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The construction of an input-to-state stability (ISS) Lyapunov function for networks of ISS system will be presented. First we construct ISS Lyapunov functions for each strongly connected component, then what remains is a cascade (or disconnected aggregation) of these strongly connected components. Using known results the constructed Lyapunov functions can be aggregated to one single ISS Lyapunov function for the whole network.

The Lyapunov function construction for the strongly connected components basically depends on two steps: The construction of a function to the positive orthant in $\mathbb{R}^n$ and the combination of the given ISS Lyapunov functions of the subsystems to a common ISS Lyapunov function for the composite system.

Keywords: Input-to-state stability (ISS), ISS Lyapunov function, networks, nonlinear stability

1 Introduction

In this paper we provide a constructive method to find an ISS Lyapunov function for a composite system, when the ISS Lyapunov functions and nonlinear gains for the subsystems are all known. This result is particularly useful, since the knowledge of a Lyapunov function directly leads to knowledge of invariant sets or allows for different controller design methods, see, e.g., [8].

In [4] a nonlinear small gain theorem for networks of ISS systems was given, but for a different formulation of ISS, namely the "$\mathcal{KL}-\mathcal{K}$"-formulation.

In [6] half part of the construction we are going to present was already carried out, but an important bit was omitted. Namely, it was shown how an ISS Lyapunov function can be constructed, if a certain function $\sigma \in \mathcal{K}_\infty^n$ exists.

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Here we are going to construct this function $\sigma \in \mathcal{K}_\infty^n$, i.e., a function satisfying $D(\Gamma(\sigma(t))) < \sigma(t)$ for all $t > 0$, that is differentiable almost everywhere and strictly increasing. Here Γ denotes the Lyapunov gain matrix of the interconnection of the subsystems, i.e., an adjacency matrix weighted by the ISS Lyapunov gains, and D is the identity plus some diagonal operator.

This essentially depends on structural properties of the matrix Γ. In Proposition 9 we construct a smooth and strictly increasing function $\sigma_s : [0,1] \to \mathbb{R}_+^n$ up to some prespecified radius, provided that Γ is irreducible. If Γ is even primitive, then this function can easily be extended to a function $\sigma \in \mathcal{K}_\infty^n$. If Γ is only irreducible, this function σ can still be defined, but under slightly stronger assumptions in another direction, see Theorem 12.

The paper is organized as follows: In Section 1 we introduce some general notation, especially for monotone operators. The Lyapunov formulation of input-to-state stability (ISS) is given in Section 3. The main result in this paper is Theorem 4, that under a small-gain condition as well as structural requirements guarantees ISS of a network of ISS systems. This will be stated and proved in Section 4. In the last section we then propose, how this result can be applied for the construction of ISS Lyapunov functions in arbitrary network topologies.

2 Notation

Let $\mathcal{K} = \{f : \mathbb{R}_+ \to \mathbb{R}_+, f \text{ is continuous, strictly increasing and } f(0) = 0\}$ and $\mathcal{K}_\infty = \{f \in \mathcal{K} : f \text{ is unbounded}\}$.

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first component and strictly decreasing to zero in the second component.

A matrix $\Gamma = (\gamma_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ defines a map on $\mathbb{R}_+^n$ via $\Gamma(s)_i = \sum_{j=1}^n \gamma_{ij}(s_j)$, for $s \in \mathbb{R}_+^n$, in analogy to matrix vector multiplication in linear algebra.

The adjacency matrix $A_\Gamma = (a_{ij})$ of a matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ is defined by $a_{ij} = 0$ if $\gamma_{ij} \equiv 0$ and $a_{ij} = 1$ otherwise. The matrix Γ is called *primitive, irreducible* or *reducible* if and only if A_Γ is primitive, irreducible or reducible. See also [1].

On $\mathbb{R}_+^n$ we have a partial order induced by the order on $\mathbb{R}$. For vectors $x, y \in \mathbb{R}_+^n$ we denote

$$
\begin{aligned}
x \geq y &\iff x_i \geq y_i \text{ for } i = 1, \dots, n, \\
x > y &\iff x_i > y_i \text{ for } i = 1, \dots, n, \text{ and} \\
x \gneqq y &\iff x \geq y \text{ and } x \neq y.
\end{aligned}
$$

A map $\Delta : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is *monotone* if $x \leq y$ implies $\Delta(x) \leq \Delta(y)$. Clearly $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ induces a monotone map. For $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$, $\Delta : \mathbb{R}_+^n \to \mathbb{R}_+^n$ we write $\Gamma \geq \Delta$ if for all $x \in \mathbb{R}_+^n$ we have $\Gamma(x) \geq \Delta(x)$. Similarly, we write $\Gamma \not\geq \Delta, \Gamma > \Delta$, respectively $\Gamma \gneqq \Delta$, if for all $x \neq 0$ we have $\Gamma(x) \not\geq \Delta(x)$, $\Gamma(x) > \Delta(x)$, respectively $\Gamma(x) \gneqq \Delta(x)$. Here $\not\geq$ means that for at least one component i the inequality $\Gamma(x)_i < \Delta(x)_i$ holds.

For monotone maps Γ on $\mathbb{R}^n_+$ we define the following sets:

$$\begin{aligned}\Omega(\Gamma) &= \{x \in \mathbb{R}^n_+ : \Gamma(x) < x\},\\ \Omega_i(\Gamma) &= \{x \in \mathbb{R}^n_+ : \Gamma(x)_i < x_i\},\\ \Psi(\Gamma) &= \{x \in \mathbb{R}^n_+ : \Gamma(x) \leq x\}, \text{ and}\\ \Psi_i(\Gamma) &= \{x \in \mathbb{R}^n_+ : \Gamma(x)_i \leq x_i\}.\end{aligned}$$

If no confusion arises we will omit the reference to Γ. Note that for general monotone maps we have $\overline{\Omega} \subsetneq \Psi$, but for $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ we have equality.

By $|\cdot|$ we denote the 1-norm on $\mathbb{R}^n$ and by S_r the induced sphere of radius r in $\mathbb{R}^n$ intersected with $\mathbb{R}^n_+$, which is an n-simplex. By $U_\varepsilon(x)$ we denote the open neighborhood of radius ε around x with respect to the Euclidean norm $\|\cdot\|$.

3 Input-to-state stability

We consider the a finite set of interconnected systems

$$\Sigma_i : \ \dot{x}_i = f(x_1, \dots, x_n, u), \quad f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}, \quad i = 1, \dots, n, \tag{1}$$

where $\sum N_i = N$.

If we consider one of the systems, indexed by i, and interpret the variables x_j, $j \neq i$, and u as unrestricted inputs, then this system is assumed to have unique solutions defined on $[0,\infty)$ for all L^∞-inputs $x_j : [0,\infty) \to \mathbb{R}^{N_j}$, $j \neq i$, and $u : [0,\infty) \to \mathbb{R}^M$.

We write the interconnection of systems (1) as

$$\Sigma : \ \dot{x} = f(x,u), \quad f : \mathbb{R}^{N+M} \to \mathbb{R}^N, \tag{2}$$

where $x = (x_1^T, \dots, x_n^T)^T$.

We will impose ISS conditions on the subsystems given by (1) and interested in conditions guaranteeing ISS of the interconnected system (2). To this end we will construct an ISS Lyapunov function for (2).

Definition 1 (ISS Lyapunov function)**.** *A smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called an ISS Lyapunov function of* (2) *if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}_\infty$, and a positive definite function α such that*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^N, \tag{3}$$

and

$$V(x) \geq \chi(|u|) \Longrightarrow \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)). \tag{4}$$

The function χ is called Lyapunov-gain. System (2) *is called input-to-state stable (ISS) if it has a ISS Lyapunov function.*

It is well known[14] that the existence of an ISS Lyapunov function is equivalent to the system being ISS in the following sense:

There exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, such that for all initial conditions $x_0 \in \mathbb{R}^N$ and all L_∞-inputs $u(\cdot)$ it holds that

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_\infty), \quad \text{for all } t \geq 0.$$

For our construction we will need the notions of proximal subgradient and non-smooth ISS Lyapunov function, c.f. [3], [2].

Definition 2. *A vector $\zeta \in \mathbb{R}^N$ is called a proximal subgradient of a function $\phi : \mathbb{R}^N \to (-\infty, \infty]$ at $x \in \mathbb{R}^N$ if there exists a neighborhood $U(x)$ of x and a number $\sigma \geq 0$ such that*

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle - \sigma |y - x|^2 \quad \forall y \in U(x).$$

The set of all proximal sub-gradients at x is called proximal sub-differential of ϕ at x and is denoted by $\partial_P \phi(x)$.

Definition 3. *A continuous function $V : \mathbb{R}^N \to \mathbb{R}_+$ is said to be a non-smooth ISS Lyapunov function of system* (2) *if*

1. *V is proper and positive-definite, that is, there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that*
$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n; \tag{5}$$
2. *there exists a positive-definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and a class $\mathcal{K}_\infty$-function χ such that*
$$\sup_{u:\, V(x) \geq \chi(|u|)} \langle f(x,u), \quad \zeta \rangle \leq -\alpha(V(x)), \tag{6}$$
for all $\zeta \in \partial_P V(x)$, and all $x \neq 0$.

See also [3, p. 188 and Theorem 4.6.3].

In analogy to Definition 1 we extend the ISS notion to the subsystems: We say that the subsystems defined by (1) are ISS, if for $i = 1, \ldots, n$ there exist smooth ISS Lyapunov functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and functions $\psi_{1i}, \psi_{2i} \in \mathcal{K}_\infty$, $\chi_{ij} \in (\mathcal{K}_\infty \cup \{0\})$, and $\chi_i \in \mathcal{K}_\infty$, and positive definite functions α_i such that

$$\psi_{1i}(|x_i|) \leq V_i(x_i) \leq \psi_{2i}(|x_i|), \quad \forall x_i \in \mathbb{R}^{N_i}, \tag{7}$$

and

$$V_i(x_i) \geq \sum_j \chi_{ij}(V_j(x_j)) + \chi_i(|u|) \Longrightarrow \nabla V_i(x_i) \cdot f_i(x,u) \leq -\alpha_i(V_i(x_i)). \tag{8}$$

The functions χ_{ij} are called ISS Lyapunov gains or simply gains, if no confusion arises.

We refer to subsystems (1) in conjunction with their ISS Lyapunov functions satisfying (7) and (8) as a network of ISS systems. The questions is, whether the composite system (2) is ISS or not.

Consider the network of ISS systems given by (1). The gain functions χ_{ij} give rise to an $n \times n$-gain matrix

$$\Gamma := (\chi_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}.$$

Associated to such a network is a graph, whose vertices are the sytems and its directed edges (i, j) correspond to inputs going from system j to system i. We will call the network strongly connected if its graph is.

4 Lyapunov type small-gain theorem for strongly connected networks

We first construct an ISS Lyapunov function under the assumption, that the network is strongly connected, or equivalently, that Γ is irreducible.

Theorem 4 (Lyapunov-type ISS small gain theorem for networks)**.** *Consider a strongly connected ISS network as in* (1), (7), *and* (8). *Assume there exists a class $\mathcal{K}_\infty$-function η such that for $D = \mathrm{diag}_n(id + \eta)$ we have*

$$D \circ \Gamma(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+, s \neq 0. \tag{9}$$

Then there exists an ISS Lyapunov function for system (2).

The proof will be given at the end of this section. It relies on two steps. First we construct a function $\sigma \in \mathcal{K}^n_\infty$ with trace in $\Omega(D\circ\Gamma)\cup\{0\}$ for a suitable diagonal operator $D = \mathrm{diag}_n(\mathrm{id} + \alpha)$, $\alpha \in \mathcal{K}_\infty$. Namely, σ satisfies

$$\sigma_i(t) > (\mathrm{id} + \alpha)\Big(\sum_{j=1}^{n} \chi_{ij}(\sigma_j(t))\Big), \quad \forall t > 0, \quad i = 1, \dots, n. \tag{10}$$

Then together with the following proposition this leads to a non-smooth ISS Lyapunov function for (2).

Proposition 5. *Consider an ISS network as in* (1), (7), *and* (8). *For each subsystem Σ_i, $i = 1, \dots, n$, let V_i be an ISS Lyapunov function satisfying (7) and (8). Assume there exists a diagonal operator $D = \mathrm{diag}_n(id + \alpha)$, $\alpha \in \mathcal{K}_\infty$, and a smooth $\sigma \in \mathcal{K}^n_\infty$, satisfying*

$$\sigma(t) \in \Omega(D \circ \Gamma), \quad \forall t > 0 \text{ and}$$
$$(\sigma_i^{-1})'(t) > 0, \quad \forall t > 0, i = 1, \dots, n.$$

Then the composite system (2) *is ISS with ISS Lyapunov function*

$$V(x) := \max_i\{\sigma_i^{-1}(V_i(x_i))\}.$$

Proof. This has essentially been proved in [6, Theorem 6]. We define

$$M_i := \Big\{(v_1, \dots, v_n)^T \in \mathbb{R}^n_+ : \sigma_i^{-1}(v_i) > \max_{j \neq i}\{\sigma_j^{-1}(v_j)\}\Big\}. \tag{11}$$

From (10) it follows that

$$\sigma_i(t) - \sum_{j=1}^{n} \chi_{ij}(\sigma_j(t)) > \eta\Big(\sum_{j=1}^{n} \chi_{ij}(\sigma_j(t))\Big) =: \rho_i(t). \tag{12}$$

Note that $\rho_i \in \mathcal{K}_\infty$, since the network is strongly connected and hence Γ has no zero rows. Now let

$$\rho(t) = \min_i \rho_i(t),$$

which is again of class $\mathcal{K}_\infty$.

Now for any $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \mathbb{R}^N$ with $(V_1(\hat{x}_1), \dots, V_n(\hat{x}_n)) \in M_i$ it follows that there is a neighborhood U of $\hat{x}$ such that $V(x) = \sigma_i^{-1}(V_i(x_i))$ holds for all $x \in U$, so that V is differentiable in $x \in U$. Again we are looking for a positive definite function $\tilde{\alpha}$ and $\phi \in \mathcal{K}$ such that $V(x) > \phi(\|u\|)$ implies $\nabla V(x) f(x,u) < -\tilde{\alpha}(V(x))$.

To derive the defining inequality of ISS Lyapunov functions consider the inequality

$$V(x) > \rho^{-1}(\chi_i(|u|)). \tag{13}$$

From this inequality it follows that $\rho(V(x)) > \chi_i(|u|)$ or using the definition of ρ

$$\sigma_i(V(x)) - \sum_{j=1}^{n} \chi_{ij}(\sigma_j(V(x))) > \chi_i(|u|),$$

or equivalently

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &> \sum_{j=1}^{n} \chi_{ij}(\sigma_j(V(x))) + \chi_i(|u|) \\ &= \sum_{j=1}^{n} \chi_{ij}(\sigma_j(\sigma_i^{-1}(V_i(x_i)))) + \chi_i(|u|) \\ &> \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \chi_i(|u|)\,, \end{aligned}$$

where we have used $(V_1(\hat{x}_1), \dots, V_n(\hat{x}_n)) \in M_i$ in the last inequality. Summarizing this shows that (13) implies

$$V_i(x_i) > \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \chi_i(|u|),$$

and hence from (8) we obtain

$$\begin{aligned} \nabla V(x) f(x,u) &= (\sigma_i^{-1})'(V_i(x_i)) \nabla V_i(x_i) f_i(x,u) \\ &\leq -(\sigma_i^{-1})'(V_i(x_i)) \alpha_i(V_i(x_i)) =: -\tilde{\alpha}_i(V(x)), \end{aligned} \tag{14}$$

where $\tilde{\alpha}_i$ is a positive definite function by definition. Now let

$$\tilde{\alpha}(t) := \min_i \alpha_i(t),$$

again a positive definite function, as desired.

It remains to treat the points where V may fail to be differentiable.

For this purpose we use some results from [3]. For smooth functions $g_i,\ i = 1,\dots,n$ it follows that $g(x,u) = \max_i \{g_i(x,u)\}$ is Lipschitz continuous and Clarke's generalized gradient of g is given by , c.f. [3],

$$\partial_{Cl} g(x) = co\Big\{ \bigcup_{i \in M(x)} \nabla_x g_i(x,u) \Big\},$$
$$M(x) = \{ i :\ g_i(x,u) = g(x) \},$$

where co denotes the convex hull. In our case

$$\partial_{Cl} V(x) = co\Big\{ (\sigma_i^{-1})'(V_i(x_i)) \nabla V_i(x_i) : \sigma_i^{-1}(V_i(x_i)) = V(x) \Big\}.$$

Note, that directly from the definitions of $\partial_P V(x)$ and $\partial_{Cl} V(x)$, see [3], e.g., it follows that $\partial_{Cl} V(x) \supset \partial_P V(x)$. Now for every extremal point of $\partial_{Cl} V(x)$ the decrease condition (14) is satisfied. By convexity, the same is true for every element of $\partial_{Cl} V(x)$. Now Theorems 4.3.8 and 4.5.5 of [3] show the strong invariance and attractivity of the set $\{x :\ V(x) \leq \gamma(\|u\|)\}$. It follows that V is an ISS-Lyapunov function for the interconnection (2). □

Before we return to the proof of Theorem 4 we develop some theory for matrices in $(\mathcal{K}_\infty \cup \{0\})^{n \times n}$.

Lemma 6. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ be such that Γ has no zero rows. Then $0 < r < s$ implies $\Gamma(r) < \Gamma(s)$.*

If Γ is primitive, then $s \lneqq t$ already implies $\Gamma^k(s) < \Gamma^k(t)$ for some $k > 0$ which does only depend on Γ.

Proof. Just compare $\Gamma(r)_i$ with $\Gamma(s)_i$. These are $\sum_{j=1}^n \gamma_{ij}(r_j)$ and, respectively, $\sum_{j=1}^n \gamma_{ij}(s_j)$. Since Γ has no zero rows, both sums are non vanishing, and from $r_j < s_j$, for $j = 1,\dots,n$, we deduce that the first sum is strictly less than the second.

For the second assertion we consider the adjacency matrix $A_\Gamma = (a_{ij})$ of Γ. Since A_Γ is primitive, there exists a $k > 0$ such that $A_\Gamma^k > 0$. It is easy to check, that this is equivalent to $t \mapsto (\Gamma^k(t \cdot e_j))_i \in \mathcal{K}_\infty$ for all $i,j = 1,\dots,n$. This proves the lemma. □

Now we state some useful properties of the sets Ψ and Ω.

Lemma 7. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ such that $\Gamma \ngeq id$. Then*

1. *$\Omega \cap S_r \neq \emptyset$ for all $r > 0$.*
2. *$\Gamma(\Psi) \subset \Psi$ and, if Γ has no zero rows, then $\Gamma(\Omega) \subset \Omega$.*

3. *If Γ has no zero rows, then $\Gamma^{k+1}(\Omega) \subset \Gamma^k(\Omega) \subset \Omega$ for all $k \geq 0$.*

4. *$\Gamma^{k+1}(\Psi) \subset \Gamma^k(\Psi) \subset \Psi$ for all $k \geq 0$. All these sets are closed. In particular, $\Psi_\infty = \Psi_\infty(\Gamma) = \bigcap_{k=0}^{\infty} \Gamma^k(\Psi)$ is non-empty, connected, and has the unboundedness property stated in 1.*

5. *If Γ is primitive, then there exists a $k > 0$ such that $(\Gamma^k(\Psi) \setminus \{0\}) \subset \Omega$.*

6. *If Γ is irreducible and there exists a $\mathcal{K}_\infty$-function α, such that for $D = \operatorname{diag}_n(id+\alpha)$ we have $\Gamma \circ D \not\geq id$, then $\Gamma(\Psi(\Gamma \circ D)) \setminus \{0\} \subset \Omega(\Gamma)$.*

Before we prove this lemma, we state a famous theorem due to Knaster, Kuratowski and Mazurkiewicz:

Theorem 8 (Knaster-Kuratowski-Mazurkiewicz, 1929)**.** *Let Δ_n denote unit n-simplex, and for a face σ of Δ_n let $\sigma^{(0)}$ denote the set of vertices of σ.*

If a family $\{A_i | i \in \Delta_n^{(0)}\}$ of subsets of Δ_n is such that all the sets are closed or all are open, and each face σ of Δ_n is contained in the corresponding union $\bigcup\{A_i | i \in \sigma^{(0)}\}$, then there is a point common to all the sets.

Proof. The original proof for closed sets was given in [9], while the formulation above is taken from [7] and was proved in [11]. □

Proof. Some of this can also be found in[5].

1. Note that S_r for $r > 0$ is a simplex with vertices $r \cdot e_i$, $i = 1, \ldots, n$. Each (nonempty) face spanned by $r \cdot e_i$, $i \in I \subset \{1, \ldots, n\}$, fulfills the assumptions of the Knaster-Kuratowski-Mazurkiewicz theorem[11],[9], i.e., it is contained in the union $\bigcup_I (\Omega_i \cap S_r)$. Then the KKM-theorem implies that $\bigcap_1^n (\Omega_i \cap S_r) \neq \emptyset$.

2. Let $s \in \Gamma(\Omega)$, i.e., $s = \Gamma(t)$ for some $t \in \Omega$, that is, $\Gamma(t) < t$. If Γ has no zero rows, then this implies $\Gamma(s) = \Gamma^2(t) < \Gamma(t) = s$, i.e., $s \in \Omega$. The other assertion is similar.

3. If $s = \Gamma^{k+1}(t)$ for some t satisfying $\Gamma(t) < t$, then writing $u = \Gamma(t)$ we have $s = \Gamma^k(u)$ clearly $\Gamma(u) = \Gamma^2(t) < \Gamma(t) = u$ by 2.

4. The nesting is proved analoguously to 3. Since Ψ is nonempty and closed, so are all $\Gamma^k(\Psi)$ by continuity of Γ. Also, by $\Gamma^{k+1}(\Psi) \subset \Gamma^k(\Psi)$ for all $k \geq 0$, the intersection $\bigcap_{k \geq 0} \Gamma^k(\Psi)$ is nonempty and closed.

   With $s \in \Psi_\infty$ each convex combination $(1-\lambda)\Gamma(s) + \lambda s$ of $\Gamma(s)$ and s, for $\lambda \in [0,1]$ is in Ψ_∞: Clearly $\Gamma(s) \leq (1-\lambda)\Gamma(s) + \lambda s \leq s$, and application of Γ gives $\Gamma^2(s) \leq \Gamma((1-\lambda)\Gamma(s) + \lambda s) \leq \Gamma(s) \leq (1-\lambda)\Gamma(s) + \lambda s \leq s$. This implies that every point is path-connected to the origin, hence Ψ_∞ is connected. The same KKM-argument as in 1. yields the unboundedness property.

5. First check, that in full analogy to adjacency matrices A, where there exsits a $k > 0$ such that the ijth entry $a_{ij}^{(k)} > 0$ if A^k is positive for every $i, j = 1, \ldots, n$, there exists a $k > 0$, such that $t \mapsto \Gamma^k(t \cdot e_j)_i$ is of class $\mathcal{K}_\infty$ for all $i, j = 1, \ldots, n$. Hence $\Gamma(s) \lneqq s$ (and hence $s \neq 0$) imlies $\Gamma^{k+1}(s) < \Gamma^k(s)$, because the strict inequality in one component gets propagated to every other component. $\square$

This will be an essential ingredient for the strict monotonicity of the path σ that we want to construct.

An intermediate result is the following, that already implies a local version of Theorem 4, where local means "on arbitrarily large compact sets around the origin".

Proposition 9. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$, $\Gamma \ngeq id$, be such that Γ has no zero rows. For every $s \in \Omega$ there exists a continuous and strictly increasing vector function $\sigma_s : [0, 1] \to (\Omega \cup \{0\}) \cap \overline{B_1(0, |s|)}$ with $\sigma_s(0) = 0$ and $\sigma_s(1) = s$. Moreover, each component function is piecewise continuously differentiable.*

Proof. Clearly $s \in \Omega$ gives $0 < \Gamma(s) < s$ and $\Gamma(s) \in \Omega$. By Lemma 6 the inequality implies $\Gamma^{k+1}(s) < \Gamma^k(s)$ for all $k \geq 1$.

From [5] we know that irreducibility and $\Gamma \ngeq$ id imply $\lim_{k\to\infty} \Gamma^k(s) = 0$[1].

Now consider $\lambda \in]0, 1[$ and let $z = (1 - \lambda)\Gamma(s) + \lambda s$. Clearly $\Gamma(s) < z < s$. Now apply Γ to obtain $\Gamma^2(s) < \Gamma(z) < \Gamma(s) < z < s$. Hence $z \in \Omega$ and by smoothly varying λ from 0 to 1 we get a smooth path from $\Gamma(s)$ to s.

So the idea is to construct $\sigma_s|_{]\frac{1}{k+2}, \frac{1}{k+1}]} \to \{z = (1 - \lambda)\Gamma^{k+1}(s) + \lambda\Gamma^k(s),\ \lambda \in]0, 1]\}$ for $k = 0, 1, 2, \ldots$ and to assign $\sigma_s(0) = 0$. For example, we can obtain

$$\sigma_s(t) = \begin{cases} 0 & \text{if } t = 0, \\ z(2 - \frac{1}{t} + \lfloor \frac{1}{t} - 1 \rfloor, \lfloor \frac{1}{t} - 1 \rfloor) & \text{if } t \in]0, 1], \end{cases}$$

where $z(\lambda, k) = (1 - \lambda)\Gamma^{k+1}(s) + \lambda\Gamma^k(s)$ and $\lfloor t \rfloor$ is the greatest integer less or equal to t. $\square$

For what follows, this will already suffice, but we note, that there can be gained more:

Corollary 10. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$, $\Gamma \ngeq id$, be such that Γ has no zero rows. For every $s \in \Omega$ there exists a continuously differentiable and strictly increasing vector function $\sigma_s : [0, 1] \to (\Omega \cup \{0\}) \cap \overline{B_1(0, |s|)}$ with $\sigma_s(0) = 0$ and $\sigma_s(1) = s$.*

Proof. Just note that instead of the previously chosen interpolation, we could also use any kind of spline interpolation in each component, to make the resulting function σ_s continuously differentiable in each component. See for example [15] for spline interpolation methods. $\square$

This gives one direction of the path, the other direction is given next.

[1]This is easy to see: We find $\Gamma^{k+1}(s) < \Gamma^k(s) < \ldots < s$, a monotone sequence. Its limit point s^* is a fixed point for Γ, hence it must be 0.

Theorem 11. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$, $\Gamma \ngeq id$, be primitive. Then there exists a piecewise continuously differentiable and strictly increasing vector function $\sigma : \mathbb{R}_+ \to \Omega \cup \{0\}$ with $\sigma(0) = 0$ and $\lim_{t\to\infty} \sigma(t) = \infty$, i.e., the component functions are of class $\mathcal{K}_\infty$.*

Proof. By Lemma 7 we have $\Psi_\infty \subset \Omega \cup \{0\}$.

Combining the results of Proposition 9 and Lemma 7 we start with $\sigma_s : [0,1] \to \Psi_\infty$, where $\sigma_s(1) = s \in \Psi_\infty$ and σ_s is piecewise $\mathcal{C}^1$ in each component.

Since we may always pick a preimage in Ψ_∞ we extend σ_s to a function σ on $\mathbb{R}_+$ by defining $\sigma|_{[0,1]} = \sigma_s$ and

$$\sigma|_{]1,\infty[}(t) = (1 - t + \lfloor t \rfloor)\Gamma^{1-\lfloor t \rfloor}(s) + (t - \lfloor t \rfloor)\Gamma^{-\lfloor t \rfloor}(s).$$

It remains to prove unboundedness of the component functions. Assume σ is bounded. Since σ is non decreasing, there must exist a limit point

$$s^* := \lim_{k\to\infty} \sigma(k) = \lim_{k\to\infty} \Gamma(\sigma(k)) = \Gamma(s^*),$$

but since $\sigma(1) > 0$ and σ is non decreasing, and hence $s^* > 0$, this contradicts $\Gamma \ngeq \mathrm{id}$.

So there exists at least one unbounded component of σ, without loss of generality this is the first one. From irreducibility (primitive matrices are also irreducible) we deduce that there exists another unbounded component and inductively we obtain that all components are unbounded.

It follows that the vector function σ constructed above fulfills $\sigma(t) \in \Omega$ for all $t > 0$ and by the same argument as in the proof of Proposition 9 the component functions of σ are strictly increasing and hence of class $\mathcal{K}_\infty$. $\square$

Note that here we used a linear interpolation, but we could also utilize spline interpolation techniques to make the curve arbitrarily smooth.

This theorem gives us a $\mathcal{K}_\infty^n$-function σ that satisfies

$$\Gamma(\sigma(t)) < \sigma(t), \quad \text{for all } t > 0,$$

for the case that Γ is primitive. Of course, primitivity is quite a restrictive assumption for the topology of the network, that we look at, not every strongly connected (irreducibility of Γ) network satisfies this assumption.

Now the aim is to extend this result to just strongly connected networks, then later to cascades of those. So we have to find such a function σ for irreducible Γ. Remember, that in Theorem 4 we are also given this diagonal operator D and the stronger assertion $\Gamma \circ D \ngeq \mathrm{id}$ instead of $\Gamma \ngeq \mathrm{id}$. (In [5] it was shown, that $D \circ \Gamma \ngeq \mathrm{id}$ and $\Gamma \circ D \ngeq \mathrm{id}$ are equivalent). This will come in handy in the next statement.

Theorem 12. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be irreducible and assume there exists a function $\alpha \in \mathcal{K}_\infty$, such that for $D = \mathrm{diag}_n(id + \alpha)$ we have $\Gamma \circ D \ngeq id$. Then there exists a continuously differentiable and strictly increasing vector function $\sigma : \mathbb{R}_+ \to \Omega(\Gamma)$ with $\sigma(0) = 0$ and $\lim_{t\to\infty} \sigma(t) = \infty$, i.e., the component functions are of class $\mathcal{K}_\infty$.*

Proof. First note that $\Psi_\infty^{\Gamma\circ D} := \bigcap_{k\geq 0}(\Gamma\circ D)^k(\Psi(\Gamma\circ D)) \subset \Omega(\Gamma)$ because of $\Psi(\Gamma\circ D)\subset \Omega(\Gamma)$. The set $\Psi_\infty^{\Gamma\circ D}$ has all the same nice properties as Ψ_∞ in Lemma 7. Hence for $s \in \Psi_\infty^{\Gamma\circ D} \subset \Omega(\Gamma)$ there exists an ascending sequence $\{z_k\}_{k\geq 0}\subset \Psi_\infty^{\Gamma\circ D}$, satisfying

$$z_0 = s \quad \text{and} \tag{15}$$
$$z_k = \Gamma\circ D(z_{k+1}) \lneqq z_{k+1} \quad \text{for all } k\geq 0. \tag{16}$$

One can easily check that this sequence is unbounded in every component (as in the proof of Theorem 11).

Again we define $\sigma|_{[0,1]} = \sigma_{z_0}$ as in Proposition 9.

In the other (unbounded) direction we first construct a path in $\Psi_\infty^{\Gamma\circ D}$ using linear interpolation:

$$\tilde\sigma(t) := (1-(t-\lfloor t\rfloor))z_{\lfloor t\rfloor+1} + (t-\lfloor t\rfloor)z_{\lfloor t\rfloor} \qquad \text{for } t>1.$$

Clearly $\tilde\sigma(t)\in\Omega$ for all $t>1$, but $\tilde\sigma$ is not necessarily strictly increasing. Next we modify $\tilde\sigma$ slightly, to get a strictly increasing $\sigma|_{]1,\infty[}$.

Since Ω is open, with the polygon $\overline{z_k z_{k+1}\dots z_{k+l}}$ for a small $\varepsilon>0$ also the neighborhood $U_\varepsilon(\overline{z_k z_{k+1}\dots z_{k+l}}) := \{x\in\mathbb{R}^n_+ : \|x-y\|<\varepsilon \text{ for some } y\in\overline{z_k z_{k+1}\dots z_{k+l}}\}$ is in Ω. Pick a minimal $l>1$ so that $z_{k+l}>z_k$. Hence for all $x\in\overline{z_{k+l}z_{k+l+1}}$ and $y\in\overline{z_k z_{k+1}\dots z_{k+l}}\setminus\{z_{k+l}\}$ we have $x>y$.

Now we pick a $\delta>0$, $\delta<\varepsilon$, and find a unique $\tilde z_{k+l}\in\mathbf{S}_\delta(z_{k+l})\cap\overline{z_{k+l}z_{k+l+1}}$, $\tilde z_{k+l}\geq z_{k+l}$, and strictly ordered points $z_{k+m}\in\overline{z_k\dots z_{k+l}}\setminus\{z_{k+l}\}$, $m=1,ldots,l-1$, satisfying

$$z_{k+m} < z_{k+m+1}, \qquad \text{for } m=0,\dots,l-1.$$

Clearly the polygon $\overline{\tilde z_k\dots\tilde z_{k+l}}\subset\Omega$, so we define

$$\sigma(k+m+\lambda) := (1-\lambda)\tilde z_{k+m} + \lambda\tilde z_{k+m+1}, \quad \text{for } m=0,\dots,l-1, \lambda\in(0,1].$$

Clearly $\sigma(t)$ is strictly increasing for $t\in(k,k+l]$ and is in Ω. The same way we proceed with the polygon $\overline{\tilde z_{k+l}z_{k+l+1}\dots z_{k+l+p}}$, where again p is minimal, such that $z_{k+l+p}>\tilde z_{k+l}$. Inductively this yields $\sigma|_{]1,\infty[}$ as desired, and together with σ_{z_0} we have $\sigma_i\in\mathcal{K}_\infty$ is strictly increasing for $i=1,\dots,n$, and $\sigma=(\sigma_1,\dots,\sigma_n)^T$ is a path in $\Omega\cup\{0\}$. ☐

Remark 13. *The functions $\sigma\in\mathcal{K}_\infty^n$ that we constructed in Theorems 11 and 12 are possibly not smooth on a discrete set in $]0,\infty[$. Nevertheless, for each $i=1,\dots,n$, the derivative σ_i' of σ_i is positive, except on this discrete set. By smoothing techniques of classical analysis (molifiiers, e.g.) these can be smoothened to to $\tilde\sigma\in\mathcal{K}_\infty^n\cap\mathcal{C}^\infty(]0,\infty[)$, satisfying $\tilde\sigma(t)\in\Omega$ for all $t>0$. This in particular implies $(\tilde\sigma_i^{-1})'(t)>0$ for all $t>0$ and $i=1,\dots,n$.*

For completeness, we state yet another result from [5]:

Lemma 14. *For $\Gamma\in(\mathcal{K}\cup\{0\})^{n\times n}$ the following are equivalent:*

1. $\exists \rho \in \mathcal{K}_\infty, D = \mathrm{diag}_n(id + \rho) : \Gamma \circ D \not\geq id$,

2. $\exists \rho \in \mathcal{K}_\infty, D = \mathrm{diag}_n(id + \rho) : D \circ \Gamma \not\geq id$,

3. $\exists \rho_1, \rho_2 \in \mathcal{K}_\infty, D_1 = \mathrm{diag}_n(id + \rho_1),\ D_2 = \mathrm{diag}_n(id + \rho_2) : D_1 \circ \Gamma \circ D_2 \not\geq id$.

Proof. Equivalence between 1 and 2 is easily established and thus omitted. The third assertion is based on the observation, that for any $\rho \in \mathcal{K}_\infty$, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$, such that

$$(\rho + \mathrm{id}) = (\rho_1 + \mathrm{id}) \circ (\rho_2 + \mathrm{id}).$$

To this end choose, e.g., $\rho_2 = \frac{1}{2}\rho$ and $\rho_1 = \frac{1}{2}\rho \circ (\frac{1}{2}\rho + id)^{-1}$. Then

$$\begin{aligned}(\rho_1 + \mathrm{id}) \circ (\rho_2 + \mathrm{id}) = \rho_1(\rho_2 + \mathrm{id}) + \rho_2 + \mathrm{id} =\\ \frac{1}{2}\rho \circ (\frac{1}{2}\rho + id)^{-1} \circ (\frac{1}{2}\rho + \mathrm{id}) + \frac{1}{2}\rho + id =\\ (\frac{1}{2}\rho \circ \mathrm{id} + \frac{1}{2}\rho + \mathrm{id}) = \rho + \mathrm{id}.\end{aligned}$$

□

Proof of Theorem 4. Just combine the statements of Proposition 5 and Theorem 12:

By Lemma 14 we have $D \circ \Gamma \not\geq \mathrm{id}$ if and only if $\Gamma \circ D \not\geq \mathrm{id}$. We may always decompose D into two diagonal operators D_1, D_2 such that $D_1 \circ D_2 = D$, whereby D_1, D_2 are also of the form $\mathrm{diag}_n(\mathrm{id} + \alpha_i)$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$.

So we have $D_1 \circ \Gamma \circ D_2 \not\geq \mathrm{id}$, which we write $\tilde{\Gamma} \circ D_2 \not\geq \mathrm{id}$. Now apply Theorem 12 to obtain a $\mathcal{K}_\infty^n$-function σ, satisfying

$$D_1 \circ \Gamma(\sigma(t)) = \tilde{\Gamma}(\sigma(t)) < \sigma(t), \text{for all } t > 0.$$

We conclude with an application of Proposition 5. □

Note, that

5 Lyapunov functions for general networks of ISS systems

In the last section we constructed ISS Lyapunov functions for strongly connected networks. But for example cascade networks are not strongly connected. Fortunately, it is already well known[13] that cascades of ISS systems are also ISS.

What is also known, is how to construct common ISS Lyapunov functions if the ISS Lyapunov functions of the subsystems together with their supply pairs are known[12].

Now, for every connected network of ISS systems the corresponding gain matrix Γ can be transformed into an upper triangular block structure, by a transformation using permutation matrices, where the blocks on the diagonal are all irreducible (or 1×1 zero blocks, which each corresponds to just one single system that does not influence any other system). From an interconnection point of view, this gives a cascade of strongly connected networks, see Figure 1.

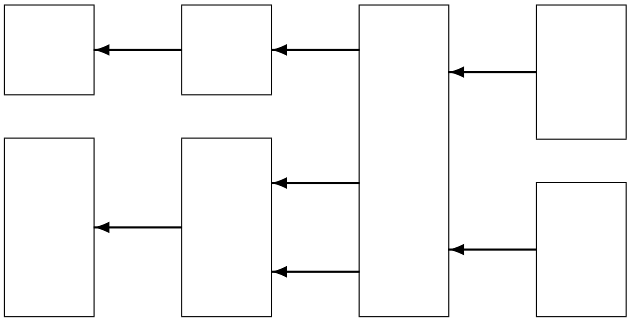

Figure 1: Cascade of strongly connected components.

Lemma 15. *For any reducible matrix $A \in \mathbb{R}^{n\times n}$ there exists a permutation matrix P such that*

$$P^T AP = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1k} \\ 0 & B_{22} & \dots & B_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & B_{kk} \end{bmatrix},$$

where the square matrices B_{ii}, $i = 1, \dots, k$, are either irreducible or 1×1 zero matrices.

Proof. See [10, p. 544]. $\square$

Note that this applies also to matrices $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$.

For non connected networks one can treat each connected component separately.

Now the method to construct an ISS Lyapunov function for a cascade of two systems is roughly as follows: We start with given ISS Lyapunov functions

$$\begin{aligned} \alpha_{1i}(|z_1|) \le V_i(z_i) \le \alpha_{2i}(|z|), \quad i = 1, 2, \\ V_1(z_1) > \gamma_1(|z_2|) \implies \nabla V_1(z_1) \cdot g_1(z_1, z_2) \le -\beta_1(|z_1|), \\ V_2(z_2) > \gamma_2(|u|) \implies \nabla V_2(z_2) \cdot g_2(z_2, u) \le -\beta_2(|u|), \end{aligned}$$

for suitable $\mathcal{K}_\infty$-functions α_{ji}, β_i, and γ_i, $i, j = 1, 2$.

This directly implies

$$\begin{aligned} \nabla V_1(z_1) \cdot g_1(z_1, z_2) &\le \gamma_1(|z_2|) - \beta_1(|z_1|), \\ \nabla V_2(z_2) \cdot g_2(z_2, u) &\le \gamma_2(|u|) - \beta_2(|u|), \end{aligned} \tag{17}$$

which is an equivalent formulation of ISS Lyapunov functions. Now in [12] the pairs (γ_1, β_1) and (γ_2, β_2) are called *supply pairs.* Note that multiplying each equation in (17) by a positive constant gives a new Lyapunov function and a new supply pair. So supply pairs are far from being unique. In [12] it was shown, that the supply pairs (γ_1, β_1) and (γ_2, β_2) can be rescaled to new supply pairs $(\tilde\gamma_1, \tilde\beta_1)$ and $(\tilde\gamma_2, \tilde\beta_2)$, such that the sum of the so obtained Lyapunov functions $\tilde V_i$, $i = 1, 2$, gives an ISS Lyapunov function for the cascade.

6 Conclusions

We constructed an ISS Lyapunov function for strongly connected networks of ISS systems and proposed a procedure to apply this method for the construction of an ISS Lyapunov function for arbitrary networks.

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