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An ISS Small-Gain Theorem for General Networks

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We provide a generalized version of the *nonlinear small-gain theorem* for the case of more than two coupled input-to-state stable (ISS) systems. For this result the interconnection gains are described in a nonlinear gain matrix and the small-gain condition requires bounds on the image of this gain matrix. The condition may be interpreted as a nonlinear generalization of the requirement that the spectral radius of the gain matrix is less than one. We give some interpretations of the condition in special cases covering two subsystems, linear gains, linear systems and an associated artificial dynamical system.

Keywords Interconnected systems – input-to-state stability – small-gain theorem – large-scale systems – monotone maps

1 Introduction

Stability is one of the fundamental concepts in the analysis and design of nonlinear dynamical systems. The notions of input-to-state stability (ISS) and nonlinear gains have proved to be an efficient tool for the qualitative description of stability of nonlinear input systems. There are different equivalent formulations of ISS: In terms of $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions (see below), via Lyapunov functions, as an asymptotic stability property combined with asymptotic gains, and others, see [15]. A more quantitative but equivalent formulation, which captures the long term dynamic behavior of the system, is the notion of input-to-state dynamical stability (ISDS), see [3].

One of the interesting properties in the study of ISS systems is that under certain conditions input-to-state stability is preserved if ISS systems are connected in cascades or feedback loops. In this paper we generalize the existing results in this area.

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In particular, we obtain a general condition that guarantees input-to-state stability of a general system described as an interconnection of several ISS subsystems.

The earliest interconnection result on ISS systems states that cascades of ISS systems are again ISS, see e.g., [11, 12, 13]. Furthermore, small-gain theorems for the case of two ISS systems in a feedback interconnection have been obtained in [3, 4, 5]. These results state in one way or another that if the composition of the gain functions of ISS subsystems is smaller than the identity, then the whole system is ISS.

The papers [3, 4, 5] use different approaches to the formulation of small-gain conditions that yield sufficient stability criteria: In [4] the proof is based on the properties of $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions. This approach requires that the composition of the gains is smaller than the identity in a robust sense, see below for the precise statement. We show in Example 12 that within the context of this approach the robustness condition cannot be weakened. The result in that paper also covers practical ISS results, which we do not treat here. An ISS-Lyapunov function for the feedback system is constructed in [5] as some combination of the corresponding ISS-Lyapunov functions of both subsystems. The key assumption of the proof in that paper is that the gains are already provided in terms of the Lyapunov functions, by which the authors need not resort to a robust version of the small-gain condition. The proof of the small-gain theorem in [3] is based on the ISDS property and conditions for asymptotic stability of the feedback loop without inputs are derived. These results will turn out to be special cases of our main result.

General stability conditions for large scale interconnected systems have been obtained by various authors in other contexts. In [8] sufficient conditions for the asymptotic stability of a composite system are stated in terms of the negative definiteness of some test matrix. This matrix is defined through the given Lyapunov functions of the interconnected subsystems. Similarly, in [9] conditions for the stability of interconnected systems in terms of Lyapunov functions of the individual systems are obtained.

In [10] Šiljak considers structural perturbations and their effects on the stability of composite systems using Lyapunov theory. The method is to reduce each subsystem to a one-dimensional one, such that the stability properties of the reduced aggregate representation imply the same stability properties of the original aggregate system. In some cases the aggregate representation gives rise to an interconnection matrix $\bar{W}$, such that quasi dominance or negative definiteness of $\bar{W}$ yield asymptotic stability of the composite system.

In [17] small-gain type theorems for general interconnected systems with linear gains can be found. These results are of the form that the spectral radius of a gain matrix should be less than one to conclude stability. The result obtained here may be regarded as a nonlinear generalization in the same spirit.

In this paper we consider a system which consists of two or more ISS subsystems. We provide conditions by which the stability question of the overall system can be reduced to consideration of stability of the subsystems. We choose an approach using estimates involving $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions to prove the ISS stability result for general interconnected systems. The generalized small-gain condition we obtain is,

that for some monotone operator $\tilde{\Gamma}$ related to the gains of the individual systems the condition

$$\tilde{\Gamma}(s) \not\geq s \tag{1.1}$$

holds for all $s \geq 0, s \neq 0$ (in the sense of the component-wise ordering of the positive orthant). We discuss interpretations of this condition in Section 4.

Although we believe our approach to be amenable to the explicit construction of a Lyapunov function given the ISS-Lyapunov functions for the subsystems, so far we have been able to prove this only for linear gains.

While the general problem can be approached by repeated application of the cascade property and the known small-gain theorem, in general this can be cumbersome and it is by no means obvious in which order subsystems have to be chosen to proceed in such an iterative manner. Hence an extension of the known small-gain theorem to larger interconnections is needed. In this paper we obtain this extension for the general case. Further, we show how to calculate the gain matrix for linear systems and give some interpretation of our result.

The paper is organized as follows. In Section 2 we introduce notation and necessary concepts and state the problem. In particular, we will need some basic properties of the positive orthant $\mathbb{R}^n_+$ interpreted as a lattice. In Section 3 we prove the main result, which generalizes the known small-gain theorem, and consider the special case of linear gains, for which we also construct an ISS-Lyapunov function. In Section 4 the small-gain condition of the main result is discussed and we show in which way it may be interpreted as an extension of the linear condition that the spectral radius of the gain matrix has to be less than one. There we also point out the connection to some induced monotone dynamical system. In Section 5 we show how the gain matrix can be found for linear systems. We conclude with Section 6.

2 Problem description

Notation By x^T we denote the transpose of a vector $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we use the following notation

$$x \geq y \;\Leftrightarrow\; x_i \geq y_i,\ i = 1, \ldots, n\,, \text{ and } x > y \;\Leftrightarrow\; x_i > y_i,\ i = 1, \ldots, n. \tag{2.2}$$

In the following $\mathbb{R}_+ := [0, \infty)$ and by $\mathbb{R}^n_+$ we denote $\{x \in \mathbb{R}^n : x \geq 0\}$. For a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{else.} \end{cases}$$

Definition 1. *(i) A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is proper, i.e., unbounded.*

(ii) A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero for $t \to \infty$.

Let $|\cdot|$ denote some norm in $\mathbb{R}^n$, and let in particular $|x|_{\max} = \max_i |x_i|$ be the maximum norm. The essential supremum norm on essentially bounded functions defined on $\mathbb{R}_+$ is denoted by $\|\cdot\|_\infty$.

Definition 2. *Consider a system*

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

such that for all initial values x_0 and all essentially bounded inputs u unique solutions exist for all positive times. We denote these solutions by $\xi(t; x_0, u)$. The system is called input to state stable (ISS), if there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$, such that the inequality

$$|\xi(t; x_0, u)| \leq \beta(|x_0|, t) + \gamma(||u||_\infty)$$

holds for all $t \geq 0, x_0 \in \mathbb{R}^n, u : \mathbb{R}_+ \to \mathbb{R}^m$ essentially bounded.

Problem statement Consider n interconnected control systems given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u) \end{aligned} \tag{2.3}$$

where $x_i \in \mathbb{R}^{N_i}, u \in \mathbb{R}^L$ and $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + L} \to \mathbb{R}^{N_i}$ is continuous and Lipschitz in the first n arguments uniformly with respect to u for $i = 1, \ldots, n$. Here x_i is the state of the i^{th} subsystem, and u is considered as an external control variable.

We may consider u as partitioned $u = (u_1, \ldots, u_n)$, such that each u_i is the input for subsystem i only. Then each f_i is of the form $f_i(\ldots, u) = \tilde{f}_i(\ldots, P_i(u)) = \tilde{f}_i(\ldots, u_i)$ with some projection P_i. So without loss of generality we may assume to have the same input for all systems.

We call the i^{th} subsystem of (2.3) ISS, if there exist functions β_i of class $\mathcal{KL}$ and γ_{ij}, γ of class $\mathcal{K}$, such that the solution $x_i(t)$ starting at $x_i(0)$ satisfies

$$|x_i(t)| \leq \beta_i(|x_i(0)|, t) + \sum_{j=1}^n \gamma_{ij}(||x_{j\,[0,t]}||_\infty) + \gamma(||u||_\infty) \tag{2.4}$$

for all $t \geq 0$.

For notational simplicity we allow the case $\gamma_{ij} \equiv 0$ and require $\gamma_{ii} \equiv 0$ for all i. The functions γ_{ij} and γ are called (nonlinear) gains. We define $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma := (\gamma_{ij}), \qquad \Gamma(s_1, \ldots, s_n)^T := \left(\sum_{j=1}^n \gamma_{1j}(s_j), \ldots, \sum_{j=1}^n \gamma_{nj}(s_j) \right)^T \tag{2.5}$$

for $s = (s_1, \ldots, s_n)^T \in \mathbb{R}_+^n$. We refer to Γ as the *gain matrix*, noting that it does not represent a linear map. Note that by the properties of γ_{ij} for $s_1, s_2 \in \mathbb{R}_+^n$ we have the implication

$$s_1 \geq s_2 \ \Rightarrow\ \Gamma(s_1) \geq \Gamma(s_2), \tag{2.6}$$

so that Γ defines a monotone map.
Assuming each of the subsystems of (2.3) to be ISS, we are interested in conditions guaranteeing that the whole system defined by $x = (x_1^T, \dots, x_n^T)^T, f = (f_1^T, \dots, f_n^T)^T$ and

$$\dot{x} = f(x, u) \tag{2.7}$$

is ISS (from u to x).

Additional Preliminaries We also need some notation from lattice theory, cf. [16] for example. Although $(\mathbb{R}_+^n, \sup, \inf)$ is a lattice, with inf denoting infimum and sup denoting supremum, it is not complete. But still one can define the upper limit for bounded functions $s : \mathbb{R}_+ \to \mathbb{R}_+^n$ by

$$\limsup_{t\to\infty} s(t) := \inf_{t\geq 0} \sup_{\tau\geq t} s(\tau).$$

For vector functions $x = (x_1^T, \dots, x_n^T)^T : \mathbb{R}_+ \to \mathbb{R}^{N_1+\dots+N_n}$ such that $x_i : \mathbb{R}_+ \to \mathbb{R}^{N_i}, i = 1, \dots, n$ and times $0 \leq t_1 \leq t_2$ we define

$$\left\| x_{[t_1,t_2]} \right\| := \begin{pmatrix} \|x_{1,[t_1,t_2]}\|_\infty \\ \vdots \\ \|x_{n,[t_1,t_2]}\|_\infty \end{pmatrix} \in \mathbb{R}_+^n.$$

We will need the following property.

Lemma 3. *Let $s : \mathbb{R}_+ \to \mathbb{R}_+^n$ be continuous and bounded. Then (setting $N_i \equiv 1$)*

$$\limsup_{t\to\infty} s(t) = \limsup_{t\to\infty} \left\| s_{[t/2,\infty)} \right\|.$$

Proof. Let $\limsup_{t\to\infty} s(t) =: a \in \mathbb{R}_+^n$ and $\limsup_{t\to\infty} \left\| s_{[t/2,\infty)} \right\| =: b \in \mathbb{R}_+^n$. For every $\varepsilon \in \mathbb{R}_+^n$, $\varepsilon > 0$ (component-wise!) there exist $t_a, t_b \geq 0$ such that

$$\forall\, t \geq t_a : \sup_{t\geq t_a} s(t) \leq a + \varepsilon \quad \text{and} \quad \forall\, t \geq t_b : \sup_{t\geq t_b} \left\| s_{[t/2,\infty)} \right\| \leq b + \varepsilon. \tag{2.8}$$

Clearly we have

$$s(t) \leq \left\| s_{[t/2,\infty)} \right\|$$

for all $t \geq 0$, i.e., $a \leq b$. On the other hand $s(\tau) \leq a + \varepsilon$ for $\tau \geq t$ implies $\left\| s_{[\tau/2,\infty)} \right\| \leq a + \varepsilon$ for $\tau \geq 2t$, i.e., $b \leq a$. This immediately gives $a = b$, and the claim is proved. □

Before we introduce the ISS criterion for interconnected systems let us briefly discuss an equivalent formulation of ISS. A system

$$\dot{x} = f(x, u), \tag{2.9}$$

with $f : \mathbb{R}^{N+L} \to \mathbb{R}^N$ continuous and Lipschitz in $x \in \mathbb{R}^N$, uniformly with respect to $u \in \mathbb{R}^L$, is said to have the *asymptotic gain property* (AG), if there exists a function

$\gamma_{AG} \in \mathcal{K}_\infty$ such that for all initial values $x_0 \in \mathbb{R}^N$ and all essentially bounded control functions $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^L$,

$$\limsup_{t \to 0} |x(t; x_0, u)| \leq \gamma_{AG}(\|u\|_\infty). \tag{2.10}$$

The asymptotic gain property states, that every trajectory must ultimately stay not far from zero, depending on the magnitude of $\|u\|_\infty$.

The system (2.9) is said to be *globally asymptotically stable at zero* (0-GAS), if there exists a $\beta_{GAS} \in \mathcal{KL}$, such that for all initial conditions $x_0 \in \mathbb{R}^N$

$$|x(t; x_0, 0)| \leq \beta_{GAS}(|x_0|, t). \tag{2.11}$$

Thus 0-GAS holds, if, when the input u is set to zero, the system (2.9) is globally asymptotically stable at $x^* = 0$.

By a result of Sontag and Wang [15] the asymptotic gain property and global asymptotic stability at 0 together are equivalent to ISS.

3 Main results

In the following subsection we present a nonlinear version of the small-gain theorem for networks. In Subsection 3.2 we restate this theorem for the case when the gains are linear functions. Here we also provide a method on how to construct an ISS-Lyapunov function for the whole network system from given ISS-Lyapunov functions of the subsystems.

3.1 Nonlinear gains

We introduce the following notation. For $\alpha_i \in \mathcal{K}_\infty, i = 1, \ldots, n$ define $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$D(s_1, \ldots, s_n)^T := \begin{pmatrix} (\mathrm{Id} + \alpha_1)(s_1) \\ \vdots \\ (\mathrm{Id} + \alpha_n)(s_n) \end{pmatrix}. \tag{3.12}$$

Theorem 4 (small-gain theorem for networks)**.** *Consider the system* (2.3) *and suppose that each subsystem is ISS, i.e., condition (2.4) holds for all* $i = 1, \ldots, n$*. Let* Γ *be given by (2.5). If there exists a mapping* D *as in (3.12), such that*

$$(\Gamma \circ D)(s) \not\geq s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}, \tag{3.13}$$

then the system (2.7) is ISS from u *to* x*.*

Remark 5. *Although looking very complicated to handle at first sight, condition (3.13) is a straightforward extension of the ISS small-gain theorem of [4]. It has many interesting interpretations, as we will discuss in Section 4.*

The following lemma provides an essential argument in the proof of Theorem 4.

Lemma 6. *Let D be as in (3.12) and suppose (3.13) holds. Then there exists a $\varphi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$,*

$$(Id - \Gamma)(w) \leq v \tag{3.14}$$

implies $|w| \leq \varphi(|v|)$.

Proof. Fix $v \in \mathbb{R}^n_+$. We first show, that for those $w \in \mathbb{R}^n_+$ satisfying (3.14) at least some components have to be bounded. To this end let

$$\begin{aligned} r^* := (D - \mathrm{Id})^{-1}(v) &= \begin{pmatrix} \alpha_1^{-1}(v_1) \\ \vdots \\ \alpha_n^{-1}(v_n) \end{pmatrix} \\ \text{and } s^* := D(r^*) &= \begin{pmatrix} v_1 + \alpha_1^{-1}(v_1) \\ \vdots \\ v_n + \alpha_n^{-1}(v_n) \end{pmatrix}. \end{aligned} \tag{3.15}$$

We claim that $s \geq s^*$ implies that $w = s$ does not satisfy (3.14). So let $s \geq s^*$ be arbitrary and $r = D^{-1}(s) \geq r^*$ (as $D^{-1} \in \mathcal{K}^n_\infty$). For such s we have

$$s - D^{-1}(s) = D(r) - r \geq D(r^*) - r^* = v\,,$$

where we have used that $(D - \mathrm{Id}) \in \mathcal{K}^n_\infty$. The assumption that $w = s$ satisfies (3.14) leads to

$$s \leq v + \Gamma(s) \leq s - D^{-1}(s) + \Gamma(s)\,,$$

or equivalently, $0 \leq \Gamma(s) - D^{-1}(s)$. This implies for $r = D^{-1}(s)$ that

$$r \leq \Gamma \circ D(r)\,,$$

in contradiction to (3.13). This shows that the set of $w \in \mathbb{R}^n_+$ satisfying (3.14) does not intersect the set

$$Z_1 := \{w \in \mathbb{R}^n_+ \mid w \geq s^*\}\,.$$

Assume now that $w \in \mathbb{R}^n_+$ satisfies (3.14). Let $s^1 := s^*$. If $s^1 \not\geq w$, then there exists an index set $I_1 \subset \{1, \ldots, n\}$, such that

$$w_i > s^1_i, \text{ for } i \in I_1 \quad \text{and} \quad w_i \leq s^1_i, \qquad \text{for } i \in I_1^c := \{1, \ldots, n\} \setminus I_1\,.$$

For index sets I and J denote by y_I the restriction

$$y_I := (y_i)_{i \in I}$$

for vectors $y \in \mathbb{R}^n_+$ and by $A_{IJ} : \mathbb{R}^{\#I}_+ \to \mathbb{R}^{\#J}_+$ the restriction

$$A_{IJ} := (a_{ij})_{i \in I, j \in J}$$

for mappings $A = (a_{ij})_{i,j \in \{1,\ldots,n\}} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$.

So from (3.14) we obtain

$$\begin{bmatrix} w_{I_1} \\ w_{I_1^c} \end{bmatrix} - \begin{bmatrix} \Gamma_{I_1 I_1} & \Gamma_{I_1 I_1^c} \\ \Gamma_{I_1^c I_1} & \Gamma_{I_1^c I_1^c} \end{bmatrix} \left(\begin{bmatrix} w_{I_1} \\ w_{I_1^c} \end{bmatrix} \right) \leq \begin{bmatrix} v_{I_1} \\ v_{I_1^c} \end{bmatrix} .$$

Hence we have in particular

$$\begin{aligned} w_{I_1} - \Gamma_{I_1 I_1}(w_{I_1}) &\leq v_{I_1} + \Gamma_{I_1 I_1^c}(s^1_{I_1^c}) \\ &\leq \underbrace{D_{I_1} \circ (D_{I_1} - \mathrm{Id}_{I_1})^{-1}}_{>\mathrm{Id}} \circ (v_{I_1} + \Gamma_{I_1 I_1^c}(s^1_{I_1^c})) =: s^2_{I_1}. \end{aligned} \tag{3.16}$$

Note that $\Gamma_{I_1 I_1}$ satisfies (3.13) with D replaced by D_{I_1}. Thus, arguing just as before, we obtain, that $w_{I_1} \geq s^2_{I_1}$ is not possible. Hence some more components of w must be bounded.

We proceed inductively, defining

$$I_{j+1} \subsetneqq I_j, \quad I_{j+1} := \{i \in I_j : w_i > s^{j+1}_i\},$$

with $I^c_{j+1} := \{1, \ldots, n\} \setminus I_{j+1}$ and

$$s^{j+1}_{I_j} := D_{I_j} \circ (D_{I_j} - \mathrm{Id}_{I_j})^{-1} \circ (v_{I_j} + \Gamma_{I_j I_j^c}(s^j_{I_j^c})).$$

Obviously this nesting will end after at most $n-1$ steps: There exists a maximal $k \leq n$, such that

$$\{1, \ldots, n\} \supsetneqq I_1 \supsetneqq \ldots \supsetneqq I_k \neq \emptyset$$

and all components of w_{I_k} are bounded by the corresponding components of $s^{k+1}_{I_k}$. For $i = 1, \ldots, n$ define

$$\zeta_i := \max\{j \in \{1, \ldots, n\} : i \in I_j\}$$

and

$$s_\zeta := (s^{\zeta_1}_1, \ldots, s^{\zeta_n}_n).$$

Clearly we have

$$w \leq s_\zeta \leq [D \circ (D - \mathrm{Id})^{-1} \circ (\mathrm{Id} + \Gamma)]^n(v)$$

and the term on the very right hand side does not depend on any particular choice of nesting of the index sets. Hence every w satisfying (3.14) also satisfies

$$w \leq [D \circ (D - \mathrm{Id})^{-1} \circ (\mathrm{Id} + \Gamma)]^n \circ \left(|v|_{\max}, \quad \ldots, \quad |v|_{\max}\right)^T$$

and taking the max-norm on both sides yields

$$|w|_{\max} \leq \varphi(|v|_{\max})$$

for some function φ of class $\mathcal{K}_\infty$. This completes the proof of the lemma. □

We proceed with the proof of Theorem 4, which is divided into two main steps. First we establish the existence of a solution of the system (2.7) for all times $t \geq 0$. In the second step we establish the ISS property for this system.

Proof. (of Theorem 4) *Existence of a solution for (2.7) for all times:* For finite times $t \geq 0$ and for $s \in \mathbb{R}^n_+$ we introduce the abbreviating notation

$$\left|\!\left| x(t) \right|\!\right| := \begin{pmatrix} |x_1(t)| \\ \vdots \\ |x_n(t)| \end{pmatrix} \in \mathbb{R}^n_+, \quad \gamma^n(\|u\|_\infty) := \begin{pmatrix} \gamma(\|u\|_\infty) \\ \vdots \\ \gamma(\|u\|_\infty) \end{pmatrix} \in \mathbb{R}^n_+ \tag{3.17}$$

$$\text{and} \quad \beta(s,t) := \begin{pmatrix} \beta_1(s_1,t) \\ \vdots \\ \beta_n(s_n,t) \end{pmatrix} : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+ . \tag{3.18}$$

Now we can rewrite the ISS conditions (2.4) of the subsystems in a vectorized form for $\tau \geq 0$ as

$$\left|\!\left| x(\tau) \right|\!\right| \leq \beta(\left|\!\left| x(0) \right|\!\right|, \tau) + \Gamma(\left\|\!\left| x_{[0,\tau]} \right|\!\right\|) + \gamma^n(\|u\|_\infty) \tag{3.19}$$

and taking the supremum on both sides over $\tau \in [0,t]$ we obtain

$$\begin{aligned} (\mathrm{Id} - \Gamma) \circ \left\|\!\left| x_{[0,t]} \right|\!\right\| = & \quad \left\|\!\left| x_{[0,t]} \right|\!\right\| - \Gamma(\left\|\!\left| x_{[0,t]} \right|\!\right\|) \\ \leq & \quad \beta(\left|\!\left| x(0) \right|\!\right|, 0) + \gamma^n(\|u\|_\infty) \end{aligned} \tag{3.20}$$

where we used (2.6). Now by Lemma 6 we find

$$\|x_{[0,t]}\|_\infty \leq \varphi\left(\left|\beta(\left|\!\left| x(0) \right|\!\right|, 0) + \gamma^n(\|u\|_\infty)\right|\right) =: s_\infty \tag{3.21}$$

for some class $\mathcal{K}$ function φ and all times $t \geq 0$. Hence for every initial condition and essentially bounded input u the solution of our system (2.7) exists for all times $t \geq 0$, since s_∞ in (3.21) does not depend on t.

Establishing ISS: We now utilize an idea from [4]: Instead of estimating $|x_i(t)|$ with respect to $|x_i(0)|$ in (2.4), we can also have the point of view that our trajectory started in $x_i(\tau)$ at time $0 \leq \tau \leq t$ and we followed it for some time $t - \tau$ and reach $x_i(t)$ at time t. For $\tau = t/2$ this reads

$$\begin{aligned} |x_i(t)| &\leq \beta_i(|x_i(t/2)|, t/2) + \sum_{j \neq i} \gamma_{ij}(\|x_{i,[t/2,t]}\|_\infty) + \gamma(u) \\ &\leq \beta_i(s_\infty, t/2) + \sum_{j \neq i} \gamma_{ij}(\|x_{i,[t/2,\infty)}\|_\infty) + \gamma(u) \qquad (3.22) \\ &= \tilde{\beta}_i(s_\infty, t) + \sum_{j \neq i} \gamma_{ij}(\|x_{i,[t/2,\infty)}\|_\infty) + \gamma(u) \qquad (3.23) \end{aligned}$$

where we again applied (2.6) to obtain (3.22) and defined

$$\tilde{\beta}_i(s_i, t) := \beta_i(s_i, t/2),$$

which is of class $\mathcal{KL}$.

To write inequality (3.23) in vector form, we define

$$\tilde{\beta}(s,t) := \begin{pmatrix} \tilde{\beta}_1(s_1,t) \\ \vdots \\ \tilde{\beta}_n(s_n,t) \end{pmatrix} \tag{3.24}$$

for all $s \in \mathbb{R}^n_+$. Denoting by $s^n_\infty := (s_\infty, \ldots, s_\infty)^T$ we obtain the vector formulation of (3.23) as

$$\left| x(t) \right| \leq \tilde{\beta}(s^n_\infty, t) + \Gamma \circ \left\| x_{[t/2,\infty)} \right\| + \gamma^n(\|u\|_\infty). \tag{3.25}$$

By the boundedness of the solution we can take the upper limit on both sides of (3.25). By Lemma 3 we have

$$\limsup_{t\to\infty} \left| x(t) \right| = \limsup_{t\to\infty} \left\| x_{[t/2,\infty)} \right\| =: l(x)\,,$$

and it follows that

$$(\mathrm{Id} - \Gamma) \circ l(x) \leq \gamma^n(\|u\|_\infty)$$

since $\lim_{t\to\infty} \tilde{\beta}(s^n_\infty, t) = 0$. Finally, by Lemma 6 we have

$$|l(x)| \leq \varphi(|\gamma^n(\|u\|_\infty)|) \tag{3.26}$$

for some φ of class $\mathcal{K}_\infty$. But (3.26) is the asymptotic gain property (2.10).

Now 0-GAS is established as follows: First note that for $u \equiv 0$ the quantity s_∞ in (3.21) is a $\mathcal{K}$ function of $|x(0)|$. So (3.21) shows (Lyapunov) stability of the system in the case $u \equiv 0$. Furthermore, (3.26) shows attractivity of $x = 0$ for the system (2.7) in the case $u \equiv 0$. This shows global asymptotic stability of $x = 0$.

Hence system (2.7) is AG and 0-GAS, which together were proved to be equivalent to ISS in [15, Theorem 1]. □

3.2 Linear gains and an ISS-Lyapunov version

Suppose the gain functions γ_{ij} are all linear, hence Γ is a linear mapping and (2.5) is just matrix-vector multiplication. Then we have the following

Corollary 7. *Consider n interconnected ISS systems as in the previous section on the problem description with a linear gain matrix Γ, such that for the spectral radius ρ of Γ we have*

$$\rho(\Gamma) < 1. \tag{3.27}$$

Then the system defined by (2.7) is ISS from u to x.

Remark 8. *For non-negative matrices Γ it is well known that (see, e.g., [1, Theorem 2.1.1, page 26, and Theorem 2.1.11, page 28])*

(i) $\rho(\Gamma)$ *is an eigenvalue of* Γ *and* Γ *possesses a non-negative eigenvector corresponding to* $\rho(\Gamma)$*,*

(ii) $\alpha x \leq \Gamma x$ *holds for some* $x \in \mathbb{R}^n_+ \setminus \{0\}$ *if and only if* $\alpha \leq \rho(\Gamma)$*.*

Hence $\rho(\Gamma) < 1$ *if and only if* $\Gamma s \not\geq s$ *for all* $s \in \mathbb{R}^n_+ \setminus \{0\}$*. Also, by continuity of the spectrum it is clear that for such* $\Gamma, \rho(\Gamma) < 1$*, there always exists a matrix* $D = diag(1+\alpha_1, \ldots, 1+\alpha_n)$ *with* $\alpha_i > 0, i = 1, \ldots, n$*, such that* $\Gamma D s \not\geq s$ *for all* $s \in \mathbb{R}^n_+ \setminus \{0\}$*.*

Remark 9. *For the case of large-scale interconnected input-output systems a similar result exists, which can be found in a monograph by Vidyasagar, cf. [17, p. 110]. It also covers Corollary 7 as a special case. The condition on the spectral radius is quite the same, although it is applied to a* test matrix*, whose entries are finite gains of products of* interconnection operators *and corresponding* subsystem operators*. These gains are non-negative numbers and, roughly speaking, defined as the minimal possible slope of affine bounds on the interconnection operators.*

Proof. (of Corollary 7) The proof is essentially the same as of Theorem 4, but note that instead of Lemma 6 we now directly have existence of

$$(\mathrm{Id} - \Gamma)^{-1} = \mathrm{Id} + \Gamma + \Gamma^2 + \ldots$$

since $\rho(\Gamma) < 1$ and from the power sum expansion it is obvious that $(\mathrm{Id}-\Gamma)^{-1}$ is a non-decreasing mapping, i.e., for $d_1, d_2 \geq 0$ we have $(\mathrm{Id}-\Gamma)^{-1}(d_1+d_2)-(\mathrm{Id}-\Gamma)^{-1}(d_1) \geq 0$.

Thus at the two places where Lemma 6 has been used we can simply apply $(\mathrm{Id} - \Gamma)^{-1}$ to get the desired estimates. □

Construction of an ISS-Lyapunov function There is another approach to describe the ISS property via so called ISS-Lyapunov functions, cf. [14].

Definition 10. *A smooth function* V *is said to be an ISS-Lyapunov function of the system* (2.9) $\dot{x} = f(x,u)$*,* $f : \mathbb{R}^{N+L} \to \mathbb{R}^N$ *if*

(i) V *is proper, positive-definite, that is, there exit functions* ψ_1, ψ_2 *of class* $\mathcal{K}_\infty$ *such that*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^{n_N}; \tag{3.28}$$

(ii) there exists a positive-definite function α*, a class* $\mathcal{K}$*-function* χ*, such that*

$$V(x) \geq \chi(|u|) \implies \nabla V(x) f(x,u) \leq -\alpha(|x|). \tag{3.29}$$

In case of linear gains a Lyapunov function for the interconnected system can be constructed, given ISS-Lyapunov functions of the subsystems.

Let $V_1(x_1), \ldots, V_n(x_n)$ be some ISS-Lyapunov functions of the subsystems (2.3), allowing for linear Lyapunov-gains γ_{ij}, i.e., there are some $\mathcal{K}_\infty$ functions ψ_{i1}, ψ_{i2} such that

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad x_i \in \mathbb{R}^{N_i}, \tag{3.30}$$

and some positive-definite functions α_i such that

$$V_i(x_i) > \max\{\max_j\{\gamma_{ij}V_j(x_j)\}, \gamma_i(|u|)\} \;\Rightarrow\; \nabla V_i(x_i)f_i(x,u) \le -\alpha_i(V_i(x_i)). \quad (3.31)$$

Consider the positive orthant $\mathbb{R}^n_+$, and let Ω_i be the subsets of $\mathbb{R}^n_+$ defined by

$$\Omega_i := \Big\{(v_1,\dots,v_n) \in \mathbb{R}^n_+ : v_i > \sum_{j=1}^n \gamma_{ij}v_j\Big\}. \quad (3.32)$$

Note that the boundaries $\partial\Omega_i$ are hyperplanes in case of linear gains. Now if (3.27) or equivalently $\Gamma s \not\ge s$, $\forall s \in \mathbb{R}^n_+$, $s \neq 0$, holds for $\Gamma = (\gamma_{ij})$, $i,j = 1,\dots,n$, then it follows that

$$\bigcup_{i=1}^n \Omega_i = \mathbb{R}^n_+ \setminus \{0\} \quad \text{and} \quad \bigcap_{i=1}^n \Omega_i \neq \emptyset. \quad (3.33)$$

The proof is the same as of Proposition 21, see below. Thus we may choose an $s > 0$ with $s \in \bigcap_{i=1}^n \Omega_i$, which implies that

$$s_i > \sum_j \gamma_{ij}s_j,\ i = 1,\dots,n; \quad (3.34)$$

see Fig. 1. If Γ is irreducible, then using Perron-Frobenius theory we see that we may choose s to be a (positive) eigenvector of Γ.

Theorem 11. *Let V_i be an ISS-Lyapunov function as in (3.31) of the i^{th} subsystem from* (2.3)*, $i = 1,\dots,n$, and s be a positive vector with* (3.34)*. Then an ISS Lyapunov function of the interconnected system* (2.3) *is given by*

$$V(x_1,\dots,x_n) := \max_i \frac{V_i(x_i)}{s_i}. \quad (3.35)$$

Proof. Let $\gamma(|u|) := \max_i \gamma_i(|u|)$ which is a $\mathcal{K}$ class function. In the following we show that there exists a positive definite function α such that:

$$V(x) \ge \gamma(|u|) \implies \nabla V(x)f(x,u) \le -\alpha(V(x)). \quad (3.36)$$

Let M_i be open domains in $\mathbb{R}^n_+$ defined by

$$M_i := \Big\{(v_1,\dots,v_n) \in \mathbb{R}^n_+ : \frac{v_i}{s_i} > \max_{j\neq i}\Big\{\frac{v_j}{s_j}\Big\}\Big\}, \quad (3.37)$$

and let P_i be the 2-dimensional planes spanned by s and the i-th axis, i.e.,

$$P_i = \Big\{v \in \mathbb{R}^n_+ \Big| \frac{v_k}{s_k} = \frac{v_j}{s_j};\ \forall k,j \neq i\Big\}. \quad (3.38)$$

Note that V defined by (3.35) is continuous in $\mathbb{R}^n_+$ and can only fail to be differentiable on the planes P_i.

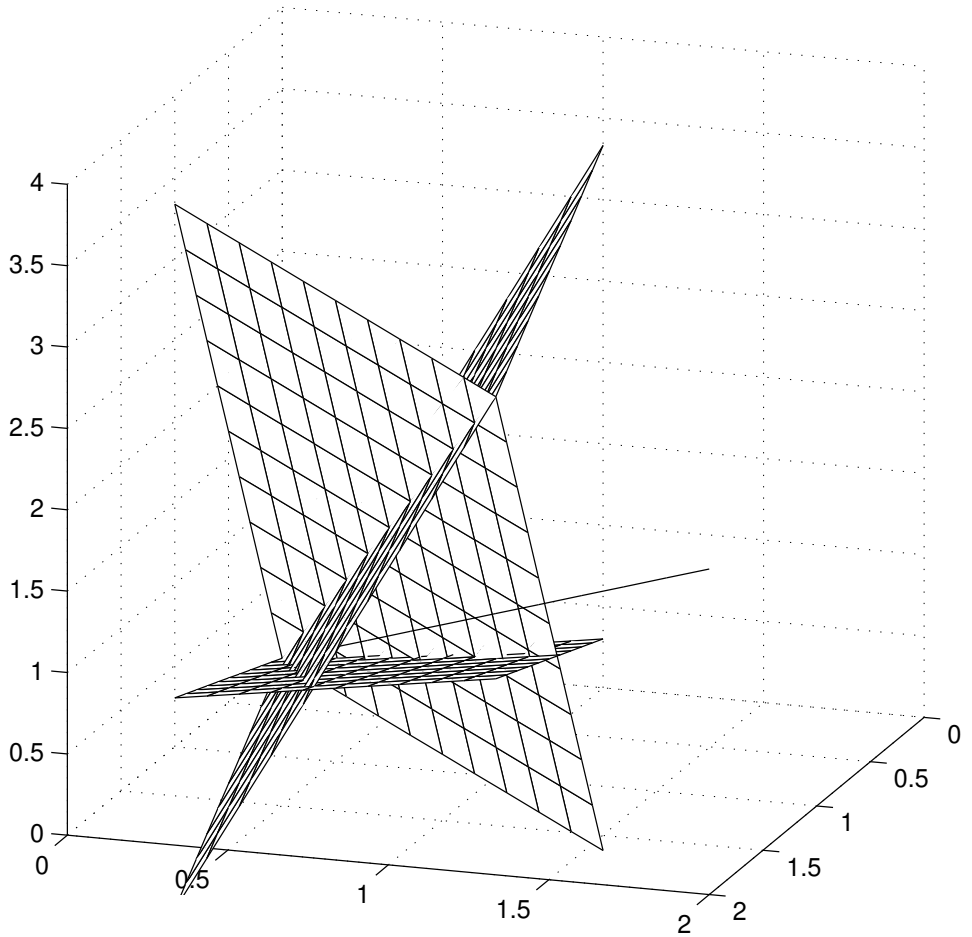

Figure 1: The sets $\partial\Omega_i$ in $\mathbb{R}^3_+$ and the eigenvector s.

Now take any $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \mathbb{R}^n$ with $(V_1(\hat{x}_1), \dots, V_n(\hat{x}_n)) \in M_i$ then it follows that in some neighborhood U of $\hat{x}$ we have $V(x) = \frac{V_i(x_i)}{s_i}$ for all $x \in U$ and

$$V_i(x_i) > \max_{j \neq i} \left\{ \frac{s_i}{s_j} V_j(x_j) \right\} > \max_{j \neq i} \{\gamma_{ij} V_j(x_j)\} \tag{3.39}$$

(the last inequality follows from(3.34)), hence by (3.31), if $V(x) = V_i(x_i)/s_i > \gamma_i(|u|)$, then

$$\nabla V(x) f(x,u) = \frac{1}{s_i} \nabla V_i(x_i) f_i(x,u) \leq -\frac{1}{s_i} \alpha_i(V_i(x_i)) < -\tilde{\alpha}_i(V(x)), \tag{3.40}$$

where $\tilde{\alpha}_i$ are positive-definite functions, since $s_i = const > 0$.

It remains to consider $x \in \mathbb{R}^n$ such that $(V_1(x_1), \dots, V_n(x_n)) \in \overline{M_i} \cap \overline{M_j}$, where $V(x)$ may be not differentiable.

For this purpose we use some results from [2]. For smooth functions $f_i,\ i = 1, \dots, n$ it follows that $f(x,u) = \max_i\{f_i(x,u)\}$ is Lipschitz and Clarke's subgradient of f is given by

$$\partial_{Cl} f(x) = co\Big\{ \bigcup_{i \in M(x)} \nabla_x f_i(x,u)\Big\}, \quad M(x) = \{i :\ f_i(x,u) = f(x)\}, \tag{3.41}$$

i.e., in our case

$$\partial_{Cl} V(x) = co\Big\{ \frac{1}{s_i}\nabla V_i(x) :\ \frac{1}{s_i} V_i(x) = V(x)\Big\}. \tag{3.42}$$

Now for every extremal point of $\partial_{Cl} V(x)$ a decrease condition is satisfied by (3.40). By convexity, the same is true for every element of $\partial_{Cl} V(x)$. Now Theorems 4.3.8 and 4.5.5 of [2] show strong invariance and attractivity of the set $\{x : V(x) \le \gamma(\|u\|)\}$. It follows that V is an ISS-Lyapunov function for the interconnection (2.3). □

See also Section 4.4 for some more considerations into this directions.

4 Interpretation of the generalized small-gain condition

In this section we wish to provide insight into the small-gain condition of Theorem 4. We first show, that the result covers the known interconnection results for cascades and feedback interconnections. We then compare the condition with the linear case.

Further we state some algebraic and graph theoretical relations and investigate some associated artificial dynamical system induced by the gain matrix Γ. We complete this section with some geometrical considerations and an overview map of all these contiguities.

4.1 Connections to known results

As an easy consequence of Theorem 4 we recover, that an arbitrarily long feed forward cascade of ISS subsystems is ISS again. If the subsystems are enumerated consecutively and the gain function from subsystem j to subsystem $i > j$ is denoted by γ_{ij}, then the resulting gain matrix has non-zero entries only below the diagonal. For arbitrary $\alpha \in \mathcal{K}_\infty$ the gain matrix with entries $\gamma_{ij} \circ (\mathrm{Id}_{\mathbb{R}_+} + \alpha)$ for $i > j$ and 0 for $i \le j$ clearly satisfies (3.13). Therefore the feed forward cascade itself is ISS.

Consider $n = 2$ in equation (2.3), i.e., two subsystems with linear gains. Then in Corollary 7 we have

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix}, \quad \gamma_{ij} \in \mathbb{R}_+$$

and $\rho(\Gamma) < 1$ if and only if $\gamma_{12}\gamma_{21} < 1$. Hence we obtain the known small-gain theorem, cf. [5] and [3].

For nonlinear gains and $n = 2$ the condition (3.13) in Theorem 4 reads as follows: There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\begin{pmatrix} \gamma_{12} \circ (\mathrm{Id} + \alpha_2)(s_2) \\ \gamma_{21} \circ (\mathrm{Id} + \alpha_1)(s_1) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

for all $(s_1, s_2)^T \in \mathbb{R}^2_+$. This is easily seen to be equivalent to

$$\gamma_{12} \circ (\mathrm{Id} + \alpha_2) \circ \gamma_{21} \circ (\mathrm{Id} + \alpha_1)(s) < s\,, \quad \forall s > 0.$$

To this end it suffices to check what happens for to the vector $[\gamma_{12} \circ (\mathrm{Id} + \alpha_2)(s_2), s_2]^T$ under Γ along with a few similar considerations. The latter is equivalent to the condition in the small-gain theorem of [4], namely, that for some $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ it should hold that

$$(\mathrm{Id} + \tilde{\alpha}_1) \circ \gamma_{21} \circ (\mathrm{Id} + \tilde{\alpha}_2) \circ \gamma_{12}(s) \leq s\,, \quad \forall s > 0\,, \tag{4.43}$$

for all $s \in \mathbb{R}_+$, hence our theorem contains this result as a particular case.

Example 12. *The condition (4.43) of [4] seems to be very similar to the small-gain condition $\gamma_{12} \circ \gamma_{21}(s) < s$ of [5] and [3], however those γ's have some different meanings in these papers. This similarity raises the question, whether the compositions with $(Id + \tilde{\alpha}_i)$, $i = 1, 2$ in (4.43) or more generally with D in* (3.13) *is necessary. The answer is positive. Namely, there is a counterexample providing a system of two ISS subsystems with $\gamma_{12} \circ \gamma_{21}(s) < s$ (γ's are defined as above) which is not ISS.*

Consider the equation

$$\dot{x} = -x + u(1 - e^{-u}), \quad x(0) = x^0 \in \mathbb{R}, u \in \mathbb{R}.$$

Integrating it follows

$$x(t) = e^{-t}x^0 + \int_0^t e^{-(t-\tau)}u(\tau)(1 - e^{-u(\tau)})\, d\tau$$

$$\leq e^{-t}x^0 + ||u||_\infty(1 - e^{-||u||_\infty}) = e^{-t}x^0 + \gamma(\|u\|_\infty), \quad \gamma(s) < s.$$

Then for a feedback system

$$\dot{x}_1 = -x_1 + x_2(1 - e^{-x_2}) + u(t), \tag{4.44}$$
$$\dot{x}_2 = -x_2 + x_1(1 - e^{-x_1}) + u(t) \tag{4.45}$$

we have ISS for each subsystem with $x_i(t) \leq e^{-t}x_i^0 + \gamma_i(||x_i||) + \eta_i(||u||)$, where $\gamma_i(s) < s$ and hence $\gamma_1 \circ \gamma_2(s) < s$ for $s > 0$, but there is a solution $x_1 = x_2 = const$, i.e.,

$$\dot{x}_1 = -x_2 e^{-x_2} + u, \quad \textit{with } u = x_2 e^{-x_2},$$

and $x_1 = x_2$ can be chosen arbitrary large with $u \to 0$ for $x_1 \to \infty$. Hence the condition $\Gamma(s) \not\geq s$, for all $s \in \mathbb{R}^n_+ \setminus \{0\}$, or for two subsystems $\gamma_{12} \circ \gamma_{21}(s) < s$, for all $s > 0$, is not sufficient for the input-to-state stability of the composite system in the nonlinear case.

4.2 Algebraic Interpretation

In this subsection we first relate the network small-gain condition (3.13) to well known properties of matrices in the linear case. This gives some idea how the new condition can be understood and what subtle differences appear in the nonlinear case. Then we extend some graph theoretical results for non-negative matrices to nonlinear gain matrices. These are needed later on.

For a start, we discuss some algebraic consequences from (3.13). Recall that for a non-negative matrix Γ the following are equivalent:

(i) $\rho(\Gamma) < 1$,

(ii) $\forall s \in \mathbb{R}^n_+ \setminus \{0\} : \Gamma s \not\geq s$,

(iii) $\Gamma^k \to 0$, for $k \to \infty$,

(iv) there exist $a_1, \dots, a_n > 0$ such that $\forall s \in \mathbb{R}^n_+ \setminus \{0\}$:

$$\Gamma(I + \operatorname{diag}(a_1, \dots, a_n))s \not\geq s.$$

Note that (iv) is the linear version of (3.13). As condition (i) is not useful in the nonlinear setting, we have turned to (ii), which we later strengthened to (3.13).

In the nonlinear case we find the obvious implication:

Proposition 13. *Condition (3.13) implies that*

$$\Gamma(s) \not\geq s \quad \textit{for any} \quad s \in \mathbb{R}^n_+ \setminus \{0\}. \tag{4.46}$$

Proof. By the monotonicity of Γ it is obvious that (3.13) implies (4.46). □

Note that the contrary is not true:

Example 14. *Let*

$$\gamma_{12} = Id_{\mathbb{R}_+}$$

and

$$\gamma_{21}(r) = r(1 - e^{-r}).$$

Since already $\lim_{r\to\infty}(\gamma_{12} \circ \gamma_{21} - Id)(r) = 0$ *there are certainly no class* $\mathcal{K}_\infty$ *functions* $\tilde{\alpha}_i, i = 1, 2$ *such that (4.43) holds.*

Remark 15. *We like to point out the connections between non-negative matrices, our gain matrix and directed graphs.*

A (finite) directed graph $G = \{V, E\}$ *consists of a set* V *of vertices and a set of edges* $E \subset V \times V$. *We may identify* $V = \{1, \dots, n\}$ *in case of* n *vertices. The adjacency matrix* $A_G = (a_{ij})$ *of this graph is defined by*

$$a_{ij} = \begin{cases} 1 & \textit{if } (i,j) \in E, \\ 0 & \textit{else.} \end{cases}$$

The other way round, given an $n \times n$-matrix A, one defines the graph $G(A) = \{V, E\}$ by $V := \{1, \dots, n\}$ and $E = \{(i,j) \in V \times V : a_{ij} \neq 0\}$.

There are several concepts and results of (non-negative) matrix theory, which are of purely graph theoretical nature. Hence the same can be done for our interconnection gain matrix Γ. We may associate a graph $G(\Gamma)$, which represents the interconnections between the subsystems, in the same manner, as we would do for matrices.

We could also use the graph of the transpose of Γ here for compatibility with our previous notation (γ_{ij} encodes whether or not subsystem j influences subsystem i) and the standard notation in graph theory (edge from i to j), then the arrows in $G(\Gamma)$ would point in the 'right' direction. But this does not affect the following results.

For instance, we say Γ is irreducible*, if $G(\Gamma)$ is* strongly connected*, that is, for every pair of vertices (i,j) there exists a sequence of edges (a* path*) connecting vertex i to vertex j. Obviously Γ is irreducible if and only if Γ^T is. Γ is called* reducible *if it is not irreducible.*

The gain matrix Γ is primitive*, if its associated graph $G_\Gamma := G(\Gamma)$ is primitive, i.e., there exists a positive integer m such that $(A_{G_\Gamma})^m$ has only positive entries.*

These definitions and the following important facts can be found in [1] and only depend on the associated graph.

If Γ is reducible, then a permutation transforms it into a block upper triangular matrix. From an interconnection point of view, this splits the system into cascades of subsystems each with irreducible adjacency matrix.

Lemma 16. *Assume the gain matrix Γ is irreducible. Then there are two distinct cases:*

a) The gain matrix $\Gamma = (\gamma_{ij}(\cdot))$, where $\gamma_{ij}(\cdot) \in \mathcal{K}$ or $\gamma_{ij} = 0$, is primitive and hence there is a non-negative integer k_0 such that Γ^{k_0} has elements $\gamma_{ij}^{k_0}(\cdot) \in \mathcal{K}$ for any i, j.

b) The gain matrix Γ can be transformed to

$$P\Gamma P^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{\nu-1,\nu} \\ A_{\nu 1} & 0 & 0 & \dots & 0 \end{pmatrix} \tag{4.47}$$

using some permutation matrix P, where the zero blocks on the diagonal are square and where Γ^ν is of block diagonal form with square primitive blocks on the diagonal.

Proof. Let A_{G_Γ} be the adjacency matrix corresponding to the graph associated with Γ. This matrix is primitive if and only if Γ is primitive. Note that the $(i,j)^{\text{th}}$ entry of $A_{G_\Gamma}^k$ is zero if and only if the $(i,j)^{\text{th}}$ entry of Γ^k is zero. Multiplication of Γ by a permutation matrix only rearranges the positions of the class $\mathcal{K}$-functions, hence this

operation is well defined. From these considerations it is clear, that it is sufficient to prove the lemma for the matrix $A := A_{G_\Gamma}$. But for non-negative matrices this result is an aggregation of known facts from the theory of non-negative matrices, see, e.g., [1] or [7]. □

4.3 Asymptotic Behavior of Γ^k

A related question to the stability of the composite system (2.7) is, whether or not the discrete positive dynamical system defined by

$$s_{k+1} = \Gamma(s_k), \quad k = 1, 2, \dots \tag{4.48}$$

with given initial state $s_0 \in \mathbb{R}^n_+$ is globally asymptotically stable. Under the assumptions we made for Theorem 4 this is indeed true for irreducible Γ.

Theorem 17. *Assume that Γ is irreducible. Then the system defined by (4.48) is globally asymptotically stable if and only if $\Gamma(s) \not\geq s$ for all $s \in \mathbb{R}^n_+ \setminus \{0\}$.*

The proof will make use of the following result:

Proposition 18. *The condition*

$$\lim_{k\to\infty} \Gamma^k(s) \to 0 \quad \textit{for any fixed} \quad s \in \mathbb{R}^n_+ \tag{4.49}$$

implies (4.46). Moreover if Γ is irreducible, then both are equivalent.

Note that the converse implication is generally not true for reducible maps Γ, such that (4.46) holds. See Example 19. But it is trivially true, if Γ is linear.

Proof. Condition (4.46) follows from (4.49), since if $\Gamma(s_0) \geq s_0$ for some $s_0 \in \mathbb{R}^n_+ \setminus \{0\}$ then $\Gamma^k(s_0) \geq \Gamma^{k-1}(s_0) \geq s_0$ for $k = 2, 3, \dots$. Hence the sequence $\{\Gamma^k(s_0)\}_{k=0}^\infty$ does not converge to 0.

Conversely, assume that (4.46) holds and that Γ is irreducible.
Step 1. First we prove that for any $s \in \mathbb{R}^n_+ \setminus \{0\}$

$$\Gamma^k(s) \not\geq s, \quad k \in \mathbb{N}. \tag{4.50}$$

Assume there exist some $k > 1$ and $s \neq 0$ with $\Gamma^k(s) \geq s$. Define $z \in \mathbb{R}^n_+$ as

$$z := \max_{l=0,\dots,k-1} \{\Gamma^l(s)\} \gneq 0\,.$$

By (2.6) and using $\Gamma^k(s) \geq s$ we have

$$\Gamma(z) \geq \max_{l=1,\dots,k} \{\Gamma^l s\} = \max_{l=0,\dots,k} \{\Gamma^l s\} \geq \max_{l=0,\dots,k-1} \{\Gamma^l s\} = z.$$

This contradicts (4.46).

Step 2. For any fixed s we prove that $\limsup_{k\to\infty}|\Gamma^k(s)|<\infty$. By Lemma 16 we have two cases. We only consider case a), then case b) follows with a slight modification.

Assume that s is such that $\limsup_{k\to\infty}|\Gamma^k(s)|=\infty$. For $t_i>0$ denote the i^{th} column of Γ^{k_0} by

$$\Gamma_i^{k_0}(t_i)=\begin{pmatrix}\gamma_{1i}^{k_0}(t_i)\\ \vdots\\ \gamma_{ni}^{k_0}(t_i)\end{pmatrix}$$

As Γ^{k_0} has no zero entries, for $i=1,\dots,n$ there are $T_i\in\mathbb{R}_+$ such that

$$\Gamma_i^{k_0}(t_i)>s\quad\text{for any}\quad t_i>T_i. \tag{4.51}$$

If $|\Gamma^k(s)|\to\infty$ there exists a k_1 and an index i_1 such that

$$\Gamma^{k_1}(s)_{i_1}\geq T_{i_1}\,.$$

The vector $\Gamma^{k_0}\circ\Gamma^k(s)$, seen as a sum of columns, is greater than the maximum over these columns, i.e.,

$$\begin{aligned}\Gamma^{k_0}\circ\Gamma^{k_1}(s)\geq\max_i\Gamma_i^{k_0}(\Gamma^{k_1}(s)_i)&\geq\Gamma_{i_1}^{k_0}(\Gamma^{k_1}(s)_{i_1})\\ &\geq\Gamma_{i_1}^{k_0}(T_{i_1})\geq s.\end{aligned} \tag{4.52}$$

This contradicts Step 1.

Step 3. So $\left\{\Gamma^k(s)\right\}_{k\geq1}$ is bounded for any fixed $s\in\mathbb{R}_+^n$. The omega-limit set $\omega(s)$ is defined by

$$\begin{aligned}\omega(s)=\Big\{x\;\big|\;&\exists\text{ subsequence }\{k_j\}_{j=1,2,\dots}\\ &\text{such that }\Gamma^{k_j}(s)\xrightarrow{j\to\infty}x\Big\}.\end{aligned} \tag{4.53}$$

This set is not empty by boundedness of $\left\{\Gamma^k(s)\right\}_{k\geq1}$. The following properties follow from this definition and boundedness of the set $\left\{\Gamma^k(s)\right\}_{k\geq1}$:

$$\forall\,x\in\omega(s)\;\Rightarrow\;\Gamma(x)\in\omega(s),$$

$$\forall\,x\in\omega(s)\;\exists\,y\in\omega(s):\;\Gamma(y)=x.$$

i.e., $\omega(s)$ is invariant under Γ. The boundedness of $\omega(s)$ allows to define a finite vector

$$z=\sup\omega(s).$$

Then for any $x\in\omega(s)$ it follows $z\geq x$ and hence $\Gamma(z)\geq\Gamma(x)$. Let $y\in\omega(s)$ be such that $\Gamma(x)=y$. Then $\Gamma(z)\geq y$. By the invariance of $\omega(s)$ it follows that

$$\Gamma(z)\geq\sup\{\Gamma(x)\,|\,\forall\,x\in\omega(s)\}=z.$$

This contradicts (4.46) if $z\neq0$, i.e., $\omega(s)=\{0\}$. This is true for any $s\in\mathbb{R}_+^n$. Hence (4.49) is proved as a consequence of (4.46), provided that Γ is irreducible. □

Example 19. *Consider the map* $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ *defined by*

$$\Gamma := \begin{bmatrix} \gamma_{11} & id \\ 0 & \gamma_{22} \end{bmatrix}$$

where for $t \in \mathbb{R}_+$

$$\gamma_{11}(t) := t(1 - e^{-t})$$

and the function γ_{22} *is constructed in the sequel. First note that* $\gamma_{11} \in \mathcal{K}_\infty$ *and* $\gamma_{11}(t) < t$, $\forall t > 0$. *Let* $\{\varepsilon_k\}_{k=1}^\infty$ *a strictly decreasing sequence of positive real numbers, such that* $\lim_{k\to\infty} \varepsilon_k = 0$ *and* $\lim_{K\to\infty} \sum_{k=1}^K \varepsilon_k = \infty$. *For* $k = 1, 2, \ldots$ *define*

$$\gamma_{22}\left(\varepsilon_k + (1 + \sum_{j=1}^{k-1} \varepsilon_j) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)}\right) := \varepsilon_{k+1} + (1 + \sum_{j=1}^{k} \varepsilon_j) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)}$$

and observe that

$$\varepsilon_k + (1 + \sum_{j=1}^{k-1} \varepsilon_j) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)} > \varepsilon_{k+1} + (1 + \sum_{j=1}^{k} \varepsilon_j) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)},$$

since $\varepsilon_k > \varepsilon_{k+1}$ *for all* $k = 1, 2, \ldots$ *and the map* $t \mapsto t \cdot e^{-t}$ *is strictly decreasing on* $(1, \infty)$.

Moreover we have by assumption, that

$$\varepsilon_k + (1 + \sum_{j=1}^{k-1} \varepsilon_j) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)} \xrightarrow[k\to\infty]{} 0.$$

These facts together imply that γ_{22} *may be extrapolated to some* $\mathcal{K}_\infty$*-function, in a way such that* $\gamma_{22}(t) < t$, $\forall t > 0$ *holds.*

Note that by our particular construction we have $\Gamma(s) \not\geq s$ *for all* $s \in \mathbb{R}^2_+ \setminus \{0\}$. *Now define* $s^1 \in \mathbb{R}^2_+$ *by*

$$s^1 := \begin{bmatrix} 1 \\ 1 + e^{-1} \end{bmatrix}$$

and for $k = 1, 2, \ldots$ *inductively define* $s^{k+1} := \Gamma(s^k) \in \mathbb{R}^2_+$.

By induction one verifies that

$$s^{k+1} = \Gamma^k(s^1) = \begin{bmatrix} 1 + \sum_{j=1}^k \varepsilon_j \\ \varepsilon_{k+1} + (1 + \sum_{j=1}^k \varepsilon_j) e^{-(1+\sum_{j=1}^k \varepsilon_j)} \end{bmatrix}.$$

By our previous considerations and assumptions we easily obtain that the second component of the sequence $\{s^k\}_{k=1}^\infty$ *strictly decreases and converges to zero as* k *tends to infinity. But at the same time the first component strictly increases above any given bound.*

Hence we established that $\Gamma(s) \not\geq s$ $\forall s \neq 0$ *in general does not imply* $\forall s \neq 0 : \Gamma^k(s) \to 0$ *as* $k \to \infty$.

Remark 20. *Note that we can even turn the constructed 2x2-*Γ *into the null-diagonal form, that is assumed in Theorem 4. Using the same notation for* γ_{ij} *as in Example 19, we just define*

$$\Gamma := \begin{bmatrix} 0 & \gamma_{11} & id & 0 \\ \gamma_{11} & 0 & 0 & id \\ 0 & 0 & 0 & \gamma_{22} \\ 0 & 0 & \gamma_{22} & 0 \end{bmatrix} \qquad \textit{and} \qquad s^1 := \begin{bmatrix} 1 \\ 1 \\ 1+e^{-1} \\ 1+e^{-1} \end{bmatrix}$$

and easily verify that $\Gamma^k(s^1)$ *does not converge to* 0.

of Theorem 17. If (4.48) is asymptotically stable, it is in particular attracted to zero, so by Proposition 18 and irreducibility of Γ we establish (4.46).

Conversely assume (4.46). Clearly $0 \in \mathbb{R}^n_+$ is an equilibrium point for (4.48) and by Proposition 18 it is globally attractive. It remains to prove stability, i.e., for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|s_0| < \delta$ implies $\Gamma^k(s_0) < \varepsilon$ for all times $k = 0, 1, 2, \ldots$

Given $\varepsilon > 0$ we can choose an $r \in \bigcap_{i=1}^n \Omega_i \cap S_\varepsilon$ where S_ε is the sphere around 0 of radius ε in $\mathbb{R}^n_+$. Define δ by

$$\delta := \sup\{d \in \mathbb{R}_+ : s < r \ \forall s \in B_d(0)\}.$$

Here $B_d(0)$ denotes the ball of radius less than d in $\mathbb{R}^n_+$ around the origin with respect to the Euclidean norm. Clearly we have $r > s_0$ for all $|s_0| < \delta$. Since $r \in \bigcap_{i=1}^n \Omega_i \neq \emptyset$ we have $r > \Gamma(r)$ and therefore $r > \Gamma(r) \geq \Gamma^2(r) \geq \ldots$ and even $\Gamma^k(r) \xrightarrow{k\to\infty} 0$ again by Proposition 18.

Hence for any s_0 such that $|s_0| < \delta$ we have $\Gamma^k(r) \geq \Gamma^k(s_0)$ for all $k = 0, 1, 2, \ldots$ by monotonicity of Γ and therefrom $\Gamma^k(s_0) < \varepsilon$ for all $k = 0, 1, 2, \ldots$ □

4.4 Geometrical Interpretation

For the following statement let us define the open domains

$$\Omega_i = \left\{ x \in \mathbb{R}^N : \ |x_i| > \sum_{j \neq i} \gamma_{ij}(|x_j|) \right\},$$

where $N = \sum_{j=1}^n N_j$ and x is partitioned to $(x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}^{N_i}, i = 1, \ldots, n$, as in (2.3).

Proposition 21. *Condition* (4.46) *is equivalent to*

$$\bigcup_{i=1}^n \Omega_i = \mathbb{R}^N \setminus \{0\} \quad \textit{and} \quad \bigcap_{i=1}^n \Omega_i \neq \emptyset. \tag{4.54}$$

Proof. Let $s \neq 0$. Formula (4.46) is equivalent to the existence of at least one index $i \in \{1, \ldots, n\}$ with $s_i > \sum_{j \neq i} \gamma_{ij}(s_j)$. This proves the first part of (4.54).

It remains to show, that (4.46) implies $\bigcap_{i=1}^n \Omega_i \neq \emptyset$. We may restrict ourselves to the positive orthant in $\mathbb{R}^n$, and the sets

$$\tilde{\Omega}_i = \left\{ s \in \mathbb{R}^n_+ : \ s_i > \sum_{j \neq i} \gamma_{ij}(s_j) \right\}$$

instead of $\Omega_i, \ i = 1, \ldots, n$.

For an index set I we define $E_I = \{s \in \mathbb{R}^n_+ : s_m = 0 \quad \text{for} \quad m \notin I\}$. Note that points of E_I can not be in $\tilde{\Omega}_m$ for $m \notin I$. Consider $\tilde{\Omega}_i$ and $\tilde{\Omega}_j$ for any $i \neq j$. The intersections $\tilde{\Omega}_i \cap E_{\{i,j\}}$ and $\tilde{\Omega}_j \cap E_{\{i,j\}}$ of this two domains with the plane $E_{\{i,j\}}$ are nonempty. The points of $\partial\tilde{\Omega}_i$ lying in this plane do not belong to $\tilde{\Omega}_k$ for any $k \neq j$, hence they are in $\tilde{\Omega}_j$. Since the domains are open it follows that the intersections $\tilde{\Omega}_i \cap \tilde{\Omega}_j \neq \emptyset$ for any $i \neq j$. Denote $\tilde{\Omega}_{ij} = \tilde{\Omega}_i \cap \tilde{\Omega}_j$ which has nonempty intersection with $E_{\{i,j\}}$ by construction. Take any $k \neq i, j$. Consider $E_{\{i,j,k\}} \supset E_{\{i,j\}}$ which has non-empty intersection with $\tilde{\Omega}_{ij}$. Let $x \in \tilde{\Omega}_{ij} \cap E_{\{i,j,k\}}$. There is some $y \in E_{\{i,j,k\}}$ with $y \notin \Omega_{ij}$ (say $y \in E_{\{k\}}$). Since $E_{\{i,j,k\}}$ is convex the segment $\overline{xy} \subset E_{\{i,j,k\}}$, hence there is some point $z \in E_{\{i,j,k\}}$ belonging to $\partial\tilde{\Omega}_{ij}$, i.e., $E_{\{i,j,k\}} \cap \partial\tilde{\Omega}_{ij}$ is non-empty.

The points of $\partial\tilde{\Omega}_{ij}$, which are not in $\tilde{\Omega}_i$, $\tilde{\Omega}_j$ and lying in $E_{\{i,j,k\}}$ can not belong to $\tilde{\Omega}_\nu, \nu \neq k$. Hence they are in $\tilde{\Omega}_k$ and it follows $\tilde{\Omega}_i \cap \tilde{\Omega}_j \cap \tilde{\Omega}_k \neq \emptyset$. By iteration the second part of (4.54) follows. □

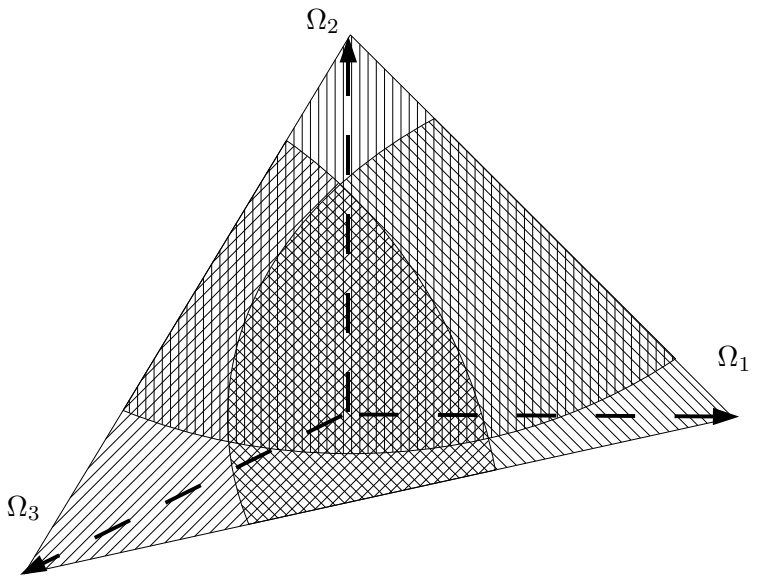

Figure 2: Overlapping of Ω_i domains in $\mathbb{R}^3$

Let us briefly explain, why the overlapping condition (4.54) is interesting: From the theory of ISS-Lyapunov functions it is known, that a system of the form (2.9) is

ISS if and only if there exists a smooth Lyapunov function V with the property

$$|x| \geq \gamma(|u|) \Rightarrow \nabla V(x) f(x,u) < -W(|x|)\,,$$

for some $W \in \mathcal{K}$. In the case of our interconnected system this condition translates to the existence of Lyapunov functions V_i for the subsystems $i = 1, \ldots, n$ with the property

$$\begin{aligned} |x_i| &\geq \sum \gamma_{ij}(|x_j|) + \gamma(|u|) \\ \Rightarrow \nabla V_i(x_i) f_i(x,u) &< -W_i(|x_i|)\,, \end{aligned} \tag{4.55}$$

Now for $u = 0$ the condition of (4.55) is simply, that $x \in \Omega_i$. Thus the overlapping condition states that in each point of the state space one of the Lyapunov functions of the subsystems is decreasing. It is an interesting problem if via this an ISS-Lyapunov function for the whole system may be constructed.

A typical situation in case of three one dimensional systems ($\mathbb{R}^3$) is presented on the Figure 2 on a plane crossing the positive semi axis. The three sectors are the intersections of the Ω_i with this plane.

4.5 Summary map of the interpretations concerning Γ

In Figure 3 we summarize the relations between various statements about Γ that were proved in section 4.

$$\begin{array}{ccccc} & & \exists D \text{ as in (3.12)} : \Gamma \circ D(s) \not\geq s & & \\ & & \Downarrow \ (\Uparrow \text{ if } \Gamma \text{ is linear}) & & \\ \Gamma^k(s) \xrightarrow{k\to\infty} 0 & \begin{array}{c}\Rightarrow \\ \Leftarrow^*\end{array} & \Gamma(s) \not\geq s & \iff & \bigcup_{i=1}^n \Omega_i = \mathbb{R}^N \setminus \{0\} \\ & & \Updownarrow \text{ if } \Gamma \text{ is linear} & & \\ & & \rho(\Gamma) < 1 & & \end{array}$$

Figure 3: Some implications and equivalences of the generalized small-gain condition. All statements are supposed to hold for all $s \in \mathbb{R}^n_+$, $s \neq 0$. The implication denoted by * holds if Γ is linear or irreducible.

5 Application to linear systems

An important special case is, of course, when the underlying systems are linear themselves. Consider the following setup, where in the sequel we omit the external input, formerly denoted by u, for notational simplicity. Let

$$\dot{x}_j = A_j x_j, \quad x_j \in \mathbb{R}^{N_j}, \quad j = 1, \ldots, n \tag{5.56}$$

describe n globally asymptotically stable linear systems, which are interconnected by the formula

$$\dot{x}_j = A_j x_j + \sum_{k=1}^{n} \Delta_{jk} x_k \quad j = 1, \ldots, n, \tag{5.57}$$

which can be rewritten as

$$\dot{x} = (A + \Delta)x, \tag{5.58}$$

where A is block diagonal, $A = \mathrm{diag}(A_j, j = 1, \ldots, n)$, each A_j is Hurwitz (i.e., the spectrum of A_j is contained in the open left half plane) and the matrix $\Delta = (\Delta_{jk})$ is also in block form and encodes the connections between the n subsystems. We suppose that $\Delta_{jj} = 0$ for all j. Define the matrix $R = (r_{jk}), R \in \mathbb{R}_+^{n \times n}$, by $r_{jk} := ||\Delta_{jk}||$. For each subsystem, there exist positive constants M_j, λ_j, such that $e^{A_j t} \leq M_j e^{-\lambda_j t}$ for all $t \geq 0$.

Define a matrix $D \in \mathbb{R}_+^{n \times n}$ by $D := \mathrm{diag}(\frac{M_j}{\lambda_j}, j = 1, \ldots, n)$.
From the last subsection we obtain

Corollary 22. *If $\rho(D \cdot R) < 1$ then (5.58) is globally asymptotically stable.*

Note that this is a special case of a theorem, which can be found in Vidyasagar [17, p. 110], see Remark 9.

Proof. Denote the initial value by x^0. Then by elementary ODE theory we have

$$x_j(t) = e^{A_j t} x_j^0 + \sum_{k \neq j} \int_0^t e^{A_j (t-s)} \Delta_{jk} x_k(s) ds \tag{5.59}$$

and by standard estimates

$$|x_j(t)| \leq M_j e^{-\lambda_j t} + \sum_{k \neq j} r_{jk} \frac{M_k}{\lambda_k} ||x_{k,[0,t]}||. \tag{5.60}$$

As one can see from (5.60), in this case the gain matrix happens to be $\Gamma = D \cdot R$. □

It is noteworthy, that this particular corollary is also a consequence of more general and precise results of a recent paper [6] by Hinrichsen, Karow and Pritchard.

6 Conclusions

We considered a composite system consisting of an arbitrary number of nonlinear arbitrarily interconnected subsystems, as they arise in applications.

For this general case we derived a multisystem version of the *nonlinear small-gain theorem*. For the special case of linear interconnection gains this is a special case of a known Theorem, cf. [17, page 110]. We also showed how our generalized small-gain theorem for networks can be applied to linear systems.

Many interesting questions remain, for instance concerning the construction of Lyapunov functions in case of nonlinear gain functions.

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