What your parents did not tell you about α-modulation spaces

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Outline

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Admissible coverings, examples

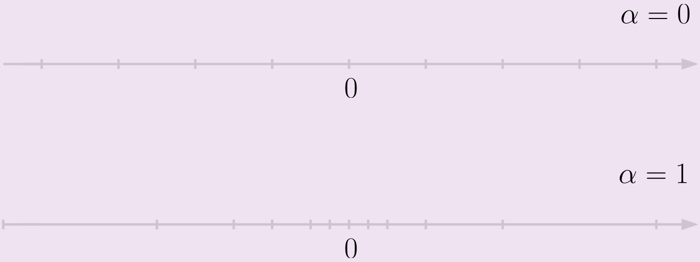
admissible coverings

A countable set $\mathcal{I}$ of subsets $I \subset \mathbb{R}$ is called an *admissible covering* if

(i) $\mathbb{R} = \bigcup_{I \in \mathcal{I}} I$,

(ii) $\#\{I \in \mathcal{I} : x \in I\} \leq 2$ for all $x \in \mathbb{R}$.

Uniform and dyadic coverings



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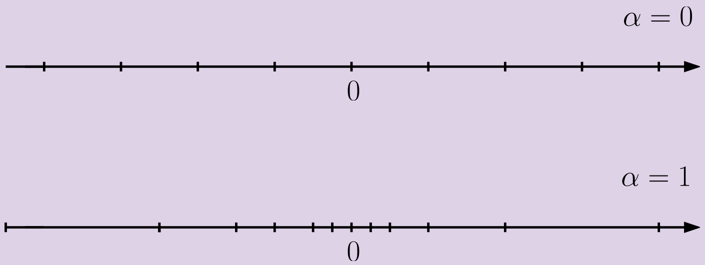

Definitions

Bounded admissible partition of unity

Given an admissible covering $\mathcal{I}$ of $\mathbb{R}$, a corresponding BAPU $(\varphi_I)_{I\in\mathcal{I}}$ is a family of functions satisfying

(i) $\operatorname{supp}(\varphi_I) \subset I$,

(ii) $\sum_{I\in\mathcal{I}} \varphi_I(x) = 1$ for all $x \in \mathbb{R}$,

(iii) $\sup_{I\in\mathcal{I}} \|\mathcal{F}^{-1}\varphi_I|L_1(\mathbb{R})\|$ is finite.

Examples: Modulation and Besov spaces

Let φ be a compactly supported C^∞ function on $\mathbb{R}$.

Uniform covering – Modulation spaces $M_{pq}(\mathbb{R}), 1 \le p, q \le \infty$

$\varphi_j(t) = \varphi(t - j)$

$$\left\| f \,|M_{pq}(\mathbb{R})\right\| = \left(\sum_{j \in \mathbb{Z}} \left\| (\varphi_j \widehat{f})^\vee | L_p(\mathbb{R}) \right\|^q \right)^{1/q}$$

Dyadic covering – Besov spaces $B^s_{pq}(\mathbb{R}), 1 \le p, q \le \infty, s > 0$

$\varphi_0(t) = \varphi$, $\varphi_1(t) = \varphi(t/2) - \varphi(t)$ and $\varphi_j(t) = \varphi_1(2^{-j+1}t), j \in \mathbb{N}$,

$$\left\| f \,|B^s_{pq}(\mathbb{R})\right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \widehat{f})^\vee | L_p(\mathbb{R}) \right\|^q \right)^{1/q}$$

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Intermediate question

Question by H. Triebel:

Probably the congruent covering is a limiting case for that purpose and of peculiar interest may be coverings which are between the congruent and the dyadic covering...

P. Gröbners answer:

uniform: $|B_{r_k}(x_k)|^{1/n} \sim |x_k|^0$

dyadic: $|B_{r_k}(x_k)|^{1/n} \sim |x_k|^1$

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α-coverings

Definition

An admissible covering is called an α-covering, $0 \leq \alpha \leq 1$, of $\mathbb{R}$ (denoted by $\mathcal{I}_\alpha$) if $|I| \sim (1+|x|)^\alpha$ for all $I \in \mathcal{I}_\alpha$.

- **position map:** $p_\alpha : \mathbb{Z} \to \mathbb{R}$,
- **size map:** $s_\alpha : \mathbb{Z} \to \mathbb{R}_+$,

$$\mathbb{Z} \to \mathcal{I}_\alpha : j \mapsto I_j := p_\alpha(j) + [0, s_\alpha(j)].$$

Example (H. Feichtinger, M. Fornasier):

Let $b > 0$ and $\alpha \in [0,1)$.

- $p_\alpha(j) = \operatorname{sgn}(j)\Big((1+(1-\alpha)\cdot b \cdot |j|)^{1/1-\alpha} - 1\Big)$,
- $s_\alpha(j) = b\Big(1+(1-\alpha)\cdot b \cdot (|j|+1)\Big)^{\alpha/1-\alpha}$.

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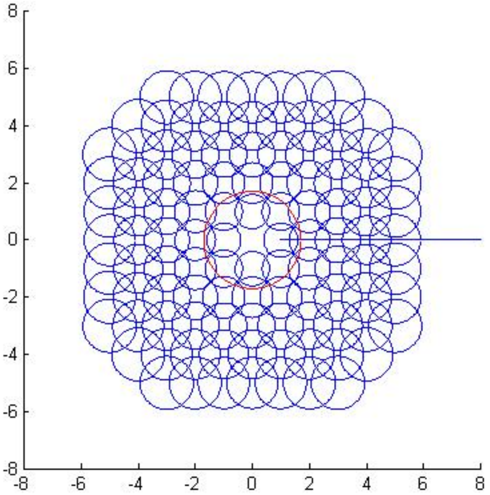
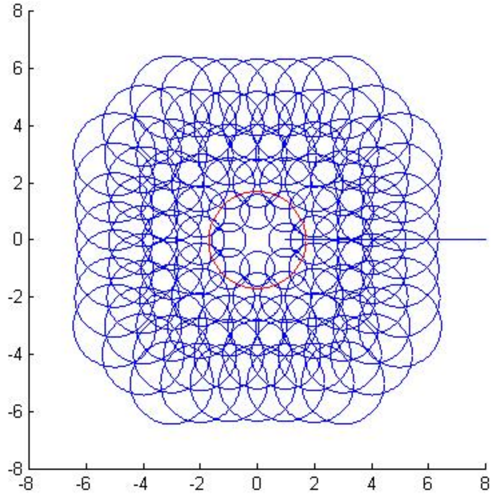
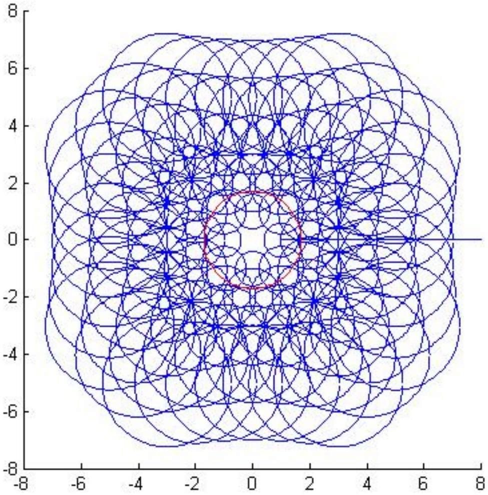
$1/4$-covering



1/2-covering



3/4-covering



Outline

α-modulation space

Definition

Given $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$, let $\mathcal{I}_\alpha$ be an α-covering of $\mathbb{R}$ and let $(\varphi_I)_{I \in \mathcal{I}_\alpha}$ be a corresponding BAPU. Then the *α-modulation space* $M^{s,\alpha}_{pq}(\mathbb{R})$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\left\| f | M^{s,\alpha}_{pq}(\mathbb{R}) \right\| = \left(\sum_{I \in \mathcal{I}_\alpha} \left\| (\varphi_I \widehat{f})^\vee | L_p(\mathbb{R}) \right\|^q (1 + |\omega_I|)^{qs} \right)^{1/q}$$

is finite. Here $\omega_I \in I$ for all $I \in \mathcal{I}_\alpha$.

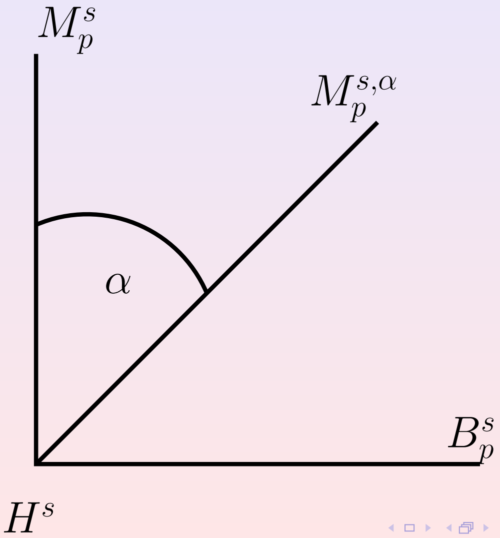
Modulation and Besov spaces

Special cases $\alpha = 0$ and $\alpha = 1$

(i) For $\alpha = 0$ the space $M^{s,0}_{pq}(\mathbb{R})$ coincides with the modulation space $M^{s}_{pq}(\mathbb{R})$. **Gabor theory !!!**

(ii) For $\alpha \to 1$ the space $M^{s,1}_{pq}(\mathbb{R})$ coincides with the Besov space $B^{s}_{pq}(\mathbb{R})$. **Wavelet theory !!!**

α-modulation space



Embeddings between α-modulation spaces

Theorem (P. Gröbner):

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq \alpha_1 < \alpha_2 \leq 1$. Then we have

$$M_{pq}^{s',\alpha_2}(\mathbb{R}^n) \subset M_{pq}^{s,\alpha_1}(\mathbb{R}^n), \quad s' = s + \frac{\alpha_2 - \alpha_1}{q}$$

$$M_{pq}^{s,\alpha_1}(\mathbb{R}^n) \subset M_{pq}^{s',\alpha_2}(\mathbb{R}^n), \quad s' = s - (1 - 1/q)(\alpha_2 - \alpha_1).$$

In particular, for $\alpha_2 = 1$ and $\alpha_1 = 0$ we get

$$B_{pq}^{s+1/q}(\mathbb{R}^n) \subset M_{pq}^{s}(\mathbb{R}^n).$$

Outline

Coorbit spaces

Feichtinger–Gröchenig theory

S: the *reservoir*, a space of functions or distributions

V: the voice transform, a linear transform which assigns to each $f \in \mathbf{S}$ a function $\mathbf{V}\, f$ on some locally compact space X.

Y: some Banach function space on X.

Then the *coorbit space* is defined to be

$$\mathrm{Co}\mathbf{Y} = \{f \in \mathbf{S} : \quad \mathbf{V}\, f \in \mathbf{Y}\}$$

equipped with the norm

$$\|f|\mathrm{Co}\mathbf{Y}\| = \|\mathbf{V}\, f|\ \mathbf{Y}\|$$

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Wavelet and Time–Frequency analysis

$$\text{Translation:} \quad T_x f(t) = f(t-x),$$
$$\text{Modulation:} \quad M_\omega f(t) = e^{2\pi i t\cdot\omega} f(t),$$
$$\text{Dilation:} \quad D_a f(t) = |a|^{-1/2} f(t/a).$$

Short Time Fourier Transform

For $f, g \in L_2(\mathbb{R})$, $(x, \omega) \in \mathbb{R} \times \mathbb{R}$,

$$V_g^0 f(x,\omega) = \langle f, M_\omega T_x g\rangle.$$

Continuous Wavelet Transform

For $f \in L_2(\mathbb{R})$ and wavelet g- radial bandlimited Schwartz function on $\mathbb{R}$, $(x, a) \in \mathbb{R} \times (0, \infty)$

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Modulation spaces and the Heisenberg group

The mixed norm space $L^{p,q}_m$, $1 \leq p, q \leq \infty$ with moderate weights

$$\|F \;|L^{p,q}_m\| = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,\omega)|^p m(x,\omega)^p \;\, \mathrm{d}x \right)^{q/p} \mathrm{d}\omega \right)^{1/q}$$

Characterization of modulation spaces

$$M^{p,q}_m := \{f \in \mathcal{S}'(\mathbb{R}) : V^0_g f \in L^{p,q}_m\}, \quad \|f \;|M^{p,q}_m\| = \|V^0_g f|\; L^{p,q}_m\|$$

$$M^{pq}_m = \mathrm{Co} L^{p,q}_m.$$

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Besov spaces and the affine group

The mixed norm space $L_s^{p,q}$ on the $ax + b$ group

$$\|F \mid L_s^{p,q}\| = \left(\int_0^\infty \left(\int_{\mathbb{R}} |F(x,a)|^p a^{-sp} \, \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}a}{a^2} \right)^{1/q}$$

Characterization of homogeneous Besov spaces

$$\dot{B}^s_{pq} = \{f \in \mathcal{S}_0' : \quad V_g^1 f \in L^{p,q}_{s+1/2-1/q}\}, \quad 1 \leq p,q \leq \infty, s \in \mathbb{R}.$$

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The affine-Heisenberg group

Bad news by B. Torresani

$$G_{\mathrm{aWH}} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{+} \times \mathbb{T}$$

No representation of the affine-Heisenberg group is ever square integrable.

Remedy of S. Dahlke et al.

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Intermediate theory

Wave packets: superposition of modulation translation and dilation

Let $\eta_\alpha(\omega) = (1 + |\omega|)^\alpha$ for $\alpha \in [0, 1]$.

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\begin{aligned}
g^\alpha_{x,\omega}(t) &= T_x M_\omega D_{\eta_\alpha(\omega)^{-1}} g(t) \\
&= (1 + |\omega|)^{\alpha d/2} e^{2\pi i \omega(t-x)}\, g((1 + |\omega|)^\alpha)(t - x)).
\end{aligned}
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Flexible Gabor-wavelet transform, α-transform

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Characterization of L_2- Sobolev spaces

Theorem (B. Nazaret, M. Holschneider)

Let $g \in \mathcal{S}(\mathbb{R})$. Assume that there is $A > 0$ such that

$$A^{-1} \leq \int_{\mathbb{R}} |\widehat{g}((1+|t|)^{-\alpha}(\omega - t))|^2 (1+|t|)^{-\alpha} \mathrm{d}t \leq A \text{ for a.e } \omega \in \mathbb{R}.$$

Then

$$f \in H^s(\mathbb{R}) \quad \text{if, and only if,} \quad V_g^\alpha f \in L^2_{(1+|\omega|^2)^{s/2}}(\mathbb{R}^2).$$

Discrete versus continuous α modulation spaces

Theorem (H. Feichtinger, M. Fornasier)

Assume $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Let $m_s(x,\omega) = (1+|\omega|)^s$

$$\mathbb{M}^{s,\alpha}_{p,q}(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : V^\alpha_g f \in L^{p,q}_{m_s}(\mathbb{R}^2)\} \quad \|f \; |\mathbb{M}^{s,\alpha}_{p,q}\| = \|V^\alpha_g f| \; L^{p,q}_{m_s}\|.$$

For a band-limited $g \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$

$$M^{s+\alpha(1/q-1/2),\alpha}_{p,q}(\mathbb{R}) = \mathbb{M}^{s,\alpha}_{p,q}(\mathbb{R}).$$

Furthermore,

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |V^\alpha_g f(x,\omega)|^p \, \mathrm{d}x\right)^{q/p} (1+|\omega|)^{sq} \mathrm{d}\omega\right)^{1/q}$$

is an equivalent norm on $M^{s+\alpha(1/q-1/2),\alpha}_{p,q}(\mathbb{R})$.

Banach frames and atomic decompositions

Gabor frames and Modulation spaces

$$\mathcal{G}_0 = \mathcal{G}_0(g, a, b) = \{T_{ak}M_{bj}g\}_{k,j\in\mathbb{Z}}.$$

Wavelet frames and Besov spaces

$$\mathcal{G}_1 = \mathcal{G}_1(\varphi, \psi) = \{T_k\varphi\}_{k\in\mathbb{Z}} \cup \{D_{2^{-j}}T_k\psi\}_{k,j\in\mathbb{Z}}.$$

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$$\mathcal{G}_\alpha = \mathcal{G}_\alpha(g, p_\alpha, s_\alpha, a) = \{M_{p_\alpha(j)}D_{s_\alpha^{-1}(j)}T_{ak}g\}_{k,j\in\mathbb{Z}}.$$

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Kohn-Nirenberg correspondence, Hörmander class

Definition

Then the operator

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\mathrm{d}\omega$$

is called the *pseudodifferential operator* with symbol σ.

Hörmander classes

$$S^N_{\rho,\delta} = \{\sigma \in C^\infty(\mathbb{R}^{2n}) : |D^\alpha_x D^\beta_\omega \sigma(x,\omega)| \leq C_{\alpha,\beta}(1+|\omega|)^{N+\delta|\alpha|-\rho|\beta|}\}$$

Boundedness of PSOs on α-modulation spaces

Theorem (L. Borup, M. Nielsen)

Suppose $N \in \mathbb{R}$, $\alpha \in [0,1]$, $\sigma \in S^N_{\rho,0}$, $\alpha \leq \rho \leq 1$, $s \in \mathbb{R}$ and $1 < p < \infty$. Then

$$K_\sigma : M^{s,\alpha}_{pq}(\mathbb{R}^n) \to M^{s-N,\alpha}_{pq}(\mathbb{R}^n).$$

THANK YOU FOR YOUR ATTENTION