Modelling mechanical systems with finite elements

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Outline

Machine tools

Figure: Kugler experimental platform

The process

Figure: Machine in action

Tool tips

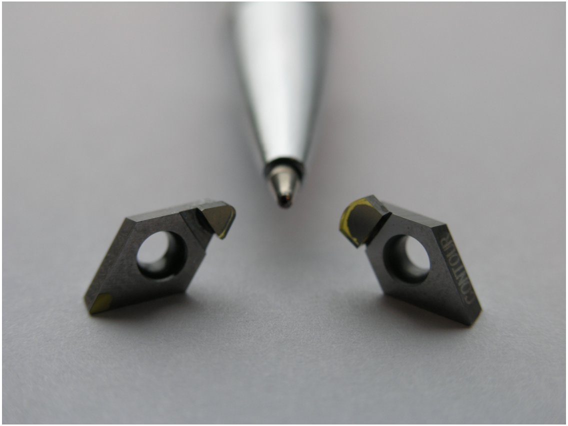

Figure: Tool tips and pen

Motion of continua

The Motion of continua may be modelled by

$$B\frac{\partial^4}{\partial x^4}w(x,t)+\mu\frac{\partial^2}{\partial t^2}w(x,t)=p(t)\,, \tag{1}$$

with some boundary conditions.

- B = stiffness
- $w(t)$ = displacement
- μ = mass
- $p(t)$ = force

Main goal

Discretize the PDE, and transform it into

$$M \frac{d^2}{dt^2} u(t) + Su(t) = p(t) \; , \tag{2}$$

by using *ansatz functions*.

- $M \;\hat{=}\;$ mass distribution
- $u(t) \;\hat{=}\;$ displacement
- $S \;\hat{=}\;$ stiffness distribution
- $p(t) \;\hat{=}\;$ force

Deformation energy

With Hooke's law define the (potential) *deformation energy*

$$V(w(x,t)) := \int_L B \left(\frac{\partial^2}{\partial x^2} w(x,t) \right)^2 \mathrm{d}x \,,$$

with some bending stiffness B (the elastic modulus).

Work of mass - and external forces

The kinetic energy consist of the

- *work of mass forces*:

$$W_m(w(x,t)) := -\int_L \mu \frac{\partial^2}{\partial t^2} w(x,t) \mathrm{d}x$$

  with some mass μ.

- *work of external forces*:

$$W_e(p(x,t)) := \int_L p(x,t) \mathrm{d}x \ .$$

Hamilton's principle

Between l_0, l_1, the true evolution $w(x,t)$ of a system is a stationary point of

$$\mathcal{S}(w(x,t)) := \int_{l_0}^{l_1} V(w(x,t)) + W_m(w(x,t)) - W_e(w(x,t))\mathrm{d}x\ .$$

Setting the derivative to zero yields that $\forall \delta w$

$$\int_0^l \delta w''(x) B \frac{\partial^2}{\partial x^2} w(x,t)\mathrm{d}x = \int_0^l \delta w(x) p(x,t)\mathrm{d}x - \int_0^l \delta w(x)\mu \frac{\partial^2}{\partial t^2} w(x,t)\mathrm{d}x\ . \tag{3}$$

Eq. (3) is called the *principle of virtual displacements (POVD)* and is equivalent to the PDE (1).

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Virtual displacements δw are

- arbitrary,
- not happening in reality,
- differentially small,
- timeless, i.e. without inertia.
- geometrically possible,

Objective: tool holder

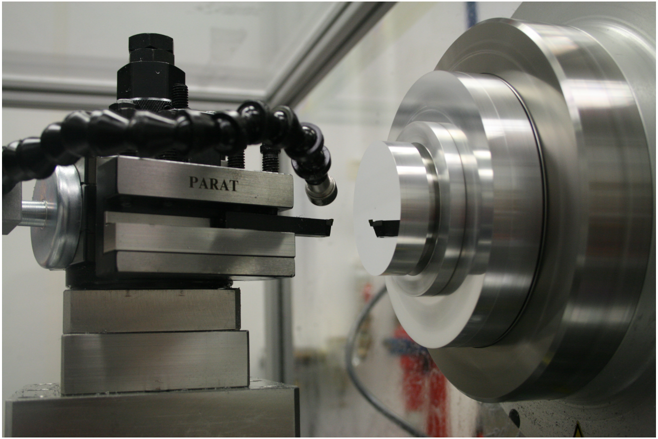

Figure: Tool holder and workpiece

From now on:

- Describing displacements
- Discretizing the POVD
- Introducing ansatz functions
- Calculating stiffness and mass matrices

Material properties and DOF

- strain stiffness D
- torsion stiffness B_x
- bending stiffnesses B_y and B_z
- mass distribution μ
- rotating mass distributions μ_{mx}, μ_{my} and μ_{mz}
- w_x, w_y, w_z - displacement in x, y- and z-axis direction, resp.
- $\beta_x, \beta_y, \beta_z$ - cross section torsion and slopes,

How to describe displacements

The *element displacement vector* for the i-th element reads

$$u_i^T(t) = [w_{x0}, w_{xl}, \beta_{x0}, \beta_{xl}, w_{y0}, \beta_{y0}, w_{yl}, \beta_{yl}, w_{z0}, -\beta_{z0}, w_{zl}, -\beta_{zl}]_i(t) \, .$$

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Where $w_{\cdot 01} \equiv \beta_{\cdot 01} \equiv 0,\ \ w_{\cdot l i-1} \equiv w_{\cdot 0i}$ and $\beta_{\cdot l i-1} \equiv \beta_{\cdot 0i},\ \ i = 2, \ldots, n\ .$
The *system displacement vector* reads

$$u^T(t) = [w_1, w_2, w_3, \beta_1, \beta_2, \beta_3, w_4, w_5, w_6, \beta_4, \beta_5, \beta_6, \ldots, w_{6n}, \ldots \beta_{6n}](t)\ .$$

Outline

Ansatz functions

For each element we have

$$\int_0^l \delta w''(x) B \frac{\partial^2}{\partial x^2} w(x,t)\mathrm{d}x = \int_0^l \delta w(x) p(x,t)\mathrm{d}x - \int_0^l \delta w(x) \mu \frac{\partial^2}{\partial t^2} w(x,t)\mathrm{d}x \, .$$

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Transform the integrals into matrices by use of the *ansatz functions*

$$\begin{aligned} f_1(\xi) &= 1 - 3\xi^2 + 2\xi^3 \,, & f_2(\xi) &= -\xi(1-\xi)^2 l \,, \\ f_3(\xi) &= 3\xi^2 - 2\xi^3 \,, & f_4(\xi) &= \xi^2(1-\xi) l \,, \\ g_1(\xi) &= 1 - \xi \,, & g_2(\xi) &= \xi \,, \end{aligned}$$

where $\xi = x/l$.

With

$$f(x) := [g_1, g_2, f_1, \ldots, f_4](x) \; ,$$

one elements displacement may now be described by the equations

$$w_i(x,t) = f^T(x) u_i(t) \quad \text{and} \quad \delta w_i(x) = \delta u_i^T f(x) \; .$$

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With these notions we obtain

$$\begin{aligned} \int_0^l \delta w_i''(x) B \frac{\partial^2}{\partial x^2} w_i(x,t) \mathrm{d}x &= B \int_0^l (\delta u_i^T f''(x)) \, (f''^T(x) u_i(t)) \mathrm{d}x \\ &= \delta u_i^T B \int_0^l (f''(x) f''^T(x)) \mathrm{d}x \; u_i(t) . \end{aligned}$$

The element stiffness matrix

It holds

$$f''f''^T = \begin{bmatrix} f_1''f_1'' & \cdots & f_1''f_4'' \\ \vdots & \ddots & \vdots \\ f_4''f_1'' & \cdots & f_4''f_4'' \end{bmatrix},$$

and due to the restriction to constant bending stiffness B we can write

$$B\int_0^{l_i} (f''f''^T)\,\mathrm{d}x_i = \frac{B}{l_i}\begin{bmatrix} 12 & -6l_i & -12 & -6l_i \\ -6l_i & 4l_i^2 & 6l_i & 2l_i^2 \\ -12 & 6l_i & 12 & 6l_i \\ -6l_i & 2l_i^2 & 6l_i & 4l_i^2 \end{bmatrix} =: S_{B_y} = S_{B_z}\,.$$

On an analogous way we obtain

$$S_D = \frac{D}{li}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad S_T = \frac{B_x}{li}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and

the element stiffness matrix

$$\hat{S}_i = \begin{bmatrix} S_D & & & \\ & S_T & & \\ & & S_{B_y} & \\ & & & S_{B_z} \end{bmatrix}.$$

The element mass matrix

$$\hat{M}_i = \begin{bmatrix} M_D & & & \\ & M_T & & \\ & & M_{B_y} & \\ & & & M_{B_z} \end{bmatrix}.$$

consisting of $M_{B_{y,z}} :=$

$$\frac{\mu l_i}{420}\begin{bmatrix} 156 & -22l_i & 54 & 13l_i \\ -22l_i & 4l_i^2 & -13l_i & -3l_i^2 \\ 54 & -13l_i & 156 & 22l_i \\ 13l_i & -3l_i^2 & 22l_i & 4l_i^2 \end{bmatrix} + \frac{\mu_{my,mz}}{30l_i}\begin{bmatrix} 36 & -3l_i & -36 & -3l_i \\ -3l_i & 4l_i^2 & 3l_i & -l_i^2 \\ -36 & 3l_i & 36 & 3l_i \\ -3l_i & -l_i^2 & 3l_i & 4l_i^2 \end{bmatrix}$$

$$M_D = \frac{\mu l_i}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad M_T = \frac{\mu_{mx} l_i}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

Re-order displacements

Two transforms of the elementwise matrices are made. Define matrix T such that

$$u_i(t) = T[w_{x0}, w_{y0}, w_{z0}, \beta_{x0}, \beta_{y0}, \beta_{z0}, w_{xl}, w_{yl}, w_{zl}, \beta_{xl}, \beta_{yl}, \beta_{zl}]_i^T(t)$$

and thus

$$S_i = T^T \hat{S}_i T \qquad M_i = T^T \hat{M}_i T \ .$$

The system matrices

We finally obtain system matrices by the transformation

$$S := \sum_i A_i^T S_i A_i \quad \text{and} \quad M := \sum_i A_i^T M_i A_i \,,$$

where A_i is chosen such that

$$u_i(t) = A_i u(t) \quad \text{and} \quad \delta u_i = A_i \delta u \,.$$