Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Wavelets From a Historical Perspective

From *Wavelets Algorithms & Applications* by Yves Meyer

Mariya Zhariy

Uttendorf 2006

Wikipedia about Wavelets

- Haar's work in the early 20th century
- Goupillaud, Grossman and Morlet's formulation of CWT (1982)
- Strömberg's early work on discrete wavelets (1983)
- Daubechies' orthogonal wavelets with compact support (1988)
- Mallat's multiresolution framework (1989) etc

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Origins of Wavelet Analysis

- At least seven different origins
- Most of the work was done around the 1930s
- Different concepts, not connected to each other
- Not related to signal or image processing
- Most of the techniques rediscovered in 1980s

Today

- Different concepts come together
- Definition of wavelet and wavelet analysis still flexible
- New aspects through new applications

Historical Overview

1909	Haar suggests an alternative system to Fourier system
1910-1920	Schauder and Faber define 'triangle functions'
1930s	Lévy analyzes Brownian motion in terms of Schauder basis
	Littlewood and Paley form 'dyadic blocks' of the Fourier coeffitients
	Franklin creates an ONB from the Schauder basis
	Lusin looks for 'atom decomposition' of Hardy spaces
1960	Atomic decomposition of L^2 given by Calderón's Identity

Rediscovering of the techniques in 1980s

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

1980s Mallat applies Littlewood-Paley theory in image processing

Holschneider, Tchamitchian use Lusin's technique to analyze the Fractal structure of the Weierstrass function

Grossmann & Morlet with CWT rediscovered the Calderón's identity

Strömberg, Meyer use Haar technique to construct regular well-localized bases

Fourier series

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Instead of entire series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

trigonimetric series (Joseph Fourier, 1807)

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \cos 2x + \dots$$

Fourier series of a 2π-periodic function f:

$$f(x) \sim \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \hat{f}_k \mathrm{e}^{ixk}, \quad \hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \mathrm{e}^{-ixk} dx$$

Paul Du Bois Reymond constructed 1873 a continuous function, whose Fourier series does not converge in classical sense.

Convergence of the Fourier series

Convergence only in few cases

- functions with left and right derivative
- Hölder functions
- functions of bounded variation

New concepts required

- new notion of function/convergence: Norm convergence(Lebesgue), a.e. convergence
- summability of series: Cesàro sums $\sigma_n = \frac{1}{n}(S_0 + \cdots + S_{n-1})$ instead of partial sums
- another orthonormal systems, able to give better representation: Haar wavelets

Haar series of a continuous function on $[0, 1]$ converge uniformly to this function.

Schauder basis

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Critics on the Haar concept: Haar 'atoms' are not continuous themselves.

Faber and Schauder(1910-1920) replaced the Haar functions by it's primitives, triangle functions:

$$\Lambda(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2 - 2x, & 1/2 < x \leq 1 \end{cases}$$

With $\Lambda_0 = x$, $\Lambda_n = \Lambda(2^j \cdot -k)$, $n = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$, triangle functions $\{1, \Lambda_0, \Lambda_1, \ldots\}$ build a Schauder basis of $C([0,1])$.

Schauder coefficients

f continuous on $[0, 1]$

$$f(x) = a + bx + \sum_{1}^{\infty} \alpha_n \Lambda_n(x).$$

The Schauder coefficients are easy to calculate: $f(0) = a$, $f(1) = a + b$. For $f_1(x) = f(x) - a - bx$ we have

$$\alpha_1 = f_1(1/2) = f(1/2) - \frac{1}{2}(f(0) + f(1)).$$

For $n = 2^j + k$, $0 \leq k < 2^j$

$$\alpha_n = f\left(\frac{k + 1/2}{2^j}\right) - \frac{1}{2}\left[f\left(\frac{k}{2^j}\right) + f\left(\frac{k+1}{2^j}\right)\right].$$

Hölder Spaces vs Schauder Basis

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Hölder spaces $C^r([0,1])$, $0 < r < 1$ are defined by

$$|f(x) - f(y)| \leq C|x-y|^r$$

Schauder coefficients for $f \in C^r$

$$|\alpha_n| \leq C 2^{-(j+1)r}$$

Note that $2^j \leq n < 2^{j+1}$

Then the Hölder spaces can be characterized in terms of Schauder coefficients as follows:

$$f \in C^r \iff |\alpha_n| \leq C n^{-r}, \quad 0 < r < 1.$$

1930s: Multifractal structures

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Pointwise calculation of the Hölder exponent r

$$|f(x) - f(x_0)| \leq C|x - x_0|^r$$

Then we look for the largest possible r, denoted by $r(x_0)$. If r varies from point to point, we have a multifractal structure.

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

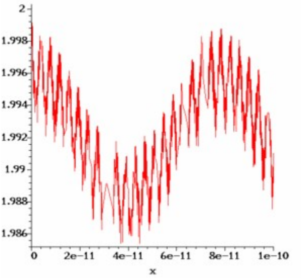
New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Weierstrass function

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 x)$$

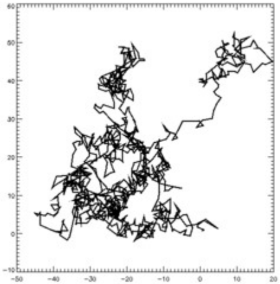


is continuous but nowhere differentiable.

It's multifractal structure, which is not evident from Fourier series, can be analyzed using wavelets of Lusin.

1930s: Analysis of Brownian motion

Brownian motion is a time-continuous stochastic process with normal distributed independent increments.
It's derivative (also known as white noise) can be discretized in time as follows:



$$\frac{d}{dt}X(t,\omega) = \sum_{i\in I} g_i(\omega) Z_i(t),$$

$\{g_i\}_{i\in I}$ – independent normal distributed random variables.

1930s: Hölder exponent of Brownian motion

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Paul Lévy took for $Z_i, i \in I$ the Haar basis.

By taking the primitives we get

$$X(t,\omega) = a_0(\omega) + b_0(\omega)t + \frac{1}{2}\sum_{1}^{\infty} 2^{-j/2} g_n(\omega)\Lambda_n(t)$$

To calculate the Hölder exponent for $X(t,\omega)$ we have to estimate it's Schauder coefficients $2^{-j/2} g_n(\omega)$.

From

$$\sup_{n\geq 2}(|g_n(\omega)|/\sqrt{\log n}) < \infty,$$

follows

$$|X(t+h,\omega) - X(t,\omega)| \leq C\sqrt{h \log \frac{1}{h}}$$

1930s: Energy distribution

Parseval Identity: The energy of 2π-periodic signal f

$$\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx$$

is equal to the squared l^2 Norm of Fourier coefficients $\hat{f}_k$.

Considering for $2 < p < \infty$

$$\|f\|_p = \left(\int_0^{2\pi}|f(x)|^p dx\right)^{1/p},$$

we can see how the energy is distributed over the interval, if it is concentrated around a few points or dispersed.

This information is not accessible by the Fourier coefficients.

1930s: Dyadic summation

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Littlewood and Paley define the dyadic sums

$$D_j f(x) = \sum_{k=2^j}^{2^{j+1}-1} (a_k \cos kx + b_k \sin kx),$$

so that the Fourier series of f is $a_0 + \sum_{j=0}^{\infty} D_j f(x)$.

D_j constitute a bank of band pass filters of length 2^j.

Fundamental result: for $1 < p < \infty$

$$\|f\|_p \sim \left\| \left(|a_0|^2 + \sum |D_j f(x)|^2 \right)^{1/2} \right\|_p,$$

and for $p = 2$ these norms are equal.

1930s: Generalization to $\mathbb{R}^d$

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Antony Zygmund creates a prototype of the mother wavelet ψ as rapidly decreasing C^∞-function with $\hat{\psi} \in C^\infty(\mathbb{R}^d)$ such that

$$\hat{\psi}(\xi) = \begin{cases} 1 & \text{if} \quad 1 + \alpha \leq |\xi| \leq 2 - 2\alpha \\ 0 & \text{if} \quad |\xi| \leq 1 - \alpha \text{ or } |\xi| \geq 2 + 2\alpha \end{cases}$$

with $0 < \alpha \leq 1/3$ and

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^{-j})|^2 = 1 \quad \text{for all} \quad \xi \neq 0.$$

The last condition ensures the energy conservation property.

1930s: Littlewood-Paley-Stein analysis

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

The dyadic theory for $\mathbb{R}^d$ proceeds by setting $\psi_j(x) = 2^{jd/2}\psi(2^j x)$ and replacing the dyadic sums by $\Delta_j(f) = f * \psi_j$.

The Littlewood-Paley-Stein function g is defined by

$$g(x) = \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)(x)|^2 \right)^{1/2}$$

It follows then for all $f \in L^p(\mathbb{R}^d)$ and $0 < p < \infty$

$$\|f\|_p \sim \|g\|_p,$$

and there is an equality for $p = 2$.

1930s: The Franklin system

The Franklin system is the Gram-Schmidt ortogonalization of the Schauder basis (in terms of L^2-inner product). The Franklin sequence $\{f_n\}_{n\geq -1}$

- is ONB for $L^2([0,1])$ unlike Schauder basis
- has two vanishing moments
- gives a characterization of Hölder spaces C^r by $|\langle f, f_n\rangle| \leq Cn^{-1/2-r}$
- has no atomic structure like Haar or Schauder bases
- has the asymptotic behaviour of Strömberg wavelets(1980)

1930s: Decomposition of Hardy spaces

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

A function $f(x+iy)$ defined in the half plane $P = \{x+iy, y>0\}$ belongs to a Hardy space $H^p(\mathbb{R})$, $1 \leq p \leq \infty$, if it is holomorphic in P and

$$\sup_{y>0} \left(\int_{\mathbb{R}} |f(x+iy)^p dx| \right)^{1/p} < \infty$$

H^p is a complex analogon of L^p.

Lusin's work concerns the analysis and synthesis of H^p functions using 'atoms' $(z-\bar{\zeta})^{-2}$ with $\zeta \in P$:

$$f(z) = \int_P (z-\bar{\zeta})^{-2} \alpha(\zeta) du\, dv,$$

where $\zeta = u + iv$.

1930s: Wavelets of Lusin

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Condition on the coefficient function $\alpha(\zeta)$: the functional $A(x)$ defined by

$$A(x) = \left(\int_{\Gamma(x)} |\alpha(u+iv)|^2 v^{-2} du\, dv \right)^{1/2},$$

with $\Gamma(x) = \{(u,v) \in \mathbb{R}^2 : v > |u-x|\}$, must be in L^p.

Then f belongs to $H^p(\mathbb{R})$ and for $1 \le p < \infty$

$$\|f\|_p \le C(p)\|A\|_p.$$

The choice of $\alpha(\zeta)$ is not unique. The natural decomposition with $\alpha(\zeta) = \frac{2i}{\pi} v f'(u+iv)$ leads to

$$\|f\|_p \sim \|A\|_p.$$

1960s: Atomic decompositions concept

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Guido Weiss and Ronald Coifman present a concept of

- atoms or simplest elements
- assembly rules

and interpret the known results for the usual function spaces: L^p, H^p.

In case of Hardy spaces the atoms and assembly rules are given by the functions $(z - \bar{\zeta})^{-2}$ and $A(x)$.

For $L^p([0,1])$-spaces with $1 < p < \infty$ the simplest example is the Haar system (Macinkiewicz, 1938).

1960s: Calderón's Identity

Let $\psi(x) \in L^2(\mathbb{R}^d)$ and it's Fourier transform satisfy

$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{a.e. on} \quad \mathbb{R}^d.$$

Let $Q_t(f) = \underline{f * \psi_t}$ with $\psi_t = t^{-d}\psi(\cdot/t)$, $Q_t^*(f) = f * \tilde{\psi}_t$, where $\tilde{\psi} = \overline{\psi(-\cdot)}$.

Calderón's identity

$$I = \int_0^\infty Q_t Q_t^* \frac{dt}{t}$$

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

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$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{a.e. on} \quad \mathbb{R}^d.$$

Let $Q_t(f) = f * \psi_t$ with $\psi_t = t^{-d}\psi(\cdot/t)$, $Q_t^*(f) = f * \tilde{\psi}_t$, where $\tilde{\psi} = \overline{\psi(-\cdot)}$.

Calderón's identity

$$I = \int_0^\infty Q_t Q_t^* \frac{dt}{t}$$

is another notation for analysis and synthesis steps of CWT.

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

1980s: Wavelets by Grossmann and Morlet

Using the same function ψ, called analysing wavelet, Grossmann and Morlet defined the ’atoms’ by

$$\psi_{a,b}(x) = a^{-d/2}\psi\Big(\frac{x-b}{a}\Big), \quad a > 0, \quad b \in \mathbb{R}^d,$$

so that the analysis step is given by

$$W(a,b) = \langle f, \psi_{a,b} \rangle,$$

and the synthesis step by

$$f(x) = \int_0^\infty \int_{\mathbb{R}^d} W(a,b)\psi_{a,b}(x)db\frac{da}{a^{d+1}}.$$

New was the application to the quantum mechanics.

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

Lusin Wavelets

In case of Hardy spaces $H^p(\mathbb{R})$, $1 \leq p < \infty$ the analysing wavelet, given by

$$\psi(z) = \frac{1}{\pi}(z+i)^{-2}$$

is holomorphic in P and belongs to all of $H^p(\mathbb{R})$. It's Fourier transform $\hat{\psi}(\xi) = -2\xi \mathrm{e}^{-\xi}$ for $\xi \geq 0$ and $\hat{\psi}(\xi) = 0$ if $\xi \leq 0$. We have

$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = \begin{cases} 1 & \text{if} \quad \xi > 0 \\ 0 & \text{if} \quad \xi \leq 0 \end{cases} .$$

This condition assures that $\psi_{a,b}$, $a > 0, b \in \mathbb{R}$ generate $H^2(\mathbb{R})$ instead of $L^2(\mathbb{R})$.

1980s: Strömberg wavelets

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps

From Fourier to Haar

Schauder Basis and Hölder Spaces

New Directions of the 1930s

Multifractal Structures and Brownian Motion

Littlewood and Paley

The Franklin System

The Wavelets of Lusin

From 1960 to 1980

Atomic decompositions

A first synthesis

Franklin system $\{f_n\}$, $n = 2^j + k$, $0 \le k < 2^j$ can be approximated by

$$f_n(x) = 2^{j/2}\psi(2^j x - k) + r_n(x),$$

where

$$\|r_n\|_2 \le C(2-\sqrt{3})^{d(n)}, \quad d(n) = \inf k, 2^j - k.$$

The function ψ was calculated by Strömberg in explicit form. Moreover, ψ

- is continuous, piecewise linear on $\mathbb{R}$
- has exponential decay
- generates orthonormal basis for $L^2(\mathbb{R})$

Definitions of Wavelet

Wavelets From a Historical Perspective

Mariya Zhariy

Introduction & Overview

First steps
From Fourier to Haar
Schauder Basis and Hölder Spaces

New Directions of the 1930s
Multifractal Structures and Brownian Motion
Littlewood and Paley
The Franklin System
The Wavelets of Lusin

From 1960 to 1980
Atomic decompositions

A first synthesis

- Grossmann & Morlet:

$$\psi \in L^2(\mathbb{R}) \quad \text{with} \quad \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \text{ a.e.}$$

- Littlewood-Paley-Stein:

$$\psi \in L^2(\mathbb{R}^d) \quad \text{with} \quad \sum_{\mathbb{Z}} |\hat{\psi}(2^{-j\xi})|^2 = 1 \text{ a.e.}$$

- Franklin & Strömberg: $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ constitute an ONB for $L^2(\mathbb{R})$

Definitions of Wavelet Analysis

Introduction & Overview

First steps
- From Fourier to Haar
- Schauder Basis and Hölder Spaces

New Directions of the 1930s
- Multifractal Structures and Brownian Motion
- Littlewood and Paley
- The Franklin System
- The Wavelets of Lusin

From 1960 to 1980
- Atomic decompositions

A first synthesis

1. Grossmann & Morlet:

$$Wf(a,b) = Q_a^*(f)(b) = \langle f, \psi_{a,b} \rangle, \quad \psi_{a,b}(x) = a^{-d/2}\psi(\frac{x-b}{a}$$

2. Littlewood & Paley replaced a by 2^{-j}

$$(Wf)_j(b) = \Delta_j^*(f)(b) = \langle f, \psi_{2^j,b} \rangle$$

3. Franklin & Strömberg replaced a by 2^{-j} and b by $k2^{-j}$

Definitions of Wavelet Analysis

1. Grossmann & Morlet:

$$Wf(a,b) = Q_a^*(f)(b) = \langle f, \psi_{a,b} \rangle, \quad \psi_{a,b}(x) = a^{-d/2}\psi\Big(\frac{x-b}{a}$$

2. Littlewood & Paley replaced a by 2^{-j}

$$(Wf)_j(b) = \Delta_j^*(f)(b) = \langle f, \psi_{2^j,b} \rangle$$

3. Franklin & Strömberg replaced a by 2^{-j} and b by $k2^{-j}$

and Synthesis

1. follows from the Calderón Identity $I = \int_0^\infty Q_a Q_a^* \frac{da}{a}$
2. follows from the fact $\Delta_j = Q_{2^j}$: $I = \sum_{\mathbb{Z}} \Delta_j \Delta_j^*$
3. is the ONB decomposition

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$