Optimal transport governed by transport equations

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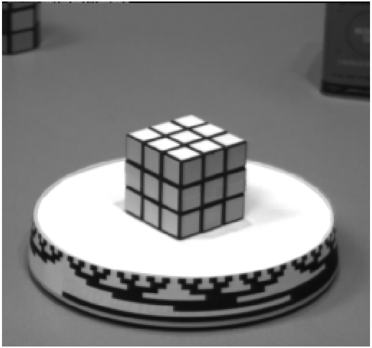
Outline

Optical flow estimation



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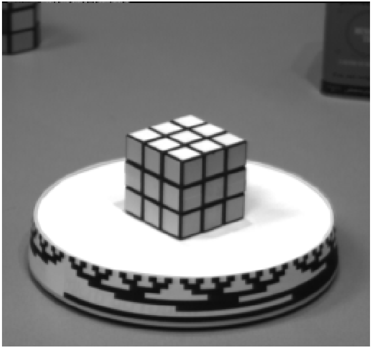
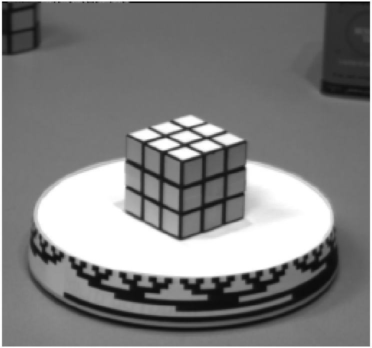
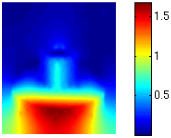


Optical flow estimation



The color plot of the velocity field.

The absolute value of the flow.



The reference color map.

Horn & Schunck method

$$\min_b E(b) = \int_\Omega (u_t + b \cdot \nabla u)^2 \mathrm{d}x\mathrm{d}y + \alpha \sum_{j=1}^{2} \int_\Omega |\nabla b_j|^2 \, \mathrm{d}x\mathrm{d}y.$$

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Euler-Lagrange-Equations:

$$\begin{cases} \Delta b_1^2 - \frac{1}{\alpha}(u_x^2 b_1 + u_x u_y b_2 + u_x u_t) = 0 \\ \Delta b_2^2 - \frac{1}{\alpha}(u_y^2 b_2 + u_x u_y b_1 + u_y u_t) = 0 \end{cases}$$

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How can we compute optical flow by optimal control?

Modeling

Given u_0, u_T and compute “best ”optical flow b by following minimizations problem

$$\min_b J(b) = \frac{1}{2} \|S(u_0, b) - u_T\|^2 + \frac{\lambda}{2} \|b\|^2$$

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governed by PDEs

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla_x u(t,x) = 0 \quad \text{in }]0,T]\times\Omega \\ u(0,x) = u_0(x) \quad \text{in } \Omega \\ \mathrm{div} b(t,\cdot) = 0 \quad \text{in } \Omega, t\in[0,T] \end{cases}$$

where $S = E_T \circ G, \quad S : (u_0, b) \to u(T)$.

Characteristic method

Let $b \in L^2([0, T]; W_0^{1,\infty}(\Omega))$, then the ODE

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(t,x) &= b(t,\Phi(t,x)) \\ \Phi(0,x) &= x \in \Omega. \end{cases}$$

has the unique solution

$$\Phi(t,x) = x + \int_0^t b(s,\Phi(s,x))ds.$$

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Because of the trivial boundary condition of b, the charateristic line lives in Ω for every $t \in [0,T]$.

Properties of the flow

- $\Phi(\cdot, x)$ is continuous w.r.t t.
- $\Phi(t, \cdot) : \Omega \to \Omega$ is bijective for every $t \in [0, T]$.
- $\Phi(t, \cdot)$ is Lipschitz continuous in Ω for every $t \in [0, T]$.
- If b is C^1 in Ω, then $\Phi(t, \cdot)$ is also C^1 in Ω.

The charateristic method

Consider the transport equation

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla_x u(t,x) = 0 \quad \text{in }]0,T] \times \Omega \\ u(0,x) = u_0(x) \quad \text{in } \Omega \end{cases} \tag{1}$$

Let $u_0 \in C^1(\Omega)$ and $b \in L^2([0,T]; W_0^{1,\infty}(\Omega))$. Then

$$u(t,x) = u_0 \circ \Phi^{-1}(t,\cdot)(x)$$

is the unique solution of (1).

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The regularity of u_0 and b is inappropriate for the modeling.

The weak solution of transport equations

Suppose $u_0 \in SBV(\Omega) \cap L^\infty(\Omega)$ and $b \in L^2([0,T]; H_0^{3,\mathrm{div}}(\Omega))$,

$$H_0^{3,\mathrm{div}}(\Omega) := \left\{ f \in H_0^3(\Omega); \mathrm{div} f = 0 \right\}.$$

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Define the weak solution of the transport equation:

$$\int_0^T \int_\Omega u\left(\partial_t \varphi + b \cdot \nabla \varphi\right) dxdt = -\int_\Omega u_0(x)\varphi(0, x)dx.$$

for every $\varphi \in C_c^\infty([0, T[\times\Omega)$.

The weak solution of transport equations

Theorem (Weak solution)

If $b \in L^2([0,T]; H_0^{3,\mathrm{div}}(\Omega))$ and $u_0 \in SBV(\Omega) \cap L^\infty(\Omega)$, then there exits a unique stable weak solution of (1)

$$\hat{u}(t,x) = u_0 \circ \Phi^{-1}(t,\cdot)(x)$$

belonging to $L^\infty([0,T]; SBV(\Omega)) \cap L^\infty([0,T] \times \Omega)$.

Sketch of proof

Existence: Define $u_\epsilon = (u_0 * \eta_\epsilon) \circ \Phi^{-1}$ and prove that (u_ϵ) is uniformly bounded in $L^\infty([0,T]; BV(\Omega))$, which is continuously embedded into $L^2([0,T]; L^2(\Omega))$ and $u_{\epsilon_k} \rightharpoonup \hat{u}$ in $L^2([0,T]; L^2(\Omega))$.

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Uniqueness: Suppose $\tilde{u} \neq \hat{u}$, then $\tilde{u} - \hat{u}$ is a weak solution with initial value $u_0 = 0$. Recall the weak solution $u_0 \circ \Phi^{-1}$ will be 0 a.e..

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Stableness: (u_ϵ) is uniformly bounded in $L^\infty([0,T] \times \Omega)$. From it we can derive that $u_{\epsilon_k} \longrightarrow \hat{u} \quad \text{in } L^p([0,T] \times \Omega)$.

Existence of a minimizer

Set up the cost functional

$$J(b) = \frac{1}{2}\,\|S(u_0, b) - u_T\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\int_0^T \|\nabla\Delta b(t,\cdot)\|^2_{L^2(\Omega)}\,dt.$$

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Theorem (Existence of a minimizer)

Suppose $\Omega \subset \mathbb{R}^2$ open, bounded and $u_0 \in SBV(\Omega) \cap L^\infty(\Omega)$, then the minimization problem

$$\inf_{b \in L^2([0,T]; H_0^{3,\mathrm{div}}(\Omega))} J(b)$$

has a solution.

Lagrange multiplier

$$L(u,b,p) = J(u,b) + \underbrace{\int_0^T \int_\Omega (u_t + b \cdot \nabla u) p \, dx dt}_{Q} .$$

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Well-definedness of Q

Suppose $p, u \in L^\infty([0,T]; SBV(\Omega)) \cap L^\infty([0,T]\times\Omega)$ and $b \in L^2([0,T]; H_0^{3,\mathrm{div}}(\Omega))$, then the following integral

$$\int_0^T \int_\Omega (pb)\cdot Du\,dt < \infty.$$

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The definition of u_t is still open.

First-order optimality condition system

$$
\begin{cases}
u_t + b\cdot\nabla u = 0 & u(0) = u_0 \\
p_t + b\cdot\nabla p = 0 & p(T) = -(u^T - u_T) \\
\mathrm{div} b = 0 & b = 0 \text{ on } \partial\Omega \\
\lambda\Delta^3 b + \nabla q = p\nabla u & b = 0, \nabla_n b = 0, \\
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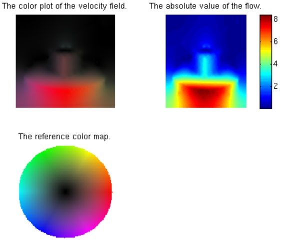
In numerical aspect we discretise the laplace operator of first order.

Numerical scheme for transport equations

Compute u and p using

- Nonlinear total variation diminishing method with flux limiter “superbee ”
- charateristic method, numerical solution of ODE using Runge-Kutta 4th order

Test for transport equations



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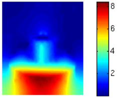
The absolute value of the flow.



The reference color map.

Discrete variational formulation of stokes equation

Find $(b_h, q_h) \in V_h \times W_h$ such that

$$\begin{cases} a(b_h, v) - c(v, q_h) &= -(p\nabla u, v) \quad \forall v \in V_h \subset H_0^1(\Omega), \\ c(b_h, w) &= 0, \quad \forall w \in W_h \subset L_0^2(\Omega), \end{cases}$$

$$a(b_h, v) := \lambda \int_\Omega \nabla b_h : \nabla v dx \qquad c(v, q_h) := \int_\Omega q_h \mathrm{div} v dx$$

$$c(b_h, w) = \int_\Omega w \mathrm{div} b_h dx \quad (p\nabla u, v) := \int_\Omega p\nabla u \cdot v dx$$

P2-P1 Triangulation

Regarding existence and uniqueness of solutions, we have to specify adequate finite element subspaces V_h and W_h such that the discrete inf-sup condition

$$\exists \beta > 0 : \sup_{v \in V_h} \frac{c(v, q_h)}{\|v\| \, \|q_h\|} \geq \beta, \forall q_h \in W_h$$

is fulfilled. E.g. The P2-P1 approximation, so-called Taylor and Hood elements.

Test for optical flow

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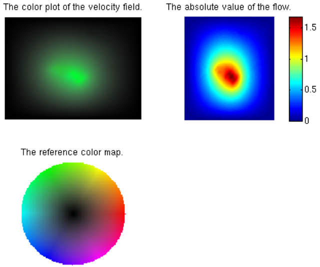

Thank you for your attention!