The marble cake problem

not an answer to Henning's question.

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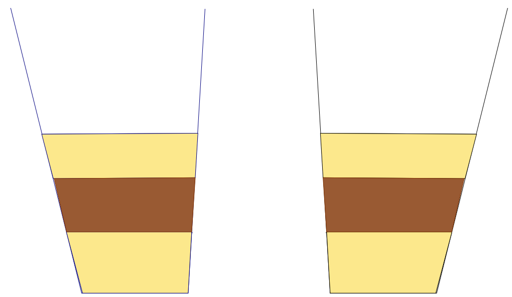
Uttendorf 2004

9.2.2004

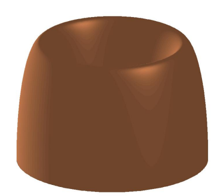
The marble cake problem

- a baking tin filled like

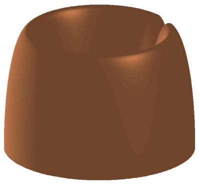


- yields

Question: What happens in the oven?



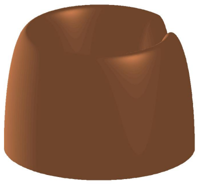
Contents



Baking Theory I

Our batter consists of

- the dry substances
  - sugar
  - salt
  - flour
  - baking powder
- the liquid substances
  - eggs: whites and yolks
  - shortening: butter and oil



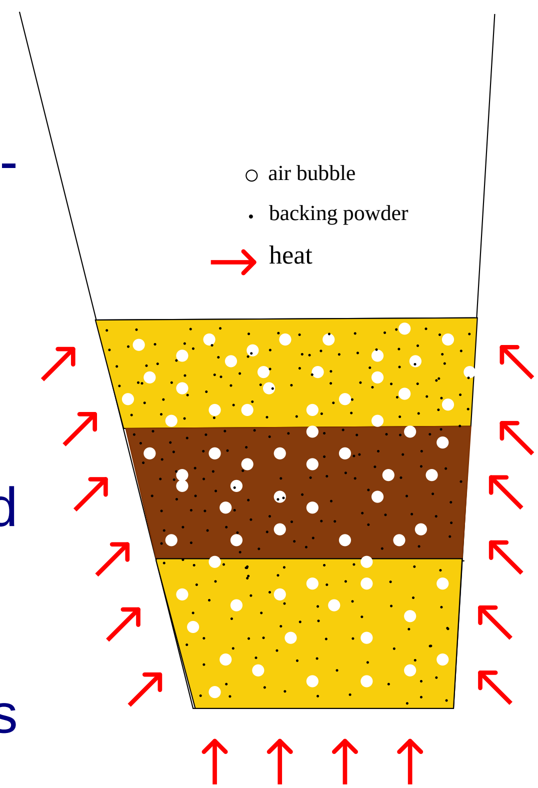
Baking Theory II

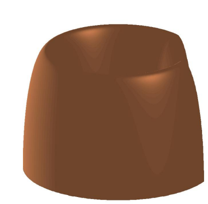
mixing process:

* integrate air bubbles
* distribute the baking powder evenly

heat:

* air bubbles grow
* baking powder reacts and suspends gas bubbles
* the batter raises and gets solid





Baking Theory III

The baking process consists of three stages

1. batter structure is stabilized:

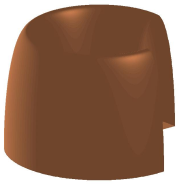
   batter temperature rises, gas cells expand, chemical leavening releases carbon dioxide

2. risen batter is set into its permanent shape by the oven heat

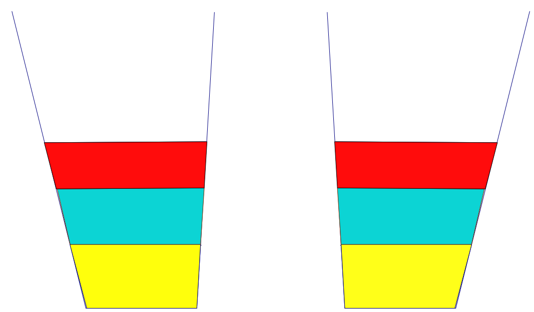
   flour, egg and milk proteins coagulate, starch gelatinizes

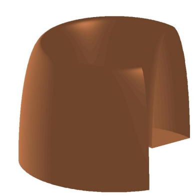
3. batter solidification is completed

   flavor-enhancing browning reactions at the surface

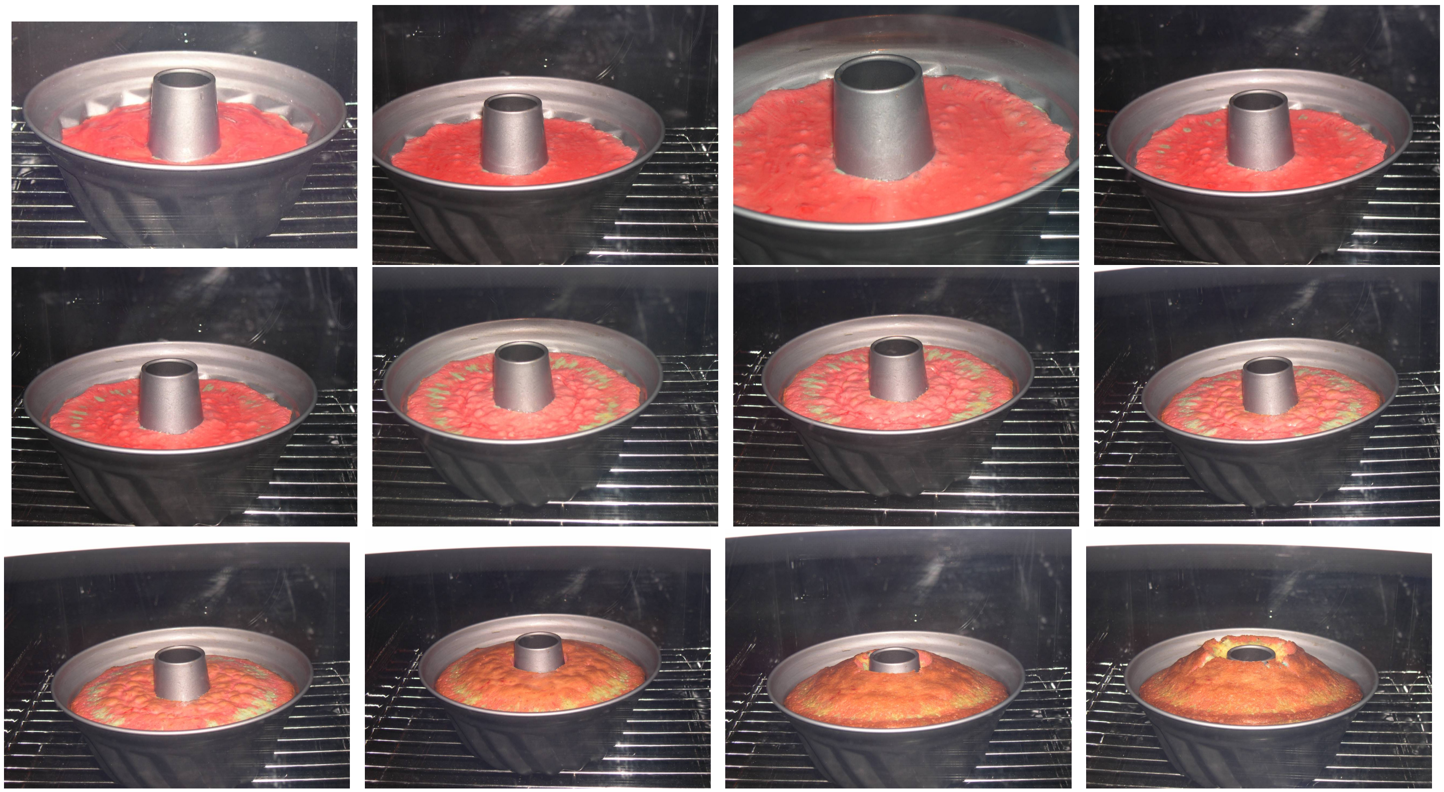


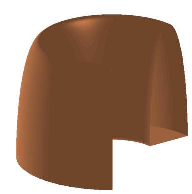
My experimental cake (31/1/2004)



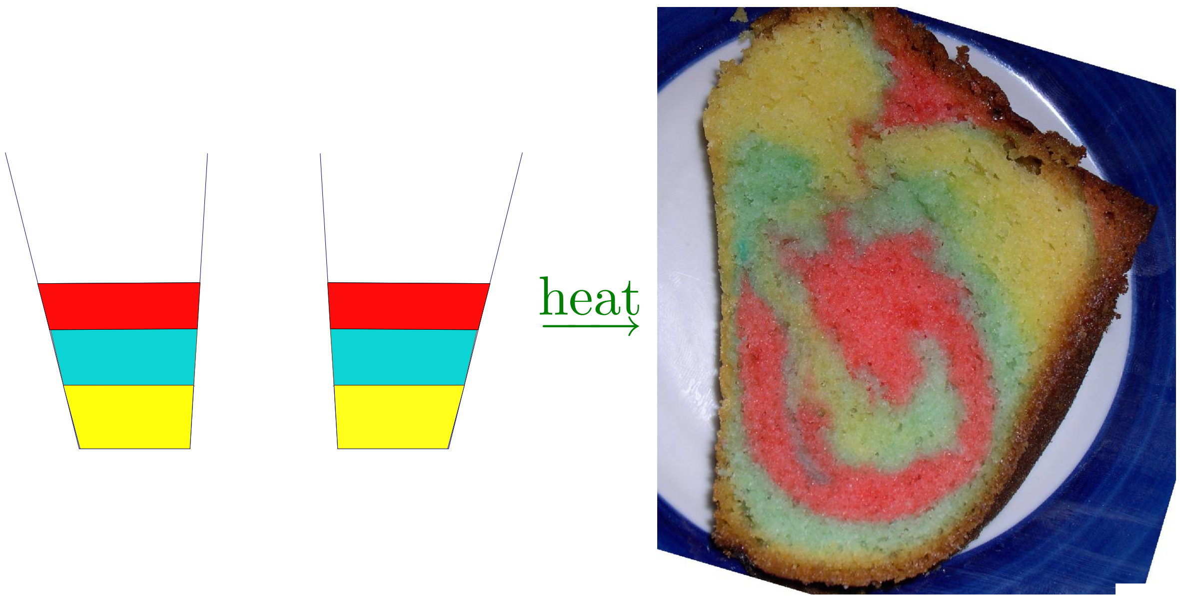


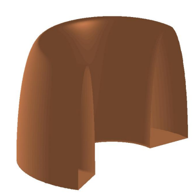
Baking snapshots





The result





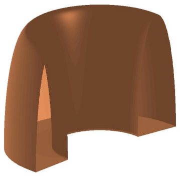
Putting the tin in the oven

- batter in the tin at 20°

$\Rightarrow$ temperature gradient

- preheated oven at 180°

- The batter is heated from the outside (tin boundary) to the inside.
  - The chemical reaction starts at the margins.
  - There must be a heat transfer.



Heat Transfer

Heat can be transferred in two ways:

via molecular mobility
(diffusion)

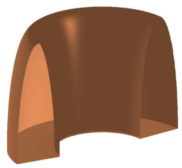
- takes always place
- kinetic energy of molecules (heat) is transferred by momentum exchange

via bulk motion of matter
(convection)

- takes often place

  e.g. Gulf stream, weather (wind)

- kinetic energy of molecules (heat) is transferred by movement of matter



Convection

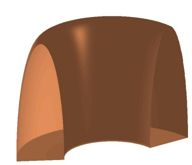
Temperature (or density) gradient
$\Rightarrow$ diffusion (necessary & sufficient)
$\not\Rightarrow$ convection (just necessary)

critical value: Rayleigh-number $\mathrm{Ra} \approx \frac{\text{buoyancy force}}{\text{viscous force}}$

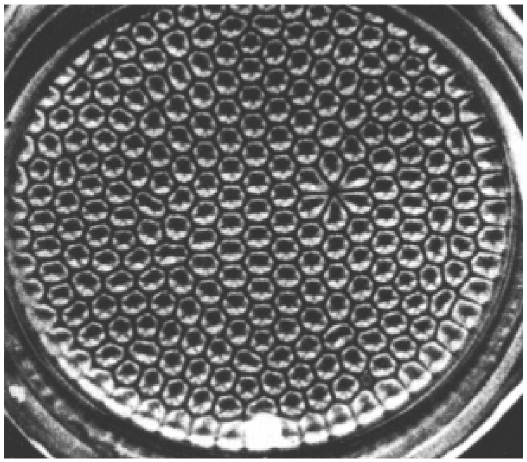
decides whether an inhomogenity (like density or temperature gradients) in a gas or a fluid results in convection

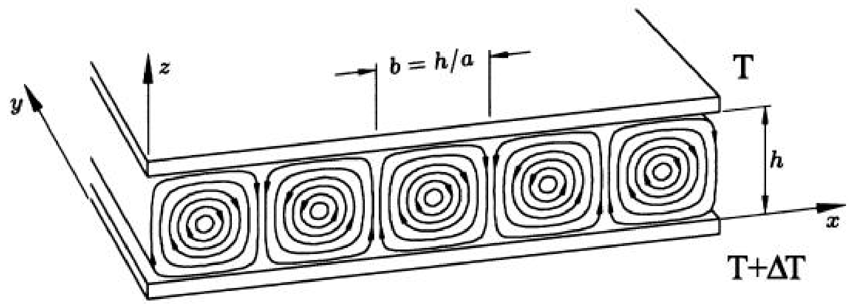
Example: heating a (motionless) viscous fluid from below, e.g. oil.

Till a specific temperature gradient the fluid is nearly motionless and just getting warmer through molecular movement. If the specific gradient is reached, regular convection cells will occure:



Bénard cells





Second experiment: fluid between two plates of different temperature. Rayleigh (1916): instability occurs if $\Gamma = -\frac{dT}{dz}$ was large enough to make the ratio $\mathrm{Ra} = \frac{\alpha g \Gamma h^4}{\kappa \nu}$ exceed a critical value.

g acceleration due to gravity, α coeff of thermal expansion, κ thermal diffusivity, ν kinematic viscosity

Fluid motion: equations

equation of continuity (expresses the conservation of matter)

$\mathbf{u}$ fluid velocity, ρ density, V volume fixed in space

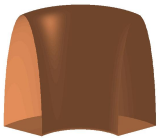
change of mass in a volume V is balanced by the flow out of V

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho \mathbf{u}\cdot\mathbf{n} dA$$

volume fixed in space, Gauss's divergence theorem, volume arbitrary

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

IFF $\rho = \mathrm{const.}$ then $\nabla \cdot \mathbf{u} = 0$ which is true for an incompressible fluid.



Equation of motion I

take element of unit volume, $\mathbf{f}$ force per unit volume, $\mathbf{a}$ acceleration, Newton's Law

$$\rho\mathbf{a} = \mathbf{f}$$

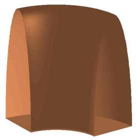
$-\nabla p$ pressure force puv, ϕ potential per unit mass of all conservative forces (e.g. gravity), $\mathbf{f}_{\text{visc}}$ viscous force

$$\rho\mathbf{a} = -\nabla p - \nabla\phi\rho + \mathbf{f}_{\text{visc}}$$

acceleration: rate of change of the velocity: $\frac{\partial \mathbf{u}}{\partial t}$ is rate change of $\mathbf{u}(\mathbf{x}, t)$ at a fixed point in space; we need this rate for a given fluid element

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \equiv \frac{D\mathbf{u}}{Dt}$$

$\frac{D}{Dt}$ is called material derivative: both space and time partial derivatives



Equation of motion II - Viscosity

Viscosity describes shear forces in a moving fluid due to internal friction of different layers. Consider a small, flat, rectangular cell in the water with its faces parallel to the flow. $\triangle F$ is the shear force

$$\frac{\triangle F}{\triangle A} = \mu \frac{\triangle u_x}{\triangle y} \rightarrow \mu \frac{\partial u_x}{\partial y}$$

general case but only half the truth (usually i get lost here)

$$\pi_{xy} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

general case of a compressible fluids needs one more term

$$\pi_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \mu' (\nabla \cdot \mathbf{u}) \delta_{ij}$$

Equation of motion II - still Viscosity

component of $\mathbf{f}_{\text{visc}}$ puv in direction of x_i

$$(\mathbf{f}_{\text{visc}})_i = \frac{\partial \pi_{ij}}{\partial x_j}$$

and for the viscous force

$$\mathbf{f}_{\text{visc}} = \mu \nabla^2 \mathbf{u} + (\mu + \mu') \nabla (\nabla \cdot \mathbf{u}$$

equation of motion

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p - \nabla \phi \rho + \mu \nabla^2 \mathbf{u} + (\mu + \mu') \nabla (\nabla \cdot \mathbf{u})$$

Outlook

- calculations
- more layers
- influence of tin shape
- influence of heating

It's magic! Create your own 1 up to 3,4,5 loops convection cake!

Finished off the cake? So where is the problem?