ISDS small-gain theorem and construction of ISDS Lyapunov functions for interconnected systems

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Abstract

We consider networks of input-to-state dynamically stable (ISDS) systems and use a small-gain condition to assure the ISDS property for their interconnection. Under this small-gain condition we provide a construction of an ISDS Lyapunov function including explicit derivation of corresponding rates and gains for the whole interconnection.

Key words: Nonlinear systems, input-to-state dynamical stability, interconnected systems, ISDS Lyapunov function, small-gain condition

1. Introduction

Consider a large scale nonlinear system of the form

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $t \in \mathbb{R}$ is the time, $\dot{x}(t)$ denotes the derivative of the state $x(t) \in \mathbb{R}^N$ with initial value x_0, the input $u(t) \in \mathbb{R}^m$ is an essentially bounded measurable function and $f : \mathbb{R}^{N+m} \to \mathbb{R}^N$, $N, m \in \mathbb{N}$. To have existence and uniqueness of a solution of (1) let the function f be continuous and locally Lipschitz in x uniformly in u. The solution is denoted by $x(t; x_0, u)$ or $x(t)$ in short.

Stability of such systems is a crucial property for applications and it is not always an easy task to check stability of a given nonlinear systems or to design it in a way that it becomes stable and robust. To solve such problems conditions of the small-gain type turn out to be helpful in many situations. An important tool to investigate stability is a Lyapunov function. However there is no general method to find a Lyapunov function for arbitrary nonlinear system.

Stability analysis of such systems can be performed in different frameworks such as passivity, dissipativity [1], input-to-state stability (ISS) [2] and its variations [3, 4, 5, 6]. We will use the notion of input-to-state dynamical stability (ISDS) introduced in [7]. This property is equivalent to ISS, however the advantage of ISDS over ISS is that the bound for the trajectories takes essentially only the recent values of the input u into account and in many cases it gives a better bound for trajectories due to the *memory*

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fading effect of the disturbance input u. Moreover the gains in the trajectory based definition of ISDS are the same as in the definition of the ISDS-Lyapunov function, which is in general not true for the ISS systems.

In many applications a large scale system of the form (1) can be considered as an interconnection of several subsystems of lower dimensions such that stability properties, for example Lyapunov functions, are known for each of the subsystems. A small-gain condition can help to check stability and to construct a Lyapunov function for the whole system.

In this paper we use a small-gain condition for interconnections of an arbitrary number of ISDS subsystems and show how an ISDS Lyapunov function can be constructed for the whole system if this small-gain condition is satisfied. To this end consider $n \geq 2$ interconnected subsystems

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)), \; i = 1, \ldots, n, \tag{2}$$

where $n \in \mathbb{N}$, $x_i(t) \in \mathbb{R}^{N_i}$, $N_i \in \mathbb{N}$, $u(t) \in \mathbb{R}^m$, $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + m} \to \mathbb{R}^{N_i}$ and assume that each subsystem is ISDS. We consider this interconnection as one large scale system (1) with $x = \left(x_1^T, \ldots, x_n^T\right)^T$, $f(x,u) = \left(f_1(x,u)^T, \ldots, f_n(x,u)^T\right)^T$ and look for a condition that assures that the whole system is ISDS with respect to the state x and the input u.

Recall that stability conditions for an interconnection of two ISS systems were developed in [8] and [9]. In [10] a small-gain theorem for $n \in \mathbb{N}$ interconnected ISS systems was proved. Since ISS Lyapunov functions are an important tool to verify the ISS property, a Lyapunov formulation of the small-gain theorem was given for two interconnected systems in [9]. There, an explicit construction of the ISS Lyapunov function of the whole system was shown. In [11, 5, 12] an explicit construction of an ISS Lyapunov function for the overall system of n interconnected subsystems was derived under a sufficient small-gain condition.

Similar to ISS systems the ISDS property of system (1) is equivalent to the existence of an ISDS Lyapunov function for system (1), see [4]. Also a 0-GAS small-gain theorem for two interconnected systems with the input $u = 0$ can be found in [4].

The purpose of this paper is to extend the mentioned results for ISS systems to the case of ISDS systems. In particular we present a small-gain theorem for $n \in \mathbb{N}$ interconnected ISDS systems of the form (2) and provide a construction of an ISDS Lyapunov function as well as the rates and gains of the ISDS estimation for the entire system consisting of $n \in \mathbb{N}$ interconnected ISDS systems under a small-gain condition. Moreover we derive decay rates of the trajectories of $n \in \mathbb{N}$ interconnected ISDS systems and the trajectory of the entire system with the external input $u = 0$.

The organisation of this paper is the following: The next section introduces necessary notions. Section 3 contains the main result of the paper. Examples are given in Section 4 and the conclusions are collected in Section 5.

2. Preliminaries

By x^T we denote the transposition of a vector $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, furthermore $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^n$ denotes the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$ where we use the standard partial order for $x, y \in \mathbb{R}^n$ given by

$$x \geq y \Leftrightarrow x_i \geq y_i,\ i = 1, \ldots, n \text{ and } x \ngeq y \Leftrightarrow \exists i : x_i < y_i.$$

$|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and the essential supremum norm of a function f is denoted by $\|f\|$. Furthermore $|x|_\infty$ denotes the maximum norm of $x \in \mathbb{R}^n$ and ∇V the gradient of a function $V : \mathbb{R}^n \to \mathbb{R}_+$. For a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{otherwise,} \end{cases} \quad t,\ s_1,\ s_2 \in \mathbb{R}_+.$$

Definition 2.1. *We define following classes of functions:*

$$\begin{aligned}
\mathcal{P} &:= \{f : \mathbb{R}^n \to \mathbb{R}_+ \mid f(0) = 0,\ f(x) > 0,\ x \neq 0\} \\
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,\middle|\, \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\right\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t, r \geq 0\} \\
\mathcal{KLD} &:= \{\mu \in \mathcal{KL} \mid \mu(r, t + s) = \mu(\mu(r, t), s), \forall r, t, s \geq 0\}
\end{aligned}$$

We will call functions of class $\mathcal{P}$ positive definite.

Remark 2.2. *The condition $\mu(r, t + s) = \mu(\mu(r, t), s)$ implies $\mu(r, 0) = r, \forall\ r \geq 0$. To show this suppose that there exists $r \geq 0$ such that $\mu(r, 0) \neq r$. Then*

$$\mu(r, 0) = \mu(r, 0 + 0) = \mu(\mu(r, 0), 0) \neq \mu(r, 0),$$

which is a contradiction. The last inequality follows from the strict monotonicity of μ with respect to the first argument. This shows the assertion.

Note that, if $\gamma \in \mathcal{K}_\infty$, then there exist the inverse function $\gamma^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma^{-1} \in \mathcal{K}_\infty$. The proof can be found in [5].

Definition 2.3. *System (1) is called input-to-state stable (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma^{ISS} \in \mathcal{K}_\infty$ such that*

$$|x(t; x_0, u)| \leq \max\{\beta(|x_0|, t), \gamma^{ISS}(\|u\|)\} \tag{3}$$

$\forall\ x_0 \in \mathbb{R}^N,\ t \in \mathbb{R}_+$ and essentially bounded and measurable inputs $u \in \mathbb{R}^m$. γ^{ISS} is called gain.

This concept has been first introduced in [2], where an equivalent formulation with sum of the both terms instead of max in (3) has been used. It is known for ISS systems that if $\limsup_{t\to\infty} u(t) = 0$ then also $\lim_{t\to\infty} x(t) = 0$ holds. However with $t \to \infty$ (3) provides only a constant positive bound for $u \not\equiv 0$. Another stability property equivalent to ISS is the following:

Definition 2.4. *System (1) is called input-to-state dynamically stable (ISDS), if there exist functions $\mu \in \mathcal{KLD}$, η, $\gamma^{ISDS} \in \mathcal{K}_\infty$ such that*

$$|x(t; x_0, u)| \leq \max\{\mu(\eta(|x_0|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\gamma^{ISDS}(|u(\tau)|), t-\tau)\} \tag{4}$$

$\forall\ t \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^N$ and essentially bounded and measurable inputs $u \in \mathbb{R}^m$. μ is called decay rate, η overshoot gain and γ^{ISDS} robustness gain.

Note that for large t the bound (4) takes essentially only the recent values of the input u into account, in particular it follows immediately from (4) that $\limsup_{t\to\infty} u(t) = 0 \ \Rightarrow\ \lim_{t\to\infty} x(t) = 0$ as stated in the following

Lemma 2.5. *If system (1) is ISDS and $\limsup_{t\to\infty} u(t) = 0$, then it holds*

$$\lim_{t\to\infty} |x(t; x_0, u)| = 0.$$

Proof. Since (1) is ISDS we have

$$\begin{aligned}
&|x(t; x_0, u)| \leq \max\{\mu(\eta(|x_0|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\gamma^{\mathrm{ISDS}}(|u(\tau)|), t-\tau)\} \\
= \quad & \max\{\mu(\eta(|x_0|), t), \operatorname*{ess\,sup}_{\tau\in[0,\frac{t}{2}]} \mu(\gamma^{\mathrm{ISDS}}(|u(\tau)|), t-\tau), \operatorname*{ess\,sup}_{\tau\in[\frac{t}{2},t]} \mu(\gamma^{\mathrm{ISDS}}(|u(\tau)|), t-\tau)\} \\
\leq \quad & \max\{\mu(\eta(|x_0|), t), \mu(\gamma^{\mathrm{ISDS}}(\|u\|_{[0,\frac{t}{2}]}), \frac{t}{2}), \operatorname*{ess\,sup}_{\tau\in[\frac{t}{2},t]} \mu(\gamma^{\mathrm{ISDS}}(|u(\tau)|), 0)\}.
\end{aligned}$$

It holds $\limsup_{t\to\infty} u(t) = 0$ and u is essentially bounded, i.e., $\exists\, K \in \mathbb{R}_+$ such that $\|u\|_{[0,t]} \leq K,\ \forall\ t > 0$. Furthermore $\forall\ \varepsilon > 0\ \exists\ T > 0$ such that $\forall\ \tau \in \left[\frac{T}{2}, T\right] : \operatorname*{ess\,sup}_{\tau\in[\frac{T}{2},T]} \gamma^{\mathrm{ISDS}}(|u(\tau)|) < \varepsilon$. With these considerations, the $\mathcal{KLD}$-property of μ and Remark 2.2 we get

$$\begin{aligned}
&\lim_{t\to\infty} |x(t; x_0, u)| \leq \lim_{t\to\infty} \max\{\mu(\eta(|x_0|), t), \mu(\gamma^{\mathrm{ISDS}}(\|u\|_{[0,\frac{t}{2}]}), \frac{t}{2}), \operatorname*{ess\,sup}_{\tau\in[\frac{t}{2},t]} \gamma^{\mathrm{ISDS}}(|u(\tau)|)\} \\
\leq \quad & \max\{\lim_{t\to\infty} \mu(\gamma^{\mathrm{ISDS}}(K), \frac{t}{2}), \lim_{t\to\infty} \operatorname*{ess\,sup}_{\tau\in[\frac{t}{2},t]} \gamma^{\mathrm{ISDS}}(|u(\tau)|)\} = 0.
\end{aligned}$$

□

Remark 2.6. *The notion of ISDS was introduced in [4] and [7]. One obtains an equivalent definition of ISDS if one replaces the euclidean norm in* (4) *by any other norm. Moreover it can be checked that all results in [4] and [7] hold true, if one uses a different norm instead of the Euclidean one.*

In the rest of the paper we assume the functions μ, η and γ^{ISDS} to be C^∞ in $\mathbb{R}_+ \times \mathbb{R}$ or $\mathbb{R}_+$ respectively. This regularity assumption is not restrictive, because for nonsmooth rates and gains one can find smooth functions arbitrarily close to the original ones, which was shown in [7], Appendix B.

An important tool for the stability analysis of system (1) are Lyapunov functions.

Definition 2.7. *Given $\varepsilon > 0$, a function $V : \mathbb{R}^N \to \mathbb{R}_+$, which is locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$ is called ISDS Lyapunov function of system (1), if there exist $\eta \in \mathcal{K}_\infty$, γ^{ISDS}, $\mu \in \mathcal{KLD}$ such that*

$$\frac{|x|}{1+\varepsilon} \leq V(x) \leq \eta(|x|), \ \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{5}$$

$$V(x) > \gamma^{ISDS}(|u|) \Rightarrow \nabla V(x) \cdot f(x,u) \leq -(1-\varepsilon)\, g(V(x)) \tag{6}$$

for almost all $x \in \mathbb{R}^N \setminus \{0\}$ and all $u \in \mathbb{R}^m$, where μ solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(r,t) = -g(\mu(r,t)), \quad r,t > 0 \tag{7}$$

for a locally Lipschitz continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

It is known that ISS implies the existence of a smooth ISS Lyapunov function for the system (1) (see [13]). A similar result for ISDS systems was proved in [4]. We use locally Lipschitz continuous Lyapunov functions, which are differentiable almost everywhere (a.e.) by the Theorem of Rademacher.

Proposition 2.8. *System (1) is ISDS with $\mu \in \mathcal{KLD}$ and η, $\gamma^{ISDS} \in \mathcal{K}_\infty$, if and only if for each $\varepsilon > 0$ there exists an ISDS Lyapunov function V, which is locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$.*

This follows by Theorem 4, Lemma 16 in [4] and Proposition 3.5.6 in [7].

Remark 2.9. *Note that for an ISDS system it holds that the decay rate μ and gains η, γ^{ISDS} in Definition 2.4 are exactly the same as in Definition 2.7. Recall that in case of ISS systems the trajectory gains are in general different from the Lyapunov ones.*

In order to have ISDS Lyapunov functions with more regularity one can use Lemma 17 in [4], which shows that for a locally Lipschitz function V there exists a smooth function $\tilde{V}$ arbitrary close to V.

Now we consider interconnected systems of the form (2).

Definition 2.10. *We call the i-th subsystem of (2) ISDS, if there exists a $\mathcal{KLD}$-function μ_i and functions η_i, γ_i^{ISDS} and $\gamma_{ij}^{ISDS} \in \mathcal{K}_\infty \cup \{0\}$, $i, j = 1, \ldots, n$ with $\gamma_{ii}^{ISDS} = 0$ such that the solution $x_i(t, x_i^0, u) = x_i(t)$ with any initial value $x_i(0) = x_i^0$ and any inputs x_j, u satisfies*

$$|x_i(t)| \leq \max\left\{\mu_i(\eta_i(|x_i^0|), t), \max_j \nu_{ij}(x_j, t), \nu_i(u,t)\right\} \tag{8}$$

for all $t \in \mathbb{R}_+$, where

$$\nu_i(u,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i(\gamma_i^{ISDS}(|u(\tau)|), t-\tau), \ \nu_{ij}(x_j,t) := \sup_{\tau\in[0,t]} \mu_i(\gamma_{ij}^{ISDS}(|x_j(\tau)|), t-\tau)$$

$i, j = 1, \ldots, n$. γ_{ij}^{ISDS}, γ_i^{ISDS} are called (nonlinear) robustness gains. The ISDS gain matrix Γ^{ISDS} is defined by $\Gamma^{ISDS} := \left(\gamma_{ij}^{ISDS}\right)$, $i, j = 1, \ldots, n$ and the map $\Gamma^{ISDS} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma^{ISDS}(s) := (\max_j \gamma_{1j}^{ISDS}(s_j), \ldots, \max_j \gamma_{nj}^{ISDS}(s_j))^T, \ s \in \mathbb{R}_+^n. \tag{9}$$

Note that by $\gamma_{ij}^{\text{ISDS}} \in \mathcal{K}_\infty \cup \{0\}$ and for $v,\ w \in \mathbb{R}_+^n$ we get

$$v \geq w \ \Rightarrow\ \Gamma^{\text{ISDS}}(v) \geq \Gamma^{\text{ISDS}}(w).$$

Definition 2.11. *For vector valued functions $x = (x_1^T, \ldots, x_n^T)^T : \mathbb{R}_+ \to \mathbb{R}^{\sum_{i=1}^n N_i}$ with $x_i : \mathbb{R}_+ \to \mathbb{R}^{N_i}$ and times $0 \leq t_1 \leq t_2,\ t \in \mathbb{R}_+$ we define*

$$\left| x(t) \right| := (|x_1(t)|, \ldots, |x_n(t)|)^T \in \mathbb{R}_+^n.$$

For $u \in \mathbb{R}^m$, $t \in \mathbb{R}_+$ and $s \in \mathbb{R}_+^n$ we define

$$\overline{\gamma}^{ISDS}(|u(t)|) := \left(\gamma_1^{ISDS}(|u(t)|), \ldots, \gamma_n^{ISDS}(|u(t)|)\right)^T \in \mathbb{R}_+^n,$$
$$\overline{\mu}(s,t) := (\mu_1(s_1,t), \ldots, \mu_n(s_n,t))^T \in \mathbb{R}_+^n,\ \ \overline{\eta}(s) := (\eta_1(s_1), \ldots, \eta_n(s_n))^T \in \mathbb{R}_+^n.$$

Now we can rewrite condition (8) in the form

$$\begin{aligned} \left| x(t) \right| \leq \max \Bigg[& \overline{\mu}\left(\overline{\eta}\left(\left| x^0 \right|\right), t\right), \sup_{\tau \in [0,t]} \overline{\mu}\left(\Gamma^{\text{ISDS}}\left(\left| x(\tau) \right|\right), t-\tau\right), \\ & \sup_{\tau \in [0,t]} \overline{\mu}(\overline{\gamma}^{\text{ISDS}}(|u(\tau)|), t-\tau) \Bigg] \end{aligned} \tag{10}$$

for all $t \in \mathbb{R}_+$. Note that the maximum, supremum and essential supremum used in (10) for vectors are taken componentwise.

If we define $N := N_1 + \ldots + N_n$, $x := (x_1^T, \ldots,\ x_n^T)^T$ and $f := (f_1^T, \ldots,\ f_n^T)^T$, then (2) becomes the system of the form (1). Now the question arises under which condition the whole system (1) is ISDS with respect to the input u and state x?

Recall that the small-gain theorem for two interconnected ISS systems was proved in [8]. This result was extended for the case of $n \geq 2$ interconnected ISS systems in [10], Theorem 4.4, where the small-gain condition is of the form

$$\Gamma(s) \not\geq s,\ \forall\ s \in \mathbb{R}_+^n \setminus \{0\}. \tag{11}$$

From (10), using the $\mathcal{KLD}$-property of μ and with $\Gamma^{\text{ISS}} := \Gamma^{\text{ISDS}}$, $\overline{\gamma}^{\text{ISS}} := \overline{\gamma}^{\text{ISDS}}$, $\overline{\beta}(r,t) := \overline{\mu}(\overline{\eta}(r), t)$ we get

$$\left| x(t) \right| \leq \max\left\{\overline{\beta}\left(\left| x^0 \right|, t\right), \Gamma^{\text{ISS}}\left(\left\| x \right\|\right), \overline{\gamma}^{\text{ISS}}(\|u\|)\right\}.$$

This implies that each subsystem of (2) is ISS and by the small-gain condition (11) their interconnection is ISS and hence ISDS, since by Proposition 3.4.4 (ii) in [7] the ISDS property is equivalent to ISS. Unfortunately by use of this equivalence we loose the quantitative information about the rate and gains of the ISDS estimation for the whole system.

In order to conserve the quantitative information of the ISDS rate and gains of the overall system we prove an ISDS small-gain theorem using ISDS Lyapunov functions in the following section.

3. Main results

In this section we provide a Lyapunov version of the ISDS small-gain theorem for $n \in \mathbb{N}$ interconnected systems, where we give an explicit construction method of an ISDS Lyapunov function and the rate and gains of the ISDS estimation for the whole system.

For the main result in this section we consider system (2) and define the ISDS Lyapunov functions of the subsystems by

Definition 3.1. *Given $\varepsilon_i \in (0,1)$, a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous on $\mathbb{R}_i^N \setminus \{0\}$ is called ISDS Lyapunov function of the i-th subsystem in (2) for $i = 1, \ldots, n$, if it satisfies:*

(i) There exists a function $\eta_i \in \mathcal{K}_\infty$ such that

$$\frac{|x_i|}{1+\varepsilon_i} \leq V_i(x_i) \leq \eta_i(|x_i|) \tag{12}$$

for all $x_i \in \mathbb{R}^{N_i} \setminus \{0\}$.

(ii) There exist functions $\mu_i \in \mathcal{KLD}$, $\gamma_i^{ISDS} \in \mathcal{K}_\infty \cup \{0\}$, $\gamma_{ij}^{ISDS} \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, n$, $i \neq j$ such that for almost all $x_i \in \mathbb{R}^{N_i}$ and all essentially bounded and measurable inputs $u \in \mathbb{R}^m$

$$\begin{aligned} V_i(x_i) > \max\{\gamma_i^{ISDS}(|u|), \max_j\{\gamma_{ij}^{ISDS}(V_j(x_j))\}\} \\ \Rightarrow \nabla V_i(x_i) f_i(x_1, \ldots, x_n, u) \leq -(1-\varepsilon_i)\, g_i(V_i(x_i)), \end{aligned} \tag{13}$$

holds, where $\mu_i \in \mathcal{KLD}$ solves the equation $\frac{d}{dt}\mu_i(r,t) = -g_i(\mu_i(r,t))$, $r, t > 0$ for some locally Lipschitz continuous function $g_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, n$.

For the proof of the main result in this section we will need the following:

Definition 3.2. *A continuous path $\sigma \in \mathcal{K}_\infty^n$ is called an Ω-path with respect to Γ if*

(i) *for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;*

(ii) *for every compact set $K \subset (0,\infty)$ there are constants $0 < c < C$ such that for all points of differentiability of σ_i^{-1} and $i = 1, \ldots, n$ we have*

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C, \ \forall r \in K;$$

(iii) *it holds $\Gamma(\sigma(r)) < \sigma(r)$, $\forall r > 0$.*

Remark 3.3. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be a gain matrix. If Γ satisfies the small-gain condition (11), then there exists an Ω-path σ with respect to Γ.*

The proof can be found in [12], Theorem 5.2, see also [14], however only the existence is proved in these works. It was noted there that if one finds a point $s \in \mathbb{R}_+^n$ with $\Gamma(s) < s$ then there is a possibility to construct a (finite) path connecting the origin to this point and satisfying the Definition 3.1 locally, i.e., in each point of the path between s and the origin. In general it is a nontrivial problem to find such s, especially in case of large n. However if Γ is defined in terms of maximization of gains as it is done in (9) the solution is very simple and one can construct a finite but arbitrary "long" Ω-path:

Proposition 3.4. *If* Γ *satisfies the small-gain condition (11), then* $\forall\, R > 0$, *and* $P \in \mathbb{R}^n_+$ *with* $P_i \geq R$ *there exist monotone and strictly increasing functions* $\sigma_i,\ i = 1, \ldots, n$ *such that* $\sigma := (\sigma_1, \ldots, \sigma_n)^T : [0, 1] \to \mathbb{R}^n_+$ *with* $\sigma(0) = 0$ *and* $\sigma(1) = P$.

Proof. Let Ω be the set of the points $s \in \mathbb{R}^n_+$ satisfying $\Gamma(s) < s$. By Remark 2.8. in [15] it follows that for any $x \in \mathbb{R}^n_+$ we have $\Gamma(Q(x)) \leq Q(x)$, where

$$Q(x) := \max\{x, \Gamma(x), \Gamma^2(x), \ldots, \Gamma^{n-1}(x)\},$$

i.e., $Q(x)$ is in Ω or it belongs to the boundary of Ω. Since Ω is an open domain it is easy to find a vector $P \in \mathbb{R}^n_+$ (searching an arbitrary small vicinity of $Q(x)$) such that $\Gamma(P) < P$. Taking $x > (R, \ldots, R)^T$ in $Q(x)$ sufficiently large we will find a P sufficiently large. In [12] and [14] it was shown that the sequence $\Gamma^k(P),\ k = 0, 1, \ldots$ converges to the origin and the linear interpolation of these points yields the desired path. □

Now we present our main result:

Theorem 3.5. *Assume that each subsystem of (2) is ISDS. This means that for each subsystem and for each* $\varepsilon_i \in (0, 1)$ *there exists an ISDS Lyapunov function* V_i, *which satisfies (12) and (13). Let* Γ^{ISDS} *be given by (9), satisfying the small-gain condition (11) and let* $\sigma \in \mathcal{K}^n_\infty$ *be an* Ω*-path from Remark 3.3 with* $\Gamma = \Gamma^{ISDS}$. *Then the whole system (1) is ISDS and its ISDS Lyapunov function is given by*

$$V(x) = \psi^{-1}\left(\max_i \left\{\sigma_i^{-1}\left(V_i(x_i)\right)\right\}\right) \tag{14}$$

with rates and gains

$$\begin{aligned} g(r) &= (\psi^{-1})'\left(\psi(r)\right) \min_i \left\{(\sigma_i^{-1})'(\sigma_i(\psi(r)))g_i(\sigma_i(\psi(r)))\right\},\ r > 0,\\ \eta(r) &= \psi^{-1}(\max_i \left\{\sigma_i^{-1}(\eta_i(r))\right\}),\ r > 0\\ \gamma^{ISDS}(r) &= \psi^{-1}(\max_i \left\{\sigma_i^{-1}(\gamma_i^{ISDS}(r))\right\}),\ r > 0, \end{aligned} \tag{15}$$

where $\psi\left(|x|\right) = \min_i \sigma_i^{-1}\left(\frac{|x|}{\sqrt{n}}\right),\ t \in \mathbb{R}_+$.

Remark 3.6. *Note that the small-gain condition (11) is equivalent to the cycle condition (see [5], Lemma 2.3.14 for details). A k-cycle in a matrix* $\Gamma = (\gamma_{ij})^n_{i,j=1}$ *is a sequence of* $\mathcal{K}_\infty$ *functions* $(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \ldots, \gamma_{i_{k-1} i_k})$ *of length k with* $i_0 = i_k$. *The cycle condition for a matrix* Γ *is that all k-cycles of* Γ *are contractions, i.e.,*

$$\gamma_{i_0 i_1} \circ \gamma_{i_1, i_2} \circ \ldots \circ \gamma_{i_{k-1}, i_k} < \mathrm{Id},$$

for all $i_0, \ldots, i_k \in \{1, \ldots, n\}$ *with* $i_0 = i_k$ *and* $k \leq n$. *See for example [5] and [6] for further details.*

The proof of Theorem 3.5 follows the idea of the proof of Theorem 5.3 in [12] and corresponding results in [11] with changes to construct the gains and rate of the whole system as in (15).

Proof. Let $0 \neq x = \left(x_1^T, \ldots, x_n^T\right)^T$. We define

$$\overline{V}(x) := \max_i \left\{\sigma_i^{-1}\left(V_i(x_i)\right)\right\}, \overline{\eta}(|x|) := \max_i \left\{\sigma_i^{-1}(\eta_i(|x|))\right\}, \psi\left(|x|\right) := \min_i \sigma_i^{-1}\left(\frac{|x|}{\sqrt{n}}\right),$$

where V_i satisfies (12) for $i = 1, \ldots, n$. Note that $\sigma_i^{-1} \in \mathcal{K}_\infty$. Let j be such that $|x|_\infty = |x_j|_\infty$ for some $j \in \{1, \ldots, n\}$, then by

$$\max_i \sigma_i^{-1}\left(\frac{|x_i|}{1+\varepsilon_i}\right) \geq \max_i \sigma_i^{-1}\left(\frac{|x_i|_\infty}{1+\varepsilon}\right) \geq \sigma_j^{-1}\left(\frac{|x_j|_\infty}{1+\varepsilon}\right) \geq \min_i \sigma_i^{-1}\left(\frac{|x|}{\sqrt{n}(1+\varepsilon)}\right) \qquad (16)$$

where $\varepsilon := \max_i \varepsilon_i$. We have

$$\psi\left(\frac{|x|}{1+\varepsilon}\right) \leq \overline{V}(x) \leq \eta(|x|). \qquad (17)$$

Note that $\overline{V}$ is locally Lipschitz continuous and hence it is differentiable almost everywhere. For any $i \in \{1, \ldots, n\}$ consider open domains $M_i \in \mathbb{R}^N \setminus \{0\}$ defined by

$$M_i := \left\{\left(x_1^T, \ldots, x_n^T\right)^T \in \mathbb{R}^N \setminus \{0\} : \sigma_i^{-1}(V_i(x_i)) > \max_{j \neq i}\left\{\sigma_j^{-1}(V_j(x_j))\right\}\right\}.$$

Now for any $\hat{x} = \left(\hat{x}_1^T, \ldots, \hat{x}_n^T\right)^T \in M_i$ it follows that there is a neighborhood U of $\hat{x}$ such that $\overline{V}(x) = \sigma_i^{-1}(V_i(x_i))$ holds for all $x \in U$. Let $\overline{\gamma}^{\text{ISDS}}(|u|) := \max_j \left\{\sigma_j^{-1}(\gamma_j^{\text{ISDS}}(|u|))\right\}$, $j = 1, \ldots, n$. Assume $\overline{V}(x) > \overline{\gamma}^{\text{ISDS}}(|u|)$. Then

$$V_i(x_i) = \sigma_i(\overline{V}(x)) > \sigma_i(\sigma_i^{-1}(\gamma_i^{\text{ISDS}}(|u|))) = \gamma_i^{\text{ISDS}}(|u|).$$

>From Definition 3.2 (iii) and $x \in M_i$ we have

$$V_i(x_i) = \sigma_i(\overline{V}(x)) > \max_{j \neq i} \gamma_{ij}^{\text{ISDS}}(\sigma_j(\overline{V}(x))) \geq \max_{j \neq i} \gamma_{ij}^{\text{ISDS}}(V_j(x_j)).$$

Thus (13) implies for almost all $x \in M_i$

$$\nabla \overline{V}(x) f(x,u) \leq -(1-\varepsilon_i)\left(\sigma_i^{-1}\right)'(V_i(x_i)) g_i(V_i(x_i)) = -(1-\varepsilon_i)\tilde{g}_i(\overline{V}(x)),$$

where $\tilde{g}_i(r) := \left(\sigma_i^{-1}\right)'(\sigma_i(r)) g_i(\sigma_i(r))$ is positive definite and locally Lipschitz. As index i was arbitrary in these considerations, with $\overline{\gamma}^{\text{ISDS}}(|u|) = \max_j \left\{\sigma_j^{-1}(\gamma_j^{\text{ISDS}}(|u|))\right\}$ and $\overline{g}(r) := \min_i \tilde{g}_i(r)$, $\varepsilon = \max_i \varepsilon_i$ the condition (6) for the function $\overline{V}$ is satisfied. From (17) we get

$$\frac{|x|}{1+\varepsilon} \leq \psi^{-1}\left(\overline{V}(x)\right) \leq \psi^{-1}\left(\overline{\eta}(|x|)\right)$$

and we define $V(x) := \psi^{-1}\left(\overline{V}(x)\right)$ as the ISDS Lyapunov function candidate with $\eta(|x|) := \psi^{-1}\left(\overline{\eta}(|x|)\right)$. Note that $\psi^{-1} \in \mathcal{K}_\infty$ and $V(x)$ is locally Lipschitz continuous. By the previous calculations for $\overline{V}(x)$ it holds

$$V(x) \geq \psi^{-1}\left(\overline{\gamma}^{\text{ISDS}}(|u|)\right) =: \gamma^{\text{ISDS}}(|u|) \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t} V(x) \leq -(1-\varepsilon) g\left(V(x)\right), \text{ a.e.,}$$

where $g(r) := (\psi^{-1})' (\psi(r)) \overline{g} (\psi(r))$ is locally Lipschitz continuous. Altogether $V(x)$ satisfies (5) and (6). Hence $V(x)$ is the ISDS Lyapunov function of the whole system and by application of Proposition 2.8 the whole system is ISDS. □

In the following we present a Corollary, which is similar to Theorem 10 in [4] for two coupled systems and covers $n \in \mathbb{N}$ coupled systems, where the rates and gains defined in Theorem 3.5 are used. We get decay rates for the trajectories of the whole system and each subsystem of n coupled systems with external input $u = 0$.

Corollary 3.7. *Consider system (2) and assume that all subsystems are ISDS with decay rates μ_i and gains η_i, γ_i^{ISDS} and γ_{ij}^{ISDS}, $i, j = 1, \dots, n$, $i \neq j$. If the small-gain condition (11) is satisfied, then the coupled system*

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix} = f(x) \tag{18}$$

is globally asymptotically stable at 0 (0-GAS) with

$$|x_j(t)| \leq |x(t)| \leq \mu \left(\psi^{-1} \left(\max_i \left\{ \sigma_i^{-1} \left(\eta_i \left(\left| x^0 \right| \right) \right) \right\} \right), t \right) \tag{19}$$

for $i, j = 1, \dots, n$, all $t \in \mathbb{R}_+$, with functions μ, σ, ψ and η_i from Theorem 3.5.

Remark 3.8. *Note that for large n function ψ in (15) becomes "small" and hence the rates and gains defined by ψ^{-1} become "large" which is not desired in applications. To avoid this kind of conservativeness one can use the maximum norm $|x|_\infty$ for the states in the above definitions and in Theorem 3.5 and Corollary 3.7. This is possible as we have noted in Remark 2.6. In this case the devision by $\sqrt{n}$ in* (16) *can be avoided and we get* (15) *with $\psi(|x|_\infty) = \min_i \sigma_i^{-1}(|x|_\infty)$. This is used in our examples below.*

Unfortunately we cannot compare directly the estimation of Theorem 10 in [4] with our estimation (19), since another approach for estimations of the trajectories for two coupled systems was used in [4]. The extension of this approach to $n > 2$ seems to be hardly possible. Our approach allows to consider n interconnected systems.

4. Examples

To compare Theorem 10 in [4] with Corollary 3.7 for the case of two subsystems we consider the Example 12 given in [4].

Example 4.1. *Consider two interconnected systems*

$$\dot{x}_1(t) = -x_1(t) + \frac{x_2^3(t)}{2}, \quad \dot{x}_2(t) = -x_2^3(t) + x_1(t).$$

As in [4] we choose $V_i = |x_i|$ and $\gamma_1(r) = \frac{2}{3}r^3$, $\gamma_2(r) = \sqrt[3]{\frac{4}{3}r}$, $\eta_1, \eta_2 = \mathrm{Id}$, $g_1(r) = \frac{1}{4}r$, $g_2(r) = \frac{1}{4}r^3$. It is easy to check that the small-gain condition is satisfied and an

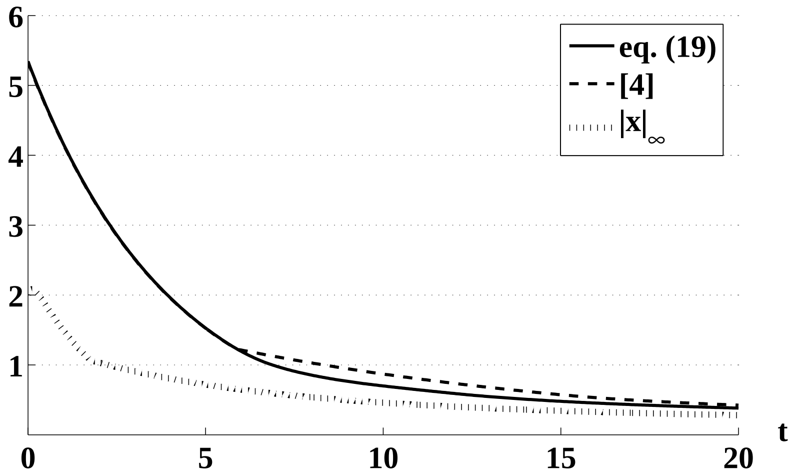

Figure 1: $|x|_\infty$ and estimations with help of Corollary 3.7 (solid curve) and Example 12 in [4] (dashed curve)

Ω-path can be chosen by $\sigma_1(r) = \mathrm{Id}$, $\sigma_2(r) = \sqrt[3]{\frac{4.49}{3}r}$. For $x_1^0 = x_2^0 = 2$ the solution x was calculated by Matlab. The plot of $|x|_\infty$ as well as its estimations by (19) *and from [4] are shown on Figure 1. To compare our estimation with [4] we plot the ISDS estimation in Example 12 in [4] with respect to the maximum norm for states using Remark 11 in [4]. The solid (dashed) curve is the estimation of $|x|_\infty$ by Corollary 3.7 ([4]).*
Both estimations tend to zero as well as the trajectory and provide nearly the same estimate for the norm of the trajectory as it should be expected.

The advantage of our approach is that it can be applied for larger interconnections. The following example illustrates the application of Theorem 3.5 for a construction of an ISDS-Lyapunov function for the case $n \geq 2$.

Example 4.2. *Consider $n \in \mathbb{N}$ interconnected systems of the form*

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + \sum_{j>1}^{n} \frac{1}{n} b_{1j} x_j^2(t) + \frac{1}{n} u(t), \\ \dot{x}_i(t) &= -a_i x_i(t) + \frac{1}{n} b_{i1} \sqrt{x_1(t)} + \sum_{j>1,\, j\neq i}^{n} \frac{1}{n} b_{ij} x_j(t) + \frac{1}{n} u(t), \ i = 2, \ldots, n, \end{aligned} \tag{20}$$

for $b_{ij} \in [0,1)$, $a_i = (1+\varepsilon_i)$, $\varepsilon_i \in (1,\infty)$ and any input $u \in \mathbb{R}^m$.
We choose $V_i(x_i) = |x_i|_\infty$ as an ISDS Lyapunov function candidate for the i-th subsystem, $i = 1, \ldots, n$ and define

$$\begin{aligned} \gamma_{1j}^{ISDS}(r) &:= b_{1j} r^2, \ j = 2, \ldots, n & \gamma_{j1}^{ISDS}(r) &:= b_{j1} \sqrt{r}, \ j = 2, \ldots, n \\ \gamma_{ij}^{ISDS}(r) &:= b_{ij} r, \ i, j = 2, \ldots, n, \ i \neq j, & \gamma_{i}^{ISDS}(r) &:= r, \ i = 1, \ldots, n, \end{aligned}$$

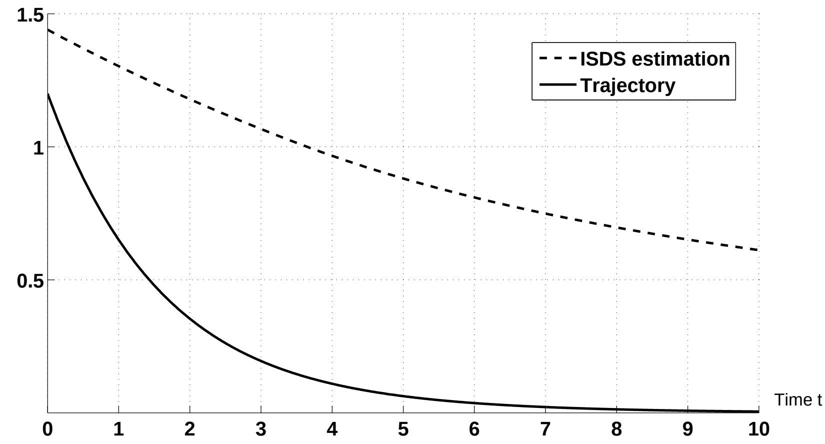

Figure 2: $|x|_\infty$ and ISDS estimation of the whole system consisting of $n = 3$ subsystems of the form (20).

$\Gamma^{ISDS} := \left(\gamma_{ij}^{ISDS}\right),\ i, j = 1, \dots, n,\ \gamma_{ii}^{ISDS} \equiv 0,\ \eta_i(r) := r$ *and* $\mu_i(r,t) = \mathrm{e}^{-\varepsilon_i t} r$ *as solution of* $\frac{\mathrm{d}}{\mathrm{d}t}\mu_i(r,t) = -g_i(\mu_i(r,t))$ *with* $g_i(r) := \varepsilon_i r$ *we obtain that* V_i *is an ISDS Lyapunov function of the i-th subsystem. To check wheather the small-gain condition is satisfied, we use the cycle condition, which is satisfied (this can be easily verified).*
We choose $\sigma(s) = (\sigma_1(s), \dots, \sigma_n(s))^T$ *with* $\sigma_1(s) := s^2$ *and* $\sigma_j(s) := s,\ j = 2, \dots, n$ *for* $s \in \mathbb{R}_+$*, which is one possibility of choosing* σ*. Then* σ *is an* Ω*-path, which can be easily checked, especially* σ *satisfies* $\Gamma^{ISDS}(\sigma(s)) < \sigma(s),\ \forall s > 0$.
Now by application of Theorem 3.5 the whole system is ISDS and the ISDS Lyapunov function is given by

$$V(x) = \psi^{-1}\left(\max_i \sigma_i^{-1}(|x_i|_\infty)\right)$$

with $\psi(r) = \min_i \sigma_i^{-1}(r) = \begin{cases} \sqrt{r}, & r \geq 1, \\ r, & r < 1 \end{cases}$ *. The gains and rates of the ISDS estimation and ISDS Lyapunov function, respectively, are given by (15). Furthermore, if* $u(t) \equiv 0$ *then by Corollary 3.7 the whole system is 0-GAS and the decay rate is given by (19).*
In the following we illustrate the trajectory and the ISDS estimation for a system consisting of subsystems of the form (20) for $n = 3$*. We choose* $a_i = \frac{11}{10},\ b_{ij} = \frac{1}{2},\ i, j = 1, 2, 3,\ i \neq j,\ u(t) = \exp(-t)$ *as input and the initial values* $x_1^0 = 0.5,\ x_2^0 = 0.8$ *and* $x_3^0 = 1.2$*. Then we calculate the ISDS estimation of the whole system as described above and get*

$$|x(t)|_\infty \leq \max\{\mu((x_3^0)^2, t), \operatorname*{ess\,sup}_{\tau \in [0,t]} \mu(\sqrt{u(\tau)}, t - \tau)\}.$$

This estimation is displayed in Figure 2 (dashed line). To verify wheather the norm of the trajectory of the whole system is below the ISDS estimation we solve the system of the form (20) for $n = 3$ *by Matlab. The norm of the resulting trajectory of the whole* *system is also displayed in Figure 2. We see, if the input $u(t)$ tends to zero the ISDS estimation tends to zero as well, whereas in the case of ISS this is not true. Also the norm of the trajectory tends to zero and is below the ISDS estimation.*

5. Conclusions

We have shown that a network of interconnected ISDS subsystems is again ISDS if the small-gain condition (11) is satisfied. In this case we provided explicit expressions for an ISDS Lyapunov function and the corresponding rates and gains of the entire interconnection. As an application of these results we investigated a system of interconnections with zero external input and derived decay rates of the subsystems and the entire system. An example with two systems taken from [4] compares the resulting estimates of the norm of a trajectory obtained by [4] and by (19). Another example with n interconnected ISDS systems illustrates the application of the our main result.

6. Acknowledgement

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