Electrical Impedance Tomography optimal currents and sparsity

Bangti Jin, Taufiquar Khan, Peter Maass, Michael Pidcock

Universität Bremen, Zentrum für Technomathematik

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Outline

Electrical impedance tomography

imaging technique: inferring conductivity or permittivity of part of the body from surface electrical measurements.

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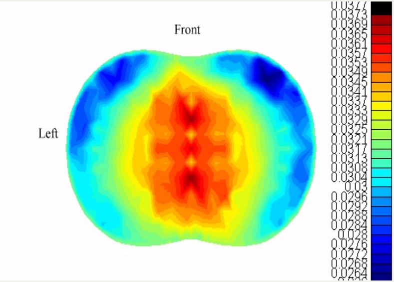
Electrical impedance tomography

medical applications: monitoring lung function, detecting skin/breast cancer, locating epileptic foci



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Mathematical formulation

gov. equation: electrical field u in the absence of source

$$-\nabla \cdot (\sigma^* \nabla u(x)) = 0 \quad \text{in } \Omega,$$

boundary condition: injected current (Neumann b.c.)

$$\sigma^* \frac{\partial u}{\partial n} = j \quad \text{on } \Gamma.$$

measurement: boundary potential (Dirichlet b.c.)

$$u = \phi \quad \text{on } \Gamma.$$

many sets of boundary measurements: NtD map

Inverse problem

Electrical impedance tomography: reconstructing the physical conductivity distribution σ^* from NtD map
uniqueness results

- Calderón (1980): linearized inverse problem
- Kohn-Vogelius: analytic σ^* with C^∞ boundary
- Nachman: $\sigma^* \in W^{2,p}$ $(p > n/2)$, Lipchitz boundary
- Astala-Päivärinta (2006): $\sigma^* \in L^\infty$ $(n = 2)$
- nonuniqueness $\Rightarrow$ cloaking (2003 –)
- ...

Review: Uhlmann G. Commentary on Calderón's paper (29) "On an inverse boundary value problem".

Numerical algorithms

- variational: Tikhonov, Kohn-Vogelius, Knowles, ...
- level-set-type: level set, phase-field, TV, Mumford-Shah, (piecewise constant conductivity)
- direct: linear sampling, factorization methods (small inclusion, spectral analysis)
- statistical approaches

common form:

$$\arg\min_{\sigma} \|\phi^\delta - F_N^\sigma j\|_Z^2 \quad \text{(boundary)}$$

$$\arg\min_{\sigma} \|F_D^\sigma \phi^\delta - F_N^\sigma j\|_Y^2 \quad \text{(domain)}$$

The choice of $j \in X$ and spaces X, Y, Z are important!

notations

- Neumann (Dirichlet) forward operator $F_N^\sigma : X \to Y,\ j \mapsto u$

$$-\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \frac{\partial u}{\partial n} = j \quad \text{on } \Gamma.$$

- Dirichlet trace operator: $\gamma_D : Y \to Z,\ u \mapsto u|_\Gamma$
- NtD operator: $\gamma_D F_N^\sigma$

notations

- $Y = \tilde{H}^1_\sigma(\Omega)$ or $L_2(\Omega)$

$$\tilde{H}^1_\sigma(\Omega) = \left\{ u \in L_2(\Omega) \middle| \int_\Omega \sigma |\nabla u|^2 dx < \infty,\ \int_\Gamma \gamma_D u ds = 0 \right\}.$$

- $Z = \tilde{H}^{\frac{1}{2}}_\sigma(\Gamma)$ or $\tilde{L}_2(\Gamma)$

$$\tilde{H}^{\frac{1}{2}}_\sigma(\Gamma) = \left\{ g \in L_2(\Gamma) \middle| \int \sigma |\nabla F^\sigma_D g| dx < \infty,\ \int_\Gamma g ds = 0 \right\}$$

- $X = \tilde{H}^{-\frac{1}{2}}_\sigma(\Gamma)$ or $\tilde{L}_2(\Gamma)$

$$\tilde{H}^{-\frac{1}{2}}_\sigma(\Gamma) = \left\{ f \middle| f = \sigma \frac{\partial}{\partial n} F^\sigma_D g,\ g \in \tilde{H}^{\frac{1}{2}}_\sigma(\Gamma) \right\}$$

Optimal currents

- Isaacson (1986): $X = Z = \tilde{L}_2(\Gamma)$

$$\max_{j,\|j\|_X=1} \|\gamma_D F_N^{\sigma_0}(j) - \gamma_D F_N^{\sigma^*}(j)\|_Z$$

- Cheney-Isaacson (1992): $X = \tilde{H}_{\sigma_0}^{-\frac{1}{2}}(\Gamma)$, $Z = \tilde{H}_{\sigma_0}^{\frac{1}{2}}(\Gamma)$
- Knowles (2004): $X = \tilde{L}_2(\Gamma)$, $Y = \tilde{H}_{\sigma_0}^{1}(\Omega)$

$$\max_{j,\|j\|_X=1} \|F_N^{\sigma_0}(j) - F_D^{\sigma_0}(\gamma_D F_N^{\sigma^*}(j))\|_Y$$

- other combinations of X and Y

Optimal currents

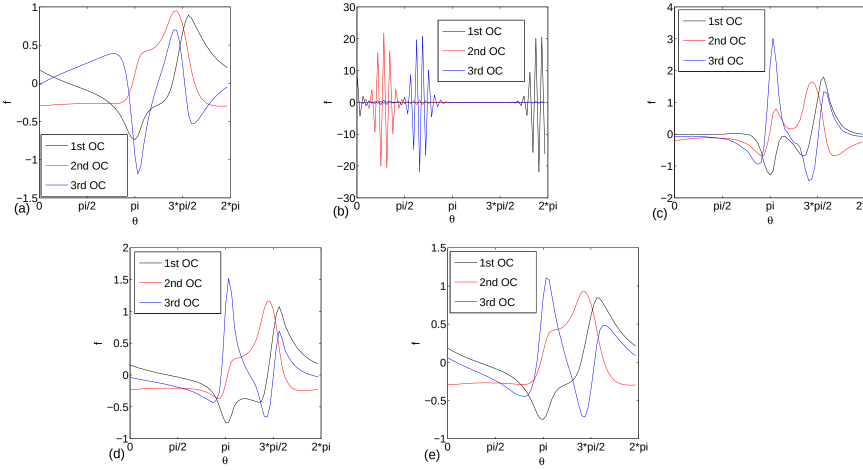
- $A_1 : \tilde{L}_2(\Gamma) \to \tilde{L}_2(\Gamma) \quad j \mapsto \gamma_D F_N^{\sigma_0}(j) - \gamma_D F_N^{\sigma^*}(j)$
- $A_2 : \tilde{H}_{\sigma_0}^{-\frac{1}{2}}(\Gamma) \to \tilde{H}_{\sigma_0}^{\frac{1}{2}}(\Gamma) \quad j \mapsto \gamma_D F_N^{\sigma_0}(j) - i_+ \circ \gamma_D F_N^{\sigma^*} \circ i_-(j)$
- $A_3 : \tilde{H}_{\sigma_0}^{-\frac{1}{2}}(\Gamma) \to \tilde{H}_{\sigma_0}^{1}(\Omega) \quad j \mapsto F_N^{\sigma_0}(j) - F_D^{\sigma_0} \circ i_+ \circ \gamma_D F_N^{\sigma^*} \circ i_-(j)$
- $A_4 : \tilde{H}_{\sigma_0}^{-\frac{1}{2}}(\Gamma) \to L_2(\Omega) \quad j \mapsto F_N^{\sigma_0}(j) - F_D^{\sigma_0} \circ i_+ \circ \gamma_D F_N^{\sigma^*} \circ i_-(j)$
- $A_5 : \tilde{L}_2(\Gamma) \to \tilde{H}_{\sigma_0}^{1}(\Omega) \quad j \mapsto F_N^{\sigma_0} - F_D^{\sigma_0} \circ i_+ \circ \gamma_D F_N^{\sigma^*}(j)$
- $A_6 : \tilde{L}_2(\Gamma) \to L_2(\Omega) \quad j \mapsto F_N^{\sigma_0} - F_D^{\sigma_0} \circ i_+ \circ \gamma_D F_N^{\sigma^*}(j)$

Optimal currents

the optimal current j is a maximal eigenfunction and can be computed by the power method: given j_0 and

$$j_{n+1} = \frac{A^* A j_n}{\|A^* A j_n\|_X},$$

- The iteration converges for any initial guess.
- Computing A^* involves computing adjoints of $F_N^\sigma, F_D^\sigma, i_+, i_-$, NtD in various function spaces (tricky & tedious)
- Evaluating these adjoints often involves solving PDEs!
- Computing several eigenfunctions by QR algorithm



(a) A_1, (b) A_2, A_3, (c) A_4, (d) A_5, (e) A_6

- optimal current depends on the choice of functionals
- $A_2 = A_3$ are not useful in practice
- A_1, A_5 and A_6 are practically identical
- A_4 is most compactly supported around the inclusion.

Sparse construction

- Knowles' functional (1998)

$$\mathcal{J}(\sigma) = \int_{\Omega} \sigma |\nabla(F_N^{\sigma} j - F_D^{\sigma} \phi)|^2 dx,$$

- The functional $\mathcal{J}(\sigma)$ does admit a minimizer in L^{∞}, i.e.

$$\arg\min_{\sigma \in \mathcal{A}} \mathcal{J}(\sigma),$$

  with admissible set

$$\mathcal{A} = \{\sigma \in L^{\infty} : \lambda^{-1} < \sigma < \lambda \text{ a.e.}, \sigma = \sigma^* \text{ a.e.} \Omega \backslash \Omega'\},$$

  does not necessary have a minimizer.

- Higher regularity on σ, e.g. BV, H^1, is need for existence.

Lemma (continuity and diff. of forward operators)

For $\sigma_0, \sigma_1 \in \mathcal{A}$, there exists a constant $Q(\lambda)$ such that for any $p > Q(\lambda)$, there holds

$$\|F_N^{\sigma_0} j - F_N^{\sigma_1} j\|_{\tilde{H}_1(\Omega)} \leq C \|\sigma_1 - \sigma_0\|_{L^p(\Omega')} \|j\|_{H^{-\frac{1}{2}}},$$
$$\|F_D^{\sigma_0} \phi - F_D^{\sigma_1} \phi\|_{\tilde{H}_1(\Omega)} \leq C \|\sigma_1 - \sigma_0\|_{L^p(\Omega')} \|\phi\|_{H^{\frac{1}{2}}}.$$

Moreover, the forward operators F_N^σ and F_D^σ are Frechét differentiable in the sense that

$$\frac{\|F_N^{\sigma+\delta\sigma} j - F_N^{\sigma} j - F_N'^{\sigma}[\delta\sigma] j\|_{\tilde{H}^1}}{\|\delta\sigma\|_{L^p(\Omega')}} \to 0 \text{ as } \delta\sigma \to 0 \text{ in } L^p(\Omega').$$
$$\frac{\|F_D^{\sigma+\delta\sigma} \phi - F_D^{\sigma} \phi - F_D'^{\sigma}[\delta\sigma] \phi\|_{\tilde{H}^1}}{\|\delta\sigma\|_{L^p(\Omega')}} \to 0 \text{ as } \delta\sigma \to 0 \text{ in } L^p(\Omega').$$

Theorem

The functional $\mathcal{J}$ is Frechét differentiable w.r.t. $L^p(\Omega')$ with its Frechét derivative given by

$$\mathcal{J}'(\sigma)[\delta\sigma] = \int_\Omega \delta\sigma(|\nabla F_D^\sigma \phi|^2 - |\nabla F_N^\sigma j|^2)dx$$

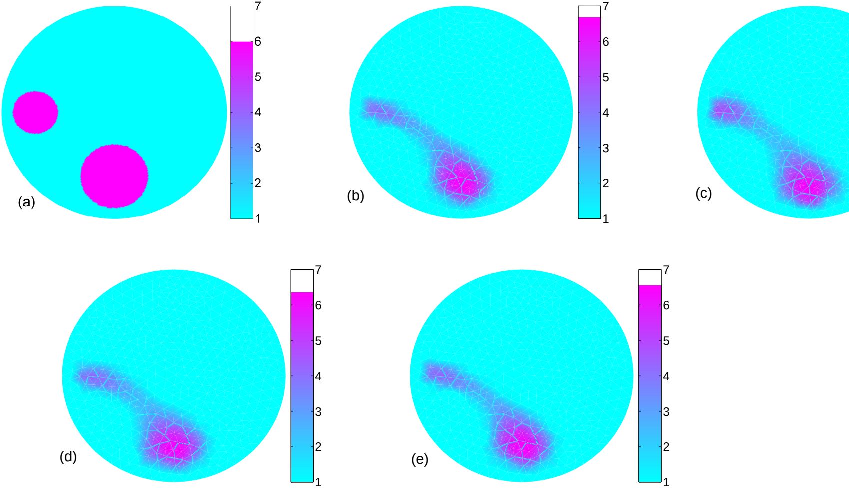
reconstruction algorithm

(i) find the optimal current j for given estimate $\delta\sigma_0$,

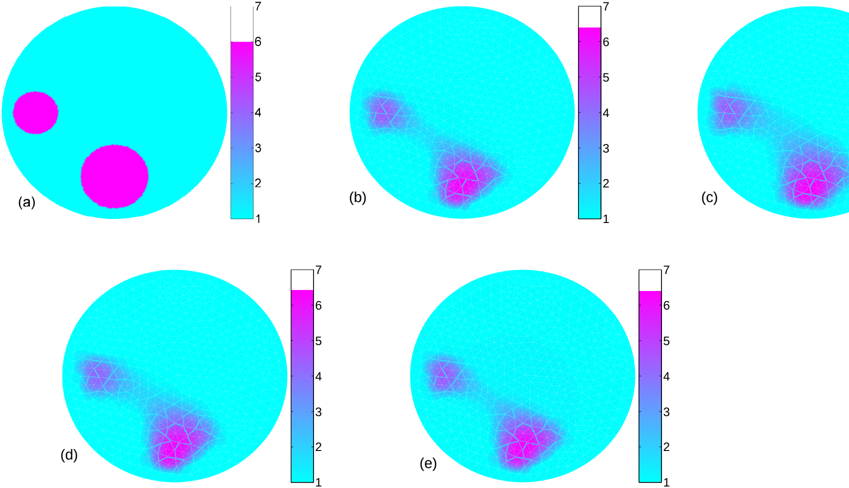
(ii) thresholded gradient descent algorithm

- compute the gradient $\mathcal{J}'(\sigma)$
- compute the Sobolev gradient $\mathcal{J}_s'(\sigma)$
- determine step size s_k with line-search algorithm
- update the inhomogeneity $\delta\sigma$ by $\delta\sigma_{k+1} = \delta\sigma_k - s_k \mathcal{J}_s'(\sigma)$
- threshold $\delta\sigma_{k+1} = T_{s_k\alpha}(\delta\sigma_{k+1})$
- check stopping criterion.

$\mathcal{J}_s'(\sigma)$: incorporating the embedding operator $H^1 \to L_2$



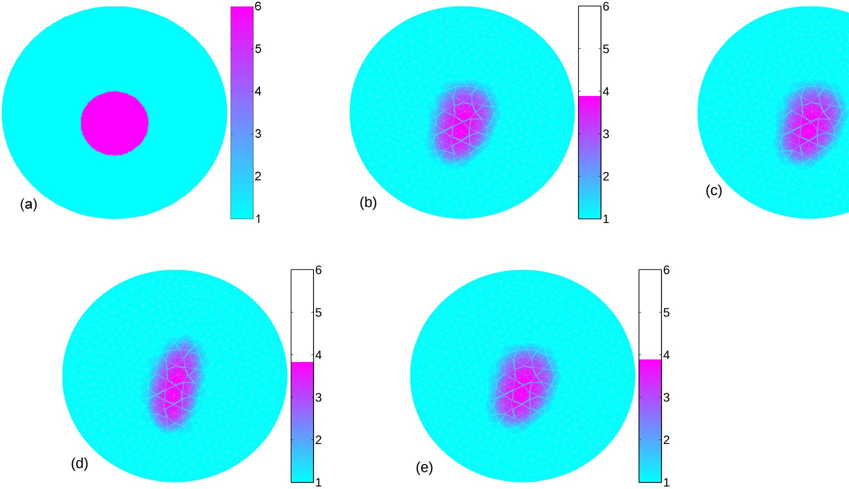
(a) exact, (b) A_1, (c) A_4, (d) A_5, (e) A_6



(a) exact, (b) A_1, (c) A_4, (d) A_5, (e) A_6

Table: The error of the reconstruction for Test 1 with 1% noise.

Case	1	2,3	4	5	6
1st OC	1.692	-	1.573	1.619	1.664
2nd OC	1.439	-	1.437	1.444	1.421



(a) exact, (b) A_1, (c) A_4, (d) A_5, (e) A_6

THANK YOU!

Next seminar: June 25

Efficient Algorithms for L^1 Data Fitting