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Angelika Bunse-Gerstner
Dorota Kubalińska
Georg Vossen
Daniel Wilczek

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h_2-NORM OPTIMAL MODEL REDUCTION FOR LARGE-SCALE DISCRETE DYNAMICAL MIMO SYSTEMS

A. BUNSE-GERSTNER, D. KUBALIŃSKA, G. VOSSEN, AND D. WILCZEK

Abstract. Modeling strategies often result in dynamical systems of very high dimension. It is then desirable to find systems of the same form but of lower complexity, whose input-output behavior approximates the behavior of the original system. Here we consider linear time invariant (LTI) discrete time dynamical systems. The cornerstone of this paper is a relation betweeen optimal model reduction in the h_2-norm and (tangential) rational Hermite interpolation. First order necessary conditions for h_2-optimal model reduction are presented for discrete Multiple-Input-Multiple-Output (MIMO) systems. These conditions lead to an optimal choice of interpolation data and a new efficient algorithm for h_2-optimal model reduction for MIMO systems. It is also shown that the conditions are equivalent to two known Lyapunov based first order necessary conditions. Numerical experiments demonstrate the approximation quality of the method.

1. Introduction

The purpose of model order reduction is to replace a large model by a smaller one, which preserves the essential behavior of the original model. For the systems considered in this paper, it can be stated as follows:

Problem: Given the Linear Time Invariant (LTI), discrete time dynamical system in state-space representation:

$$\Sigma: \qquad \begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k \end{aligned} \tag{1.1}$$

or equivalently, in the frequency domain, represented by its transfer function

$$H(s) := C(sI_N - A)^{-1}B \tag{1.2}$$

where $A \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{N\times m}$ and $C \in \mathbb{C}^{p\times N}$. The vectors $x_k \in \mathbb{C}^N$, $y_k \in \mathbb{C}^p$ and $u_k \in \mathbb{C}^m$ are the state, output and the input of the system at time t_k, respectively and N is very large. It will be assumed throughout the paper that the system is stable, that is, all eigenvalues of A lie inside the unit circle, observable and reachable.

Construct a reduced-order system

$$\hat{\Sigma}: \qquad \begin{aligned} \hat{x}_{k+1} &= \hat{A}\hat{x}_k + \hat{B}u_k, \\ \hat{y}_k &= \hat{C}\hat{x}_k \end{aligned} \tag{1.3}$$

with transfer function

$$\hat{H}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} \tag{1.4}$$

Key words and phrases. Model Reduction, Rational Interpolation, Tangential Interpolation, discrete Sylvester Equation, h_2 Approximation.

where $\hat{A} \in \mathbb{C}^{n\times n}$, $\hat{B} \in \mathbb{C}^{n\times m}$, $\hat{C} \in \mathbb{C}^{p\times n}$ and $n \ll N$, whose input-output behavior approximates the input-output behavior of the large system. The quality of this approximation could be measured by the closeness of the transfer functions, i.e.

$$\|H(s) - \hat{H}(s)\| < \varepsilon$$

for a given accuracy ε and a suitable norm.

Discrete single-input-single-output (SISO) and multiple-input-multiple-output (MIMO) dynamical systems arise quite frequently in varius fields of applications in which the physical or technical systems are modelled by suitable systems of differential or difference equations. For a simple example, consider the discretization of the 1-D heat equation $y_t = y_{xx}$. It is well-known that a semi-discretization using the method of lines can lead to stability problems if the discretization in space is too fine. Hence, one often prefers a full discretization via a Crank-Nicolson scheme which provides a discrete system. One could imagine a boundary time dependent control which in our context would lead to single or double input, or a distributed control which implies as many controls as states. Furthermore, the measured state (or output) can be on one or both sides of the space domain but also the temperature distribution on the whole domain can be interesting in the model. For more challenging examples we refer for instance to Verlaan [19] or Lawless et al. [12]. In all cases, modeling leads to systems with a very high–dimensional state which makes model reduction an important and essential task.

Existing reduction methods can be divided into two groups. On the one hand, there are truncation methods, using singular value decompositions to select the important part of the system and neglecting the rest. A well-known method in this group is balanced truncation (see e.g. Mullis/Roberts [16] or Moore [15]). The advantage of this technique is that it preserves stability and that global error bounds can be derived. The complexity of the computation however is of order N^3. Therefore the method is by far too expensive for very large systems. New approaches try to approximately compute the transformation matrix with lower costs (see for example [5], [4], [17] and references therein), but then the global error bounds are lost. On the other hand, we have Krylov-interpolation based methods which can handle large dimensions in the computation of the reduced system, but often cannot guarantee the preservation of stability and do not have computable global error bounds. There are also methods, which are a combination of these two approximation methods. A recent and rather complete study of model reduction techniques with an emphasis on Krylov and SVD-Krylov methods can be found in [1] and references therein.

In this paper we will focus on the question "Which reduced order system minimizes the approximation error $H - \hat{H}$ in an appropriate measure?" Here we investigate this problem taking the h_2-norm of the system as the measure. This norm is defined as

$$||\Sigma||_{h_2}^2 = ||H||_{h_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{trace}\left(H(e^{\mathrm{i}w})^* H(e^{\mathrm{i}w}) \right) dw \right)^{1/2} \tag{1.5}$$

(cf. [1]). We will show that a local minimizer for this problem satisfies certain Hermite interpolation conditions. Hence, this method is an interpolation based model reduction. We stress that although such methods are widely considered as providing reduced systems with only a good local approximation around the a priori

chosen interpolation points, here we are aiming for an optimal approximation in the h_2-norm. The crucial point here is the correct choice of the interpolation data. Similar ideas have been developed also for SISO continuous (see [10]) and MIMO continuous systems (see [20], [2]).

The paper is organized as follows. In Section 2 we will give a simple proof for a useful alternative expression of the h_2-norm which was also presented (and proven differently) in [1]. This formula will be used in Section 3 to formulate interpolation based necessary optimality conditions for the minimization problem stated above. Furthermore, other necessary conditions will be discussed in this section. Initially, we will prove the so-called Wilson conditions which are due to Wilson [22] who stated (and proved) them only for continuous systems. We will show the equivalence of Wilson's conditions to our new interpolation based ones as well as to the conditions of Hyland and Bernstein [11]. Similar equivalence proofs for SISO continuous systems have been given by Gugercin, Antoulas and Beattie [10]. In Section 4 we will briefly review some ideas of interpolation and tangential interpolation for fixed interpolation data. Based on these ideas, a numerical algorithm, MIRIAm (MIMO Iterative Rational Interpolation Algorithm), will be presented. We illustrate its efficiency on two numerical examples.

2. h_2-NORM FOR DISCRETE SYSTEMS

Throughout the paper we will assume for ease of presentation that the system matrix A has N pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_N$ which will be referred to as (simple) poles of the system. The results presented can be extended to multiple poles (similar to those for continuous systems, cf. [20]) but the formulas become rather complex.

2.1. Properties of transfer functions.

Without loss of generality we assume that A is in diagonal form $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ where $\lambda_i \neq \lambda_j$ for $i \neq j$. Otherwise, a state space transformation $x = S\tilde{x}$ where the columns of S are the eigenvectors of A yield an equivalent system with $\tilde{A} = S^{-1}AS$, $\tilde{B} = S^{-1}B$ and $\tilde{C} = CS$ where $\tilde{A}$ is diagonal. We denote

$$B = (b_{ij})_{i,j} = [b_1^* \ \ldots \ b_N^*]^*, \quad C = (c_{ij})_{i,j} = [c_1 \ \ldots \ c_N], \tag{2.1}$$

with row vectors $b_k = [b_{k1} \ldots b_{km}]$ and column vectors $c_k = [c_{1k} \ldots c_{pk}]^T$ for $k = 1, \ldots, n$. Note that column vector $Be_l \in \mathbb{C}^N$ represents the l-th input for $l = 1, \ldots, m$ and row vector $e_q^T C \in C^N$ represents the q-th output for $q = 1, \ldots, p$ of the system, where e_l denotes the l-th unit vector. The transfer function H is a $(p \times m)$-dimensional matrix-valued function with components H_{ql}, $l = 1, \ldots, m$, $q = 1, \ldots, p$:

$$H(s) = \begin{pmatrix} H_{11}(s) & \ldots & H_{1m}(s) \\ \ldots & \ddots & \ldots \\ H_{p1}(s) & \ldots & H_{pm}(s) \end{pmatrix}, \quad H_{ql}(s) = e_q^T C(sI_N - A)^{-1} Be_l, \tag{2.2}$$

and each H_{ql} can be interpreted as a SISO transfer function with input Be_l and output $e_q^T C$. The numbers λ_k, $k = 1, \ldots, N$, are called poles of the transfer function.

By a partial fraction expansion of each H_{ql}, we obtain

$$H_{ql}(s) = \sum_{k=1}^{n} \frac{\phi_k^{ql}}{s - \lambda_k}. \tag{2.3}$$

Comparison of coefficients in (2.2) with those in (2.3) yields

$$\phi_k^{ql} = c_{qk} b_{kl}, \qquad l = 1, \dots, m, \quad q = 1, \dots, p, \quad k = 1, \dots, N. \tag{2.4}$$

Remark 2.1. We note that for MIMO systems with $p > 1$ or $m > 1$ some ϕ_k^{ql} can be zero in each component H_{ql} whereas for SISO systems all $\phi_k := \phi_k^{11}$, $k = 1, \dots, N$, are nonzero. In this case, ϕ_k is the residue of H in the pole λ_k denoted by $\mathrm{Res}(H, \lambda_k)$. It is well-known that a SISO system can be written in a canonical form such that $A = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ is diagonal and $B = (1, \dots, 1)^* \in \mathbb{C}^N$. In this canonical form, we have $C_k = \phi_k$.

2.2. **h_2-norm.** Before stating the main result, we will give some auxiliary results which will be used later in the proof. By substituting $\exp(\mathrm{i}w) \mapsto z$ in (1.5), the h_2-norm can be written as a complex line integral as follows

$$||\Sigma||_{h_2}^2 = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{z} \mathrm{trace} \left(H\Big(\frac{1}{z^*}\Big)^* H(z) \right) dz \tag{2.5}$$

where $\gamma(t) := \exp(\mathrm{i}t)$, $t \in [0, 2\pi]$, is a parameterization of the unit cycle. This integral will be calculated via the well-known residue theorem using the following auxiliary result (cf, e.g. [7]) for calculating residues at simples poles.

Proposition 2.2. Let H be a meromorphic complex function and λ be a simple pole of H with residue ϕ. Let furthermore G be a complex function which is holomorphic in λ. Then, we have $\mathrm{Res}(H \cdot G, \lambda) = \mathrm{Res}(H, \lambda) G(\lambda) = \phi G(\lambda)$.

The following reformulation of $||\Sigma||_{h_2}$ is obvious.

Lemma 2.3.

$$||\Sigma||_{h_2}^2 = ||H||_{h_2}^2 = \sum_{l=1}^{m} \sum_{q=1}^{p} ||H_{ql}||_{h_2}^2. \tag{2.6}$$

We are now ready to state the main result of this section.

Lemma 2.4. *Given a stable system Σ where A is diagonal with pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_N$. Let B and C be partitioned as in (2.1). Then the h_2-norm of Σ is given by*

$$||\Sigma||_{h_2}^2 = \sum_{k=1}^{N} \frac{1}{\lambda_k^*} \mathrm{trace} \left(H\left(\frac{1}{\lambda_k^*}\right) b_k^* c_k^* \right) \tag{2.7}$$

Proof. Consider first the SISO case. Then (2.5) simplifies to

$$||\Sigma||_{h_2}^2 = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{z} H\Big(\frac{1}{z^*}\Big)^* H(z) dz$$

We will first show that $G(z) := H(1/z^*)^*/z$ is well-defined and even holomorphic for all $z \in D := \{z \in \mathbb{C} : |z| < 1\}$. For $z \in D$ we have

$$G(z) = \frac{1}{z} H\Big(\frac{1}{z^*}\Big)^* = \frac{1}{z} \sum_{k=1}^{N} \phi_k^* \frac{1}{\frac{1}{z} - \lambda_k^*} = \sum_{j=k}^{N} \phi_k^* \frac{1}{1 - z\lambda_k^*}$$

with $1 - z\lambda_k^* \neq 0$ for all $z \in D$ due to $\lambda_k^* \in D$. Hence, G is holomorphic for $|z| < 1$ and the poles of $F(z) := G(z)H(z)$ in D are the poles of H (which, due to stability, are all in D). Finally, from the residue theorem and Proposition 2.2 we conclude

$$\begin{aligned} ||\Sigma||_{h_2}^2 &= \sum_{\lambda \text{ pole of } F \text{ in } D} \operatorname{Res}(F, \lambda) = \sum_{k=1}^{N} \operatorname{Res}(F, \lambda_k) \\ &= \sum_{k=1}^{N} \operatorname{Res}(H, \lambda_k) G(\lambda_k) = \sum_{k=1}^{N} \phi_k \frac{1}{\lambda_k} H\Big(\frac{1}{\lambda_k^*}\Big)^* \\ &= \sum_{k=1}^{N} \phi_k^* \frac{1}{\lambda_k^*} H\Big(\frac{1}{\lambda_k^*}\Big) \end{aligned} \tag{2.8}$$

The last equality follows from the fact that the norm is a real number and hence we have $\|\cdot\| = \|\cdot\|^*$. This proves the SISO case in view of (2.4). For MIMO systems, we will use this formula for each H_{ql} which can be interpreted as a SISO transfer function. It is important to note that although some ϕ_k^{ql} can be zero, the h_2-norm formula remains correct for each component H_{ql}. The corresponding summands in the last expression in (2.8) vanish. Hence, (2.6) and (2.8) yield

$$||\Sigma||_{h_2}^2 = \sum_{l=1}^{m} \sum_{q=1}^{p} ||H_{ql}||_{h_2}^2 = \sum_{l=1}^{m} \sum_{q=1}^{p} \sum_{k=1}^{N} \Big(\phi_k^{ql}\Big)^* \frac{1}{\lambda_k^*} H_{ql}\Big(\frac{1}{\lambda_k^*}\Big)$$

which, together with (2.4), can be written as (2.7). □

The quantities $\frac{1}{\lambda_k^*}$ are called *mirror images of the poles of the system*. They play an important role in h_2-optimal model reduction. The new expression (2.7) for the h_2-norm was first obtained by Antoulas [1]. We presented a new proof based on the residue theorem. As mentioned above, this result can be generalized to multiple poles (see [20] for continuous systems). Using Lemma 2.4 we can now easily prove the following result on the h_2-norm of the error system.

Lemma 2.5. *Let $\Sigma = (A, B, C)$ and $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C})$ be state space representations of the original and reduced systems, respectively. Assume without loss of generality that the state matrices A and $\hat{A}$ are diagonal. Let H and $\widehat{H}$ be the corresponding transfer functions. Then the h_2-norm of the error system, denoted by $\mathcal{J}$ can be represented by*

$$\begin{aligned} \mathcal{J} = \|H - \hat{H}\|_{h_2}^2 &= \sum_{j=1}^{N} \left(\tfrac{1}{\lambda_j^*} \operatorname{trace}\left(\left[H\left(\tfrac{1}{\lambda_j^*}\right) - \hat{H}\left(\tfrac{1}{\lambda_j^*}\right) \right] b_j^* c_j^* \right) \right) \\ &+ \sum_{j=1}^{n} \left(\tfrac{1}{\hat{\lambda}_j^*} \operatorname{trace}\left(\left[\hat{H}\left(\tfrac{1}{\hat{\lambda}_j^*}\right) - H\left(\tfrac{1}{\hat{\lambda}_j^*}\right) \right] \hat{b}_j^* \hat{c}_j^* \right) \right) \end{aligned} \tag{2.9}$$

where $\frac{1}{\lambda_k^*}$, $\frac{1}{\hat{\lambda}_k^*}$ are the mirror images of the poles of Σ and $\hat{\Sigma}$, respectively.

Proof. The system matrices of the error system Σ_e are given by

$$A_e = \begin{pmatrix} A & 0 \\ 0 & \hat{A} \end{pmatrix} \qquad B_e = \begin{pmatrix} B \\ \hat{B} \end{pmatrix} \qquad C_e = \begin{pmatrix} C & -\hat{C} \end{pmatrix} \tag{2.10}$$

with transfer function $H_e = H - \hat{H}$. As A_e is also diagonal, the poles of Σ_e are $\lambda_1, \ldots, \lambda_N, -\hat{\lambda}_1, \ldots, -\hat{\lambda}_n$. Hence, using formula (2.7) directly yields (2.9). □

3. h_2 optimal model reduction for MIMO systems

Searching for the global minimum is too hard a task even for SISO systems; therefore here the aim is to find reduced order systems, which satisfy first order necessary optimality conditions. For notational convenience, we just use the hat superscript for such candidates.

After formulating the minimization problem, we will give three different types of necessary conditions in this section. Initially, we will present and prove new interpolation based conditions for discrete h_2-optimal model reduction. Then we will prove Wilson's conditions (cf. [22]) which has not yet been proven for discrete systems. For the sake of completeness, we will finally cite the conditions of Hyland and Bernstein. In the last paragraph we will then prove the equivalence of all three conditions. For continuous SISO systems the equivalence of the corresponding conditions is proven in [10].

3.1. Problem statement.

Given a large order system (1.1) of dimension N. Construct a reduced order system (1.3) of a fixed dimension n which minimizes the h_2-norm of the error system. Using (2.9), this means that we consider the following minimization problem:

$$\text{Minimize} \quad \mathcal{J}(\tilde{v}) = \|H - \hat{H}\|_{h_2}^2 \tag{3.1}$$

with optimization variable $\tilde{v} = (\Re v^T, \Im v^T)^T \in \mathbb{R}^{2(n+nm+pn)}$, where

$$v = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n, \hat{b}_1, \ldots, \hat{b}_n, \hat{c}_1^T, \ldots, \hat{c}_n^T)^T,$$

with row vectors $\hat{b}_k = [\hat{b}_{k1} \ldots \hat{b}_{km}]$ and column vectors $\hat{c}_k = [\hat{c}_{1k} \ldots \hat{c}_{pk}]$ for $k = 1, \ldots, n$. Here $\Re$ and $\Im$ denote the real and imaginary part, respectively. This is an unconstrained, smooth optimization problem with respect to $\tilde{v}$.

3.2. First order necessary optimality conditions.

3.2.1. *Interpolation conditions.* We obtain the following results on the interpolation data satisfying first order necessary conditions:

Theorem 3.1. *Given the original large order system of the form (1.1) with transfer function $H(s)$. Let $\hat{H}(s)$ be the transfer function of the reduced order system (1.3) given in an eigenvector basis $\hat{A} = \operatorname{diag}\left(\hat{\lambda}_1, \ldots, \hat{\lambda}_n\right)$, $\hat{B} = \left[\hat{b}_1^*, \ldots, \hat{b}_n^*\right]^*$ and $\hat{c} = [\hat{c}_1, \ldots, \hat{c}_n]$. If $\hat{H}(s)$ solves the optimal h_2-problem (3.1) then the following*

conditions are satisfied

$$\left.\begin{aligned} \hat{c}_j^* H\left(\tfrac{1}{\hat{\lambda}_j^*}\right) &= \hat{c}_j^* \hat{H}\left(\tfrac{1}{\hat{\lambda}_j^*}\right), \\ H\left(\tfrac{1}{\hat{\lambda}_j^*}\right)\hat{b}_j^* &= \hat{H}\left(\tfrac{1}{\hat{\lambda}_j^*}\right)\hat{b}_j^*, \\ \hat{c}_j^* H'\left(\tfrac{1}{\hat{\lambda}_j^*}\right)\hat{b}_j^* &= \hat{c}_j^* \hat{H}'\left(\tfrac{1}{\hat{\lambda}_j^*}\right)\hat{b}_j^*, \end{aligned}\right\} \quad \text{for } j = 1, \ldots, n, \tag{3.2}$$

where $\frac{1}{\hat{\lambda}_{j^*}}$ *are the mirror images of the poles of* $\hat{\Sigma}$, $\hat{b}_j$ *is the* j^{th} *row of* $\hat{B}$ *and* $\hat{c}_j$ *is the* j^{th} *column of* $\hat{C}$

For the proof of this theorem the following lemma will be helpful. It can be proven by direct calculations.

Lemma 3.2. *(i) For* $q = 1, \ldots, n$ *and* $l = 1, \ldots, m$ *we have*

$$\frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{b}_{ql})} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{b}_{ql})} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{b}_{ql})} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{b}_{ql})} = 0\,, \tag{3.3}$$

$$\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{b}_{ql})} = -i\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{b}_{ql})} = \frac{\hat{c}_q e_l^* \lambda_j^*}{1 - \lambda_j^* \hat{\lambda}_q} \quad \textit{and} \tag{3.4}$$

$$\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{b}_{ql})} = -i\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{b}_{ql})} = \frac{\hat{c}_q e_l^* \hat{\lambda}_j^*}{1 - \hat{\lambda}_j^* \hat{\lambda}_q}\,. \tag{3.5}$$

(ii) For $q = 1, \ldots, n$ *and* $l = 1, \ldots, p$ *we have*

$$\frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{c}_{lq})} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{c}_{lq})} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{c}_{lq})} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{c}_{lq})} = 0\,, \tag{3.6}$$

$$\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{c}_{lq})} = -\mathrm{i}\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{c}_{lq})} = \frac{e_l \hat{b}_q \lambda_j^*}{1 - \lambda_j^* \hat{\lambda}_q} \quad \textit{and} \tag{3.7}$$

$$\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{c}_{lq})} = -\mathrm{i}\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{c}_{lq})} = \frac{e_l \hat{b}_q \hat{\lambda}_j^*}{1 - \hat{\lambda}_j^* \hat{\lambda}_q}\,. \tag{3.8}$$

(iii) For $q = 1, \ldots, n$ *we have*

$$\frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{\lambda}_q)} = \frac{\partial H\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{\lambda}_q)} = 0\,, \tag{3.9}$$

$$\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Re(\hat{\lambda}_q)} = -\mathrm{i}\frac{\partial \hat{H}\left(\frac{1}{\lambda_j^*}\right)}{\partial \Im(\hat{\lambda}_q)} = \frac{\hat{c}_q \hat{b}_q \left(\lambda_j^*\right)^2}{\left(1 - \hat{\lambda}_q \lambda_j^*\right)^2}\,, \tag{3.10}$$

$$\frac{\partial \hat{H}\left(\frac{1}{\hat{\lambda}_j^*}\right)}{\partial \Re(\hat{\lambda}_q)} = -\mathrm{i}\frac{\partial \hat{H}\left(\frac{1}{\hat{\lambda}_j^*}\right)}{\partial \Im(\hat{\lambda}_q)} = \begin{cases} \frac{\hat{c}_q \hat{b}_q \left(\hat{\lambda}_j^*\right)^2}{\left(1-\hat{\lambda}_q \hat{\lambda}_j^*\right)^2} & \text{for } j \neq q \\ \frac{-1}{\left(\hat{\lambda}_q^*\right)^2}\hat{H}'\left(\frac{1}{\hat{\lambda}_q^*}\right) + \frac{\hat{c}_q \hat{b}_q \left(\hat{\lambda}_q^*\right)^2}{\left(1-\hat{\lambda}_q \hat{\lambda}_q^*\right)^2} & \text{for } j = q \end{cases} \tag{3.11}$$

$$\frac{\partial H\left(\frac{1}{\hat{\lambda}_j^*}\right)}{\partial \Re(\hat{\lambda}_q)} = -\mathrm{i}\frac{\partial H\left(\frac{1}{\hat{\lambda}_j^*}\right)}{\partial \Im(\hat{\lambda}_q)} = \begin{cases} 0 & \text{for } j \neq q \\ \frac{-1}{\left(\hat{\lambda}_q^*\right)^2}H'\left(\frac{1}{\hat{\lambda}_q^*}\right) & \text{for } j = q \end{cases} \tag{3.12}$$

Proof. (Theorem 3.1): In view of Lemma 3.2 and Equations (3.3)-(3.5), differentiating $\mathcal{J}$ with respect to $\Re(\hat{b}_{ql})$ yields

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \Re(\hat{b}_{ql})} &= \operatorname{trace}\left\{ \sum_{j=1}^{N} \frac{-\hat{c}_q e_l^*}{1-\lambda_j^* \hat{\lambda}_q} b_j^* c_j^* + \sum_{j=1}^{n} \frac{\hat{c}_q e_l^*}{1-\hat{\lambda}_j^* \hat{\lambda}_q} \hat{b}_j^* \hat{c}_j^* + \frac{1}{\hat{\lambda}_q^*}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) e_l \hat{c}_q^* \right\} \\
&= \operatorname{trace}\left\{ \left[\left(\sum_{j=1}^{N} \frac{-c_j b_j}{1-\lambda_j \hat{\lambda}_q^*} + \sum_{j=1}^{n} \frac{\hat{c}_j \hat{b}_j}{1-\hat{\lambda}_j \hat{\lambda}_q^*} \right) e_l \hat{c}_q^* \right]^* + \frac{1}{\hat{\lambda}_q^*}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) e_l \hat{c}_q^* \right\} \\
&= \operatorname{trace}\left\{ \left[\frac{1}{\hat{\lambda}_q^*}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) e_l \hat{c}_q^* \right]^* + \frac{1}{\hat{\lambda}_q^*}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) e_l \hat{c}_q^* \right\} \\
&= 2\Re\left(\operatorname{trace}\left\{ \frac{1}{\hat{\lambda}_q^*}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) e_l \hat{c}_q^* \right\} \right)
\end{aligned}$$

Analogously, we obtain that

$$\frac{\partial \mathcal{J}}{\partial \Im(\hat{b}_{ql})} = -2\Im\left(\operatorname{trace}\left\{ \frac{1}{\hat{\lambda}_q^*}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) e_l \hat{c}_q^* \right\} \right)$$

Thus, due to stability of the reduced order system we have

$$\frac{\partial \mathcal{J}}{\partial \Re(\hat{b}_{ql})} = 0 \text{ and } \frac{\partial \mathcal{J}}{\partial \Im(\hat{b}_{ql})} = 0 \Leftrightarrow \operatorname{trace}\left\{ \left[\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right] e_l \hat{c}_q^* \right\} = 0 \tag{3.13}$$

The first condition of Equation (3.2) follows directly from the right hand side of Equivalence (3.13) by substituting $q \mapsto j$.

Using Equations (3.6)-(3.8), we show that the following holds:

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \Re(\hat{c}_{lq})} &= \operatorname{trace}\left\{ \sum_{j=1}^{N} \frac{-e_l^* \hat{b}_q}{1-\lambda_j^* \hat{\lambda}_q} b_j^* c_j^* + \sum_{j=1}^{n} \frac{e_l^* \hat{b}_q}{1-\hat{\lambda}_j^* \hat{\lambda}_q} \hat{b}_j^* \hat{c}_j^* + \frac{1}{\hat{\lambda}_q}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) \hat{b}_q^* e_l^* \right\} \\
&= \operatorname{trace}\left\{ \left[\sum_{j=1}^{N} \frac{-c_j b_j}{1-\lambda_j \hat{\lambda}_q^*} \hat{b}_q^* e_l^* + \sum_{j=1}^{n} \frac{\hat{c}_j \hat{b}_j}{1-\hat{\lambda}_j \hat{\lambda}_q^*} \hat{b}_q^* e_l^* \right]^* + \frac{1}{\hat{\lambda}_q}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) \hat{b}_q^* e_l^* \right\} \\
&= \operatorname{trace}\left\{ \left[\frac{1}{\hat{\lambda}_q}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) \hat{b}_q^* e_l^* \right]^* + \frac{1}{\hat{\lambda}_q}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) \hat{b}_q^* e_l^* \right\} \\
&= 2\Re\left(\operatorname{trace}\left\{ \frac{1}{\hat{\lambda}_q}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) \hat{b}_q^* e_l^* \right\} \right)
\end{aligned}$$

In a similar way, we prove that

$$\frac{\partial \mathcal{J}}{\partial \Im(\hat{c}_{lq})} = -2\Im\left(\operatorname{trace}\left\{ \frac{1}{\hat{\lambda}_q}\left(\hat{H}\left(\frac{1}{\hat{\lambda}_q^*}\right) - H\left(\frac{1}{\hat{\lambda}_q^*}\right)\right) \hat{b}_q^* e_l^* \right\} \right)$$

Therefore as for (3.13) we get

$$\frac{\partial \mathcal{J}}{\partial \Re(\hat{c}_{lq})} = 0 \text{ and } \frac{\partial \mathcal{J}}{\partial \Im(\hat{c}_{lq})} = 0 \Leftrightarrow \quad \operatorname{trace}\left\{ \left[H(-\hat{\lambda}_q^*) - \hat{H}(-\hat{\lambda}_q^*)\right] \hat{b}_q^* e_l \right\} = 0 \tag{3.14}$$

The right hand side of Equivalence (3.14) directly leads to the second part of Conditions (3.2).

For the third part, we first note that the following equalities hold

$$\begin{aligned} \sum_{j=1}^{N} \frac{-c_j b_j \lambda_j \left(\hat{\lambda}_q^*\right)^2}{\left(1-\hat{\lambda}_q^* \lambda_j\right)^2} &= \tfrac{1}{\hat{\lambda}_q^*} H'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) + H\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \\ \sum_{j=1}^{n} \frac{-\hat{c}_j \hat{b}_j \lambda_j \left(\hat{\lambda}_q^*\right)^2}{\left(1-\hat{\lambda}_q^* \hat{\lambda}_j\right)^2} &= \tfrac{1}{\hat{\lambda}_q^*} \hat{H}'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) + \hat{H}\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \end{aligned} \tag{3.15}$$

Equations (3.9)-(3.12) lead to

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial \Re(\hat{\lambda}_q)} = \operatorname{trace} \Bigg\{ & \sum_{j=1}^{N} \tfrac{-1}{\lambda_j^*} \frac{\hat{c}_q \hat{b}_q \left(\lambda_j^*\right)^2}{\left(1-\hat{\lambda}_q^* \lambda_j^*\right)^2} b_j^* c_j^* + \sum_{j=1}^{n} \tfrac{1}{\hat{\lambda}_j^*} \frac{\hat{c}_q \hat{b}_q \left(\hat{\lambda}_j^*\right)^2}{\left(1-\hat{\lambda}_q \hat{\lambda}_j^*\right)^2} \hat{b}_j^* \hat{c}_j^* \\ & + \tfrac{1}{\hat{\lambda}_q^*} \left(\tfrac{-1}{\left(\hat{\lambda}_q^*\right)^2} \hat{H}'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) + \tfrac{1}{\left(\hat{\lambda}_q^*\right)^2} H'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^* \hat{c}_q^* \\ & - \tfrac{1}{\left(\hat{\lambda}_q^*\right)^2} \left(\hat{H}\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - H\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^* \hat{c}_q^* \Bigg\} \end{aligned}$$

Thus, from (3.15) we obtain

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial \Re(\hat{\lambda}_q)} = \operatorname{trace} \Bigg\{ & \left[\tfrac{1}{\left(\hat{\lambda}_q^*\right)^2} \left(\tfrac{1}{\hat{\lambda}_q^*} H'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) + H\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - \tfrac{1}{\hat{\lambda}_q^*} \hat{H}'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - \hat{H}\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^* \hat{c}_q^* \right]^* \\ & + \tfrac{1}{\left(\hat{\lambda}_q^*\right)^2} \left(-\tfrac{1}{\hat{\lambda}_q^*} \hat{H}'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) + \tfrac{1}{\hat{\lambda}_q^*} H'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - \hat{H}\left(\tfrac{1}{\hat{\lambda}_q^*}\right) + H\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^* \hat{c}_q^* \Bigg\} \\ = 2\Re \Bigg(\operatorname{trace} \Bigg\{ & \tfrac{1}{\left(\hat{\lambda}_q^*\right)^3} \left(H'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - \hat{H}'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^* \hat{c}_q^* \Bigg\} \Bigg) \\ + 2\Re \Bigg(\operatorname{trace} \Bigg\{ & \tfrac{1}{\left(\hat{\lambda}_q^*\right)^2} \underbrace{\left(H\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - \hat{H}\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^*}_{=0} \hat{c}_q^* \Bigg\} \Bigg) \end{aligned}$$

where the second summand vanishes due to the already proven first part of Conditions (3.2). Analogously we obtain that

$$\frac{\partial \mathcal{J}}{\partial \Im(\hat{\lambda}_q)} = -2\Im \left(\operatorname{trace} \left\{ \tfrac{1}{\left(\hat{\lambda}_q^*\right)^3} \left(H'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) - \hat{H}'\left(\tfrac{1}{\hat{\lambda}_q^*}\right) \right) \hat{b}_q^* \hat{c}_q^* \right\} \right) \tag{3.16}$$

From the considerations above it is easy to see that

$$\frac{\partial \mathcal{J}}{\partial \Re(\hat{\lambda}_q)} = 0 \text{ and } \frac{\partial \mathcal{J}}{\partial \Im(\hat{\lambda}_q)} = 0 \quad \Leftrightarrow \quad \operatorname{trace} \left\{ \left[H'(-\hat{\lambda}_q^*) - \hat{H}'(-\hat{\lambda}_q^*) \right] \hat{b}_q^* \hat{c}_q^* \right\} = 0 \tag{3.17}$$

Conditions (3.2) directly follow from the right hand sides of Equivalences (3.17). □

Expression (2.9) for the h_2-norm of the error system together with Conditions (3.2) directly implies the following result for the h_2-error of an optimal reduced system.

Remark 3.3. Suppose the system matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ solve the h_2-norm optimal model reduction problem (3.1). Then the h_2-norm of the error system is given by

$$\|H - \hat{H}\|_{h_2}^2 = \sum_{j=1}^{N} \left(\tfrac{1}{\lambda_j^*} \operatorname{trace} \left(\left[H\left(\tfrac{1}{\lambda_j^*}\right) - \hat{H}\left(\tfrac{1}{\lambda_j^*}\right) \right] b_j^* c_j^* \right) \right)$$

3.2.2. *Wilson conditions.* The Wilson optimality conditions for discrete systems involve the gramians of the error system which can be defined via discrete Sylvester equations also called Stein equations. Therefore it is necessary to introduce some notations and relations. Consider the two Stein equations of the discrete error system (2.10)

$$A_e \mathcal{P}_e A_e^* + \tilde{B}_e = \mathcal{P}_e \quad \text{with} \quad \tilde{B}_e := B_e B_e^* \tag{3.18}$$

$$A_e^* \mathcal{Q}_e A_e + \tilde{C}_e = \mathcal{Q}_e \quad \text{with} \quad \tilde{C}_e := C_e^* C_e \tag{3.19}$$

where the symmetric matrices $\mathcal{P}_e$ and $\mathcal{Q}_e$ are the reachability and observability gramians of the error system, respectively. Partition $\mathcal{P}_e$ and $\mathcal{Q}_e$ as

$$\mathcal{P}_e = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \qquad \mathcal{Q}_e = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix},$$

where $\mathcal{P}_{11}, \mathcal{Q}_{11} \in \mathbb{C}^{N,N}; \mathcal{P}_{12}, \mathcal{P}_{21}^*, \mathcal{Q}_{12}, \mathcal{Q}_{21}^* \in \mathbb{C}^{N,n}$ and $\mathcal{P}_{22}, \mathcal{Q}_{22} \in \mathbb{C}^{n,n}$. The full-rank submatrices $\mathcal{P}_{11}, \mathcal{Q}_{11}, \mathcal{P}_{22}$ and $\mathcal{Q}_{22}$ solve the Stein equations

$$A\mathcal{P}_{11}A^* + BB^* = \mathcal{P}_{11}$$

$$A^*\mathcal{Q}_{11}A + C^*C = \mathcal{Q}_{11}$$

$$\hat{A}\mathcal{P}_{22}\hat{A}^* + \hat{B}\hat{B}^* = \mathcal{P}_{22} \tag{3.20}$$

$$\hat{A}^*\mathcal{Q}_{22}\hat{A} + \hat{C}^*\hat{C} = \mathcal{Q}_{22}\,. \tag{3.21}$$

Hence, they are the gramians of the original and the reduced system

$$\mathcal{P}_{11} = \mathcal{P} \qquad \mathcal{Q}_{11} = \mathcal{Q} \qquad \mathcal{P}_{22} = \hat{\mathcal{P}} \qquad \mathcal{Q}_{22} = \hat{\mathcal{Q}}\,.$$

As gramians are symmetric we obtain $\mathcal{P}_{12} = \mathcal{P}_{21}^*$ and $\mathcal{Q}_{12} = \mathcal{Q}_{21}^*$ and both matrices are solutions of the following Stein equations

$$A\mathcal{P}_{12}\hat{A}^* + B\hat{B}^* = \mathcal{P}_{12} \tag{3.22}$$

$$A^*\mathcal{Q}_{12}\hat{A} - C^*\hat{C} = \mathcal{Q}_{12}\,. \tag{3.23}$$

Finding an h_2-norm optimal reduced model for a real system Σ requires to determine the first derivatives of the error functional $\mathcal{J}(\hat{A}, \hat{B}, \hat{C})$. In [1] it has been shown that

$$\mathcal{J}(\hat{\Sigma}) = ||\Sigma_e||_{h_2}^2 = \operatorname{trace}\left[C_e \mathcal{P}_e C_e^*\right] = \operatorname{trace}\left[\mathcal{P}_e \tilde{C}_e\right] \tag{3.24}$$

The derivatives of $\mathcal{J}$ with respect to the elements of $\hat{A}, \hat{B}$ and $\hat{C}$ namely $\hat{a}, \hat{b}$ and $\hat{c}$ result in the following necessary conditions in [22] given for continuous systems.

Theorem 3.4 (Wilson Conditions for discrete systems)**.** *Let $\hat{\Sigma}$ minimize (3.24). Then the following holds*

$$\mathcal{Q}_{12}^* A \mathcal{P}_{12} + \mathcal{Q}_{22}\hat{A}\mathcal{P}_{22} = 0 \tag{3.25}$$

$$\mathcal{Q}_{12}^* B + \mathcal{Q}_{22}\hat{B} = 0 \tag{3.26}$$

$$\hat{C}\mathcal{P}_{22} - C\mathcal{P}_{12} = 0\,. \tag{3.27}$$

Proof. The first part of the following proof follows partly the ideas in Wilsons proof for continuous systems. Differentiate Equation (3.18) with respect to the elements of $\hat{A}$ and $\hat{B}$. Let β be any parameter appearing in the elements of these matrices

$$\frac{\partial}{\partial\beta}\left(A_e\mathcal{P}_eA_e^* + \tilde{B}_e - \mathcal{P}_e\right) = 0\ .$$

It follows that

$$\frac{\partial A_e}{\partial\beta}\mathcal{P}_eA_e^* + A_e\frac{\partial\mathcal{P}_e}{\partial\beta}A_e^* + A_e\mathcal{P}_e\frac{\partial A_e^*}{\partial\beta} + \frac{\partial\tilde{B}_e}{\partial\beta} - \frac{\partial\mathcal{P}_e}{\partial\beta} = 0\ .$$

Postmultiplying by $\mathcal{Q}_e$ and taking the trace we get

$$\begin{aligned}&\operatorname{trace}\left(\frac{\partial A_e}{\partial\beta}\mathcal{P}_eA_e^*\mathcal{Q}_e\right) + \operatorname{trace}\left(\frac{\partial A_e^*}{\partial\beta}\mathcal{Q}_eA_e\mathcal{P}_e\right) + \operatorname{trace}\left(\frac{\partial\tilde{B}_e}{\partial\beta}\mathcal{Q}_e\right) =\\ &\operatorname{trace}\left(\frac{\partial\mathcal{P}_e}{\partial\beta}\mathcal{Q}_e\right) - \operatorname{trace}\left(\frac{\partial\mathcal{P}_e}{\partial\beta}A_e^*\mathcal{Q}_eA_e\right)\ .\end{aligned} \tag{3.28}$$

Differentiating the error functional (3.24) with respect to the elements of $\hat{A}$, $\hat{B}$ and $\hat{C}$ yields

$$\frac{\partial}{\partial\beta}\mathcal{J}(\hat{\Sigma}) = \frac{\partial}{\partial\beta}\operatorname{trace}\left(\mathcal{P}_e\tilde{C}_e\right) = \operatorname{trace}\left(\frac{\partial\mathcal{P}_e}{\partial\beta}\tilde{C}_e\right) + \operatorname{trace}\left(\mathcal{P}_e\frac{\partial\tilde{C}_e}{\partial\beta}\right)\ .$$

Together with Equation (3.19) and (3.28) it follows that

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathcal{J}(\hat{\Sigma}) &= \operatorname{trace}\left(\frac{\partial\mathcal{P}_e}{\partial\beta}\mathcal{Q}_e\right) - \operatorname{trace}\left(\frac{\partial\mathcal{P}_e}{\partial\beta}A_e^*\mathcal{Q}_eA_e\right) + \operatorname{trace}\left(\mathcal{P}_e\frac{\partial\tilde{C}_e}{\partial\beta}\right)\\ &= \operatorname{trace}\left(\frac{\partial A_e}{\partial\beta}\mathcal{P}_eA_e^*\mathcal{Q}_e\right) + \operatorname{trace}\left(\frac{\partial A_e^*}{\partial\beta}\mathcal{Q}_eA_e\mathcal{P}_e\right) +\\ &\quad + \operatorname{trace}\left(\frac{\partial\tilde{B}_e}{\partial\beta}\mathcal{Q}_e\right) + \operatorname{trace}\left(\frac{\partial\tilde{C}_e}{\partial\beta}\mathcal{P}_e\right)\ .\end{aligned}$$

The first two summands of the last equation are equal if the system is real. Because of the derivatives of A_e^*, $\tilde{B}_e$ and $\tilde{C}_e$ with respect to β the functional $\mathcal{J}$ is not complex differentiable. Thus we have to devide the variable β into its real and imaginary part. For the (i,j)-th element $\hat{a}_{ij}$ of $\hat{A}$ we get

$$\begin{aligned}\frac{\partial\hat{A}}{\partial\Re\{\hat{a}_{ij}\}} = \frac{\partial\hat{A}^*}{\partial\Re\{\hat{a}_{ij}\}} &= \begin{cases}1 & \text{in the } (i,j)\text{-th element}\\ 0 & \text{elsewhere}\end{cases}\\ \frac{\partial\hat{A}}{\partial\Im\{\hat{a}_{ij}\}} = -\frac{\partial\hat{A}^*}{\partial\Im\{\hat{a}_{ij}\}} &= \begin{cases}i & \text{in the } (i,j)\text{-th element}\\ 0 & \text{elsewhere.}\end{cases}\end{aligned}$$

Hence, we have the following equations

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \Re\{\hat{a}_{ij}\}} &= \operatorname{trace}\left(\begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial \hat{A}}{\partial \Re\{\hat{a}_{ij}\}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^* & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & \hat{A}^* \end{bmatrix} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^* & \mathcal{Q}_{22} \end{bmatrix}\right) \\
&\quad + \operatorname{trace}\left(\begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial \hat{A}}{\partial \Re\{\hat{a}_{ij}\}} \end{bmatrix} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^* & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^* & \mathcal{P}_{22} \end{bmatrix}\right) \\
&= 2 \operatorname{trace}\left(\frac{\partial \hat{A}}{\partial \Re\{\hat{a}_{ij}\}} \Re\{\mathcal{Q}_{12}^* A \mathcal{P}_{12} + \mathcal{Q}_{22} \hat{A} \mathcal{P}_{22}\}\right) \\
\frac{\partial \mathcal{J}}{\partial \Im\{\hat{a}_{ij}\}} &= 2 \operatorname{trace}\left(i \frac{\partial \hat{A}}{\partial \Im\{\hat{a}_{ij}\}} \Im\{\mathcal{Q}_{12}^* A \mathcal{P}_{12} + \mathcal{Q}_{22} \hat{A} \mathcal{P}_{22}\}\right) ,
\end{aligned}$$

which gives (3.25) by considering $\frac{\partial \mathcal{J}}{\partial \hat{a}_{ij}} = 0$ for all elements $\hat{a}_{ij}$ of $\hat{A}$. The other two necessary conditions (3.26) and (3.27) can be proven in a similar manner by computing the derivatives with respect to the elements of $\tilde{B}_e$ and $\tilde{C}_e$. □

Remark 3.5. Generally reduced order models are constructed by an oblique projection $\Pi = VZ^*$ with $V, Z \in \mathbb{C}^{N,n}$ and $Z^*V = I_n$, where I_n is the $n \times n$ identity matrix.

$$\hat{\Sigma} = \left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right) = \left(\begin{array}{c|c} Z^*AV & Z^*B \\ \hline CV & 0 \end{array}\right) . \tag{3.29}$$

The projection Π could be deduced directly from the Wilson conditions (3.25)-(3.27). For instance (3.25) yields $\hat{A} = -\mathcal{Q}_{22}^{-1}\mathcal{Q}_{12}^* A \mathcal{P}_{12}\mathcal{P}_{22}^{-1}$ and we get $Z := -\mathcal{Q}_{12}\mathcal{Q}_{22}^{-1}$ and $V := \mathcal{P}_{12}\mathcal{P}_{22}^{-1}$.

3.2.3. *Hyland-Bernstein conditions.* Similar to the Wilson conditions we provide the Hyland-Bernstein conditions [11] by means of the gramians and the Stein equations. The reduced order model could in principle be constructed by an oblique projection $\Pi = VZ^*$ as demonstrated in (3.29).

Theorem 3.6 (Hyland-Bernstein Conditions)**.** *Suppose the system matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ solve the h_2-norm optimal model reduction problem (3.1). Then there exist two nonnegative-definite matrices $\mathcal{P}, \mathcal{Q} \in \mathbb{C}^{N,N}$ and a positive-definite matrix $M \in \mathbb{C}^{n,n}$ such that*

$$\mathcal{P}\mathcal{Q} = VMZ^* \tag{3.30}$$

$$\operatorname{rank}(\mathcal{P}) = \operatorname{rank}(\mathcal{Q}) = \operatorname{rank}(\mathcal{P}\mathcal{Q}) . \tag{3.31}$$

The reduced model is constructed equivalently to (3.29) and furthermore the following two conditions are satisfied

$$\begin{aligned}
\Pi\left[A\mathcal{P}A^* + BB^* - \mathcal{P}\right] &= 0 \\
\left[A^*\mathcal{Q}A + C^*C - \mathcal{Q}\right]\Pi &= 0
\end{aligned}$$

with the projection matrix $\Pi = VZ^*$.

3.3. **Equivalence of first order necessary conditions.** For SISO continuous systems with simple poles it has been shown [10] that the Lyapunov based first order necessary conditions of Hyland-Bernstein [11] and Wilson [22] are equivalent to some interpolation conditions [14] which are similar to the first order necessary Conditions (3.2). We proved [21] that the equivalences are also true for continuous MIMO systems and multiple poles. This leads to similar but more complex interpolation conditions.

We point out that although the first order necessary conditions are equivalent computing systems satisfying these conditions is to costly for large systems. The computational costs are much smaller in case of interpolation based techniques. Methods presented in [11] and [23] locate solutions that satisfy first order necessary conditions, however they require solving a series of large Lyapunov equations and therefore are not applicable for high dimensional systems. In Section 4 we present an effective numerical algorithm which produces a reduced order model that satisfies the tangential interpolation based first order necessary Conditions (3.2) and is suitable for very large systems.

3.3.1. *Equivalence between Interpolation and Wilson conditions.* The equivalence between the interpolation based and Wilson conditions will be verified by a proper analysis of the projection $\Pi = VZ^*$. The following two lemmas reveal the structur of the coloumnspan of the projection matrices V and Z. The intermediate result leads to two further lemmas which finally yield the interpolation condition.

Lemma 3.7. *The following statements are equivalent.*

$$\begin{array}{lrl} \textit{(i)} & V = & \mathcal{P}_{12}\mathcal{P}_{22}^{-1} \\ \textit{(ii)} & \operatorname{Ran} V = & \operatorname{colspan}\{(-A+\frac{1}{\hat{\lambda}_1^*}I)^{-1}B\hat{b}_1^*, \ldots, (-A+\frac{1}{\hat{\lambda}_n^*}I)^{-1}B\hat{b}_n^*\} \end{array}$$

Proof. Without loss of generality we assume that $\hat{A} = \operatorname{diag}[\hat{\lambda}_1, \ldots, \hat{\lambda}_n]$. We reformulate Equation (3.22)

$$\begin{bmatrix} a_{11} & \ldots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \ldots & a_{NN} \end{bmatrix} \begin{bmatrix} p_{11} & \ldots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{N1} & \ldots & p_{Nn} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_1^* & & \\ & \ddots & \\ & & \hat{\lambda}_n^* \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \begin{bmatrix} \hat{b}_1^* & \ldots & \hat{b}_n^* \end{bmatrix}$$
$$= \begin{bmatrix} p_{11} & \ldots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{N1} & \ldots & p_{Nn} \end{bmatrix}.$$

where b_k is the k-th row of B with $k = 1, \ldots, N$; $\hat{b}_l$ is the l-th row of $\hat{B}$ with $l = 1, \ldots, n$ and p_{kl} is the k,l-th component of $\mathcal{P}_{12}$. Splitting this into single equations yields the following structure of n blocks each with N equations

$$\left\{ \begin{array}{rcccc} (a_{11}p_{11} + a_{12}p_{21} + \ldots + a_{1N}p_{N1})\,\hat{\lambda}_1^* & + & b_1\hat{b}_1^* & = & p_{11} \\ (a_{21}p_{11} + a_{22}p_{21} + \ldots + a_{2N}p_{N1})\,\hat{\lambda}_1^* & + & b_2\hat{b}_1^* & = & p_{21} \\ \vdots & & \vdots & & \vdots \\ (a_{N1}p_{11} + a_{N2}p_{21} + \ldots + a_{NN}p_{N1})\,\hat{\lambda}_1^* & + & b_N\hat{b}_1^* & = & p_{N1} \end{array} \right\} \cdots$$

$$\cdots \left\{ \begin{array}{ccccc} (a_{11}p_{1n} + a_{12}p_{2n} + \ldots + a_{1N}p_{Nn})\,\hat{\lambda}_n^* & + & b_1\hat{b}_n^* & = & p_{1n} \\ (a_{21}p_{1n} + a_{22}p_{2n} + \ldots + a_{2N}p_{Nn})\,\hat{\lambda}_n^* & + & b_2\hat{b}_r^* & = & p_{2r} \\ \vdots & & \vdots & & \vdots \\ (a_{N1}p_{1n} + a_{N2}p_{2n} + \ldots + a_{NN}p_{Nn})\,\hat{\lambda}_n^* & + & b_N\hat{b}_n^* & = & p_{Nn} \end{array} \right\} .$$

Stating this in matrix form and under the assumption that $(A - \frac{1}{\hat{\lambda}_j^*}I)$ is invertible for $j = 1, \ldots, n$ we can easily dissolve each equation with respect to the columns p_k of $\mathcal{P}_{12}$ $(k = 1, \ldots, n)$.

$$\left. \begin{array}{c} (A - \frac{1}{\hat{\lambda}_1^*}I)\hat{\lambda}_1^* p_1 + B\hat{b}_1^* = 0 \\ \vdots \\ (A - \frac{1}{\hat{\lambda}_n^*}I)\hat{\lambda}_n^* p_n + B\hat{b}_n^* = 0 \end{array} \right\} \implies \left\{ \begin{array}{c} p_1 = \frac{1}{\hat{\lambda}_1^*}(-A + \frac{1}{\hat{\lambda}_1^*})^{-1}B\hat{b}_1^* \\ \vdots \\ p_n = \frac{1}{\hat{\lambda}_n^*}(-A + \frac{1}{\hat{\lambda}_n^*})^{-1}B\hat{b}_n^* \end{array} \right.$$

Up to this point we just did simple algebraic reformulations. For the proof of the implication $(i) \implies (ii)$ the Wilson condition (3.27) is used. Recall that $V := \mathcal{P}_{12}\mathcal{P}_{22}^{-1}$ and $\mathcal{P}_{22}^{-1}$ has full-rank. Hence, it holds that the image of $\mathcal{P}_{12}$ ist a subset of the image of $\mathcal{P}_{22}^{-1}$ and therefore is equal to the image of V which is an intersection of both

$$\begin{aligned} \operatorname{Ran} V &= \operatorname{Ran} \mathcal{P}_{12} \cap \operatorname{Ran} \mathcal{P}_{22}^{-1} \underset{\operatorname{Ran}\mathcal{P}_{12} \subseteq \operatorname{Ran}\mathcal{P}_{22}^{-1}}{=} \operatorname{Ran} \mathcal{P}_{12} \\ &= \operatorname{colspan}\{p_1, \ldots, p_n\} \\ &= \operatorname{colspan}\{(-A + \tfrac{1}{\hat{\lambda}_1^*}I)^{-1}B\hat{b}_1^*, \ldots, (-A + \tfrac{1}{\hat{\lambda}_n^*}I)^{-1}B\hat{b}_n^*\} . \end{aligned}$$

For the proof of the reverse implication $(ii) \implies (i)$ we have to show that

$$\operatorname{Ran} V = \operatorname{colspan}\{p_1, \ldots, p_n\} \quad \implies \quad V = \mathcal{P}_{12}\mathcal{P}_{22}^{-1} .$$

If $\operatorname{Ran} V = \operatorname{colspan}\{p_1, \ldots, p_n\}$, then $V = \mathcal{P}_{12} * K$ where $K \in \mathbb{C}^{n,n}$ is a nonsingular matrix. Premultiply Equation (3.22) by Z^* yields

$$Z^*A\mathcal{P}_{12}\hat{A}^* + Z^*B\hat{B}^* = Z^*\mathcal{P}_{12} .$$

Because V and Z describe an oblique projection we get the following results

$$\begin{aligned} Z^*V = I_n \quad &\implies \quad Z^*\mathcal{P}_{12} = K^{-1} \\ Z^*AV = \hat{A} \quad &\implies \quad Z^*A\mathcal{P}_{12} = \hat{A}K^{-1} \end{aligned}$$

Thus we obtain

$$\hat{A}K^{-1}\hat{A}^* + \hat{B}\hat{B}^* = K^{-1}$$

which is indeed the Stein equation (3.20) for the reachability gramian of the reduced system. Consequently $K^{-1} = \mathcal{P}_{22}$ which completes the proof. □

Equivalently the projection matrix Z could be determined with the following lemma.

Lemma 3.8. *The following statements are equivalent.*

$$\begin{array}{lrl} (i) & Z = & -\mathcal{Q}_{12}\mathcal{Q}_{22}^{-1} \\ (ii) & \operatorname{Ran} Z^* = & \operatorname{rowspan}\{\hat{\mathrm{c}}_1^*\mathrm{C}(-\mathrm{A} + \tfrac{1}{\hat{\lambda}_1^*}\mathrm{I})^{-1}, \ldots, \hat{\mathrm{c}}_\mathrm{n}^*\mathrm{C}(-\mathrm{A} + \tfrac{1}{\hat{\lambda}_\mathrm{n}^*}\mathrm{I})^{-1}\} \end{array}$$

Proof. Analogues to the preceeding proof using (3.19) instead of (3.18) we get similar expression for the columns of $\mathcal{Q}_{12}$ and therefore

$$\operatorname{Ran} Z = \operatorname{colspan}\{(-A^* + \tfrac{1}{\hat{\lambda}_1} I)^{-1} C^* \hat{c}_1, \ldots, (-A^* + \tfrac{1}{\hat{\lambda}_n} I)^{-1} C^* \hat{c}_n\} \, .$$

On the other hand the equations above lead to $Z = \mathcal{Q}_{12} L$, where $L \in \mathbb{C}^{n,n}$ is a nonsingular matrix. A similar discussion yields $-L^{-1} = \mathcal{Q}_{22}$. □

On basis of these results we could now infer the connection to the interpolation condition. The following result was proven in [18]. We cite it here, as it will be a crucial ingredient of our algorithm in the numerical section.

Lemma 3.9. *Let $V \in \mathbb{C}^{N,n}$ and $Z \in \mathbb{C}^{N,n}$ be matrices of full rank n such that $Z^*V = I_n$. Let $\sigma_j \in \mathbb{C}$, $\ell_j \in \mathbb{C}^{1\times p}$ and $r_l \in \mathbb{C}^{m\times 1}$ be given points and vectors, respectively. If $(\sigma_j I - A)^{-1} B r_j \in \operatorname{colspan}(V)$ and $(\sigma_j^* I - A^*)^{-1} C^* \ell_j \in \operatorname{colspan}(Z)$ holds for $j \in 1, \ldots, n$ the following tangential Hermite interpolation conditions are satisfied*

$$\begin{aligned} H(\sigma_j^*) r_j &= \widehat{H}(\sigma_j^*) r_j \\ \ell_j H(\sigma_j^*) &= \ell_j \widehat{H}(\sigma_j^*) \qquad k = 1, 2, \ldots, n \\ \ell_j H'(\sigma_j^*) r_j &= \ell_j \widehat{H}'(\sigma_j^*) r_j \end{aligned} \tag{3.32}$$

Hence, choosing $\sigma_j = 1/\hat{\lambda}_j^*$, $r_j = \hat{b}_j^*$ and $l_j = \hat{c}_j^*$ leads to the following result.

Lemma 3.10. *Let $V \in \mathbb{C}^{N,n}$ and $Z \in \mathbb{C}^{N,n}$ be matrices of full rank n such that $Z^*V = I_n$. If for all $j \in 1, \ldots, n$*

$$v_j := (-A + \tfrac{1}{\hat{\lambda}_j^*} I)^{-1} B \hat{b}_j^* \in \operatorname{colspan}(V) \quad \textit{and}$$

$$z_j := (-A^* + \tfrac{1}{\hat{\lambda}_j} I)^{-1} C^* \hat{c}_j \in \operatorname{colspan}(Z)$$

then the interpolation conditions (3.2) are satisfied.

The reverse direction of the preceeding lemma completes the equivalence proof between the Wilson and the Interpolation conditions.

Lemma 3.11. *Let $\hat{\Sigma}$ be a reduced system which satisfies the interpolation conditions (3.2). $\hat{\Sigma}$ can always be derived from Σ by a projection $\Pi = VZ^*$ with* $\operatorname{Ran} \mathrm{V} = \textit{colspan}\{\tilde{v}_1, \ldots, \tilde{v}_n\}$ *and* $\operatorname{Ran} \mathrm{Z} = \textit{colspan}\{\tilde{z}_1, \ldots, \tilde{z}_n\}$ *and*

$$\begin{aligned} \tilde{v}_j &= (-A + \tfrac{1}{\hat{\lambda}_j^*} I)^{-1} B \hat{b}_j^* \\ \tilde{z}_j &= (-A^* + \tfrac{1}{\hat{\lambda}_j} I)^{-1} C^* \hat{c}_j \qquad \textit{for } j = 1, \ldots, n \, . \end{aligned}$$

Proof. The system $\hat{\Sigma}$ is completely described by its matrix valued transfer function $\hat{H}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ and hence, under assumption that $\hat{A}$ is diagonal, comprises $n(m+p+1)$ specific elements, namely the entries of its system matrices. Thus the interpolation conditions (3.2) supply $n(m+p+1)$ constraints which can be met by the same number of restrictions imposed on the columnspaces of the projection matrices V and Z. See also [13]. □

Summarize all previous results yields the following theorem.

Theorem 3.12. *The necessary interpolation based Conditions (3.2) for simple poles given in Theorem 3.1 are equivalent to the Wilson conditions (3.25) - (3.27) presented in Theorem 3.4.*

3.3.2. *Equivalence between Hyland-Bernstein and Wilson conditions.* The idea for the proof of the following theorem can be found in [10] for continuous systems.

Theorem 3.13. *Let $\mathcal{P}$, $\mathcal{Q}$ and M be symmetric positive-definit matrices. $\mathcal{P}_{22}$, $\mathcal{Q}_{22}$, $\mathcal{P}_{12}$ and $\mathcal{Q}_{12}$ are solutions of the Equations (3.20) - (3.23), respectively. Then the following necessary conditions of Wilson (Theorem 3.4) and Hyland-Bernstein (Theorem 3.6) are equivalent.*

(i) $V = \mathcal{P}_{12}\mathcal{P}_{22}^{-1} \quad Z = -\mathcal{Q}_{12}\mathcal{Q}_{22}^{-1}$

(ii) $\Pi\left[A\mathcal{P}A^* + BB^* - \mathcal{P}\right] = 0 \qquad \left[A^*\mathcal{Q}A + C^*C - \mathcal{Q}\right]\Pi = 0$

$$\mathcal{P}\mathcal{Q} = VMZ^* \qquad \operatorname{rank}(\mathcal{P}) = \operatorname{rank}(\mathcal{Q}) = \operatorname{rank}(\mathcal{P}\mathcal{Q})$$

Proof. First we will proof the direction *(ii)* $\Longrightarrow$ *(i)*. In [11] it was already shown that $\mathcal{P} = \Pi\mathcal{P}$ for Π and $\mathcal{P}$ as in the Hyland-Bernstein conditions. It follows

$$\Pi\left[A\Pi\mathcal{P}A^* + BB^* - \Pi\mathcal{P}\right] = V\left[Z^*AVZ^*\mathcal{P}A^* + Z^*BB^* - Z^*\mathcal{P}\right] = 0\,.$$

Because V is a full-rank matrix the equation above leads to

$$\hat{A}Z^*\mathcal{P}A^* + \hat{B}B^* - Z^*\mathcal{P} = 0\,.$$

Transposing this equation yields to equation (3.22)

$$A\mathcal{P}Z\hat{A}^* + B\hat{B}^* - \mathcal{P}Z = 0\,.$$

Therefore $\mathcal{P}Z = \mathcal{P}_{12}$. Together with $\mathcal{P} = VZ^*\mathcal{P}$ and a premultiplication by Z^* we get

$$Z^*AVZ^*\mathcal{P}Z\hat{A}^* + Z^*B\hat{B}^* - Z^*\mathcal{P}Z = 0$$
$$\hat{A}Z^*\mathcal{P}Z\hat{A}^* + \hat{B}\hat{B}^* - Z^*\mathcal{P}Z = 0$$

which is identical to the Lyapunov Equation (3.20). Since $Z^*\mathcal{P}Z = Z^*\mathcal{P}_{12} = \mathcal{P}_{22}$ we get $Z^*\mathcal{P}_{12}\mathcal{P}_{22}^{-1} = I_n = Z^*V$ and finally $V = \mathcal{P}_{12}\mathcal{P}_{22}^{-1}$. Similar arguments together with $\mathcal{Q} = \mathcal{Q}\Pi$ yields $Z = -\mathcal{Q}_{12}\mathcal{Q}_{22}^{-1}$.

For the proof of the reverse implication *(i)* $\Longrightarrow$ *(ii)* we insert $\mathcal{P}_{12} = V\mathcal{P}_{22}$ in Equation (3.22).

$$AV\mathcal{P}_{22}V^*A^*Z + BB^*Z - V\mathcal{P}_{22} = 0$$

We transpose this equation and premultiply with V

$$VZ^*\left[AV\mathcal{P}_{22}V^*A^* + BB^* - V\mathcal{P}_{22}V^*\right] = 0\,.$$

Now we could define $\mathcal{P} := V\mathcal{P}_{22}V^*$ and obtain

$$\Pi\left[A\mathcal{P}A^* + BB^* - \mathcal{P}\right] = 0\,.$$

Analogically for $\mathcal{Q}_{12} = -Z\mathcal{Q}_{22}^{-1}$ we define $\mathcal{Q} := Z\mathcal{Q}_{22}Z^*$ and therefore

$$\left[A^*\mathcal{Q}A + CC^* - \mathcal{Q}\right]\Pi = 0\,.$$

As long as V, Z and the gramians are full-rank matrices condition (3.31) holds by construction. Moreover it is still left to proof condition (3.30)

$$\mathcal{P}\mathcal{Q} = V\mathcal{P}_{22}V^*Z\mathcal{Q}_{22}Z^* = V\mathcal{P}_{22}\mathcal{Q}_{22}Z^*\,.$$

The gramians $\mathcal{P}$ and $\mathcal{Q}$ are positive definite and consequently also M is a positive definite matrix. □

It is worth to note that Theorem 3.12 and Theorem 3.13 directly imply the following result.

Remark 3.14. All three necessary conditions given in this section are equivalent.

4. Algorithm and numerical examples

4.1. **MIRIAm, an efficient numerical algorithm.** As seen in (3.2), the reduced system is characterized by satisfying certain tangential interpolation condition. Up to a constant these conditions involve the moments of H and $\hat{H}$ at the mirror images $1/\hat{\lambda}_j^*$. It is well known that computing these moments explicitly leads to numerical instabilities, especially for large dimensions. The remedy for this problem are Krylov subspace based techniques that impose tangential moment matching conditions without computing the moments explicitly (cf., [9], [24],[25] for SISO systems and [8], [18] for MIMO systems). However, the interpolation data, i.e., the interpolation points and the directions of the tangential interpolation, are a priori unknown. Hence, we will now describe an iterative algorithm which in each iteration step derives a reduced system for a fixed set of interpolation data.

Let $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a set of interpolation points and let $\ell = \{\ell_1, \ldots, \ell_n\}$ and $r = \{r_1, \ldots, r_n\}$ be sets of left and right tangential directions, respectively. Let the columns of $V = [v_1, \ldots, v_n]$ and $\tilde{Z} = [\tilde{z}_1, \ldots, \tilde{z}_n]$ be

$$v_j = (-A + \sigma_j^* I)^{-1} B r_j^* \tag{4.1}$$

$$\tilde{z}_j = (-A^* + \sigma_j I)^{-1} C^* \ell_j \,. \tag{4.2}$$

for $j = 1, \ldots, n$ and choose $Z := \tilde{Z}(\tilde{Z}^* V)^{-*}$. Then, due to Lemma 3.9 the projection $\Pi = VZ^*$ leads to a reduced order system $\hat{\Sigma}$ with $\hat{A} = Z^*AV$, $\hat{B} = Z^*B$ and $\hat{C} = CV$ satisfying the tangential interpolation conditions (3.32) for the fixed interpolation data σ, l and r. For the next iteration step, let us take the mirror images of the poles of $\hat{A}$ as new interpolation points (i.e. $\sigma^{next} = eig(\hat{A})^{-*}$), the rows of $\hat{C}^*$ as left tangential directions (i.e. $\ell_j^{next} = \widehat{C}_j^*$) and the columns of $\widehat{B}^*$ as right tangential directions (i.e. $r_j^{next} = \widehat{B}_j^*$), where $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are the system matrices of $\hat{\Sigma}$ represented in the eigenvector basis. Now let $f(\Pi)$ denote the projection for this new set of interpolation points σ^{next} and new tangential directions ℓ^{next} and r^{next}. Note that a search for a reduced order system satisfying the first order necessary h_2-optimality Conditions (3.2) can be formulated as the search for a fix point of the function $f(\Pi) = \Pi$. A **M**IMO **I**terative **R**ational **I**nterpolation **A**lgorith**m** (MIRIAm) presented below, if it converges, solves this problem. The algorithm is a generalization of Iterative Rational Krylov Algorithm [10], applicable only for SISO systems, to the MIMO case.

A MIMO Iterative Rational Interpolation Algorithm (MIRIAm)

(1) Select an initial set of shifts $\sigma = \{\sigma_1, \ldots, \sigma_n\}$
(2) Select an initial set of directions $\ell = \{\ell_1, \ldots, \ell_n\}$ and $r_j = \{r_1, \ldots, r_n\}$
(3) while (not converged)
   a) Compute V via (4.1)
   b) Compute Z via (4.2)
   c) $Z = Z(Z^*V)^{-*}$ to guarantee $Z^*V = I_n$
   d) $\hat{A} = Z^*AV,\ \hat{B} = Z^*B,\ \hat{C} = CV$

e) Compute the eigenvalue decomposition of $\widehat{A} = X\Omega X^{-1}$, where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$
f) Assign $\sigma_j = 1/\omega_j^*$, $\ell_j = j^{th}$ column of $(\widehat{C}X)^*$, $r_j = j^{th}$ row of $(X^{-1}\widehat{B})^*$

4.2. **Numerical examples.** We demonstrate the performance of our algorithm, MIRIAm, on two examples. For both examples we compute reduced order models by our method and for comparison also by balanced truncation. We compare the errors in each the h_2-norm as well as in the h_∞-norm., which is the norm used for the error bound in balanced truncation.

For MIRIAm we use here the following four sets of interpolation data as starting data:

(1) BTbased We first computed the reduced order models using balanced truncation and then transforme the resulting system such that A is diagonal. We then choose as starting interpolation points the mirror images of the poles of this reduced system and its columns of $\widehat{B}^*$ and rows of $\widehat{C}^*$ as right and left tangential directions, respectively.
(2) Ordered EV - We transform the original system such that A is diagonal and its eigenvalues appear in descending order of magnitude. Let n be the dimension of the reduced order system. We choose as starting interpolation points the mirror images of the first n eigenvalues of A i.e. poles of the system. We use the conjugate transposes of the first n rows of B and the first n columns of C as right and left tangential directions, respectively.
(3) Rand-Complex - Interpolation points are randomly chosen complex points lying outside the unit circle. Complex random vectors were chosen as left and right tangential directions
(4) Rand-Real - Interpolation points are randomly chosen real points lying outside the interval [-1, 1], tangential directions are purely real and generated randomly with values varying between -1 and 1.

The first set of interpolation data was chosen in order to investigate the possible h_2- improvement of results obtained by balanced truncation and the corresponding changes of the h_∞-error. From our extensive earlier experiments we learned that eigenvalues of reduced systems received by balanced truncation or satisfying the first order necessary h_2-norm optimality conditions are often very good approximation of eigenvalues of the original system lying closest to the unit circle. Therefore it may seem possible to get a good approximation of the original system by computing approximations to such eigenvalues and just do one interpolation with their mirror images. We demontrate on the first example below that there are cases for which choosing these specific mirror images, i.e. "Ordered EV", not necessarily leads to good approximations nor does it improves the convergence of our method in such cases.

Example 4.1. ISS (International Space Station)
This is a model of component 1r (Russian service module) of the International Space Station. It has 270 states, 3 inputs and 3 outputs, i.e. N=270, m=p=3 (for details see e.g. [3],[6] and references therein). The discrete system was obtained by bilinear transformation of the continuous time system. The bilinear transformation is defined by $z = \frac{1+s}{1-s}$, and maps the open left-half of the complex plane onto the inside of the unit disc, and the imaginary axis onto the unit circle.

In this example the system is of the form

$$\Sigma : \quad \begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}$$

where we have an additional direct *feedthrough term* Du_k, which does not affect the performance of model reduction techniques. The system matrices $A,\ B,\ C$ and D stem from matrices $A_c,\ B_c,\ C_c,\ D_c$ of the original continuous system as follows:

$$\begin{aligned} A &= (I + A_c)(I - A_c)^{-1} \\ B &= \sqrt{2}(I - A_c)^{-1}B_c \\ C &= \sqrt{2}C_c(I - A_c)^{-1} \\ D &= D_c + C(I - A_c)^{-1}B_c \end{aligned} \tag{4.3}$$

For the bilinear transformation and other transformations between continuous and discrete systems, see [1] and references therein.

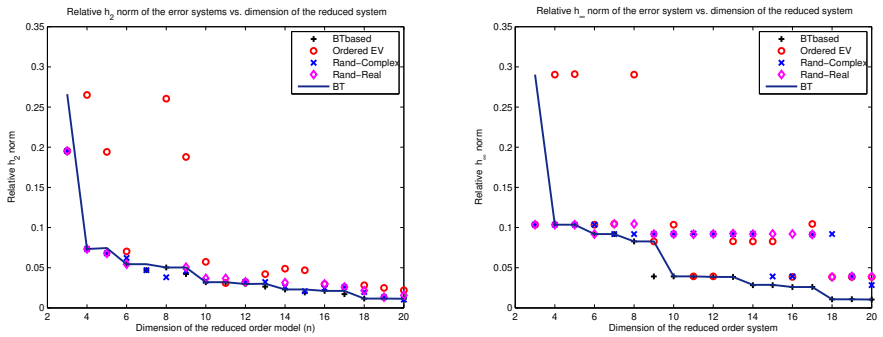

FIGURE 1. The relative h_2-norm of the error system vs the dimension of the reduced system (CD player)

In Figure 1 we present a comparison of relative h_2- and h_∞-norms of the error systems, i.e. the quotient of the norm of the error system over the norm of the original one, obtained by balanced truncation and MIRIAm for dimensions of the reduced order varying from 20 to 3. The figure clearly ilustrates that the quality of our method strongly depends on the choice of starting data. In general we get results very close to the ones obtained by balanced truncation, although with inappropriate selection of starting data we run into local minima, which do not give a good approximation. The best results, usually after very few iterations, were obtained when "BTbased" interpolation data were taken. We observe in this case that the improvement of h_2-norm does not cause big changes in the h_∞-norm. Taking random, real or complex, choices of data usually leads to results close, in the h_2-norm sense, to the ones obtained by balanced truncation. We also observe that often the h_∞-norm of the reduced system is significantly worse. We can easily see that for this example the worst results in both norms were obtained when "Ordered EV" were chosen. A closer look at the eigenvalue distribution of the original system explains this behaviour. In Figure 2 it is seen on the left that the eigenvalues of the original system are all close to the unit circle, the largest in magnitude being close to -1. In such a situation we cannot hope to find the suitable ones by just taking the largest in magnitude. The right hand side graphic shows the selected "Ordered EV", which are all very close to -1 and the eigenvalues of the balanced

truncation reduced system.They also mainly cluster close to -1, but spread out more and there are a few of them far away but still close to the circle.

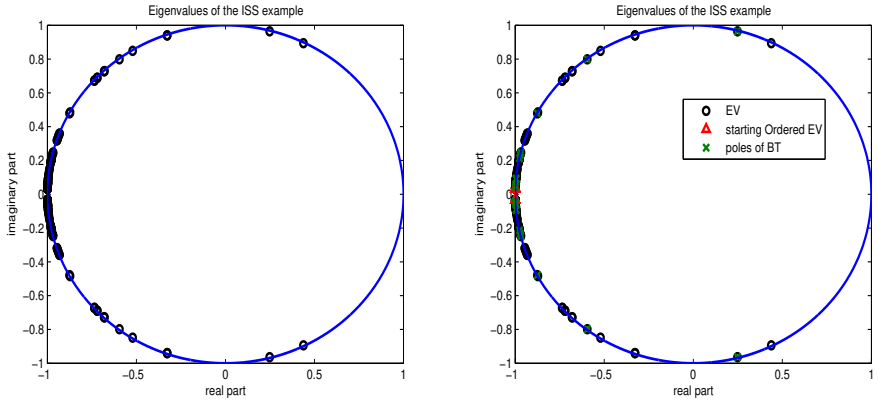

FIGURE 2. Eigenvalues(left) and starting data

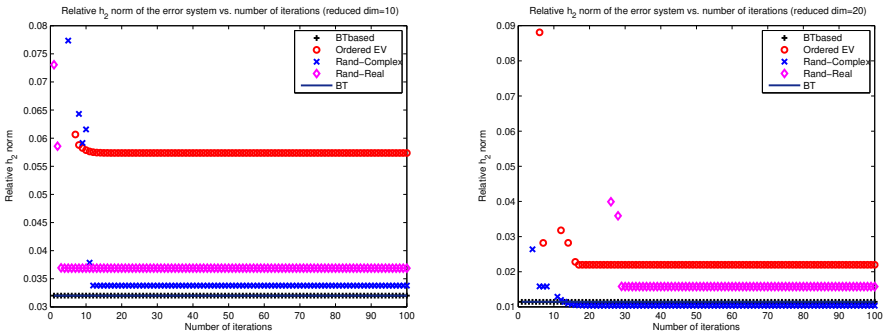

FIGURE 3. The relative h_2-norm of the error system vs. the number of iterations for the reduced systems of order 10(left) and 20(right) (ISS)

In order to examine the convergence of the proposed algorithm we reduce the order of the system and in each step of the iteration we compute the relative h_2-norm of the error system. Figure 3 shows the relative h_2-norm of the error system versus the number of iterations for size $n, n \in \{10, 20\}$ of the reduced systems. We see that the algorithm converges after a small number of iterations. If the size of the reduced system is odd, e.g. $n \in \{5, 15\}$, as presented in Figure 4, convergence is attained only after quite a large number of iterations. Therefore we advice to choose as orders of the reduced systems an even number.

Example 4.2. This is a randomly generated discrete time dynamical system. It has 400 states, 4 inputs and 4 outputs, i.e. N=400, m=p=4. The poles and Hankel Singular Values of the system are presented in the Figure 5.

In Table 1 we present a comparison of the relative h_2- and h_∞-norm of the error systems obtained by balanced truncation and MIRIAm for even dimensions of the reduced system varying from 30 to 4. BT-based starting initial data were used.

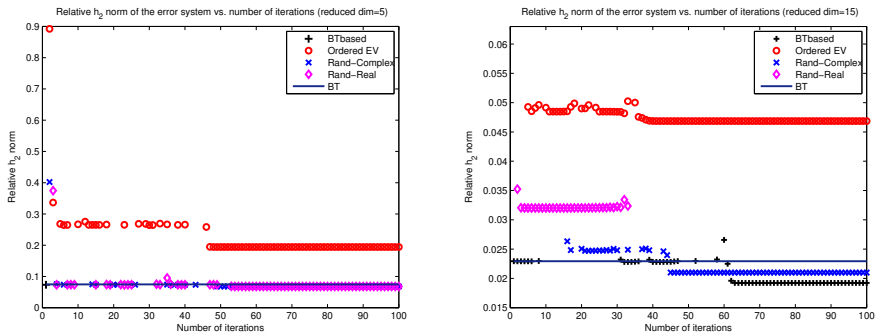

FIGURE 4. The relative h_2-norm of the error system vs. the number of iterations for the reduced systems of order 5(left) and 15(right) (ISS)

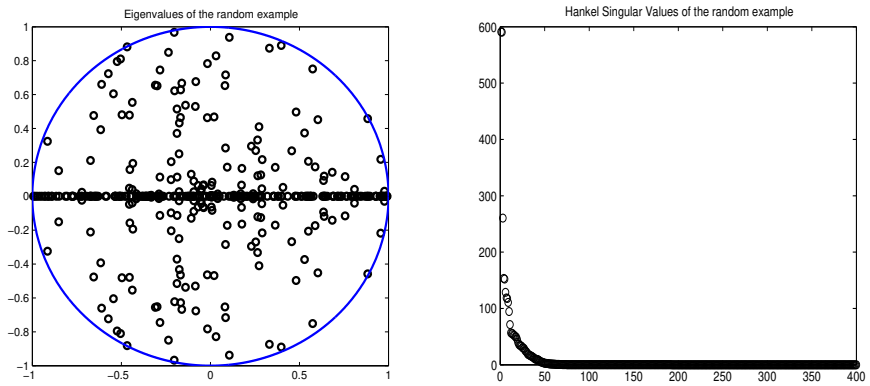

FIGURE 5. Eigenvalues(left) and Hankel Singular Values (right) of the random example

We observe that, in most cases, after very few iterations with MIRIAm we obtain results, which are better approximations in the h_2-norm than balanced truncation. Table 1 illustrates also that the h_∞-norms of reduced order models obtained with MIRIAm is slightly worse than ones obtained by balanced truncation, as expected.

For MIMO systems we have local minima more often than for SISO systems. In Figure 6 the dimension of the reduced system is plotted versus the relative h_2-norm of the error systems obtained by balanced truncation and by MIRIAm. In the latter case we chose our four different starting interpolation sets as before. Figure 6 reveals that all selection strategies work quite well. Indeed, MIRIAm with complex random starting data gives better results than balanced truncation for almost all sizes of the reduced systems n, except $n \in \{14, 16, 18\}$. the reduced systems n, except $n \in \{14, 16, 18\}$. For real random starting data results are obtained only for $\{n = 12, 16, 20, 24\}$. Starting with balanced truncation based interpolation data, we end up with a reduced system whose h_2-norm is either equal or smaller than the norm of analogous system obtained by balanced truncation. The last few of this selection of initial interpolation data also leads to reduced order systems approximating the original one better than balanced truncation.

n	BT	MIRIAm
30	2.052×10^{-1}	1.843×10^{-1}
28	2.513×10^{-1}	2.102×10^{-1}
26	3.105×10^{-1}	2.369×10^{-1}
24	3.364×10^{-1}	2.655×10^{-1}
22	3.691×10^{-1}	3.011×10^{-1}
20	4.021×10^{-1}	3.407×10^{-1}
18	4.696×10^{-1}	3.659×10^{-1}
16	4.852×10^{-1}	4.002×10^{-1}
14	5.473×10^{-1}	4.378×10^{-1}
12	5.987×10^{-1}	5.102×10^{-1}
10	6.850×10^{-1}	5.651×10^{-1}
8	7.548×10^{-1}	6.217×10^{-1}
6	8.031×10^{-1}	7.338×10^{-1}
4	8.582×10^{-1}	7.652×10^{-1}

ns	BT	MIRIAm
30	2.591×10^{-2}	5.181×10^{-2}
28	3.643×10^{-2}	7.080×10^{-2}
26	3.743×10^{-2}	6.837×10^{-2}
24	4.371×10^{-2}	6.575×10^{-2}
22	5.434×10^{-2}	7.508×10^{-2}
20	6.003×10^{-2}	7.619×10^{-2}
18	6.722×10^{-2}	7.935×10^{-2}
16	7.552×10^{-2}	8.329×10^{-2}
14	8.983×10^{-2}	1.966×10^{-1}
12	9.159×10^{-2}	1.881×10^{-1}
10	1.327×10^{-1}	1.969×10^{-1}
8	1.689×10^{-1}	2.096×10^{-1}
6	2.139×10^{-1}	2.269×10^{-1}
4	2.635×10^{-1}	2.647×10^{-1}

TABLE 1. A comparison of the relative h_2- (left) and h_∞- norm (right) of the error systems obtained by balanced truncation and MIRIAm

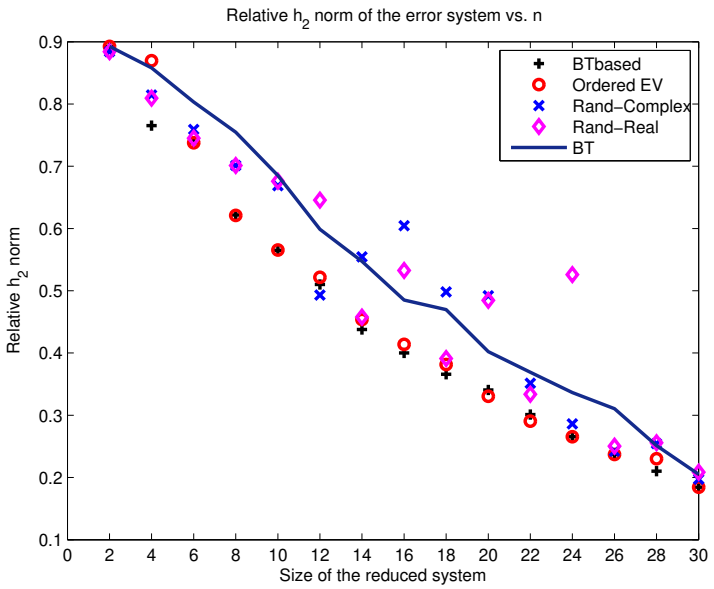

FIGURE 6. The relative h_2-norm of the error system vs the dimension of the reduced system (Random)

We also observe that "BTbased" behaves always better than balanced truncation on this example. We tried several different examples and taking the data coming from balanced truncation as initial selection always leads, if the algorithm converges, to a reduced order system whose relative h_2-norm of error system was smaller or equal to the one obtained by balanced truncation. For this example, also "Ordered EV" gives very good results.

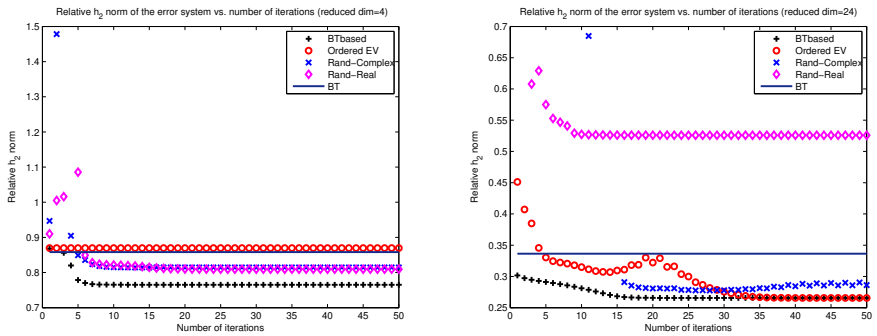

FIGURE 7. The relative h_2-norm of the error system vs. the number of iterations for reduced order systems of dimesion n=4(left) and n=24 (right) (Random)

As before we examine the convergence of our algorithm for this example by checking in each step of the iteration the relative h_2-norm of the error system. Figure 7 shows the relative h_2-norm of the error system versus the number of iterations for the reduced systems of size $n = 4$ and $n = 24$. We see that MIRIAm converges fast for all chosen sets of initial data. Figure 8 illustrates that some choices of initial data lead to situations, were we need to run many iterations until the algorithm converges.

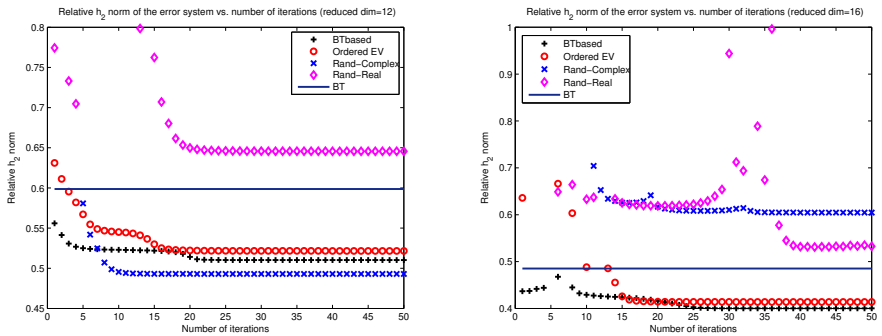

FIGURE 8. The relative h_2-norm of the error system vs. the number of iterations for reduced order systems of dimension n=12(left) and n=16 (right) (Random)

5. CONCLUSIONS

In this paper, several topics in h_2-optimal model reduction for large-scale discrete linear dynamical MIMO systems have been discussed. For the problem of finding a reduced system such that the h_2-norm of the error system is minimized, i.e. the h_2-optimal model reduction problem, we presented three different types of first order necessary conditions. The first condition involves certain tangential interpolation conditions for the transfer function of the original and the reduced system.The proof for these conditions uses a reformulation of the h_2-norm involving the poles and residues of the system which was received by basic results from the theory of complex functions. The conditions also implied an h_2-error expression for an

optimal reduced system. The second necessary condition was proven via another well-known representation of the h_2-norm involving the state-space matrices and turned out to be identical to the Wilson condition which has only been proven so far for continuous systems. The third condition was already presented and proven by Hyland and Bernstein. As for Wilson's condition (and in contrast the interpolation condition) it is identical for continuous and discrete systems. We proved equivalence for these three necessary conditions. As mentioned already, the interpolation conditions in this paper are only valid for simple poles but an adaptation to multiple poles (similar as for continuous systems, cf. [20]) is possible. In this case the conditions involve higher derivatives of the transfer functions.

In the numerical section we presented MIRIAm, an alogrithm which, if it converges, provides a reduced system that satisfies the interpolation based first order necessary conditions for the h_2-problem. As these conditions involve interpolation data which stem from the unknown optimal reduced system, our algorithm works iteratively by constructing a reduced system which satisfies interpolation conditions for fixed interpolation data in each step. On the one hand, this algorithm is numerically efficient also for large-scale systems as it comprises only solutions of some large linear systems. No solutions of any large-scale Sylvester or Lyapunov equation (like in many SVD-based model reduction methods) is required. On the other hand, we demonstrated that our algorithm can indeed provide reduced systems where the h_2-error is up to 20 % lower than for balanced truncation which means that systems obtained by balanced truncation can be "further away" from being h_2-optimal.

The h_2-error formula presented in this paper (Remark 3.3) can not be efficiently computed as it involves the calculation of all eigenvalues of the original system. Hence, future work should involve the search for a computable error bound for the h_2-problem similar to the h_∞-error bound for balanced truncation as well as the development of the convergence results. Furthermore, convergence results for our algorithm are still missing.

References

[1] A.C. Antoulas, *Approximation of large-scale dynamical systems*, Advances in Design and Control **6**, SIAM, Philadelphia, 2005.

[2] A.C. Antoulas, *private communication*.

[3] A.C. Antoulas, D.C. Sorensen and S. Gugercin, *A survey of model reduction methods for large-scale systems*, Contemporary Mathematics **280** (2001) 193-219.

[4] U. Baur and P. Benner, *Factorized Solution of Lyapunov Equations Based on Hierarchical Matrix Arithmetic*, Computing **78**:3 (2006) 211-234.

[5] P. Benner, V. Mehrmann and D.C. Sorensen, *Dimension reduction of large-scale systems*, Proceedings of a workshop held in Oberwolfach, Germany, 19-25.10.2003 Berlin, Springer, 2005 (Lecture notes in Computational Science and Engineering **45**) J86-BEN-460.

[6] Y. Chahlaoui and P. van Dooren, *A collection of Benchmark examples for model reduction of linear time invariant dynamical systems*, SLICOT Working Note 2002-2, February 2002.

[7] J.B. Conway, *Functions of one complex variable*, Springer, 1978.

[8] K.A. Gallivan, A. Vandendorpe and P. van Dooren, *Model Reduction of MIMO Systems via Tangential Interpolation*, SIAM J. Matrix Anal. Appl. **26**:2 (2004) 328-349.

[9] E.J. Grimme, *Krylov projection methods for model reduction*, PhD Thesis, ECE Department, University of Illinois, Urbana-Champaign , 1997.

[10] S. Gugercin, C. Beattie and A.C. Antoulas, *Rational Krylov Methods for Optimal $\mathcal{H}_2$ Model Reduction*, ICAM Technical Report, Virginia Tech, 2006 and submitted to SIAM Journal on Matrix Analysis and Applications.

[11] D.C. Hyland and D.S. Bernstein, *The optimal projection equations for model reduction and the relationship among the methods of Wilson, Skelton and Moore*, IEEE Trans. Automatic Control. **30** (1985) 1201-1211.
[12] A.S. Lawless, N.K. Nichols, C. Boess, A. Bunse-Gerstner, *Using model reduction methods within incremental four-dimensional variational data assimilation*, to appear in onthly Weather Review.
[13] A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra Appl. (2007), doi:10.1016/j.laa.2007.03.008
[14] L. Meier and D.G. Luenberger, *Approximation of Linear Constant Systems*, IEEE Trans. Automatic Control. **12** (1967) 585-588.
[15] B.C. Moore, *Principal component analysis in linear systems: controllability, observability and model reduction*, IEEE Trans. Automatic Control **26**:1 (1981) 17-31.
[16] C.T. Mullis and R.A. Roberts, *Synthesis of minimum roundoff noise fixed point digital filters*, IEEE Trans. Circuits Syst. **23** (1976) 551-562.
[17] Q. Su, V. Balakrishnan and C.K. Koh, *Efficient Approximate Balanced Truncation of General Large-Scale RLC Systems via Krylov Methods*, Proc. 15th International Conference on VLSI Design and 7th Asia and South Pacific Design Automation Conference, January 2002, pp. 311-316.
[18] A. Vandendorpe, *Model reduction of linear systems, an interpolation point of view*, PhD Thesis, Universite Catholique De Louvain, December 2004.
[19] M. Verlaan, *Efficient Kalman Filtering Algorithms for Hydrodynamic Models*, TU Delft, The Netherlands, 1998.
[20] G. Vossen, A. Bunse-Gerstner, D. Kubalinska and D. Wilczek, *Necessary optimality conditions for $\mathcal{H}_2$-optimal model reduction*, ZeTeM Technical Report, University of Bremen, to appear.
[21] D. Wilczek, A. Bunse-Gerstner, D. Kubalinska and G. Vossen, *Equivalences of necessary optimality conditions for $\mathcal{H}_2$-optimal model reduction*, ZeTeM Technical Report, University of Bremen, to appear.
[22] D.A. Wilson, *Optimum Solution of Model Reduction Problem*, Proc. Inst. Elec. Eng. **117**:6 (1970) 1161-1165.
[23] D.A. Wilson, *Model reduction for multivariable systems*, Int. J. Control **20**:1 (1974), 57-64.
[24] A. Youssuff and R.E. Skelton, *Covariance equivalent realizations with applications to model reduction of large-scale systems*, Control and Dynamic Systems 22, C.T. Leondes, ed., Academic Press, 273-348, 1985.
[25] A. Youssuff, D.A. Wagie and R.E. Skelton, *Linear system approximation via covariance equivalent realizations*, Journal of Math. Anal. and App. **196** (1985) 91-115.

Center for Industrial Mathematics, University of Bremen, Bremen, Germany
E-mail address: Bunse-Gerstner@math.uni-bremen.de

Center for Industrial Mathematics, University of Bremen, Bremen, Germany
E-mail address: dorota@math.uni-bremen.de

Center for Industrial Mathematics, University of Bremen, Bremen, Germany
E-mail address: vosseng@math.uni-bremen.de

Center for Industrial Mathematics, University of Bremen, Bremen, Germany
E-mail address: wilczek@math.uni-bremen.de

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