Boundary value problems for PDE with operator type coefficients

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Content:

1. Introduction.[1,2] Let G be a bounded domain in R^2 with $\partial G = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, and D (D does not depend on (x,t)) a domain which is everywhere dense in H. Let $A(x,t)$, $B(x,t)$ be linear operators (possibly unbounded) with domain D, and $u(x,t)$, $((x,t) \in G)$ a function with values in the space H. Let $u(x,t)$ satisfy the equation

$$A(x,t)Lu(x,t) = B(x,t)u(x,t) + f(x,t,u,u_t), \quad (x,t) \in G, \tag{1}$$

$$Lu(x,t) \equiv u_{tt} + a_{11}u_{xx} + a_1 u_t + a_2 u_x, \tag{2}$$

$$a_{11} \in C^2(G)(a_{11} > 0), a_1(x,t), a_2(x,t) \in C^1(G),$$

with boundary conditions in Γ_1

$$\frac{\partial u}{\partial n} \mid_{\Gamma_1} = g, \quad u(x,t) \mid_{\Gamma_1} = f_1. \tag{3}$$

Definition 1. By a solution of the equation (1) we called a two times smooth differeniable function which belongs to the domain of operators A, B for every

$(x,t) \in G$ and satisfies equation (1). Cauchy problem is to find the solution of equation (1) satisfies condition (3) with

$$f, g \in D(A) \cap D(B).$$

Cauchy problem (1),(3) is not in general well posed in the sense of Hadamard. We will prove theorems of uniqueness and stability of Cauchy problem, construct approximate solutions.

2.1. **Subsidary facts.** Cosider equation

$$Ax = f, \ x \in X, \ f \in F, \tag{4}$$

where X and F are Banach spaces, and A is a compact operator. Let $M \subset X$.

Theorem 1.(Tichonov A.N., 1943) Asuume that a solution of the equation (4) is unique and M is a compact set. Then A^{-1} is continuous in M, i.e. there exists a function $\omega(\varepsilon)(\varepsilon \geq 0, \omega(\varepsilon)$ is continuous in 0 and $\omega(0) = 0)$ such that the following inequality

$$\| x_1 - x_2 \| \leq \omega(\| Ax_1 - Ax_2 \|)$$

is valid for all $x_1, x_2 \in M$, where $\| \ . \ \|$ is corresponding norm in X and F, respectively.

Definition 2. (Lavrent'ev M.M.,1959). If there exists a function $\omega(\varepsilon)(\varepsilon \geq 0, \omega(\varepsilon)$ is continuous in 0 and $\omega(0) = 0)$ such that

$$\| x_1 - x_2 \| \leq \omega(\| Ax_1 - Ax_2 \|)$$

for $\forall x_1, x_2 \in M$, then the problem

$$Ax = f, \quad A : D(A) \longrightarrow F, \quad D(A) \subseteq X$$

is called correct in the sense of Tichonov in $M \subseteq D(A)$.

Definition 3.(Bukhgeim A.L., 1971) Problem

$$Ax = f$$

is called l-correct problem, if for $\forall \delta > 0$ there exists a positive constant $c(\delta)$ such that

$$\| x \| \leq \delta l(x) + c(\delta) \| Ax \|$$

for $\forall x \in D(A) \cap D(l)$, where l is a functional corresponding to the problem (4).

Theorem 2. The problem $Ax = f$ is l- correct if and only if it is correct in the sense of Tichonov in any set

$$M_s = \{x \in D(l) \mid l(x) \leq s, \quad s > 0\}.$$

2. Cauchy Problem for ODE with operator coefficients.
2.1. First order equation. Consider (1) with:

$$u(x,t) = u(t), \quad Lu(t) \equiv u_t(t),$$
$$A(x,t) = A(t), \quad B(x,t) = B(t)$$

and $G = [0, T]$, then we have the following Cauchy problem

$$Au_t = Bu + f(t,u), \quad u(0) = u_0. \tag{5}$$

Theorem 3.(S.G.Krein, 1957). Let $A \equiv I, \; B = B^*$ be constant operators, $f = 0$, then for a solution of the equation (5) we have

$$\| u(t) \| \leq \| u(0) \|^{(T-t)/t} \| u(T) \|^{t/T},$$

where the $\| \, . \, \|$ is norm in the Hilbert space H.
The same result is valid in case $A \equiv I, \; BB^* = B^*B$ (B is constant normal operator).
From Theorem 3 follows Cauchy problem for the equation (5) is correct in the sense of Ticchonov in

$$M = \{u : \| u(T) \| \leq m\}.$$

H.A.Levine (1970) proved similar results for A and B which are not constant (symmetric and others). We have given only references that are close to our results. There are a lot of results one can find in corresponding articles. For example, general character results about Cauchy problem belongs to L.Hermander, A.P. Calderon, L.Nirenberg and others.
2.2. **Approximate solution** (K.S.Fayazov, 1992) Let $u(t)$ be a function of a scalar argument t, $0 \leq t \leq T$, with values in a Hilbert space H. Consider the differential equation

$$Au_t = Bu, \tag{6}$$

where B is a positive self-adjoint operator with domain $D(B)$ dense in H and A is a self-adjoint operator establishing an izomorfism of H onto H; moreover, $E^+ + E^- = I$, where E^+ and E^- are the spectral ptojections corresponding to the positive and negative parts of the spectrum of the operator A. To construct approximate solution the corresponding Cauchy problem we use the solution of the following spectral problem:

$$Bv = \lambda Av \tag{7}$$

that was studied by S.Pyatkov. Let φ_k^+ and φ_k^- be the eigenfunctions of problem (7) corresponding to the positive λ_k^+, and negative λ_k^- eigenvalues. It was proved by S.Pyatkov that the eigenfunctions of problem (7) form a Riesz basis for H. Assume that a solution to the problem Cauchy for the equation (6) exists, and

$$u \in \{\| u(T) \|_0 \leq M\}, \; \| f - f_\varepsilon \|_0 \leq \varepsilon,$$

where

$$\| u(t) \|_0^2 = \sum_{i=1}^{\infty} \{| (Uu, \varphi_i^+ |^2 + | (Uu, \varphi_i^- |^2\},$$

$$U = E^+ - E^-.$$

Then we construct an approximate solution to the given problem as follows:

$$u_{N_\varepsilon} = \sum_{k=1}^{N} \{u_{k\varepsilon}^+ \varphi_k^+ + u_{k\varepsilon}^- \varphi_k^-\},$$

where

$$u_{k\varepsilon}^{\pm} = exp(\lambda_k^{\pm} t) f_{k\varepsilon}^{\pm}.$$

We have

$$\| u(t) - u_{N_\varepsilon}(t) \| \leq \varepsilon e^{\lambda_N^+ t} + M e^{(\lambda_{N+1}^+ (t-T))} + \nu(N),$$

where $\nu(N) \longrightarrow 0$ as $N \longrightarrow \infty$.

Remark 1. One can use quazi-inverse method to approximatly solve this problem too. Approximate solution of considering problem will be

$$u_{\alpha\varepsilon}(t) = \sum_{n=1}^{\infty} \{u_{k\varepsilon}^+ \varphi_k^+ + u_{k\varepsilon}^- \varphi_k^-\} e^{-\alpha(\lambda_k^+)^2 t}.$$

Example 1. Let B be a self-adjoint positively defined in $L_2(-1, 1)$ operator which is generated by the differential expression

$$Bu = -\frac{\partial^2 u}{\partial x^2}$$

with bondary conditions

$$u |_{x=-1} = u |_{x=1} = 0.$$

We difine the operator A as the operator of multiplication with the function $\text{sign}(x)$.

2.2. Second order equation.

Consider (1) with:

$$u(x,t) = u(t),\ \ Lu(t) \equiv u_{tt}(t),$$

$$A(x,t) = A(t), B(x,t) = B(t)$$

and $G = [0, T]$, then we have

$$Au_{tt} = Bu + f(t, u, u_t) \tag{8}$$

and corresponding Cauchy problem.

Theorem 4. (S.G.Krein, 1957). Let $A \equiv I$, $B = B^*$ be constant operators, $f = 0$, then for a solution of the equation (8) the following is true

$$\| u(t) \|^2 \leq$$

$$\leq c(t)\{\| u(0) \|^2 + | a |\}^{(T-t)/T}\{\| u(T) \|^2 + | a |\}^{t/T} - | a |,$$

where $\| . \|$ is norm in the Hilbert space H,

$$c(t) = e^{2t(T-t)}, \ a = [(Au(0), u(0)) - (u'(0), u'(0))].$$

The solution of the Cauchy problem is unique and conditionly stable if $A \equiv I$, $B = B^*$.

H.A.Levine has found some estimates for the case when $A(t), B(t)$ are linear symmetric (and others too) operators and right side of equation can be nonlinear function. The uniqueness and stability of the solution of Cauchy problem follow from his results in many cases and he gave many intersting and significanf examples from mathematical physics.

Approximate solution. By the same conditions on operators A and B one can show that the approximate solution for the Cauchy problem is

$$u_{N_\varepsilon} = \sum_{i=1}^{N}\{u_{k\varepsilon}^{+}\varphi_k^{+} + u_{k\varepsilon}^{-}\varphi_k^{-}\},$$

where

$$u_{k\varepsilon}^{+} = ch\sqrt{\lambda_k^{+}}tf_{k\varepsilon}^{+} + g_{k\varepsilon}^{+}sh\sqrt{\lambda_k^{+}}t/\lambda_k^{+},$$

$$u_{k\varepsilon}^{-} = \cos\sqrt{| \lambda_k^{-} |}tf_{k\varepsilon}^{-} + g_{k\varepsilon}^{-}\sin\sqrt{| \lambda_k^{-} |}t/ | \lambda_k^{-} |,$$

$$f_{k\varepsilon}^{\pm} = \pm(Uf_{1\varepsilon}, \varphi_k^{\pm}),$$

$$g_{k\varepsilon}^{\pm} = \pm(Ug_{\varepsilon}, \varphi_k^{\pm}).$$

Remark 2. This problem can be approximatly solved by quazi-inverse method too.

Example 2. Boundary value (ill-posed) problem for mixed (hyperbolik-elliptic) type partial differential equation.

3. Cauchy Problems for elliptic type differential equations with:

3.1. Constant coeffisients.[1,2]

Let $u(x, y)$ be a function of points $(x, y) \in G$ (G is a bounded simply connected domain in R^2 with piecewise smooth boundary ∂G) with values in a Hilbert space H. Consider the equation

$$\Delta u(x, y) = Bu(x, y), \ (x, y) \in G, \tag{9}$$

where B is a linear operator with values in H, and domain $D(B)$ is everywhere dense in H. Morover, let

$$\partial G = \bar{\Gamma_1} \cup \bar{\Gamma_2}, \Gamma_1 \cap \Gamma_2 = \emptyset,$$

$$\frac{\partial u}{\partial n} \mid_{\Gamma_1} = g,\ u(x,y) \mid_{\Gamma_1} = f_1, \tag{10}$$

where $f_1 \in C^1(\Gamma_1; H)$ and $g \in C(\Gamma_1; H)$.
Before stating a theorem on l- correctness of initial problem, we impose some conditions on the operator B and the domain G.
1. Suppose that the operator B in problem is normal operator, i.e. $BB^* = B^*B$. Assume that the operator B_1 ($B = B_1 + iB_2$) posses a complete orthonormal system of eigenfunctions $\{\varphi_k\}$, $k = 1, 2, ...$, and let $\{\lambda_k\}$ and $\{\mu_k\}$ be the corresponding systems of eigenvalues of the operators B_1 and B_2, respectively; moreover,

$$\mid \gamma_1 \mid \leq \mid \gamma_2 \mid \leq \mid \gamma_3 \mid \leq \cdots \leq \mid \gamma_k \mid \leq ...,$$

where $\gamma_k = \lambda_k + i\mu_k$, $k = 1, 2, \cdots$. Grounding on the state conditions, we can rewrite problem (9),(10) as follows:

$$\Delta u_k(x,y) - \gamma_k u_k(x,y) = 0,\ (x,y) \in G \tag{11}$$

$$\frac{\partial u_k}{\partial n} \mid_{\Gamma_1} = g_k,\ u_k(x,y) \mid_{\Gamma_1} = f_{1k},\ k = 1, 2, ..., \tag{12}$$

where

$$u_k(x,y) = (u, \varphi_k),\ f_k = (f, \varphi_k),\ q_k = (q, \varphi_k);$$

moreover,

$$\parallel u(x,y) \parallel^2_{L_2(G;H)} = \sum_{n=1}^{\infty} \parallel u_n(x,y) \parallel^2_{L_2(G)} .$$

2. Suppose that the function $\varphi(z) = s(x,y) + it(x,y)$ $(0 < c_1 < \mid \varphi'(z) \mid < c_2)$ executes a conformal mapping from the domain G into the domain Ω_T; moteover, the part Γ_2 of the part G goes into Γ_2' and Γ_1 into Γ_1' and Ω_t correspondes to G_{xy}. Under such transformation, problem (11), (12) takes the form

$$\Delta U_k(s,t) - a_k(s,t)U_k(s,t) = 0,\ (s,t) \in \Omega_T,$$

$$\frac{\partial U_k(s,t)}{\partial n} \mid_{\Gamma_1'} = Q_k,\ U_k(s,t) \mid_{\Gamma_1'} = F_{1k},\ k = 1, 2, ...,$$

where

$$U_k(s,t) = u_k(x(s,t), y(s,t)), a_k(s,t) = \gamma_k \mid \varphi'(z) \mid^{-2} .$$

Assign $l(U) = \parallel U(s,t) \parallel^2_{L_2(\Omega_T; D(B))}$.

Theorem 5.[1,2] Assume that conditions 1 and 2 are satisfied and $u(x,y)\mid_{\Gamma_1}=0$. Then every solution of problem (9),(10) in the space

$$C^1(\bar{G};H)\cap C^2(G;D(B))\cap L_2(G;D(B))$$

satisfies the inequality

$$\| u(x,y) \|^2_{L_2(G;H)} \leq c_1 l(U) \mid \gamma_N \mid^{-1} + \delta c_N(t),$$

where

$$\delta = \gamma^{1-\omega(t)}\{\| U(s,t) \|^2_{L_2(\Omega_T;H)} + \gamma\}^{\omega(t)},$$

c_1 is constant,

$$\gamma = \theta\{\| U_t(s,t) \|^2 + \| U_s(s,t) \|^2\}_{\Gamma_1'},$$

$\theta \geq 0$ is constant,

$$c_N(t) = exp\{k_N(\omega(t)(T-t_0)+(t_0-t))/k_1\},$$

$k_N \geq 0$ and depends on $N; k_1$ is constant depending on$T$.

Theorem 6.[1,2] A solution of the Cauchy problem in the space

$$C^1(\bar{G};H)\cap C^2(G;D(B))\cap L_2(G;D(B))$$

is unique; moreover, if

$$u(x,y)\mid_{\Gamma_1}=0$$

and $l(u) \leq m$ then the solution satisfies the estimate

$$\| u(x,y) \|^2_{L_2(G;H)} \leq \omega_m(\delta),$$

where $\omega_m(\delta) \sim mc_1^{-1}\{c_2/ln(1/\delta)\}^{1/2}$ for small $\delta \longrightarrow 0$.

Theorem 7.[1,2] Let

$$Lu \equiv u_{xx} + u_{tt},\ A \equiv I$$

and B be a self-adjoint constant operator. Suppose $\omega(x,t)$ satisfies

$$\Delta\omega = B\omega + f(t,x,\omega,\omega_t)$$

and $v \in C^1(\bar{\Omega}_T;H)\bigcap C^2(\Omega_T;H)\bigcap L_2(\Omega_T;D)$ is such that the following

$$\Delta v = Bv + f(t,x,v,v_t) - \varepsilon(v)$$

is defined, and $\omega\mid_{\Gamma_1}=0,\ v\mid_{\Gamma_1}=0$. Let $u=\omega-v$, and

$$\| f(t,x,v,v_t) - f(t,x,\omega,\omega_t) \|^2 \leq$$

$$\leq c_1\int_{t_0}^{t} \| u(\tau,x) \|^2 d\tau + c_2\int_{t_0}^{t} \| u_\tau(\tau,x) \|^2 d\tau,$$

where c_1, c_2 are positive constants. Then there exist $\theta_i \geq 0, i = 1, 2$ such that

$$\iint_{\Omega_t} \| u(\tau, s) \|^2 \, ds d\tau \leq$$

$$\leq c(t)\gamma^{1-\omega(t)}\{\iint_{\Omega_T} \| u(\tau, s) \|^2 \, ds d\tau + \gamma\}^{\omega(t)} - \gamma,$$

where

$$\gamma = \theta_1\{\max_{\Gamma_1}(\| u_t \|^2 + \| u_s \|^2)\} + \theta_2 \| \varepsilon(v) \|_{L_2(\Omega_T;H)^2},$$

$$\omega(t) = \{e^{-pt_0} - e^{-pt}\}/\{e^{-pt_0} - e^{-pT}\},$$

$$c(t) = \exp\{q(\omega(t)T - t)/p\},$$

q, p are some constants.

Theorem 8.[1,2] The solution of the Cauchy problem for the equation (1) is unique in the space

$$C^1(\bar{\Omega}_T; H) \cap C^2(\Omega_T; H) \cap L_2(\Omega_T; D).$$

Theorem 9.[1,2] The solution of the Cauchy problem for the equation (1) is stable in the space

$$C^1(\bar{\Omega}_T; H) \cap C^2(\Omega_T; H) \cap L_2(\Omega_T; D),$$

if

$$u \in \{u : \iint_{\Omega_T} \| u(\tau, s) \|^2 \, ds d\tau \leq M\}, \; u \mid_{\Gamma_1} = 0.$$

3.2. Variable coefficients.[1,2] Let A be a constant self-adjoint operator and $(Au, u) > 0$ for all $u \neq 0$; $(Au, u) = 0 \longrightarrow u = 0$. Let $B(x, t)$ be self-adjoint operator for every $(x, t) \in \Omega_T$ and

$$(B_t u, u) \geq -c(Bu, u),$$

with

$$c = \max\{\max_{(x,t)\in\Omega_T}(| a_{11_t} | + | a_1/2 | + | a_{11_x}/2 |)/a_{11};$$

$$\max_{(x,t)\in\Omega_T}(| a_{11_x} | + | a_1/2 | + | a_2 | + 2c_1)\}.$$

Let ω satisfy

$$AL\omega(x, t) = B\omega(x, t) + f(x, t, \omega, \omega_t),$$

and v satisfy

$$ALv(x, t) = Bv(x, t) + f(x, t, v, v_t) - \varepsilon(v).$$

Let $u = \omega - v$ and

$$\alpha = f(x, t, \omega, \omega_t) - f(x, t, v, v_t),$$

$$\| \alpha \|^2 \leq c_1 \| u \|^2 + c_2 \| u_t \|^2,$$

then

$$ALu(x,t) = Bu(x,t) + \alpha + \varepsilon(v), \tag{13}$$

where c_1, c_2 are constants.

Theorem 10.[1,2] If the coefficients a_{11}, a_1, a_2 satisfy the condition (2), the solution of equation (13) is equal zero on Γ_1 and satisfies the inequality

$$\iint_{\Omega_T} (u, Au) ds d\tau \leq M,$$

then for

$$u \in C^1(\bar{\Omega}_T; H) \cap C^2(\Omega_T; H) \cap L_2(\Omega_T; D)$$

the following inequality is true

$$\iint_{\Omega_T} (u, Au) ds d\tau \leq \gamma^{1-\omega(t)} (M + \gamma)^{\omega(t)} c(t) - \gamma,$$

where

$$\gamma = \theta_1 \max_{\Gamma_1} \{(u_t, Au_t) + u_x, Au_x)\} + \theta_2 \| \varepsilon(v) \|^2_{L_2(\Omega_t; H)} .$$

Remark 3. A similar results one can obtain for arbitrary second order elliptic type operators

$$Lu(x,t) \equiv u_{tt} + \sum_{i,j01}^{n} a_{ij}(x,t) u_{x_i x_j} + \sum_{i=1}^{n} a_i(x,t) u_{x_i} + a_0(x,t) u_t.$$

Example 3. We cosider the equation

$$sign(y)(u_{tt}(t,x,y) + u_{xx}(t,x,y)) = -u_{yy}(t,x,y)$$

in the region $Q = (-1,1) \times \Omega_T$. The equation is a mixed type equation. We will consider the problem: Find the solution of equation in $Q(y \neq 0)$ which satisfies the following boundary conditions:

1.

$$\frac{\partial u(t,x,y)}{\partial n} |_{\Gamma_1} = g, u(t,x,y) |_{\Gamma_1} = f_1;$$

2.

$$u(t,x,-1) = 0, \ u(t,x,1) = 0, \ (x,t) \in \Omega_T;$$

3.

$$u(t,x,-0) = u(t,x,+0), \ u_y(t,x,-0) = u_y(t,x,+0).$$

Theorem 11.[1,2] If a solution of this problem becomes zero on the surface Γ_1 and satisfies

$$\int_0^T \int_{\varphi_1(t)}^{\varphi_2(t)} \int_{-1}^{1} u_y^2(t,x,y) dy dx dt \leq M,$$

then

$$\int_0^t \int_{\varphi_1(t)}^{\varphi_2(t)} \int_{-1}^{1} u_y^2(t,x,y)dydxdt \le \gamma^{1-\omega(t)}(M+\gamma)^{\omega(t)}c(t)-\gamma,$$

where

$$c_1(t) = \exp\{t(T-t)/2\},\ \omega(t) = 1-t/T,$$

$$\gamma = \theta \max_{\Gamma_1} \int_{-1}^{1} \{u_{yx}^2 + u_{yt}^2 + u_{yy}^2\}dy,$$

(θ is constant that depends on T and Γ_1).
From this theorem one can easily see that the uniqueness and stability of the solution of this problem follows.

4.Cauchy problem for degenerate DE and higher order PDE.

4.1. Degenerate DE . Let the following inequality be valid

$$\| t^{2p}D_t^{2n} + Au \|^2 \le$$

$$\le \chi(n) \sum_{k=1}^{\frac{n-1}{2}} C_{2k+1} t^{2p-4k-2} \| D_t^{2n-2k-1}u \|^2 +$$

$$+\chi(n) \sum_{k=1}^{\frac{n-1}{2}} \bar{C}_{2k+1} t^{p-4k-2} \| D_t^{n-2k-1}u \|^2 +$$

$$+(1-\chi(n)) \sum_{k=1}^{\frac{n}{2}} C_{2k+1} t^{2p-4k} \| D_t^{2n-2k}u \|^2 +$$

$$+(1-\chi(n)) \sum_{k=1}^{\frac{n}{2}} \bar{C}_{2k+1} t^{p-4k} \| D_t^{n-2k}u \|^2 +$$

$$+ \sum_{k=n+1}^{2n} C_k t^{2p-2k} \| D_t^{2n-2k}u \|^2 + f(t), \tag{14}$$

where $C_i, \bar{C}_i$ are constants, p is a parametr $(0 \le p < 2n)$, and

$$\chi(n) = \{0, \text{if n is even}; 1, \text{if n is odd}\}.$$

We assume $u(t),\ 0 \le t \le T$, takes value from Hilbert space H, and A has everywhere dense domain in H, moreover:

1.$A$is symmetric, i.e.$(Au,v) = (u,Av),\ v,u \in D(A)$;

2.$(Au,u) \ge 0,\ u \in D(A)$;

3. A is constant operator.

Initial data are given in the form: for any

$$\omega > 0, \delta > 0 \ \ (i = 1, \dots, 2n), \ (j = 1, \dots, n)$$

$$t^{\frac{(2n-1)p}{2n} - 2i + 2} \parallel D_t^{2n-i} u \parallel^2, \ \ t^{\frac{p}{2n} - 2j + 2} \parallel A^{1/2} D_t^{n-j} u \parallel^2 \leq M t^{2\omega + \delta}. \qquad (15)$$

Definition 4. Function $u(t) : [o, T] \longrightarrow H$ is called the solution of inequality (14)if:

$$1. u(t), u_t(t), u_{tt}(t), \dots, D_t^n u, \dots, D_t^{2n} \text{are continuous in} [0, T]$$

;

$$2. D_t^i u \in D(A), \ i = 0, 1, \dots, 2n-2;$$

3. $u(t)$ satisfies (14) and (15).

Theorem 12. (stability). There exits $\omega > 0, \xi_i > 0$ $(i = 2, \dots, 2n)$, and $\eta_i > 0$ $(j = 2, \dots, n)$ such that the inequalities

$$1. C_j < \xi_i (2n - p) \omega^{2i-1}, \ C_j' < \eta_i (2n - p) \omega^{2j-1};$$

$$2. t^{\frac{(-2n+1)p}{2n} + 1} \mid f(t) \mid \leq \varepsilon^2 t^{2\omega + \delta} \ (\varepsilon > 0)$$

and inequality (15) imply

$$\int_0^T t^{-2\omega + \frac{(2n-1)p}{2n} - 2n + 1} \{ \xi_{2n} (2n - p) z^{2n-1} - C_{2n} \} \parallel u \parallel^2 dt \leq$$

$$\leq C \exp\{(-1/2) \ln (\frac{\ln T'^{\ln \frac{M}{\varepsilon}}}{\ln T \ln \ln \frac{M}{\varepsilon}}) \},$$

where $0 < T' < T$, C depends on T,δ, ω; $\varepsilon \leq \varepsilon_0$, ε_0 depends on p and ω; $z \geq \omega$.

Theorem 13. (Uniqueness) Let $f(t) = 0$. If the condition of the above theorem is valid then $u(t) \equiv 0$.

Example 4. We can take as examples to our inequality (14) equation with operator A in the form: $A = \Delta^{2k}$ or $A = -\Delta^{2k-1}$ and corresponding boundary conditions:

1). when $A = \Delta^{2k}$, $2n = 2k$, we get elliptic case $t^p D_t^{2n} u + \Delta^{2k} u = \dots$;

2).When $A = \Delta^{2k}$, $2n \leq (\neq) 2k$, we get parabolic case;

3).If $A = -\Delta^{2k-1}$, $2n \leq (\neq) 2k - 1$ we get parabolic equation;

4). If $n = k = 1$, we get hyperbolic case: $t^p D_t^2 u - \Delta u = \dots$.

Similar type of problem has been studied by H.Cordes, N.Aronszain, S.Alihac, S.Schischatsky, A.Avdeev and others. For example S.Schischatsky (1982) studied Cauchy problem for the following inequality

$$\parallel t^p u_t + A(t) u \parallel^2 \leq b_1 t^{p-1} [u, u] + b_2 t^{2p-2} \parallel u \parallel^2 + f(t),$$

where $0 < p < 1$.

Let

$$(Au, u) \geq 0,\ u \in D(A).$$

We will consider inequality

$$\| t^p u_{tttt} + Au \|^2 \leq$$

$$\leq t^{p/4-1} \ln^{-2} \frac{1}{t} \{c_2 t^p \| u_{tt} \|^2 + \bar{c}_2 \| A^{1/2} u \|^2 + c_3 t^{p/2} \| u_t \|^2 + c_4 \| u \|^2\}, \quad (16)$$

where $c_2, \bar{c}_2, c_3, c_4$ are constant and p is parametr ($p \geq 4$).
Definition 5. The function $u(t) : [0, T] \longrightarrow H$ is called a solution of the inequality (16), if the following hold:
1). u, u_t, u_{tt}, u_{ttt} are continuous for $0 \leq t \leq T$ and $u, u_t, u_{tt}, u_{ttt} \in D(A)$;
2). u_{tttt} exists for $0 \leq t \leq T$;
3). $u(t)$ satisfies (16) for $0 \leq t \leq T$.
Theorem 14. Let $u(t)$ is a solution of the inequality (16), with $p = 4$. If for any $\omega \geq (\neq) 0$

$$t^{-\omega} \| u_{ttt} \|, t^{-\omega} \| u_{tt} \|, t^{-\omega} \| u_t \|, t^{-\omega} \| u \| \longrightarrow 0$$

as $t \longrightarrow 0$, then $u(t) = 0$.
Theorem 15. Let $u(t)$ is a solution of the inequality (16), with $p > 4$. If for any $\omega > 0$

$$\exp(\omega t^{(-p/4+1)}) \{\| u_{ttt} \| + \| u_{tt} \| + \| u_t \| + \| u \|\} \longrightarrow 0$$

as $t \longrightarrow 0$, then $u(t) = 0$.

4.2. Higher order PDE.

a).Let $D(D$ does not depend of $t)$ be an every where dense domain in H, and $A(t)$, $B(t)$, $C(t)$ are linear operators (possibly unbounded) with domain D. Let $u(t)$ be a function with values in the space H. Let $u(t)$ satisfy the equation

$$(B(t)\frac{d}{dt} - A(t))^l (\frac{d}{dt} - C(t))^k u(t) = f,$$

where l, k are given natural numbers and $\frac{d^i u(t)}{dt^i} \mid_{t=0} 0 g_i$, $i = 0, 1, \ldots, l + k - 1$. We have proved the theorems of uniqueness and stability for some of linear symmetric operators.

b). Let $\varphi_t \in C^2(t \geq 0)(i = 1, 2), t \in [0, T]$, and $|\varphi'(t)| t^{1/2} < \mu, (i = 1, 2)$, where μ is a constant. Let Ω_T be a bounded simply connected region in R^2 defined as follows:

$$\Omega_T = \{(x, t) : 0 < t < T, \varphi_1(t) < x < \varphi_2(t), \varphi_1(0) = \varphi_2(0)\},$$

and $\partial\Omega_T = \overline{\Gamma_1} \cup \overline{\Gamma_2}, \Gamma_1 \cap \Gamma_2 = \emptyset$, where

$$\Gamma_1 = \{(x, t) : x = \varphi_i(t), (i = 1, 2), 0 \leq t < T\},$$

$$\Gamma_2 = \{(x, t) : t = T, \varphi_1(t) < x < \varphi_2(t)\}.$$

Let D (does not depend of (x,t)) be everywhere dense domain in Hilbert space H, and $A(x,t)$, $B(x,t)$, $C(x,t)$ and $E(x,t)$ are family of linear operators (possibly unbounded) with domain D, $u(x,t)((x,t)\in\Omega_T)$ is a function with values in the space H. Let $u(x,t)$ satisfies the equation

$$(AL_1-B)(CL_2-E)u(x,t)=0,\ (x,t)\in\Omega_T, \tag{17}$$

where

$$L_iu(x,t)\equiv u_{tt}+a_{11}^iu_{xx}+a_2^iu_x+a_1^iu_t,\ \ (i=1,2)$$

with $a_{11}^i\in C^4(\Omega_T)(a_{11}^i>0)$, a_2^i, $a_1^i\in C^1(\overline{\Omega}_T)$, $(i=1,2)$, and in the part Γ_1 of the bound $\partial\Omega_T$ the boundary values are given

$$\frac{\partial^iu(x,t)}{\partial n^i}\mid_{\Gamma_1}=g_i,\ \ i=0,1,2,3. \tag{18}$$

Similar initial problems one can see in cases:
1) L_1 is defined as above, $L_2u(x,t)\equiv u_{tt}$ or $L_2u(x,t)\equiv u_t$;
2)$L_1u(x,t)\equiv u_{tt}$ (or $L_1u(x,t)\equiv u_t$) and L_2 defined as above.
We prove theorems of uniqueness and stability of problem (17)-(18) using the following lemmas.

Lemma 1. Let A is constant self adjoint positive defined operator, and $s>0$. Then for solution of the following Cauchy problem

$$u'(t)-Au(t)=v(t),\ u(0)=0$$

the following inequality is true

$$\int_0^T\parallel u(\tau)\parallel^2d\tau\le\frac{(u(T),Au(T))}{s}+\frac{exp(2sT^2)}{s}\int_0^T\parallel v(\tau)\parallel^2d\tau.$$

Lemma 2. Let $A$and B be self adjoint constant operators. Let $(Bu,u)\ge\lambda(u,u)$ $(\lambda>0)$ and A^{-1} exists. Then for solution of the equation

$$A\frac{d^2u}{dt^2}=Bu$$

is valid inequality

$$(Bu,u)\le h(t)((Bu(T),u(T))+\mid\alpha\mid)^{t/T}((Bu(0),u(0))+\mid\alpha\mid)^{1-t/T},$$

where

$$h(t)=exp(2t(T-t)),\alpha=\frac{1}{2}(Bu'(0),u'(0))-(Bu(0),\ A^{-1}Bu(0))).$$

Lemma 3.[1, 2] Let $A=C=I$, B and E are constant self adjoint operators. Let $\omega(x,t)$ satisfies equation

$$\Delta\omega-B\omega=v$$

and $\omega \in C^1(\bar{\Omega_T}; H) \cap L_2(\Omega_T; D)$, and $v(x,t)$ satisfies equation

$$\Delta v = Cv$$

and $v \in C^1(\bar{\Omega_T}; H) \cap C^2(\Omega_T; H) \cap L_2(\Omega_T; D)$, and let $\omega \mid_{\Gamma_1} = 0, v \mid_{\Gamma_1} = 0$. Then there are constants $K_i \geq 0$ $(i = 1, 2, 3)$ such that functions $\psi_1(t) = \ln(\int\int_{\Omega_t} \| \omega \|^2 ds d\tau + \gamma_1)$, $\psi_2(t) = \ln(\int\int_{\Omega_t} \| v \|^2 ds d\tau + \gamma_2)$ satisfy the following differential inequalities, respectively,

$$\psi_i''(t) + p_i \psi_i'(t) + q_i \geq 0, (p_i, q_i \geq 0),$$

where

$$\gamma_1 = K_1 \max_{\Gamma_1}(\| \omega_t \|^2 + \| \omega_x \|^2) + K_2 \int_{t_0}^{t} \| \ v \|^2 d\tau,$$

$$\gamma_2 = K_3 \max_{\Gamma_1}(\| v_t \|^2 + \| v_x \|^2).$$

3) Let $L_1 u(x,t) \equiv u_{tt}$ and $L_2 u(x,t) \equiv u_{tt} + u_{xx}$, then using Lemma 2,3 one can get:

Theorem 16. Let $A = C = I$, B and E are selfadjoint constant operators. The solution of the problem (17)-(18) is unique on the space $C^3(\overline{\Omega_T}; H) \cap C^4(\Omega_T; H) \cap L_2(\Omega_T; D)$.

Theorem 17. Let $A = C = I$, B and E are self adjoint constant operators. Let

$$\frac{\partial^i u}{\partial u^i} \mid_{\Gamma_1} = 0, \ \ i = 0, 1, 2, \ \ \frac{\partial^3 u}{\partial u^3} \mid_{\Gamma_1} = g,$$

and

$$\int\int_{\Omega_T} \{ \mid (Bu_{\tau\tau}, u_{\tau\tau}) \mid + \mid (Bu_{ss}, u_{ss}) \mid \} ds d\tau \leq M.$$

Then for solution of the problem (1)-(3) is valid the inequality

$$\int\int_{\Omega_t} \mid u(s,\tau) \mid^2 ds d\tau \leq \gamma_1^{1-w(t)} \cdot (M + \gamma_1)^{w(t)} c_1(t),$$

where

$$\gamma_1 = \Theta_1 \max_{\Gamma_1} |(Bg, g)| \ \ c_1(t) = h(t) \cdot c(t) \cdot \Theta_2,$$

$h(t)$, $c(t)$ are function defined above, Θ_1 and Θ_2 nonnegative constants dependent of $dim \Omega_T$.

Example. We consider the equation

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} \right) \left(sign(y) \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0$$

in the region $Q = (-1, 1) \times \Omega_T$ (Ω_T is defined as above). We will consider the problem:

Finding the solution of equation in $Q(y \neq 0)$ which satisfies the following boundary conditions

1) $\frac{\partial^i u}{\partial n^i}(t,x,u)\Big|_{\Gamma_1'} = g_i,\ \ i = 0,1,2,3$, where

$\Gamma_1' = \Gamma_1 \times (-1,1)$, $\Gamma_2' = \Gamma_2 \times (-1,1)$;

2)$u_y^i(t,x,-1) = 0,\ \ u_y^i(t,x,1) = 0,\ \ (t,x) \in \Omega_T, i = 0,1.$

3) $u(t,x,-0) = u(t,x,+0),\ \frac{\partial u(t,x,-0)}{\partial y} = \frac{\partial u(t,x,+0)}{\partial y}$

Here E is a selfadjoint positive defined in $L_2(-1,1)$ operator which is generated by the differential expression

$$Eu \equiv -\frac{\partial^2}{\partial y^2}$$

And with boundary conditions $u\ |_{y=-1} = u\ |_{y=1} = 0$. We define the operator C as the operator of multiplication with the function $\mathrm{sgn}(y)$. A, B are identity operators. This problem is ill-posed problem in the sense of Hadamard, since continuous dependence of the solution from the data is absent in it.

References.

1.M.M.Lavrent'ev, K.S.Fayazov. Cauchy problem for partial differential equations with operator coefficients in spase./ J. Inverse and Ill-posed problems. vol.2,N 4,pp.283-296(1994).

2.K.S.Fayazov, E.Schock.Boundary value problem for second order partial differential equation with operator coefficients/ Abstract and Applied Analysis.2001, vol.6.issue 5, pages 253-266.