On problem of bounding infinity norm of matrix inverse

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SCiE seminar, Bremen, 10. April 2010

Outline

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Motivations

In many situations in numerical analysis, one is involved with a problem of the form

$$AS = D$$

- A: matrix
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Estimations of error

$$\|S - S_e\| \leq c\|D - D_e\|$$

can be established by the bound for matrix inverse

$$\|A^{-1}\| \leq c$$

Main problem

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For some reasons, **the main interest is the upper bound for the infinity norm of matrix inverse.**

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Ahlberg-Nilson-Varah bound for SDD matrices[Varah 1975]

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_i\{|a_{ii}| - \sum_{j \neq i} |a_{ij}|\}} \tag{1}$$

Bound (1) was firstly mentioned in [Ahlber-Nilson 1963] and then was reproved and extended to the block case by [Varah 1975]

Definition

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Ahlber-Nilson-Varah bound (1975) for block case

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_i\{\|A_{ii}^{-1}\|_\infty^{-1} - \sum_{j\neq i}\|A_{ij}\|_\infty\}} \tag{2}$$

Definition

- M-matrix: non-positive off-diagonal entries(Z-matrix) and positive real part of eigenvalues
- H-matrix: If its collocation matrix, $\mu(A)$, is an M-matrix
- Given H-matrix $A = (a_{ij})$

$$u_A = \{u > 0 : \mu(A)u > 0, \|u\|_\infty = 1\}$$
$$f_A(u) = \min_i \{(\mu(A)u)_i\}, u \in u_A$$

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Note

- An SDD matrix is an H-matrix
- f_A continuous on $\overline{u_A}$ and let $f_A(u_0) = \max\{f_A(u), u \in \overline{u_A}\} > 0.$

Vagar bound for H-matrix[Vagar 1976]

If A is non-singular H-matrix, then

$$\|A^{-1}\|_\infty \le \frac{1}{\max\{f_A(u), u \in \overline{u_A}\}} = \frac{1}{f_A(u_0)} \tag{3}$$

where

$$u_0 = \frac{\mu^{-1}(A)z}{\|\mu^{-1}(A)z\|}, z = [1, \cdots, 1]^T \tag{4}$$

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Note

If A is SDD, $\xi = [1, \cdots, 1]^T \in \overline{u_A}$ and therefore

$$\frac{1}{\min_i\{|a_{ii}| - \sum_{j\neq i}|a_{ij}|\}} = \frac{1}{f_a(\xi)} \geq \frac{1}{\max\{f_A(u), u \in \overline{u_A}\}}$$

Notation

$A = (a_{ij}) \in \mathbb{C}^{nxn}; N = \{1, 2, \cdots, n\}, S \in \mathcal{P}(N)\backslash\{N, \emptyset\};$
$\overline{S} = N\backslash S; r_i(A) = \sum_{j\in N\backslash\{i\}} |a_{ij}|; r_i^S(A) = \sum_{j\in S\backslash\{i\}} |a_{ij}|;$
$T(A) = \{i \in N, |a_{ii}| \leq r_i(A)\};$
$J(A) = \{j \in N, |a_{jj}| - r_i(A) = min_{i\in N}(|a_{ii} - r_i(A) > 0)\}$

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Definition

A is called an S-SDD, with a given proper subset $S \subset N$, if

$$|a_{ii}| > r_i^S(A) \qquad \forall i \in S$$
$$(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\overline{S}}(A)) > r_i^{\overline{S}}(A) r_j^S(A) \qquad \forall i \in S, j \in \overline{S}$$

Moraca bound for S-SDD and SDD matrices[Moraca 2007]

Let A be an S-SDD(SDD) matrix for some proper set S of N s.t. $\emptyset \subset T(A) \subset \overline{S}(J(A) \subset \overline{S} \subset N)$. Then

$$\|A^{-1}\|_\infty \leq \max_{i \in S, j \in \overline{S}} \frac{|a_{ii}| - r_i^S(A) + r_j^S(A)}{(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\overline{S}}(A)) - r_i^{\overline{S}}(A) r_j^S(A)}. \tag{5}$$

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Comments

- Bound (5) is tighter and have a wider range of application than (1).
- Bound (5) is looser but easier to calculate than (3).

Definition

- Denote by $r_i(A)$ the i^{th} row sum of A. Let $\langle m \rangle = \bigcup_{k=1}^{m} N_k$ be a partitioning of the index set N. Then we have $N_1 x N_2 x \ldots x N_m$ aggregated matrices of order m

$$A^{(i_1,\ldots,i_m)} = \begin{bmatrix} r_{i_1}(A_{11}) & \cdots & r_{i_1}(A_{1m}) \\ \ldots & \ddots & \ldots \\ r_{i_m}(A_{m1}) & \cdots & r_{i_m}(A_{mm}) \end{bmatrix}, \; A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \ldots & \ddots & \ldots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}$$

- A is a PM-matrix if A is a Z-matrix and all $A^{(i_1,\ldots,i_m)}$ are non-singular M-matrices.
- A is PH-matrix if $\mu(A)$ is a PM-matrix.

Notes

- If A is a PM-matrix(PH-matrix) with partitioning $\langle m \rangle$ then A is a PM-matrix(PH-matrix) with all partitionings which are finer than $\langle m \rangle$.
- With the finest partitioning, A is a PM-matrix(PH-matrix) iff A is a M-matrix(H-matrix).
- With the coarsest partitioning, A is a PH-matrix iff A is SDD.

★ SDD matrices $\subset$ PH-matrices $\subset$ H-matrices; PM-matrices $\subset$ M-matrices;

Kolotilina's bound for PM- and PH-matrices [Kolotilina 2009]

If A is a PM-matrix(PH-matrix) with given partitioning, then A is non-singular and

$$\|A^{-1}\|_\infty \leq \max_{i_1,\cdots,i_m} \|A^{(i_1,\cdots,i_m)^{-1}}\|_\infty. \tag{6}$$

$$(\|A^{-1}\|_\infty \leq \max_{i_1,\cdots,i_m} \|\mu(A)^{(i_1,\cdots,i_m)^{-1}}\|_\infty). \tag{7}$$

Furthermore, these bounds are monotone w.r.t. the underlying partitioning.

Notes

- If A is an H-matrix, (7) sharpens the Ostrowski result ($|A^{-1}| \leq \mu(A)^{-1}$).
- If A is SDD, i.e. a PH-matrix with the coarsest partitioning, (7) reduces to the ANV bound.
- When the partitioning consists of 2 subsets, (7) reduces to Moraca's bound for S-SDD matrices.

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Conclusion

Review some well-known results on the bound for infinity norm of matrix inverse

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Review some well-known results on the bound for infinity norm of matrix inverse

Open problems

- Relating to scaling technique and the iterative Varah bound for SDD matrix mentioned in [Moraca 2007]
- On the effect of scaling technique on the upper bound of [Kolotilina 2009]

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