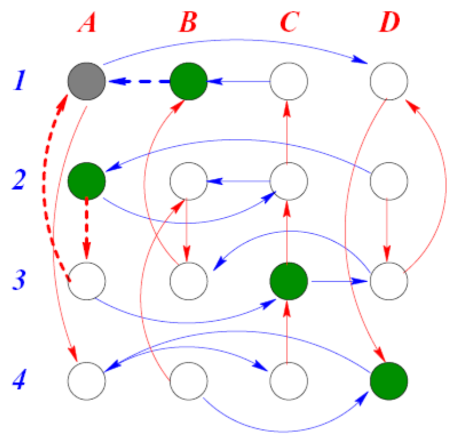
The Stable Marriage Problem



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Each of the men has given you a complete list of the women, ranked similarly.

Your job: Arrange N ”happy”(stable) marriages!

By stable we mean that once the matchmaker has arranged the marriages, there should be no man who says to another woman,

"You know, I love you more than the woman I was matched with - let's run away together!"

where the woman agrees, because she loves the man more than her husband.

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In the spirit of equality, no woman should make such a successful proposal to a man: should she so propose, we want the man to respond,

"Madam, I am flattered by your attention, but I am married to someone I love more than you, so i am not interested."

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Question: Is it always possible for the matchmaker to arrange such a group of stable marriages, regardless of the preference lists of the men and women ?

Application

- Assignment of students to college places
- Medicine students where assigned to hospitals in USA until 1982 (Students prepared a list of favourite hospitals and hospitals wrote a list of favourite students after interviews)
- Lists were sent to a central computer who computed the assignment

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A marriage is a bijective mapping $H : M \longrightarrow F$.
We write

$$(m, f) \in H,\ H(m) = f,\ H^{-1}(f) = m.$$

Definition 1. *A marriage is instable, if there exists* $m \in M, \ f \in F$ *such that*

(1) $(m, f) \notin H$, *that is,* m *and* f *are not married*

(2) m *would prefer to be married with* f *instead of his wife* $H(m)$

(3) f *would prefer to be married with* m *instead of her husband* $H^{-1}(f)$

Example 2. $M = \{anton, bernd, christoph\}$,
$F = \{Anna, Bettina, Carola\}$

anton(a): Bettina, Anna, Carola
bernd(b): Carola, Anna, Bettina
christoph(c): Carola, Bettina, Anna

Anna(A): anton, bernd, christoph
Bettina(B): bernd, christoph, anton
Carola(C): anton, christoph, bernd

One instable marriage(because of anton and Anna):

$$\{aC, bA, cB\}$$

The Algorithm of GALE and SHAPLEY(1962)

while ($\nexists$ a married man $m \in M$){
  m proposes to his favourite f on his list;
  if (f is not married)
    marry m and f;
 else if (f prefers m instead of her husband m'{
    divorce (f, m');
    marry f, m;
    m' removes f from his list; }
  else (f is happy with m')
    m removes f from his list;
}

Test of the algorithm for our example

a proposes to $B \rightarrow \{aB\}$

a	B	A	C
b	C	A	B
c	C	B	A
A	a	b	c
B	b	c	a
C	a	c	b

Test of the algorithm for our example

a proposes to $B \to \{aB\}$
b proposes to $C \to \{aB, bC\}$

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a : satisfied, b : prefers just C, c : satisfied

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Proposition 3. *The algorithm stops after finite steps.*

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Proposition 3. *The algorithm stops after finite steps.*

In each step either a woman is married or an entry is removed from a list. A woman that is married once will be married forever.

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2. The marriage is stable.
   Suppose, H is instable, i.e. $\exists(m, f)$ which prefer each other more than their partners $H(m)$ and $H^{-1}(f)$.
   $\Longrightarrow m$ has proposed to f before $H(m)$ but was rejected because of $H^{-1}(f)$.
   Since the women can only improve their partnership this is a contradiction.

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Definition 6. *A woman f is called unreachable for a man m if there exists no stable marriage H' with $(m, f) \in H'$.*

Proposition 7. *H is men–optimal, i.e. each man marries the best reachable woman. That is, there exists no other stable marriage, where m obtains a better woman.*

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Case2 m' prefers f rather than f'

$$m' : \; \ldots <_{m'} f <_{m'} \ldots <_{m'} f' <_{m'} \ldots$$
$$f : \; \ldots <_f m' <_f \ldots <_f m <_f \ldots$$

$\Longrightarrow$ contradiction to stability of H'

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Proposition 9. *H is "women–pessimal" (opposition of optimal).*

Let the stable marriage $H' \neq H$ be the worst possible arrangement for the women.
Let $(f, m) \in H$, $(f, m') \in H'$.
m : unreachable women $<_m f <_m \ldots$
$f : \ldots m <_f \ldots <_f m' \ldots$
Contradiction to stability of H'!

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- can be modelled as a graph problem $\longrightarrow$ Perfect Matching Problem (A matching is a subset of edges where each edge is connected with at most one knot. A perfect matching is a subset of edges where each edge is connected with exactly one knot.)
- Marry to obtain the most happiness $\longrightarrow$ Largest Perfect Matching Problem
- List uncomplete $\longrightarrow$ yet another solution (Theorem of HALL)

Questions

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- One can find an example where the men–optimal stable marriage is as much as possible worser than the optimal solution of the Perfect Matching Problem.
- Is the solution of the Perfect Matching Problem always stable ?

Conclusion

Back to application: The algorithm prefers the hospitals. In 1981 appeared an article in the New England Journal of Medicine about this fact. Before that, people assumed that the assignment is fair for both.