A Continuous Model for Production Networks

Joint work with:

Junprof. Dr. M. Herty, C. Kirchner, Prof. Dr. A. Klar (Kaiserslautern)
Dr. A. Fügenschuh, Prof. Dr. A. Martin (Darmstadt)
Prof. Dr. P. Degond (Toulouse)

Simone Göttlich

Technische Universität Kaiserslautern

Workshop Math. Modelle in der Transport- und Produktionslogistik
Bremen, 12. Januar 2007

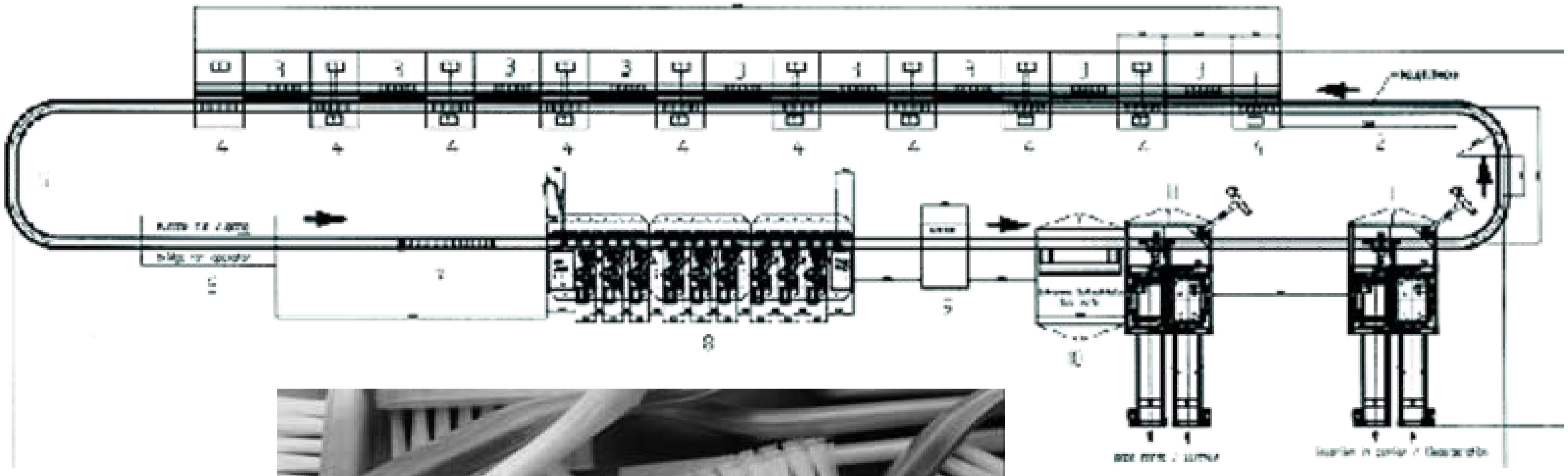
Outline

- Modeling Production Networks
  - Model of Armbruster, Degond and Ringhofer
  - Network Formulation
  - Priority Network Model
- Optimal Control Problems
  - Mixed Integer Programming Model
  - Adjoint Calculus
- Outlook

Modeling Fundamentals

- processors and unbounded queues
- one product flow
- conservation of mass
- instationary flows
- deterministic

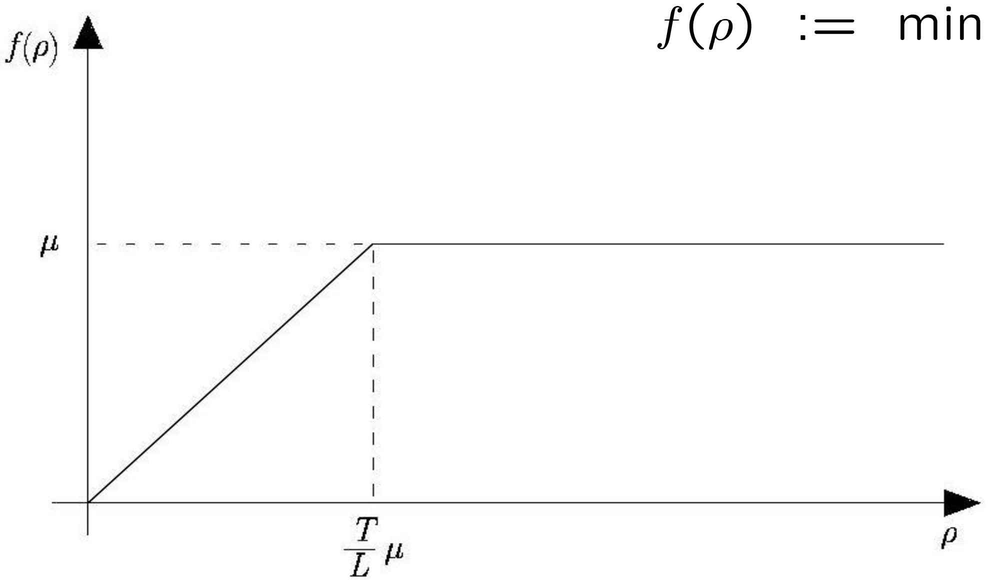
Production Line for Toothbrush Manufacturing



Idea of Supply Chain Modeling

See **Armbruster, Degond** and **Ringhofer**: *A model for the dynamics of large queuing networks and supply chains*

$$\begin{aligned} \partial_t \rho + \partial_x f(\rho) &= 0, \quad \forall x \in [a,b],\ t \geq 0 \\ f(\rho) &:= \min\left\{v\rho, \mu\right\} \end{aligned}$$



ρ : density of parts

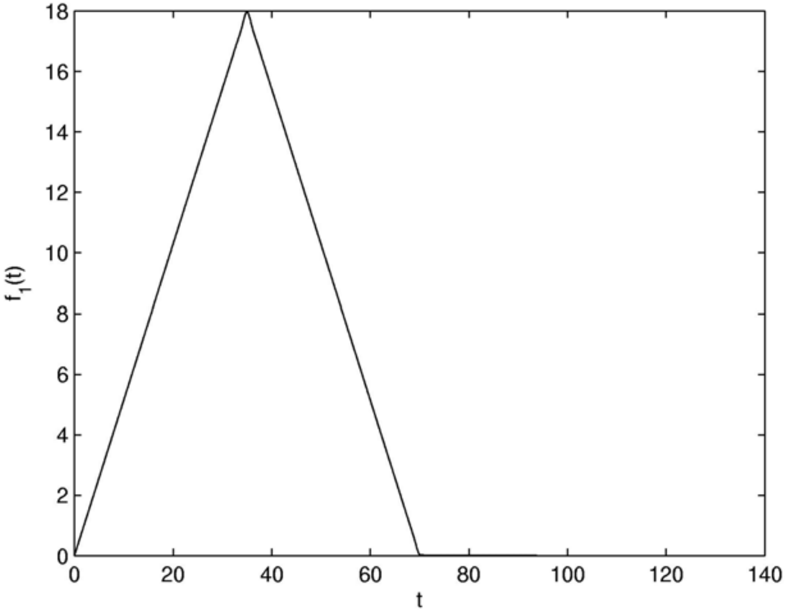
μ : maximum capacity

v : processing velocity

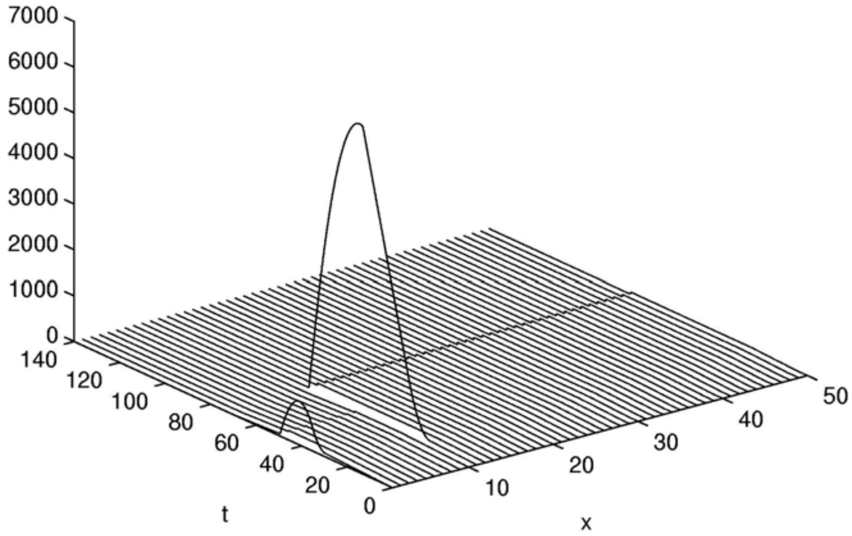
Idea of Supply Chain Modeling

j	x_j	μ_j	v_j
1	$[0, 10]$	15	0.2
2	$[10, 40]$	10	0.2
3	$[40, 50]$	15	0.2

Inflow profile



Density



Network Model

Idea:

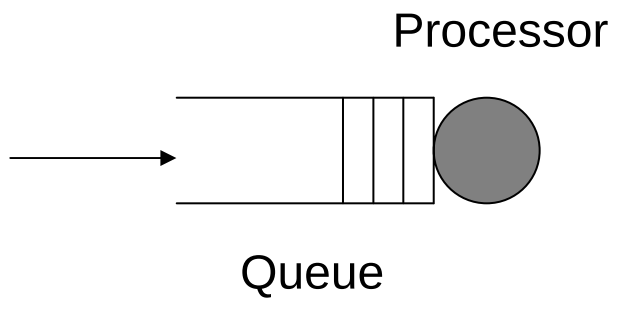
- Each processor is described by one arc
- Use above equations to describe the dynamics of the processor
- Add equations for queues in front of the processor.

Advantage:

- Standard treatment of equations
- Straightforward definitions for complex networks

Network Model

See **Göttlich, Herty**, **Klar**: *Network models for supply chains*



We assume that the dynamics of the processor are described by

$$\partial_t \rho^e + \partial_x f^e(\rho^e) = 0 \quad \text{with} \quad f^e(\rho^e) = v^e \rho^e.$$

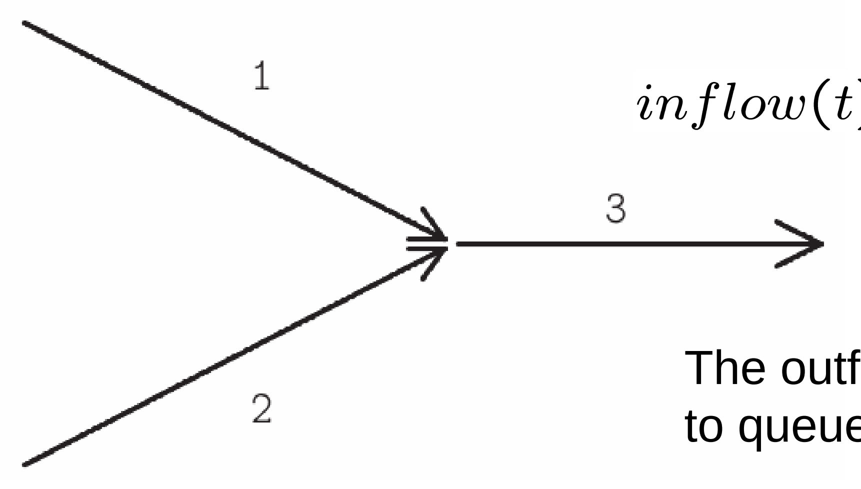
The flow in a queue is subject to

$$\partial_t q^e(t) = inflow(t) - outflow(t).$$

Flow conservation through nodes!

Coupling Conditions I

In a network, inflow to and outflow of a particular queue depends on the kind of vertex.



The inflow to queue 3 is given by the output of processors 1 and 2:

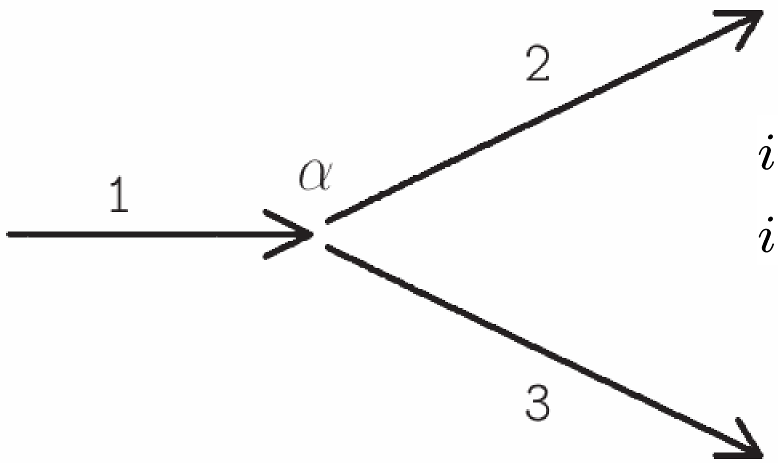
$$inflow(t) = f^1(\rho^1(b^1,t)) + f^2(\rho^2(b^2,t))$$

The outflow of queue 3 depends on the inflow to queue 3 and the actual content of queue 3.

$$outflow(t) = f^3(\rho^3(a^3,t)) = \begin{cases} \min\{inflow(t), \mu^3\} & q^3(t) = 0 \\ \mu^3 & q^3(t) > 0 \end{cases}$$

Coupling Conditions II

The inflow to queue 2 and 3 is given by the output of processor 1 and an adjustable $\alpha \in [0,1]$:



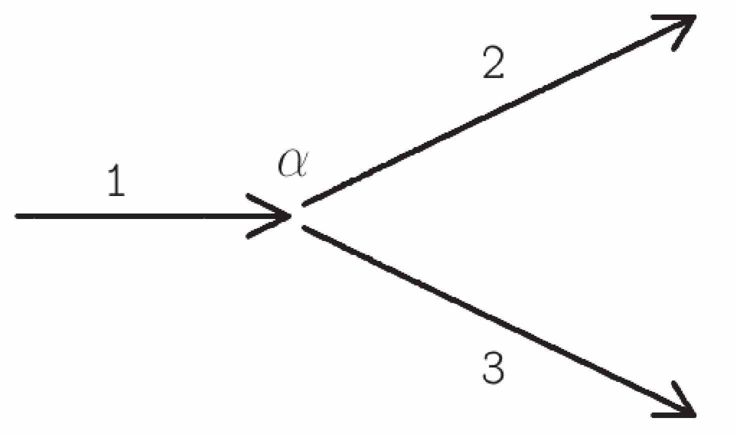
$$\begin{aligned} inflow^2(t) &= \alpha \cdot f^1(\rho^1(b^1,t)) \\ inflow^3(t) &= (1-\alpha) \cdot f^1(\rho^1(b^1,t)) \end{aligned}$$

The outflow of queues 2 and 3 depend on the inflow to queues 2 and 3 and the actual contents of queues 2 and 3.

$$e = 2,3$$

$$outflow^e(t) = f^e(\rho^e(a^e,t)) = \begin{cases} \min\{inflow^e(t), \mu^e\} & q^e(t) = 0 \\ \mu^e & q^e(t) > 0 \end{cases}$$

Network Coupling



δ_v^+: Set of all outgoing arcs for a fixed node **v** (here 2 and 3)

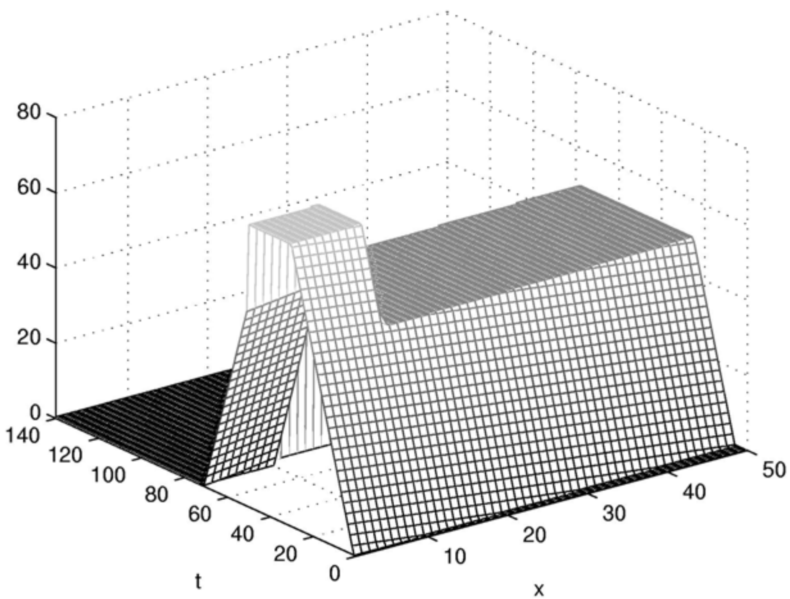
$\alpha_v^e(t)$: Time-dependent distribution rates satisfying

$$\alpha_v^e(t) \in [0,1] \text{ and } \sum_{e \in \delta_v^+} \alpha_v^e(t) = 1$$

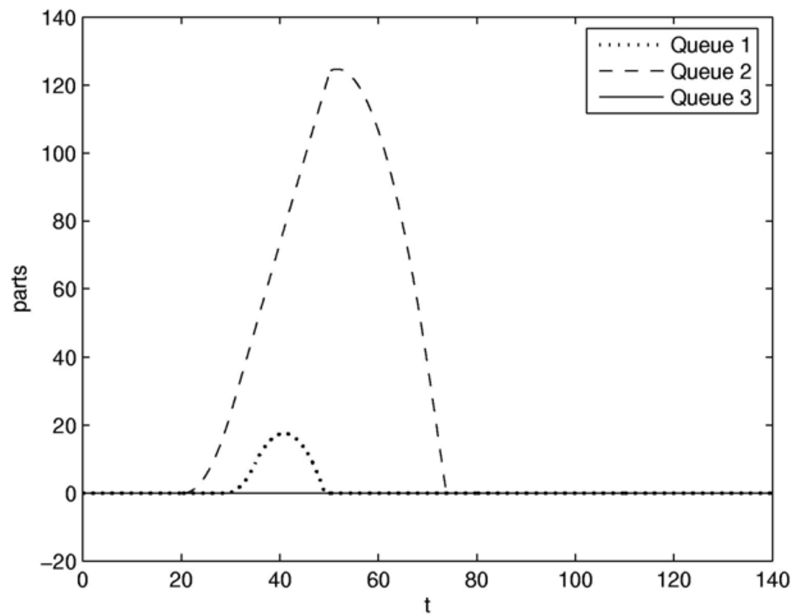
$\alpha_v^e(t)$ will be obtained as solutions of the optimization problem

Network Model

Density



Queues



Approximations

See **Armbruster, de Beer, Freitag, Jagalski, Ringhofer** : *Autnonous Control of Production Networks using a Pheromone Approach*

The quantity $outflow(t)$ is discontinous in $q^e(t)$.

$$outflow(t) = \begin{cases} \min\{inflow(t), \mu^e\} & q^e(t) = 0 \\ \mu^e & q^e(t) > 0 \end{cases}$$

Hence, we use the following relaxation

$$outflow(t) := \min\{\mu^e, \frac{q^e(t)}{\epsilon}\} \approx \psi_\epsilon(q^e(t))$$

ψ_ϵ is a smooth function and $\epsilon << 1$ is the relaxation parameter.

Note: Necessary assumption for applying adjoint calculus !

Network Models with multiple policies

See **Armbruster, Degond, Ringhofer**: *Kinetic and fluid Models for supply chains supporting policy attributes*

Idea: Fluxes with a higher priotity (e.g. time to due-date) have to be produced first. For example, two-phase flow (*K=2*) and $Y_1 < Y_2$:

$$\partial_t \rho_k + \partial_x q_k = 0,$$
$$\partial_t(\rho_k Y_k) + \partial_x q_k Y_k = 0,$$

with the following definition of q_k :

1. If $\mu < \rho_1 v_1$, then $q_1 = \mu$ and $q_2 = 0$.
2. If $\rho_1 v_1 < \mu < \rho_1 v_1 + \rho_2 v_2$, then $q_1 = \rho_1 v_1$ and $q_2 = \mu - \rho_1 v_1$.
3. If $\rho_1 v_1 + \rho_2 v_2 \leq \mu$, then $q_k = \rho_k v_k$, for $k = 1, 2$.

Network Models with multiple policies

See **Degond, Göttlich, Herty, Klar**: *A network model for supply chains with multiple policies*

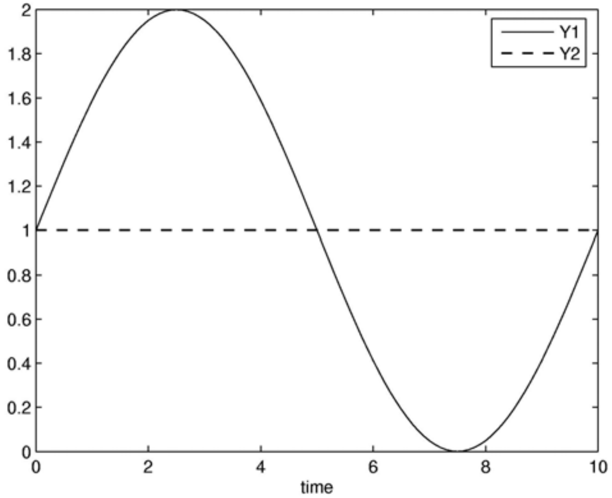
Idea: Use above equations to describe one processor and define again **coupling conditions**

New: Introduce a pointer-function $\mathcal{Y}^\nu(t)$

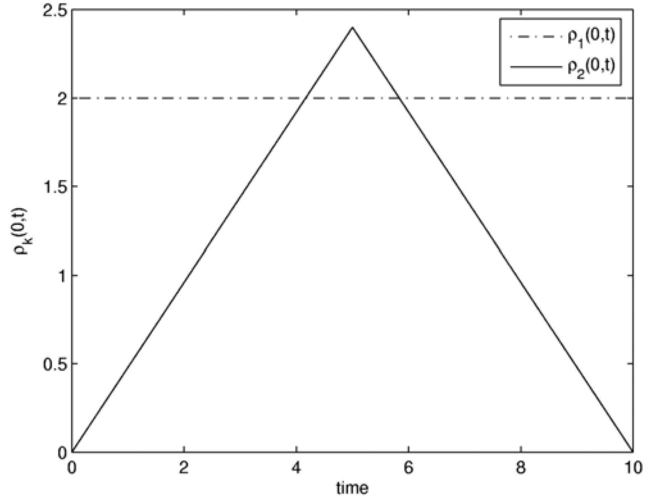
$$\mathcal{Y}^\nu(t) = \begin{pmatrix} Y_1 & \text{if } \pi_1^e(t) \neq 0 \text{ or if } q_1^{e-1}(x_\nu^{e-1},t) > \mu^e \\ Y_2 & \text{if } \pi_1^e(t) = 0,\ \pi_2^e(t) \neq 0 \text{ or if} \\ & q_1^{e-1}(x_\nu^{e-1},t) + q_2^{e-1}(x_\nu^{e-1},t) \geq \mu^e > q_1^{e-1}(x_\nu^{e-1},t) \\ +\infty & \text{if } \pi_1^e(t) = \pi_2^e(t) = 0 \text{ and if} \\ & q_1^{e-1}(x_\nu^{e-1},t) + q_2^{e-1}(x_\nu^{e-1},t) \leq \mu^e \end{pmatrix}.$$

Example: One processor with $\mu = 2$ and v=1

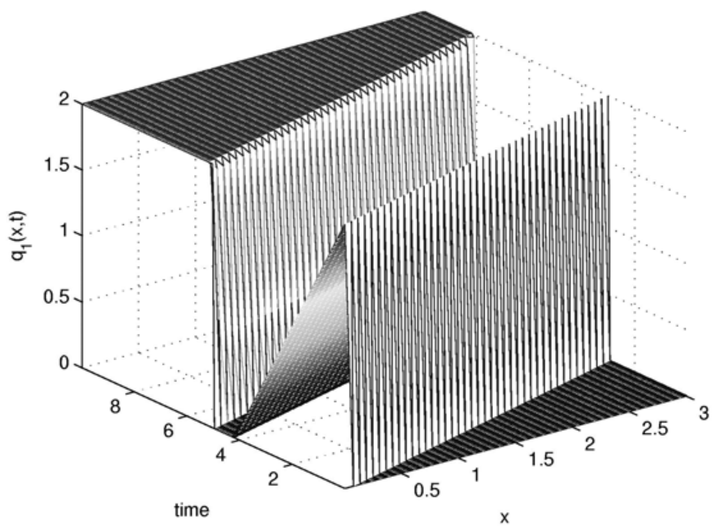
Priorities



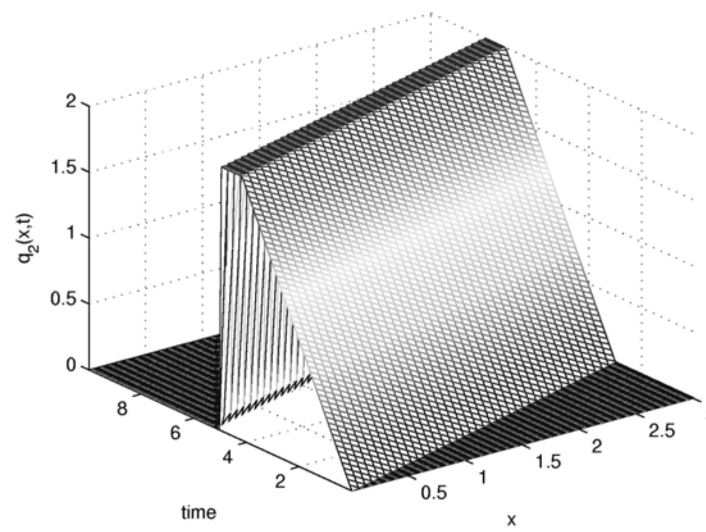
Boundary cond.



q1

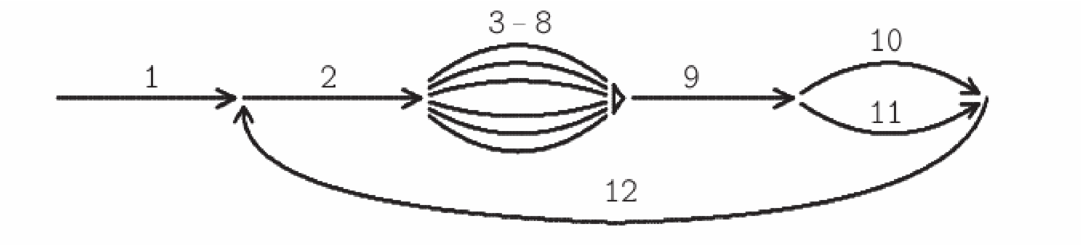


q2



Optimization Issues

Optimal routing of goods through the network



We need to specify an objective function:

- Minimize of parts in queues
- Maximize the output
- Combinations of both etc.

Continuous Optimization Problem

maximize flux and minimize queues

$$\min_{\alpha_v^e(t)} \sum_{e\in\mathcal{A}} \int_0^T \int_{a^e}^{b^e} \mathcal{F}\Big(\rho^e(x,t), q^e(t)\Big)\, dx\, dt$$

subject to

flux through processor

$$\partial_t \rho^e(x,t) + \partial_x f^e(\rho^e(x,t)) = 0$$

coupling at vertex

$$\partial_t q^e(t) = \alpha_v^e(t) \sum_{\bar{e}\in\delta_v^-} f^{\bar{e}}(\rho^{\bar{e}}(b^{\bar{e}},t)) - f^e(\rho^e(a^e,t))$$

queue outflow

$$f^e(\rho^e(a^e,t)) = \min\{\mu^e, \tfrac{q^e(t)}{\epsilon}\}$$

initial conditions

$$q^e(0) = 0,\ \rho^e(x,0) = 0$$

Remarks

Optimal control problem with PDE as constraints!

How to solve ?

Mixed Integer Programming

Adjoint Calculus

Mixed Integer Program (MIP)

See **Fügenschuh, Göttlich, Herty, Klar, Martin**: *A Discrete Optimization Approach to Large Scale Networks based on PDEs*

A mixed-integer program is the minimization/maximization of a **linear** function subject to **linear** constraints.

Problem: Non-linear description of queue-outflow

$$outflow(t) := \min\{\mu^e, \frac{q^e(t)}{\epsilon}\} \approx \psi_\epsilon(q^e(t))$$

Idea: Introduce binary variables (decision variables) δ_p

Simplified Model

- Step 1: Two point Upwind discretization on each arc

$$\mathsf{f}_t^e := f^e(\rho^e(a^e,t)), \quad \mathsf{g}_t^e = f^e(\rho^e(b^e,t)), \quad \mathsf{q}_t^e := q^e(t) \;\forall e,t$$

$$\mathsf{g}_{t+1}^e = \mathsf{g}_t^e + \frac{\Delta t}{L^e} v^e \left(\mathsf{f}_t^e - \mathsf{g}_t^e\right), \quad \forall e,t$$

- Step 2: Reformulate regularization condition

Introduce binary variables $\zeta_t^e \in \{0,1\}$ for all e,t

$$\begin{aligned} \mu^e \zeta_t^e \; &\leq \mathsf{f}_t^e \leq \; \mu^e, \\ \frac{q_t^e}{\epsilon} - M\zeta_t^e \; &\leq \mathsf{f}_t^e \leq \; \frac{q_t^e}{\epsilon}, \qquad M >> 1 \end{aligned}$$

Simplified Model

- Step 3: Reformualtion of coupling conditions

$$h^e : \text{total inflow to arc } e$$

$$\sum_{e \in \delta_v^+} \mathsf{h}_t^e = \sum_{e \in \delta_v^-} \mathsf{g}_t^e \quad \forall v, t$$

$$\mathsf{q}_{t+1}^e = \mathsf{q}_t^e + \Delta t \, (\mathsf{h}_t^e - \mathsf{f}_t^e) \, , \quad \forall e, t$$

$$0 \le \mathsf{f}_t^e \le \mu^e, \; 0 \le \mathsf{g}_t^e \le \mu^e, \; 0 \le \mathsf{q}_t^e$$

- Step 4: Trapezoidal rule for discretization of cost functional (linear)

$$\sum_{e,t} \Delta t \, \frac{L^e}{2} \Big(\mathcal{F}(\mathsf{f}_t^e / v^e, \mathsf{q}_t^e) + \mathcal{F}(\mathsf{g}_t^e / v^e, \mathsf{q}_t^e) \Big)$$

Result: Mixed Integer Programming Model (MIP)

Discrete Optimization Problem

maximize flux and minimize queues

$$\min_{\alpha(t)} \sum_{e\in\mathcal{A}} \sum_{t=1}^{T} \Delta t \, \frac{L^e}{2} \Big(\mathcal{F}(\mathsf{f}_t^e / v^e, \mathsf{q}_t^e) + \mathcal{F}(\mathsf{g}_t^e / v^e, \mathsf{q}_t^e) \Big)$$

subject to

flux through processor

$$\mathsf{g}_{t+1}^e = \mathsf{g}_t^e + \frac{\Delta t}{L^e} v^e \left(\mathsf{f}_t^e - \mathsf{g}_t^e \right)$$

coupling at vertex

$$\sum_{e\in\delta_v^+} \mathsf{h}_t^e = \sum_{e\in\delta_v^-} \mathsf{g}_t^e$$

$$\mathsf{q}_{t+1}^e = \mathsf{q}_t^e + \Delta t \left(\mathsf{h}_t^e - \mathsf{f}_t^e \right)$$

queue outflow

$$\mu^e \zeta_t^e \le \mathsf{f}_t^e \le \mu^e$$

$$\frac{\mathsf{q}_t^e}{\epsilon} - M\zeta_t^e \le \mathsf{f}_t^e \le \frac{\mathsf{q}_t^e}{\epsilon}, \qquad \zeta_t^e \{0,1\}$$

initial conditions

$$\mathsf{q}_0^e = 0, \; \mathsf{f}_0^e = 0, \; \mathsf{g}_0^e = 0$$

Model Extensions

Praxis-relevant constraints can be easily added to the MIP model:

- Bounded queues: $\mathsf{q}_t^e \leq \overline{\mathsf{q}}^e$
- Optimal input: $\max \sum_{e=1,t} \mathsf{f}_t^e$
- Maintenance shut-down:

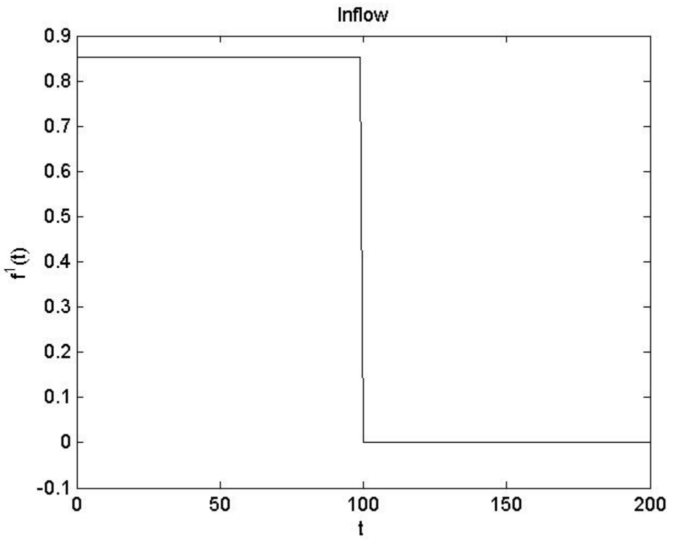
$$\phi_t^{\tilde{e}} \in \{0,1\},\ \forall t,\ \forall l = 0,\ldots,N-1$$

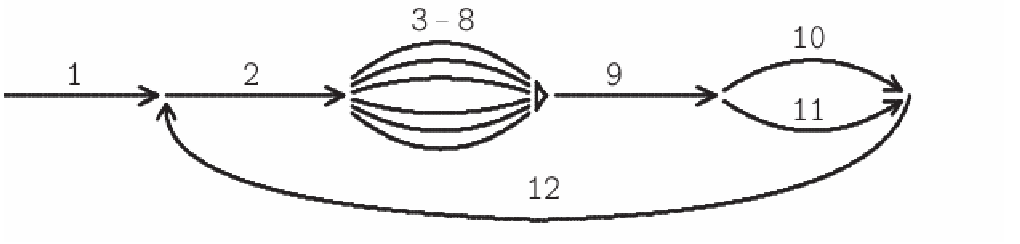
$$\mathsf{h}_{t+l}^{\tilde{e}} \leq \max\{\mu^e : e \in E\}|E| \cdot (1-\phi_t^{\tilde{e}})$$

$$\sum_{t=1}^{T} \phi_t^{\tilde{e}} = 1$$

Example

Solved by ILOG CPLEX 10.0





e	μ^e	$\overline{q}^e$
1	100	100
2	0.71	20
3 - 8	0.013	10
9	0.71	1
10 - 11	0.12	1
12	0.71	1

Maximization of outflow, i.e. optimizing the amount of parts passing processor 12

$$\min -\sum_t \frac{1}{t+1} g_t^{12}$$

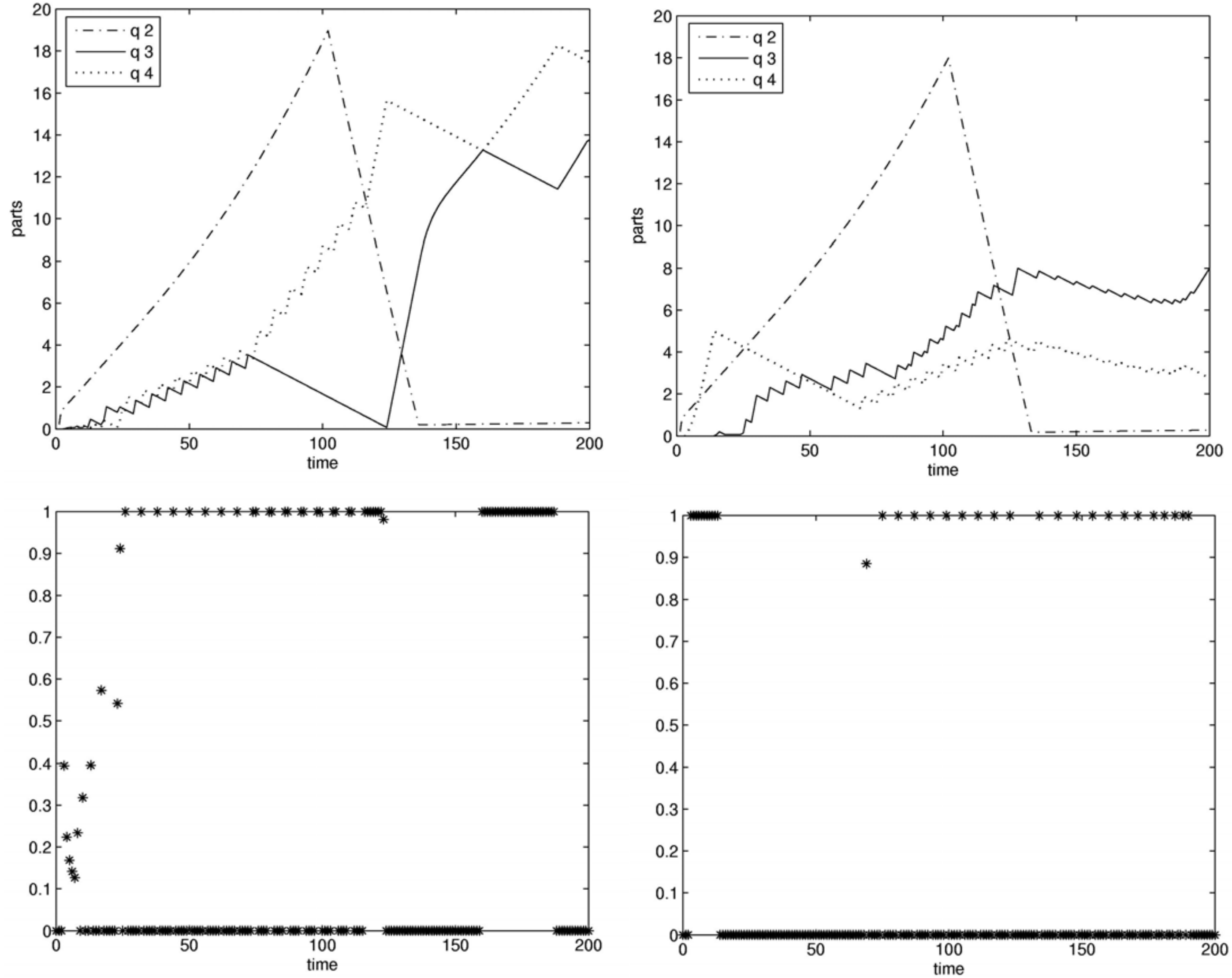
Example I

$$q_t^e \leq \overline{q}^e$$

unbounded **bounded**

Parts in queues 2,3,4

Distribution rates on arc 3

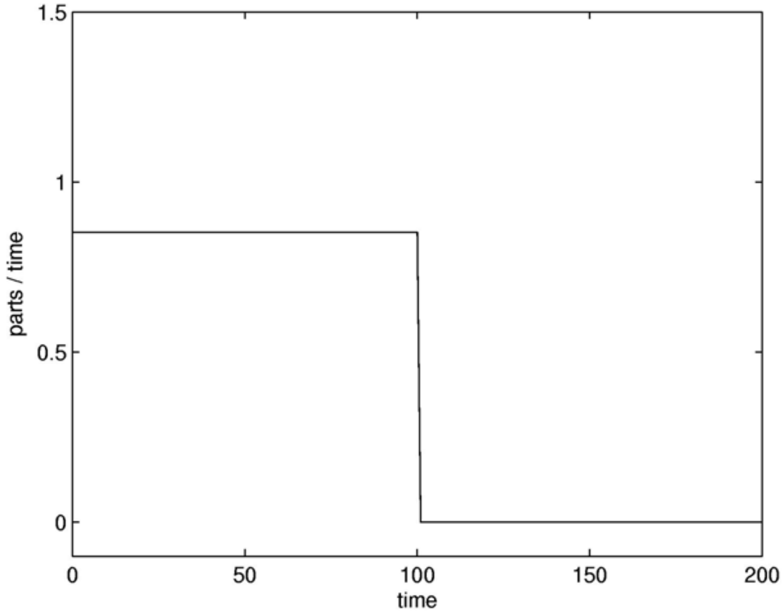


Example II

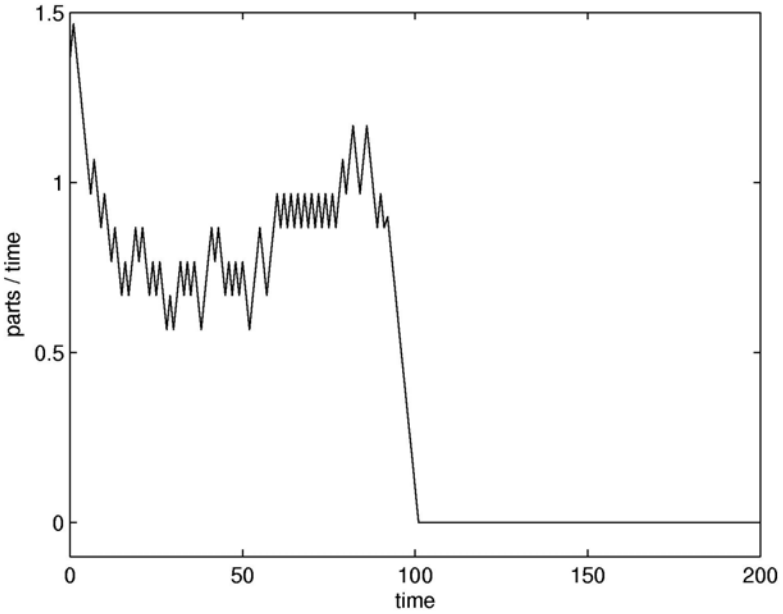
Inflow profile optimization with bounded queues

$$\max \sum_{e=1,t} f_t^e$$

user-defined inflow



optimal inflow



Adjoint Calculus: General Setup

See **Göttlich, Herty, Kirchner, Klar**: *Optimal Control for Continuous Supply Network Models*

$$\min_{y \in Y, u \in U} J(y,u)$$

$$\text{subject to} \qquad c(y,u) = 0, \quad u \in U_{ad}, \quad y \in Y_{ad}$$

$J : Y \times U \mapsto \mathbb{R}$ is the objective function.

$c : Y \times U \mapsto Z$ is the state operator.

$U_{ad} \subset U$ is the set of admissible controls.

$Y_{ad} \subset Y$ is the set of admissible states.

J and c are continuously Fréchet differentiable.

Adjoints: Main Idea

We introduce the **Lagrangian function**

$$l(y,u,p) \;=\; J(y,u) + \langle p, c(y,u)\rangle$$

The Lagrangian and the reduced functional are related:

$$\hat{J}(u) \;=\; J(y(u),u) = l(y(u),u,p)$$

Formally this leads to

$$\nabla_u \hat{J}(u) \;=\; \nabla_u J(y(u),u) = \nabla_u\, l(y(u),u,p)$$

Adjoints: Main Idea

A necessary first order optimality condition is

$$0 \;=\; \nabla_u \hat{J}(u) = \nabla_u \, l(y(u), u, p)$$

The derivative on the right hand side can be computed.

One obtains the so-called **adjoint equation**

$$c_y(y(u), u)\, p \;=\; -J_y(y(u), u)$$

Furthermore, the **gradient** is determined as

$$\nabla_u J(y(u), u) \;=\; J_u(y(u), u) + c_u(y(u), u)\, p$$

Adjoints: Main Idea

Lagrangian function

$$L(\vec{\rho}^e, \vec{A}^v, \vec{q}^e, \vec{\Lambda}^e, \vec{P}^e) = \sum_{e\in\mathcal{A}} \int_0^T \int_{a^e}^{b^e} v^e \rho^e dx dt + \int_0^T q^e dt -$$

$$\sum_{e\in\mathcal{A}} \int_0^T \int_{a^e}^{b^e} \Lambda^e \partial_t \rho^e + \Lambda^e v^e \partial_x \rho^e dx dt -$$

$$\sum_{e\in\mathcal{A}} \int_0^T P^e \left(\partial_t q^e - h^e(\vec{\rho}^e, \vec{A}^v) + \psi^e(q^e) \right) dt$$

with **Lagrange multipliers** Λ^e and P^e

Adjoints: Main Idea

Optimality Sytem

$$\partial_t \rho^e + v^e \partial_x \rho^e = 0,\ \rho^e(x,0) = 0,\ v^e \rho^e(a,t) = \psi^e(q^e),$$
$$\partial_t q^e = h^e(\vec{\rho}^e, \vec{A}^v) - \psi^e(q^e),\ q^e(0) = 0,$$

$$-\partial_t \Lambda^e - v^e \partial_x \Lambda^e = v^e,\ \Lambda^e(x,T) = 0,$$
$$v^e \Lambda^e(b,t) = \sum_{\bar{e} \in \delta_v^+ \text{ s.t. } e \in \delta_v^-} P^{\bar{e}}(t) \frac{\partial}{\partial \rho^{\bar{e}}} h^{\bar{e}}(\vec{\rho}^e, \vec{A}^v),$$
$$-\partial_t P^e = 1 - \left(P^e - \Lambda^e(a,t)\right)(\psi^e)'(q^e),\ P^e(T) = 0,$$

$$\sum_{e \in \delta_v^+} P^e \frac{\partial}{\partial A^{v,\bar{e}}} h^e(\vec{\rho}^e, \vec{A}^v) = 0.$$

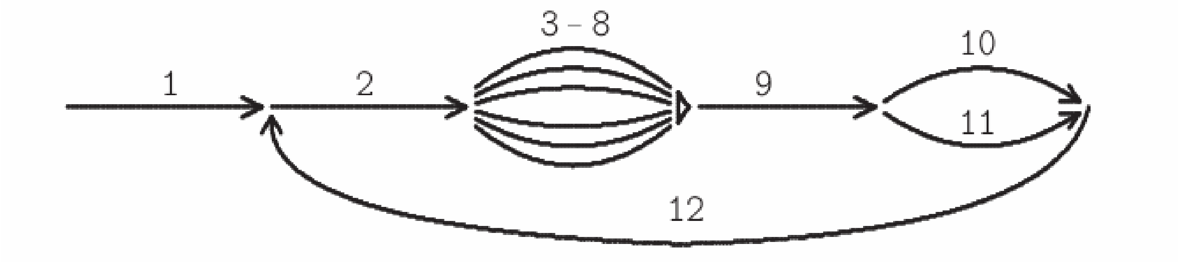
Optimization Algorithm

We can use a (projected) steepest descent method:

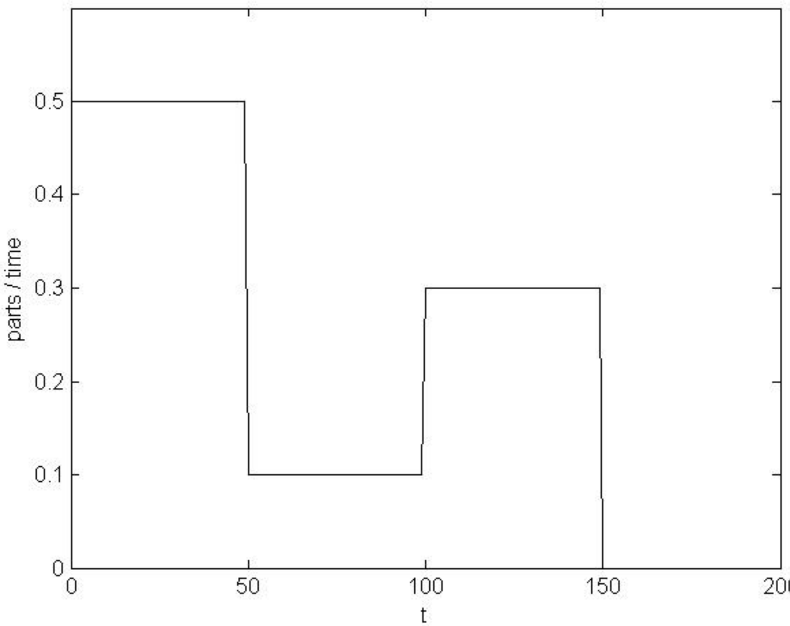
1. Choose $u_0 \in U_{ad}$
2. Compute for u_0 the solution of state and adjoint eq.
3. Compute $\nabla_u J(y(u), u)$. If it is zero, then STOP.
4. Update $u_0 := P(u_0 - \sigma \nabla_u J)$ for a suitable choice of $\sigma > 0$
5. Go to 2.

Numerical Results

Compare results of the adjoint approach to the MIP:



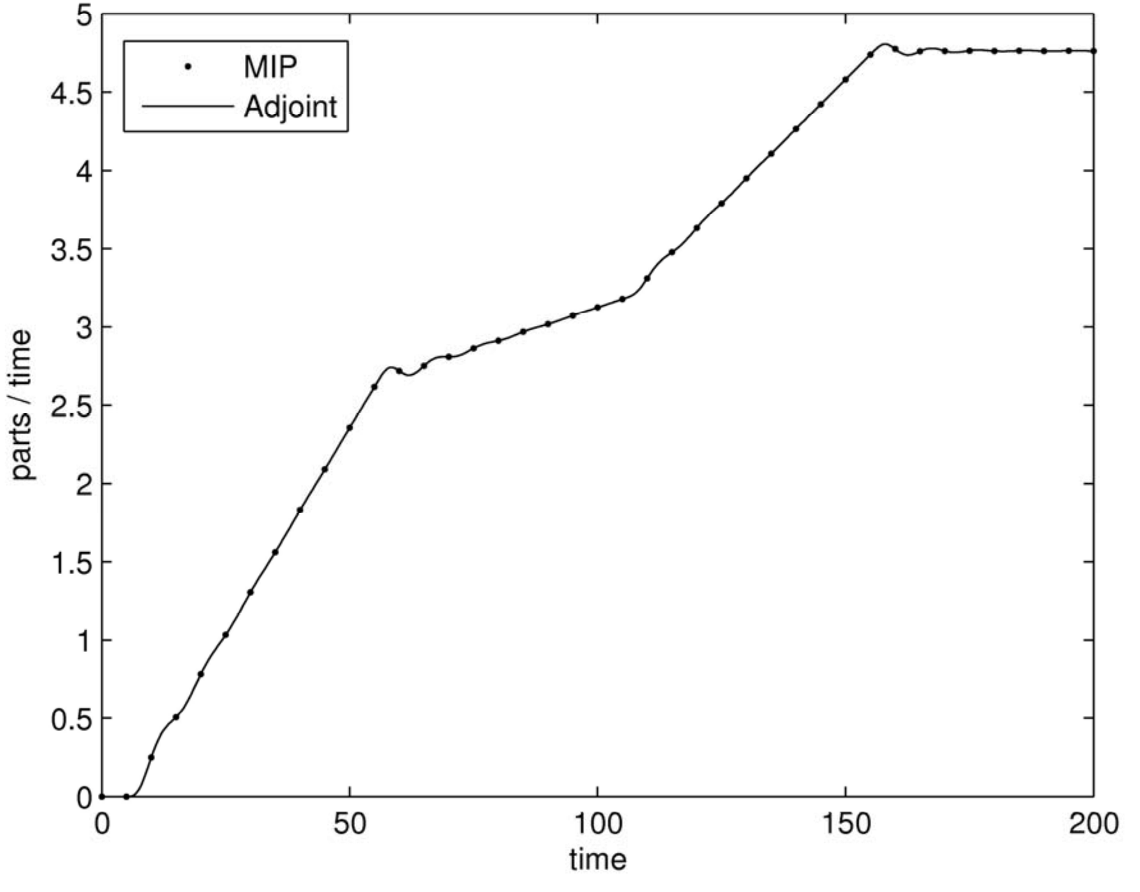
Inflow profile



Objective function:

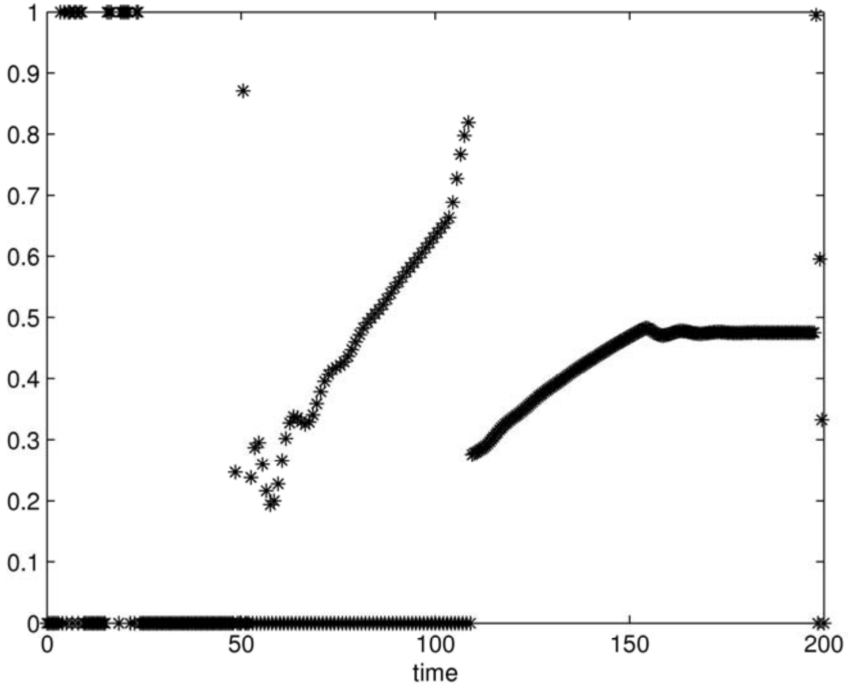
$$J(\vec{\alpha}) = -\sum_{j=1}^{NT+1} \frac{g_j^{12}}{j+1}$$

Optimal outflow profile

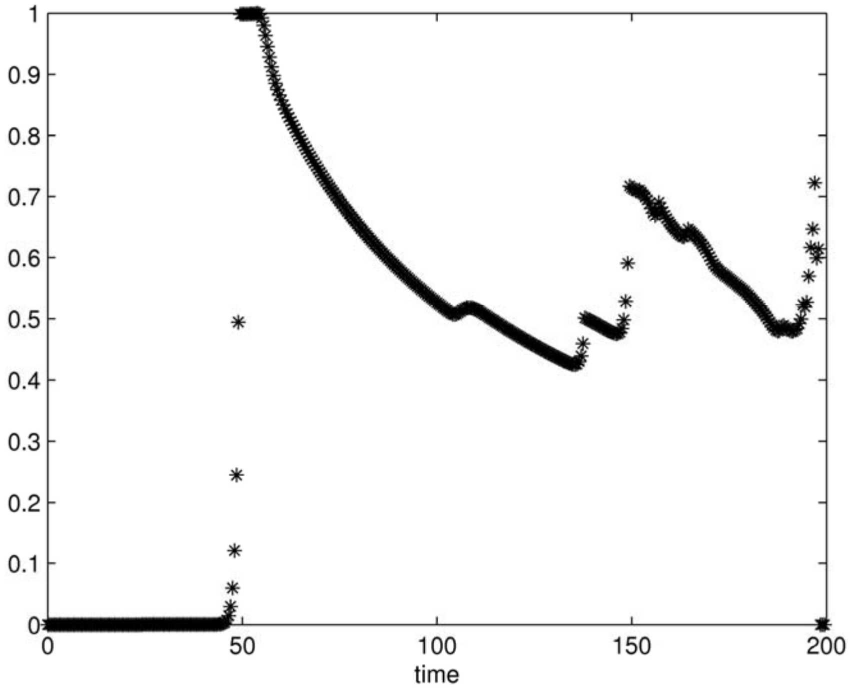


Optimal control

$\alpha^{9,10}(t)$ from MIP



$\alpha^{9,10}(t)$ from adjoint calculus



Outlook and Current Work

- Realistic Problems
  - Real-World Objective Funtions
  - Contraints
- Multi-commodity model
- Stochastic aspects
  - Random Breakdows of processors

Thanks you for your attention …

… Questions?