet differentials

General necessary and sufficient conditions for the existence of a local minimum follow from general results for real functions on Banach spaces (see for instance Showalter (1997)). However, the difficulty consists in verifying these conditions in concrete applications, in particular in infinitely-dimensional spaces. Nevertheless, we provide them here in short. We only formulate a result for the cost functional $J_\alpha^{(1)}$. Clearly, under Fréchet differentiability it holds for $J_\alpha^{(2)}$, too.

Theorem 4.13. (**Necessary condition for a local minimum**) Let the assumptions of theorem 2.10 and (2.96) - (2.100) be given. Let the cost functional $J_\alpha^{(1)}$ given by (4.1), (4.3), (4.4), (4.7). Moreover, let be

$$(4.67)\qquad J_\alpha^{(1)} \,:\, D(J_\alpha^{(1)} \to [0,\infty[\qquad \text{Fréchet differentiable.}$$

If $J_\alpha^{(1)}$ has a local minimum in $\kappa_0 \in K_{ad}$, then there holds

$$(4.68)\qquad \forall\, \kappa \in K_{ad} \quad : \quad \frac{\partial J_\alpha^{(1)}}{\partial \kappa}(\kappa_0)(\kappa - \kappa_0) \geq 0.$$

Remarks 4.14. (i) This general result also remains valid under the weaker assumption of Gâteaux differentiability (see Showalter (1997), Zeidler (1986), e.g.).

(ii) For inner points, i.e., for $\kappa_0 \in \overset{\circ}{K}_{ad}$ (4.68) renders the necessary condition in form of an equation:

$$(4.69) \qquad \forall\, \kappa \in K \qquad : \qquad \frac{\partial J_\alpha^{(1)}}{\partial \kappa}(\kappa_0)\kappa = 0.$$

By means of the first and second Fréchet differentials a sufficient criterion for a local minimum can be formulated.

Theorem 4.15. (**Sufficient condition for a local minimum**) Under the assumptions of theorem 4.13 let the functional $J_\alpha^{(1)}$ be twice Fréchet differentiable. Than $J_\alpha^{(1)}$ has a local minimum in $\kappa_0 \in \overset{\circ}{K}_{ad}$, if, besides (4.68) the following condition holds.

$$(4.70) \qquad \exists\, c_0 > 0 \qquad \forall\, \kappa \in K \qquad : \qquad \frac{\partial^2 J_\alpha^{(1)}}{\partial \kappa^2}(\kappa_0)(\kappa, \kappa) \geq c_0 \, \|\kappa\|^2_{L^\infty(\Omega)}.$$

The representations in (4.51) and (4.52) of ${}^{\partial^2 J_\alpha^{(1)}}/{}_{\partial \kappa^2}(\kappa_0)(\kappa, \kappa)$ for the case of linear parameter dependence (2.19) do *not* show how to apply practically this sufficient condition in a convenient way.

5 Outlook

In this study, we have dealt with special inverse problems arising from boundary-value problems for linear second-order equations with a parameter. Applications in stationary linear elasticity and in electrical impedance tomography, e.g. may lead to such problems. We have given a sound basis in the framework of weak solution theory under mild assumptions.

In the focus are two prototypical problems (2.1) - (2.4) and (2.5) - (2.8) only differing in the boundary condition at some part of the boundary. Hence, each prototypical problem has a uniquely determined (or "calculated") solution. The comparison of these solutions within the domain may be the basis of inverse problems for identification of the parameter (see subsection 4.1).

The properties of the solution operators have been studied in detail (continuity, higher integrability of the gradient of weak solutions, Fréchet differentiability). At many places we have followed Jin and Maaß (2012a), Jin et al. (2012) with modifications and extensions.

Special items in this study are:

(i) non-linear parameter dependence within the PDE,

(ii) mixed boundary conditions,

(iii) use of higher integrability of the gradient of weak solutions up to the boundary.

Analogously as Jin and Maaß (2012a), Jin et al. (2012), the Lipschitz continuity of the Fréchet derivatives of the solution operators w.r.t. some L^s topology for $s < \infty$ has been established. This is essential for further investigation concerning convergence rates of regularization methods, e.g. This is not the topic in this study, we refer to Jin and Maaß (2012a), Hofmann et al. (2007) and for the references cited therein.

Furthermore, we have introduced some cost functionals comparing the two calculated solutions within the domain. This is based on approaches due to Knowles (1998) as well as to Yun and Shang (2011).

Detailed studies on existence of minimizers, convergence rates of regularization methods and applications, to problems in elasticity, e.g., remain for future work.

References

Abdel-Karim, M. (2005). Numerical integration method for kinematic hardening rules with partial activation of dynamic recovery term, *Int. J. of Plast.* **21**: 1303–1321.

Adams, R. A. and Fournier, J. J. F. (2003). *Sobolev Spaces*, Vol. 140 of *Pure and Applied Mathematics*, 2nd edn, Academic Press, Elsevier Science Ltd.

Alder, H., Hömberg, D. and Weiss, W. (2006). Simulationsbasierte Regelung der Laserhärtung von Stahl, *HTM Z. Werkst. Waermebeh. Fertigung* **61**(2): 103 – 108.

Amann, H. (1993). Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, *Function spaces, differential operators and nonlinear analysis*, Springer, pp. 9–126.

Amann, H. and Escher, J. (2006). *Analysis II. Grundstudium Mathematik. Zweite Auflage*, Birkhäuser.

Andrieux, S., Baranger, T. and Abda, A. B. (2006). Solving cauchy problems by minimizing an energy-like functional, *Inverse problems* **22**(1): 115.

Andrieux, S. and Baranger, T. N. (2008). An energy error-based method for the resolution of the cauchy problem in 3d linear elasticity, *Computer Methods in Applied Mechanics and Engineering* **197**(9): 902–920.

Avril, S., Bonnet, M., Bretelle, A.-S., Grédiac, M., Hild, F., Ienny, P., Latourte, F., Lemosse, D., Pagano, S., Pagnacco, E. and Pierron, F. (2008). Overview of identification methods of mechanical parameters based on full-field measurements, *Experimental Mechanics* **48**: 381 – 402.

Avril, S. and Pierron, F. (2007). General framework for the identification of constitutive parameters from full-field measurements in linear elasticity, *International Journal of Solids and Structures* **44**(14): 4978–5002.

Avril, S., Pierron, F., Pannier, Y. and Rotinat, R. (2008a). Stress reconstruction and constitutive parameter identification in plane-stress elasto-plastic problems using surface measurements of deformation fields, *Experimental Mechanics* **48**: 403 – 419.

Banerjee, B., Walsh, T. F., Aquino, W. and Bonnet, M. (2013). Large scale parameter estimation problems in frequency-domain elastodynamics using an error in constitutive equation functional, *Computer methods in applied mechanics and engineering* **253**: 60–72.

Banks, H. and Kunisch, K. (1989). *Estimation Techniques for Distributed Parameter Systems*, Birkhäuser, Boston.

Bard, Y. (1974). *Nonlinear Parameter Estimation*, Academic Press, New York.

Bari, S. and Hassan, T. (2000). Anatomy of coupled constitutive models for ratcheting simulation, *Int. J. of Plast.* **16**: 381–409.

Bensoussan, A. and Frehse, J. (2013). *Regularity results for nonlinear elliptic systems and applications*, Vol. 151, Springer Science & Business Media.

Bertram, A. and Glüge, R. (2015). *Festkörpermechanik (überarbeitete Ausgabe)*, Otto-von-Guericke-Universität Magdeburg.

Bonnet, M. and Constantinescu, A. (2005). Inverse problems in elasticity, *Inverse Problems* **21**: R1 – R50.

Brown, R. M. and Mitrea, I. (2009). The mixed problem for the lamé system in a class of lipschitz domains, *Journal of Differential Equations* **246**(7): 2577–2589.

Bui, H. and Tanaka, M. (eds) (1994). *Inverse Problems in Engineering Mechanics*, Balkema, Rotterdam.

Ciarlet, P. G. (1988). *Mathematical Elasticity*, Vol. 1 of *Three Dimensional Elasticity*, Elsevier Science Publishers Ltd.

Constantinescu, A. (1995). On the identification of elastic moduli from displacement-force boundary measurements, *Inverse Problems in Engineering* **1**(4): 293–313.

Cooreman, S., Lecompte, D., Sol, H., Vantomme, J. and Debruyne, D. (2007). Elasto-plastic material parameter identification by inverse methods: Calculation of the sensitivity matrix, *International Journal of Solids and Structures* **44**: 4329 – 4341.

Dahlberg, B., Kenig, C., Verchota, G. et al. (1988). Boundary value problems for the systems of elastostatics in lipschitz domains, *Duke Math. J* **57**(3): 795–818.

Dahlke, S., Novak, E. and Sickel, W. (2010). Optimal approximation of elliptic problems by linear and nonlinear mappings iv: Errors in l2 and other norms, *Journal of Complexity* **26**(1): 102–124.

Daubechies, I., Defrise, M. and De Mol, C. (2004). An iterative thresholding algorithm for linear inverse problems with a sparsity constraint, *Communications on pure and applied mathematics* **57**(11): 1413–1457.

Djimli, L., Taleb, L. and Meziani, S. (2010). The role of the experimental data base used to identify material parameters in predicting the cyclic plastic response of an austenitic steel, *Int. J. of Pressure Vessels and Piping* **87**: 177 – 186.

Edwards, R. E. (1965). *Functional Analysis: Theory and Applications*, Holt, Rinehart and Winston, New York.

Engl, H. W., Hanke, M. and Neubauer, A. (1996). *Regularization of inverse problems*, Vol. 375, Kluwer Academic Pub.

Feissel, P. and Allix, O. (2007). Modified constitutive relation error identification strategy for transient dynamics with corrupted data: the elastic case, *Computer Methods in Applied Mechanics and Engineering* **196**(13): 1968–1983.

Fiaschi, A., Knees, D. and Reichelt, S. (2013). Global higher integrability of minimizers of variational problems with mixed boundary conditions, *Journal of Mathematical Analysis and Applications* **401**(1): 269–288.

Frerichs, F., Sander, S., Lübben, T., Schüttenberg, S. and Fritsching, U. (2014). Determination of heat transfer coefficients in high speed quenching processes, *Materials Performance and Characterization* **3**(4): 163–183.

Geymonat, G. and Pagano, S. (2003). Identification of mechanical properties by displacement field measurement: a variational approach, *Meccanica* **38**(5): 535–545.

Giaquinta, M. (2016). *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.(AM-105)*, Vol. 105, Princeton University Press.

Gockenbach, M., Jadamba, B. and Khan, A. (2008). Equation error approach for elliptic inverse problems with an application to the identification of lamé parameters, *Inverse Problems in Science and Engineering* **16**(3): 349–367.

Gockenbach, M. S. and Khan, A. A. (2005). Identification of lamé parameters in linear elasticity: a fixed point approach, *Journal of Industrial and Management Optimization* **1**(4): 487.

Gockenbach, M. S. and Khan, A. A. (2007). An abstract framework for elliptic inverse problems: Part 1. an output least-squares approach, *Mathematics and Mechanics of Solids* **12**(3): 259–276.

Gockenbach, M. S. and Khan, A. A. (2008). An abstract framework for elliptic inverse problems: Part 2. an augmented lagrangian approach, *Mathematics and Mechanics of Solids* .

Grédiac, M. and Pierron, F. (2006). Applying the virtual fields method to the identification of elasto-plastic constitutive parameters, *International Journal of Plasticity* **22**(4): 602–627.

Gröger, K. (1989). A $w^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations, *Math. Ann.* **283**: 679 – 687.

Gröger, K. and Rehberg, J. (1989). Resolvent estimates inw- 1, p for second order elliptic differential operators in case of mixed boundary conditions, *Mathematische Annalen* **285**(1): 105–113.

Haller-Dintelmann, R. and Rehberg, J. (2008). Maximal parabolic regularity for divergence operators including mixed boundary conditions, *WIAS Preprint 1288*, WIAS Berlin.

Haupt, P. (2002). *Continuum Mechanics and Theory of Materials, 2nd Ed.*, Springer-Verlag.

Herzog, R., Meyer, C. and Wachsmuth, G. (2011). Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions, *Journal of Mathematical Analysis and Applications* **382**(2): 802–813.

Herzog, R., Meyer, C. and Wachsmuth, G. (2012). C-stationarity for optimal control of static plasticity with linear kinematic hardening, *SIAM Journal on Control and Optimization* **50**(5): 3052–3082.

Herzog, R., Meyer, C. and Wachsmuth, G. (2013). B-and strong stationarity for optimal control of static plasticity with hardening, *SIAM Journal on Optimization* **23**(1): 321–352.

Hofmann, B., Kaltenbacher, B., Poeschl, C. and Scherzer, O. (2007). A convergence rates result for tikhonov regularization in banach spaces with non-smooth operators, *Inverse Problems* **23**(3): 987–1010.

Hömberg, D. and Weiss, W. (2006). PID control of laser surface hardening of steel, *IEEE Transactions on Control Systems Technology* **14**(5): 896 – 904.

Imanuvilov, O. Y., Uhlmann, G. and Yamamoto, M. (2012). On uniqueness of lamé coefficients from partial cauchy data in three dimensions, *Inverse Problems* **28**(12): 5002.

Isakov, V. (2006). *Inverse problems for partial differential equations*, Vol. 127, Springer.

Ito, K. and Kunisch, K. (2008). *Lagrange multiplier approach to variational problems and applications*, Vol. 15, Society for Industrial and Applied Mathematics.

Jin, B., Khan, T. and Maass, P. (2012). A reconstruction algorithm for electrical impedance tomography based on sparsity regularization, *International Journal for Numerical Methods in Engineering* **89**(3): 337–353.

Jin, B., Khan, T., Maass, P. and Pidcock, M. (2011). Function spaces and optimal currents in impedance tomography, *Journal of Inverse and Ill-Posed Problems* **19**(1): 25–48.

Jin, B. and Maaß, P. (2012a). An analysis of electrical impedance tomography with applications to tikhonov regularization., *ESAIM Control Optim. Calc. Var.* **18**: 1027–1048.

Jin, B. and Maass, P. (2012b). Sparsity regularization for parameter identification problems, *Inverse Problems* **28**(12): 1–70.

Kajberg, J., Sundin, K. G., Melin, L. G. and Ståhle, P. (2004). High strain-rate tensile testing and viscoplastic parameter identification using microscopic high-speed photography, *International Journal of Plasticity* **20**: 561 – 575.

Kajberg, J. and Wikman, B. (2007). Viscoplastic parameter estimation by high strain-rate experiments and inverse modelling - speckle measurements and high-speed photography, *International Journal of Solids and Structures* **44**: 145–164.

Kern, D. (2011). *Analysis and numerics for a thermomechanical phase transition model in steel*, PhD thesis, TU Berlin.

Khan, A. and Rouhani, B. (2007). Iterative regularization for elliptic inverse problems, *Computers & Mathematics with Applications* **54**(6): 850–860.

Knabner, P. and Angermann, L. (2000). *Numerik partieller Differentialgleichungen*, Springer.

Knowles, I. (1998). A variational algorithm for electrical impedance tomography, *Inverse Problems* **14**(6): 1513.

Kohn, R. V. and McKenney, A. (1990). Numerical implementation of a variational method for electrical impedance tomography, *Inverse Problems* **6**(3): 389.

Kohn, R. V. and Vogelius, M. (1987). Relaxation of a variational method for impedance computed tomography, *Communications on Pure and Applied Mathematics* **40**(6): 745–777.

Kufner, A., John, O. and Fucik, S. (1977). *Function spaces*, Academia, Prague.

Laurin, F., Charrier, J.-S., Lévêque, D., Maire, J.-F., Mavel, A. and Nuñez, P. (2012). Determination of the properties of composite materials thanks to digital image correlation measurements, *Procedia IUTAM* **4**: 106–115.

Le, T. M., Fatahi, B. and Khabbaz, H. (2015). Numerical optimisation to obtain elastic viscoplastic model parameters for soft clay, *International Journal of Plasticity* **65**: 1–21.

Lecompte, D., Smits, A., Sol, H., Vantomme, J. and Van Hemelrijck, D. (2007). Mixed numerical–experimental technique for orthotropic parameter identification using biaxial tensile tests on cruciform specimens, *International Journal of Solids and Structures* **44**(5): 1643–1656.

Liščić, B. (2009). Heat transfer control during quenching, *Materials and Manufacturing Processes* **24**(7-8): 879–886.

Liščić, B., Singer, S. and Beitz, H. (2011). Dependence of heat transfer coefficient at quenching on diameter of cylindrical workpieces, *International Heat Treatment & Surface Engineering* **5**(3): 101–105.

Lukaschewitsch, M., Maass, P., Pidcock, M. and Sebu, C. (2009). The asymptotic behaviour of weak solutions to the forward problem of electrical impedance tomography on unbounded three-dimensional domains, *Mathematical Methods in the Applied Sciences* **32**(2): 206–222.

Mahnken, R. (2004). *Encyclopedia of Computational Mechanics*, Vol. 2, John Wiley & Sons, Ltd.

Mahnken, R. and Dammann, C. (2013). Simulation of strain induced anisotropy for polymers with weighting functions, *submitted to Archive of Applied Mechanics* .

Mahnken, R. and Stein, E. (1996a). Parameter identification for viscoplastic models based on analytical derivatives of a least-square functional and stability investigations, *Int. J. of Plast.* **12**: 451 – 479.

Mahnken, R. and Stein, E. (1996b). A unified approach for parameter identification of inelastic material models in the frame of the finite element method, *Comput. Methods Appl. Mech. Engrg.* **136**: 225–258.

Meidner, D. (2008). *Adaptive Space-Time Finite Element Methods for Optimization Problems Governed by Nonlinear Parabolic Systems*, PhD thesis, Ruprecht-Karls-Universität Heidelberg, Germany.

Naumann, J. and Wolff, M. (1991). On a global lq-gradient estimate on weak solutions of nonlinear elliptic systems, *University of Catania (Preprint 1991)* .

Naumann, J. and Wolff, M. (1995). A global Lp-gradient estimate on weak solutions of nonlinear parabolic systems under mixed boundary conditions, Preprint der Humboldt Universität Berlin Nr. 952.

Noman, M., Clausmeyer, T., Barthel, C., Svendsen, B., Huétink, J. and van Riel, M. (2010). Experimental characterization and modeling of the hardening behavior of the sheet steel lh800, *Materials Science and Engineering: A* **527**(10): 2515–2526.

Ott, K. A. and Brown, R. M. (2013). The mixed problem for the lamé system in two dimensions, *Journal of Differential Equations* **254**(12): 4373–4400.

Pham, Q. M. (2015). Reconstructing conductivity coefficients based on sparsity regularization and measured data in electrical impedance tomography, *Inverse Problems in Science and Engineering* **23**(8): 1366–1387.

Pierron, F., Avril, S. et al. (2010). Extension of the virtual fields method to elasto-plastic material identification with cyclic loads and kinematic hardening, *International Journal of Solids and Structures* **47**(22): 2993–3010.

Pierron, F., Vert, G., Burguete, R., Avril, S., Rotinat, R. and Wisnom, M. (2007). Identification of the orthotropic elastic stiffnesses of composites with the virtual fields method: sensitivity study and experimental validation, *Strain* **43**(3): 250–259.

Pietryga, M. P., Vladimirov, I. N. and Reese, S. (2012). A finite deformation model for evolving flow anisotropy with distortional hardening including experimental validation, *Mechanics of Materials* **44**: 163–173.

Pottier, T., Toussaint, F. and Vacher, P. (2011). Contribution of heterogeneous strain field measurements and boundary conditions modelling in inverse identification of material parameters, *European Journal of Mechanics-A/Solids* **30**(3): 373–382.

Rahimi, M. R., Yun, G. J. and Shang, S. (2012). Inverse estimations of dynamic stiffness of highway bridge embankment from earthquake records, *Journal of Bridge Engineering* .

Rossi, M. and Pierron, F. (2012). Identification of plastic constitutive parameters at large deformations from three dimensional displacement fields, *Computational Mechanics* **49**(1): 53–71.

Sadegh Zadeh, K. and Montas, H. J. (2014). Parametrization of flow processes in porous media by multiobjective inverse modeling, *Journal of Computational Physics* **259**: 390–401.

Salençon, J. (2001). *Handbook of Continuum Mechanics: General Concepts - Thermoelasticity*, Springer Science & Business Media.

Scherzer, O. (2009). *Variational methods in imaging*, Vol. 167, Springer.

Schuster, T., Kaltenbacher, B., Hofmann, B. and Kazimierski, K. S. (2012). *Regularization methods in Banach spaces*, Vol. 10, de Gruyter.

Shang, S. and Yun, G. J. (2012). Identification of elasto-plastic constitutive parameters by self-optimizing inverse method: Experimental verifications, *Computers Materials and Continua* **27**(1): 55.

Shang, S., Yun, G. J., Kim, B.-R. and Lee, H.-K. (2013). Parameter estimation of a rate-dependent damage constitutive model for damage-tolerant brittle composites by self-optim analyses, *International Journal of Damage Mechanics* **22**(5): 699–718.

Shi, P. and Wright, S. (1994). Higher integrability of the gradient in linear elasticity, *Mathematische Annalen* **299**: 435–448.

Showalter, R. E. (1997). *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, Vol. 49 of *Mathematical Surveys and Monographs*, American Mathematical Society.

Suchocki, C., Pawlikowski, M., Skalski, K., Jasiński, C. and Morawiński, Ł. (2013). Determination of material parameters of quasi-linear viscoelastic rheological model for thermoplastics and resins, *Journal of Theoretical and Applied Mechanics* **51**(3): 569–580.

Taleb, L. and Cailletaud, G. (2010). An updated version of the multimechanism model for cyclic plasticity, *Int. Journal of Plast.* **26**(6): 859 – 874.

Thielecke, F. (1998). *Parameteridentifizierung von Simulationsmodellen f ür das viskoplastische Verhalten von Metallen - Theorie, Numerik, Anwendung -*, Braunschweiger Schriften zur Mechanik, Nr. 34.

Tortorelli, D. A. and Michaleris, P. (1994). Design sensitivity analysis: overview and review, *Inverse problems in Engineering* **1**(1): 71–105.

Tröltzsch, F. (2010). *Optimal control of partial differential equations: theory, methods, and applications*, Vol. 112, American Mathematical Soc.

Vexler, B. (2004). *Adaptive Finite Element Methods for Parameter Identification Problems*, PhD thesis, Ruprecht-Karls-Universität Heidelberg.

Weaver, J. (2015). *The Self-Optimizing Inverse Methodology for Material Parameter Identification and Distributed Damage Detection*, PhD thesis, The University of Akron, US.

Widany, K.-U. and Mahnken, R. (2012). Adaptivity for parameter identification of incompressible hyperelastic materials using stabilized tetrahedral elements, *Computer Methods in Applied Mechanics and Engineering* **245-246**: 117 – 131.

Wolff, M. (1996). A global L^q-gradient estimate on weak solutions of nonlinear stationary Navier-Stokes equations under mixed boundary conditions, Preprint der Humboldt Universität Berlin Nr. 96-3.

Wolff, M., Böhm, M., Bökenheide, S., Lammers, D. and Linke, T. (2012). An implicit algorithm to verify creep and trip behavior of steel using uniaxial experiments, *Z. Angew. Math. Mech.* **92**: 355–379.

Wolff, M., Bökenheide, S., Schlasche, J., Büsing, D., Karsch, T., Böhm, M. and Zoch, H.-W. (2013). An extended approach to multi-mechanism models in plasticity - theory and parameter identification, *Technical Report 13-06*, Berichte aus der Technomathematik, FB 3, Universität Bremen.

Yosida, K. (1965). *Functional analysis*, Springer-Verlag Berlin.

Yun, G. J. and Shang, S. (2016). A new inverse method for the uncertainty quantification of spatially varying random material properties, *International Journal for Uncertainty Quantification* **6**(6).

Yun, G. and Shang, S. (2011). A self-optimizing inverse analysis method for estimation of cyclic elasto-plasticity model parameters, *International Journal of Plasticity* **27**: 576 – 595.

Zeidler, E. (1986). *Nonlinear Functional Analysis and its Applications I - Fixed-Point Theorems*, Springer.

Zeidler, E. (1990a). *Nonlinear Functional Analysis and its Applications II/A - Linear Monotone Operators*, Springer.

Zeidler, E. (1990b). *Nonlinear Functional Analysis and its Applications II/B - Nonlinear Monotone Operators*, Springer.

Zeidler, E. (1997). *Nonlinear Functional Analysis and its Applications IV - Applications to Mathematical Physics*, Springer.