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Abstract

Numerous problems from natural sciences and engineering as well as from other disciplines lead to optimal control problems governed by systems of time-dependent partial differential equations (PDE). Control functions have to be determined such that a given performance index is optimized subject to additional constraints. These optimal control problems are solved numerically by time–consuming methods. Moreover, if the optimal controls are to be pursued in a real process, the possibility of data disturbances force recomputing the optimal controls in real–time to preserve constraints and optimality, at least approximately. For this purpose, a numerical method based on the parametric sensitivity analysis of nonlinear optimization problems is suggested to calculate higher order approximations for the optimal solution of the perturbed optimal control problems in real-time without solving the PDE system explicitely. By this method computing times can be reduced to a few nanoseconds on a typical one processor personal computer. The method is illustrated by the real–time optimal control of the nonlinear Burgers equation.

Keywords: perturbed optimal control problems; nonlinear programming methods; partial differential equations; parametric sensitivity analysis; real–time control

1 Introduction

This paper is concerned with the real–time approximation of perturbed optimal control problems governed by time-dependent partial differential equations and subject to control constraints. In practice, control problems are often subject to disturbances or perturbations in the system data. In mathematical terms, perturbations are expressed by means of parameter fluctuations that enter the dynamics, boundary conditions or control constraints. *Stability* and *sensitivity analysis* are concerned with the behavior of optimal solutions under parameter perturbations. The so-called parametric sensitivity derivatives are a helpful tool in real-time optimal control applications, see e.g. Pesch [19] and Büskens and Maurer [6]–[9] for optimal control problems governed by ordinary differential equations.

Although it is well–known that there exists a gap between the computational and theoretical aspects of optimal control problems with PDEs, the need on numerical methods capable to solve these problems is given today. Naturally this difficulty goes over to the computational and theoretical aspects of parametric sensitivity analysis of optimal control problems with PDEs, if numerical solution techniques are used. Numerical aspects of solving optimal control problems subject to partial differential equations by e.g. SQP methods are commonly discussed, e.g. by Heinkenschloss and Sachs [14], Casas, Tröltzsch and Unger [10, 11], Kunisch and Volkwein [15], Maurer and Mittelmann [17, 18] to name only some references.

In this paper the method of lines is used to discretize the partial differential equations, transforming the original system into a system of ordinary differential equations. To compute sensitivity differentials via nonlinear optimization methods a subsequent discretization of the ODE control problem is accomplished. This leads to so-called *direct optimization methods* which have proved

to be powerful tools for solving ODE optimal control problems; cf., e.g., Büskens [3], Büskens and Maurer [6]–[9] and the references cited therein. These methods use only control and state variables as optimization variables and completely dispense with adjoint variables. Alt [1], Dontchev and Hager [12] and Malanowski, Büskens and Maurer [16] prove the convergence of the discretized problem to the continuous solution for ODE optimal control problems. For general optimal control problems involving partial differential equations, these results do not yet exist. Hence, we tacitly assume the convergence to the presumed unique continuous solution as the mesh size tends to zero.

In Büskens and Griesse [5] the *numerical computation* of sensitivity differentials for optimal control problems for PDEs is discussed. The proposed NLP-based method is capable of computing approximations to sensitivity differentials for the state, control and adjoint variables in parametric PDE optimal control problems. These sensitivity differentials allow an easy access to real–time optimal control approximations of perturbed problems. In case of deviations in the perturbation parameters the method presented in this paper is able to calculate higher order real–time approximations of the perturbed PDE optimal control problem without solving the PDE system again. To the authors knowledge this is this first time that approximations of perturbed PDE optimal control problems can be calculated in real–time.

The general mathematical structure of perturbed PDE optimal control problems is outlined in Section 2. Discretization details for the PDE can be found in Section 3. A short overview of basic results on sensitivity analysis and solution differentiability for perturbed finite-dimensional nonlinear optimization problems is offered in Section 4. By re-transforming the numerical solution and sensitivity quantities, information about the optimal solution and the sensitivity of the original PDE optimal control problem can be obtained. This will be covered in Section 5. Finally, Section 6 presents three numerical examples for PDE optimal control problems with control constraints.

2 Parametric PDE optimal control problems

We consider the following perturbed generally coupled time-dependent PDE optimal control problem **POCP(p)** with control constraints:

$$
\begin{array}{rcll}
\text{Minimize} & & & \\
F(y,u,p) & = & \displaystyle\int_\Gamma f_1(y(x,t),u(x,t),x,t,p)\,dx\,dt & \\
 & & \displaystyle+\int_\Omega f_2(y(x,t),u(x,t),x,t,p)\,dx\,dt & \\
\text{subject to} & & & \\
y_t(x,t) & = & f(y(x,t),y_x(x,t),y_{xx}(x,t),u(x,t),x,t,p), & \\
y(x,t_0) & = & y_0(x,p), & \text{for} \quad x\in\Omega_x \\
y(x,t) & = & y_D(u(x,t),x,t,p), & \text{for} \quad x\in\Gamma_x,\ t\in\Omega_t \\
\partial_\nu y(x,t) & = & y_N(y(x,t),u(x,t),x,t,p), & \text{for} \quad x\in\Gamma_x,\ t\in\Omega_t \\
0 & \geq & C(u(x,t),x,t,p), & \text{for} \quad (x,t)\in\Omega.
\end{array}
\tag{1}
$$

Herein $t \in [t_0,t_f] = \Omega_t \subset \mathbb{R}$ denotes the *time variable*, while $x \in \Omega_x \subset \mathbb{R}^{n_x}$ of dimension n_x denotes the *spatial variables*. Further, Ω_x is a bounded domain with piecewise smooth boundary $\Gamma_x = \partial\Omega_x$. Herewith $\Omega = \Omega_x \times \Omega_t$ and $\Gamma = \Gamma_x \times \Omega_t$ are defined. Moreover, let $y : \Omega \to \mathbb{R}^{n_y}$ be a vector function of dimension n_y of which $y_t(x,t)$ denotes the first derivative w.r.t. the time variable t. Likewise the first and second partial derivatives w.r.t. the spatial variables x are denoted by $y_x(x,t)$ and $y_{xx}(x,t)$. The control function $u : \Omega \to \mathbb{R}^{n_u}$ has components defined either on Ω *(distributed control)* or on the boundary Γ *(boundary control)*. *Perturbations*, which may appear in all functions of (1), are characterized by a parameter vector $p \in \Omega_p \subset \mathbb{R}^{n_p}$. A solution of the PDE system in (1) depends on the spatial variable x, the time variable t, the control function u and the fixed parameter vector p. While we mainly have a parabolic PDE system in mind, hyperbolic systems can be considered, too. Always providing that the PDE system in (1) is well-defined and uniquely solvable for given u and p, combinations of Dirichlet or Neumann conditions are defined by the functions $y_D : \mathbb{R}^{n_u} \times \Gamma_x \times \Omega_t \times \Omega_p \to \mathbb{R}^{n_y}$ or $y_N : \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \Gamma_x \times \Omega_t \times \Omega_p \to \mathbb{R}^{n_y}$. Herein derivatives in the direction of the outward unit normal ν of Γ_x are denoted by ∂_ν in (1). The vector function $C : \times\mathbb{R}^{n_u} \times \Omega_x \times \Omega_t \times \Omega_p \to \mathbb{R}^{n_c}$, $n_c \geq 0$, allows for additional inequality control constraints. We point out that in principle state constraints can be taken into account, too.

Nevertheless they might be to time consuming in view of the real–time approximations discussed later.

The problem is to determine a control vector — containing boundary or distributed control elements or both — that minimizes the functional F in (1) subject to the given restrictions. Note that (1) can be extended to contain more general terms, for example higher order derivatives or time–spatial derivatives.

3 Discretization of the PDE control problem

The main idea of discretization is to transform the PDE optimal control problem (1) into a finite dimensional nonlinear optimization problem (NLP). On the other hand for an efficient evaluation of the higher order real–time approximations discussed later on it is stringently necessary to find a discretization where the constraints are independent on the state $y(x,t)$ itself. To fulfill these requirements we proceed as follows:

First the method of lines is used to transform the partial differential equation into a system of ordinary differential equations by discretizing all functions with respect to the spatial variable x. Possible choices for the spatial discretization are finite differences, finite elements or others. The method of lines transforms the partial differential equation into a system of ordinary differential equations. Since the method is well known we will not discuss it in detail and tacitly assume that there exist a transcription to the perturbed optimal ODE control problem **(OCP(p))**:

$$\begin{array}{lrcl} \text{Minimize} & F(w,v,p) = \displaystyle\int_{t_0}^{t_f} f_0(w(t),v(t),t,p)\ dt & & \\ \text{subject to} & \dot{w}(t) & = & \tilde{f}(w(t),v(t),t,p), \\ & w(t_0) & = & w_0(p), \\ & \tilde{C}(v(t),t,p) & \leq & 0, \quad t \in [t_0,t_f]. \end{array} \tag{2}$$

We assumed, that the Dirichlet or Neumann conditions can be substituted directly into the ODE system in (2). Here, $w(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$ denote the state of the system and the control in a given time interval $[t_0,t_f] = \Omega_t$, respectively. Data perturbations are again modeled by a parameter $p \in \Omega_p$. The functions $f_0 : \mathbb{R}^{n+m+1} \times \Omega_p \to \mathbb{R}$, $\tilde{f} : \mathbb{R}^{n+m+1} \times \Omega_p \to \mathbb{R}^n$, $w_0 : \Omega_p \to \mathbb{R}^n$, and $\tilde{C} : \mathbb{R}^{n+m+1} \times \Omega_p \to \mathbb{R}^k$ are assumed to be sufficiently smooth on appropriate open sets. The admissible class of control functions is the class of piecewise continuous controls. The final time t_f can be either fixed or free. Note that the formulation in (2) contains only *pure control constraints* and that the former state variable $y(x,t)$ can be identified by the components of $w(t)$ depending on the discretization accomplished with the method of lines. Likewise the former control variable $u(x,t)$ can be identified by the components of $v(t)$.

As a second step a suitable *discretization* of the the control problem (2) is used by which it is transformed into a nonlinear optimization problem (NLP). These techniques are well developed and there exist a number of excellent methods. In principal these methods can be divided into two classes. The first type of methods is characterized by the fact that both the discretized state *and* control variables are taken as optimization variables. Unfortunately this approach leads to a high number of state depending constraints and hence are not suitable for the higher order real–time approximations presented in this paper. For the second class of NLP methods, only the discretized control variables are considered as optimization variables whereas the state variables are calculated as functions of the control variables using appropriate numerical integration methods. One obtains a NLP problem where the constraints are independent on the state of the system. We use one variation of the code NUDOCCCS of Büskens [2, 3] and reflect the main idea for the simple Euler method subsequently. Without loss of generality let $t_0 = 0$. For notational simplicity we choose equidistant mesh points $t_i := (i-1)h$, $i = 1,\ldots,N_t$, $h := \frac{t_f}{N_t - 1}$. Let $u^i \in \mathbb{R}^m$ denote approximations for $v(t_i)$. Then for given $z := (v^1,\ldots,v^{N_t}) \in \mathbb{R}^{mN_t}$ state approximations $w^i \in \mathbb{R}^n$ of the values $w(t_i)$ can be archived recursively as functions of the control variables:

$$\begin{array}{rcl} w^1(z,p) & := & w_0(p), \\ w^{i+1}(z,p) & := & w^i(z,p) + h\tilde{f}(w^i(z,p),v^i,t_i,p), \quad i = 1,\ldots,N_t - 1. \end{array} \tag{3}$$

Hereby, the control problem (2) is replaced by:

$$\begin{array}{ll} \min\limits_{z} & h \cdot \sum\limits_{i=0}^{N-1} f_0(w^i(z,p), v^i, t_i, p) \\ \text{subject to} & \tilde{C}(v^i, t_i, p) \quad \leq \quad 0, \quad i = 1, \ldots, N_t. \end{array} \tag{4}$$

Note that a free final time t_f can be handled as an additional variable in z and that due to the chosen discretization the state $w(z,p)$ appears in the objective of (4) but not in the constraints. Hence the numerical calculation of the objective is expensive due to the implicitly considered PDE problem while the calculation of the constraints is not. Problem (4) defines a perturbed NLP problem $\mathbf{NLP_1(p)}$ of form

$$\begin{array}{ll} \min\limits_{z} & H(z,p), \\ \text{subject to} & G_i(z,p) \leq 0, \quad i = 1, \ldots, N_c. \end{array} \tag{5}$$

which can be solved by standard techniques, e.g. SQP methods, if we use suitable definitions for N_c and the functions H and G_i.
All calculations described hereafter were performed by the code NUDOCCCS of Büskens [2, 3] in which also various higher order approximations of the state and control variables are implemented. The treatment of stiff ODEs, grid refinement techniques and a numerical check of second order sufficient optimality conditions can also be found in this code, see [3]. Recently, the convergence of solutions discretized via Euler's method to solutions of the continuous control problem has been proved in Malanowski, Büskens and Maurer [16].
By solving the NLP problem (5) we obtain an estimate of the *continuous* control and state variables (u, y) of (1) at appropriate $x_l \in \Omega_x$, $t_j \in \Omega_t$ depending on the applied discretization. Likewise, all other variables and functions of the continuous problem (1) can be determined approximately.

4 Parametric sensitivity analysis of perturbed NLP problems

In Section 3 a method to transform a perturbed control problem into a parametric NLP problem has been discussed. It should be mentioned that the results hereafter do not depend on the discretization technique used. After solving (5) we know the set and the number N_a of active constraints, i.e. those constraints in (5) with $G_i(t_i; z) = 0$. Let $G^a = (G^a_1, \ldots, G^a_{N_a})^T$ denote the collection of these active constraints. Then the solution of (5) is the same as the solution of $\mathbf{NLP_2(p)}$

$$\begin{array}{ll} \min\limits_{z} & H(z,p) \\ \text{subject to} & G^a(z,p) = 0, \end{array} \tag{6}$$

since inactive constraints have no impact on the optimal solution. For this, we restrict the subsequent discussion to the formulation (6). The Lagrangian for (6) is defined as

$$L(z, \eta, p) = H(z,p) + \eta^\top G^a(z,p)$$

with Lagrange multiplier $\eta = (\eta_1, \ldots, \eta_{N_a})^\top$. The following theorem states sufficient conditions for the differentiability of an optimal solution $z(p)$ w.r.t. p.

Theorem 1: *Let H and G^a be twice continuously differentiable w.r.t. z and p. Let z_0 be a strong regular local solution of (6) for a fixed parameter p_0 with Lagrange multiplier η_0, i.e. $G^a(z_0, p_0) = 0$ and*

- *z_0 is regular in the sense $rg\,(\nabla_z G^a(z_0, p_0)) = N_a$, i.e., the gradients $\nabla_z G^a_i(z_0, p_0)$ are linearly independent,*
- *the first order necessary optimality conditions hold at z_0, i.e., $\nabla_z L(z_0, \eta_0, p_0) = 0, \eta_0^\top G^a(z_0, p_0) = 0$,*
- *the strict complementary condition holds at z_0, i.e., $(\eta_0)_i > 0$ for $i = 1, \ldots, N_a$,*

- *the second order sufficient conditions hold at z_0, i.e., $v^\top \nabla^2_{zz} L(z_0, \eta_0, p_0) v > 0$, $\forall v \in ker(\nabla_z G^a(z_0, p_0)(z_0, p_0))$, $v \neq 0$.*

Then there exists a neighborhood $\mathcal{P}(p_0)$ such that (6) possesses a unique strong regular local solution $z(p)$ and $\eta(p)$ for all $p \in \mathcal{P}(p_0)$. Furthermore, $z(p)$ and $\eta(p)$ are continuously differentiable functions of p in $\mathcal{P}(p_0)$ and it holds

$$\begin{pmatrix} \nabla^2_{zz} L(z_0, \eta_0, p_0) & \nabla_z G^a(z_0, p_0)^\top \\ \nabla_z G^a(z_0, p_0)(z_0, p_0) & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{dz}{dp}(p_0) \\ \frac{d\eta}{dp}(p_0) \end{pmatrix} = - \begin{pmatrix} \nabla^2_{zp} L(z_0, \eta_0, p_0) \\ \nabla_p G^a(z_0, p_0) \end{pmatrix}. \tag{7}$$

Notice, that the left matrix in (7) is non-singular under the assumptions of Theorem 1. Hence, (7) points out a way to compute the sensitivity differentials dz/dp and $d\eta/dp$ at p_0 explicitly by solving the linear equation system. The proof of the theorem is based on the implicit function theorem and can be found in Fiacco [13] or Büskens [3]. The assumptions in Theorem 1 can be checked numerically by use of the projected or reduced Hessian, compare Büskens and Maurer [8] or Büskens [3]. In the following section these results are applied to real–time approximations.

5 Higher Order Admissible Real-Time Approximations

In the preceeding sections methods were devoted to calculate the nominal solution and the corresponding sensitivity differentials. In case of a deviation Δp in the parameter p a first order Taylor approximation for $z(p_0 + \Delta p)$ is given by

$$z(p) := z(p_0 + \Delta p) \approx \tilde{z}(p) := z(p_0) + \frac{dz}{dp}(p_0)\Delta p. \tag{8}$$

Since the quantities $z(p_0)$ and $\frac{dz}{dp}(p_0)$ in (8) can be computed *offline*, the benefit of (8) is that only a matrix-vector multiplication and a vector-vector addition has to be performed *online* to approximate $z(p_0 + \Delta p)$ extremely fast. Note that the calculation of $\tilde{z}(p)$ in (8) is independent on the PDE system, hence (8) is particularly suitable for time critical processes. We use (8) as a first *real-time approximation*. It holds, cf. Büskens [4]:

Theorem 2: *Let the assumptions of Theorem 1 hold and let the functions H and G^a in (6) be three times continuously differentiable w.r.t. to z and p. Then there exists a neighborhood $U(p_0)$ of p_0 with*

$$\begin{aligned} \|z(p) - \tilde{z}(p)\| &= \mathcal{O}(\|\Delta p\|^2), & (9)\\ \|H(z(p), p) - H(\tilde{z}(p), p)\| &= \mathcal{O}(\|\Delta p\|^2), & (10)\\ \|G^a(\tilde{z}(p), p)\| &= \mathcal{O}(\|\Delta p\|^2). & (11) \end{aligned}$$

In the unconstraint case, i.e., $N_a = 0$, we have

$$\|H(z(p), p) - H(\tilde{z}(p), p)\| = \mathcal{O}(\|\Delta p\|^3). \tag{12}$$

According to Theorem 2, Formula (8) yields, in view of optimality and admissibility, real–time approximations for small perturbations Δp for many problems. Nevertheless, the approximation $\tilde{z}(p)$ is generally not admissible, i.e., $\|G^a(\tilde{z}(p), p)\| = \mathcal{O}(\|\Delta p\|^2)$.

In particular, for larger perturbations Δp approximation $\tilde{z}(p)$ may be not acceptable in view of admissibility. To overcome this problem, we introduce an artifical perturbation q. Instead of (6) we treat the problem $\mathbf{NLP_3(p, q)}$

$$\begin{array}{ll} \min\limits_{z} & H(z, p) \\ \text{subject to} & G^a(z, p) - q = 0. \end{array} \tag{13}$$

Obviously problem $\mathbf{NLP_2(p)}$ is equivalent to $\mathbf{NLP_3(p, q_0)}$ if the nominal perturbation is chosen to $q = q_0 = 0$. Moreover problem (13) fulfills the assumptions of Theorem 1, if (6) does, and hence we are able to calculate the sensitivities $\frac{dz}{dq}(0)$ and $\frac{d\eta}{dq}(0)$ similar to (7). Herewith we are able to formulate the following corrector iteration method to achieve admissibility for the constraints without loss of its optimality:

(i) Choose $\varepsilon^\infty > 0$ and initialize $\tilde{z}^{[1]}(p) := \tilde{z}(p)$. Set $k := 1$.

(ii) If $\|G^a(\tilde{z}^{[k]}(p), p)\| < \varepsilon^\infty$, then STOP.

(iii) Calculate

$$\tilde{z}^{[k+1]}(p) := \tilde{z}^{[k]} - \frac{dz}{dq}(0) G^a(\tilde{z}^{[k]}(p), p), \tag{14}$$

and set $k := k + 1$.

(iv) Go to (ii).

Iteration (14) achieves admissibility in the active constraints and additionally improves the optimality as shown by the following theorem, cf. Büskens [4].

Theorem 3: *Let the assumptions of Theorem 1 hold and let the functions H and G^a in (6) be three times continuously differentiable w.r.t. to z and p. Then there exists a neighborhood $U(p_0)$ of p_0 and a vector $v \in \mathbb{R}^{N_z}$ with $v \in ker(\nabla_z G^a(z_0, p_0))$ and $\|v\| = \mathcal{O}(\|\Delta p\|^2)$ such that for all $p \in U(p_0)$ the sequence $\tilde{z}^{[k]}(p)$ in (14) converges to a fixed point $\tilde{z}^{[\infty]}(p)$ with*

$$\begin{aligned} \|z(p) - \tilde{z}^{[\infty]}(p)\| &= \|v\| + \mathcal{O}(\|\Delta p\|^3), & (15)\\ \|H(z(p), p) - H(\tilde{z}^{[\infty]}(p), p)\| &= \mathcal{O}(\|\Delta p\|^3), & (16)\\ \|G^a(\tilde{z}^{[\infty]}(p), p)\| &= 0. & (17) \end{aligned}$$

Iteration (14) can be interpreted as follows: Approximation $\tilde{z}^{[k]}(p)$ causes a deviation in G^a which is identified as the perturbation q. For this perturbation an additional correction step of type (8) is performed. Of course, the new approximation will again cause a deviation in G^a and the corrector iteration method is born.
Note, that the fixed point in Theorem 3 is not unique, cf. Büskens [4]. Nevertheless, any fixed point of iteration (14) fulfills (15)–(17), especially the higher order of optimality in the objective. Now it becomes evident why the discretizations in (2)–(5) were chosen in such manner that the constraints are independent on the state: Iteration (14) needs no objective calculation and hence can be performed without solving the expensive underlying PDE system. Only the control constraints have to be computed, which is generally cheap.

6 Example: The Burgers Equation

In order to illustrate the performance of the algorithms presented in Sections 2–5 the nonlinear one-dimensional viscous Burgers equation is investigated subject to a tracking-type PDE optimal control problem with control constraints, cf. Volkwein [20]. All computations were performed on a 1GHz PIII personal computer using the code NUDOCCCS.

$$\begin{aligned} &\text{Minimize}\\ F(y,u,p) &= \frac{\alpha}{2}\left(\int_\Omega [y(x,t) - 0.035]^2\,dx\,dt + (1+p_2)\sigma \int_{t_0}^{t_f} [u_1(t)^2 + u_2(t)^2]\,dt\right),\\ &\text{subject to}\\ y_t(x,t) &= (1+p_1)\nu y_{xx}(x,t) - y(x,t)y_x(x,t) + p_6 t,\\ y(x,t_0) &= (1+p_5)x^2(1-x)(1-x),\\ y_x(0,t) &= u_1(t),\\ y_x(1,t) &= u_2(t),\\ 0 &\geq C(u_1(t), u_2(t), p_3, p_4), \end{aligned} \tag{18}$$

with $n_x = n_y = 1$, $n_u = 2$, $n_p = 6$, $\Omega_t = [0,1]$, $\Omega_x = (0,1)$, $\sigma = 0.01$, $\nu = 0.1$, $\alpha = 1000$. For the method of lines, the first derivative $y_x(x_l, t)$ at a discretized spatial point x_l in the interior of Ω_x is approximated by the second order formula

$$y_x(x_l, t) \approx \frac{y(x_l + \triangle x, t) - y(x_l - \triangle x, t)}{2\triangle x}, \tag{19}$$

while second order derivatives are approximated by the second order formula

$$y_{xx}(x_l,t) \approx \frac{y(x_l+\triangle x,t)-2y(x_l,t)+y(x_l-\triangle x,t)}{\triangle x^2}. \tag{20}$$

The state variables on the boundaries $y(0,t)$ and $y(1,t)$, entering the right hand side of the PDE when (18) approximating $y_x(x,t)$ and $y_{xx}(x,t)$, are calculated by the second order approximations

$$\begin{aligned} y(0,t) &\approx \frac{4y(\triangle x,t)-y(2\triangle x,t)-2\triangle x u_1(t)}{3}, \\ y(1,t) &\approx \frac{4y(1-\triangle x,t)-y(1-2\triangle x,t)+2\triangle x u_2(t)}{3}. \end{aligned} \tag{21}$$

All computations are performed with 18 lines for the spatial variable x which leads to $\triangle x = \frac{1}{16}$ and a system of 16 first order ordinary differential equations if the Neumann conditions are directly inserted into all functions. An explicit fourth order Runge–Kutta scheme and a linear interpolation of the control variable is used for the integration in time. In order to be able to use this explicit Runge–Kutta scheme, it is necessary to choose a sufficiently high number $N_t = 51$ of grid points for the discretized time interval. Similar results can be achieved by using an implicit Runge–Kutta method with fewer time steps, at the expense of additional numerical cost. The spatial integral in the objective is approximated by the trapezoidal method.
The perturbation parameter p_1 is connected to the viscosity coefficient ν while perturbations in the objective are concerned with p_2. The perturbations p_3 and p_4 will be used to simulate deviations in the constraints defined later on. The parameter p_5 allows for deviations in the initial values. Finally parameter p_6 produces a forcing term on the right hand side which grows with time.
The optimal solutions calculated hereafter are obtained from the discretized formulation (5) for the nominal perturbation parameter $p_0 = (0,0,0,0,0,0)^T$.

6.1 The unconstrained case

First we consider the unconstraint case with $n_c = 0$. After about 2.5 seconds of computational time, the optimal nominal solution is obtained with an objective value $F(y(p_0),u(p_0),p_0) \approx 2.176303 \cdot 10^{-2}$. The nominal state in the interior and the unperturbed optimal control functions are depicted in Figure 1.

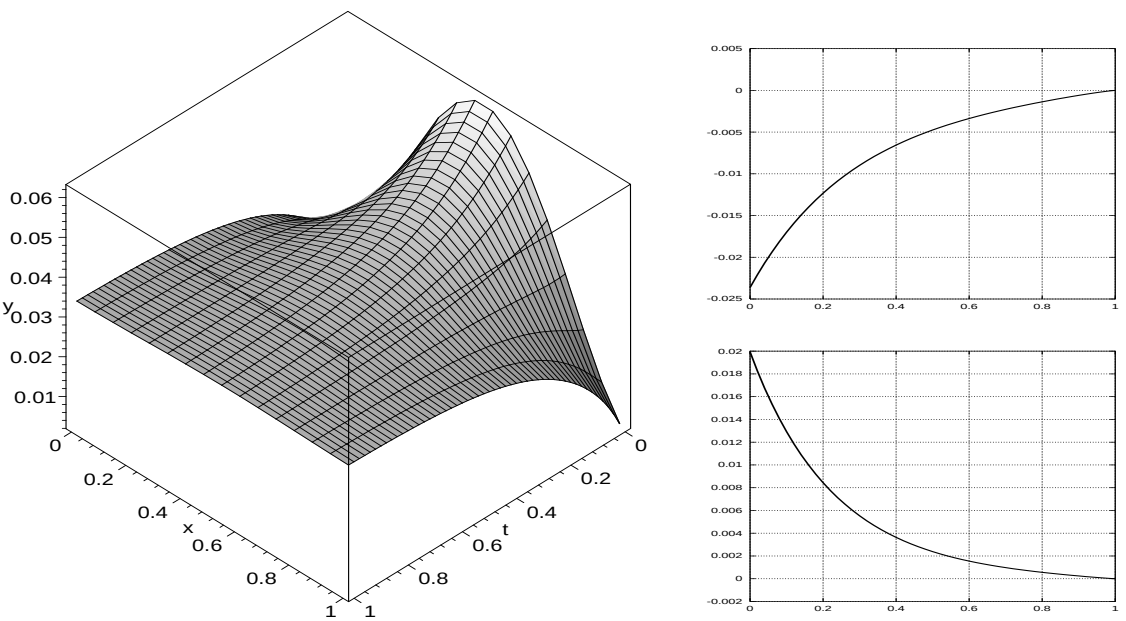

Figure 1: Nominal state $y(x,t)$ (left), nominal optimal controls $u_1(t)$, $u_2(t)$ (right) for the unconstrained case.

All assumptions of Theorem 1 have been carefully checked numerically for the discretized problem, and the Hessian of the Lagrangian has been found to be positive definite. Hence the sensitivity

differentials of the control variables for p_1, p_2, p_5 and p_6, can be obtained from expression (7) and are depicted in Figure 2.

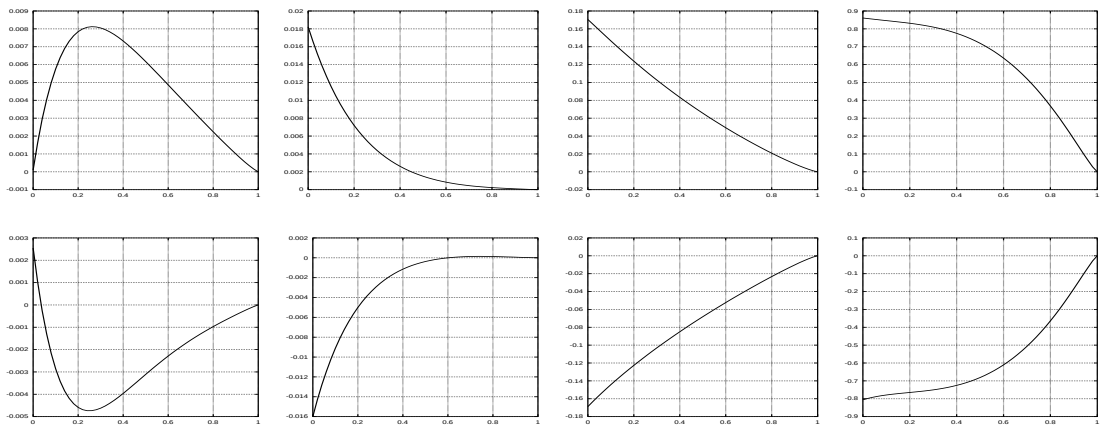

Figure 2: Sensitivity differentials $\frac{du_1(t)}{dp}(p_0)$ (top) and $\frac{du_2(t)}{dp}(p_0)$ (bottom) of the control variables for the unconstrained case.

In order to judge the quality of the real–time approximation (8) for the unconstrained Burgers problem, we set up the following Table 1 which lists the relative errors

$$\zeta_H^{[k]}(p) := \frac{H(\tilde{z}^{[k]}(p), p) - H(z(p), p)}{H(z(p), p)}, \qquad k = 0, 1, 2, \ldots \tag{22}$$

of the objective for different perturbations p. Herein and in the following $\zeta_H^{[0]}$ denotes the relative error of the objective obtained after an integration of the perturbed system using the nominal control variables.

	$\zeta_H^{[0]}(p)$	$\zeta_H^{[1]}(p)$
$p = (0, 0, 0, 0, -0.1, -0.1)$	$4.38 \cdot 10^{-01}$	$3.18 \cdot 10^{-05}$
$p = (0, 0, 0, 0, 0.1, 0.1)$	$4.27 \cdot 10^{-01}$	$3.05 \cdot 10^{-05}$
$p = (-0.1, -0.1, 0, 0, 0, 0)$	$5.77 \cdot 10^{-04}$	$8.22 \cdot 10^{-06}$
$p = (-0.02, 0.02, 0, 0, -0.02, 0.02)$	$7.95 \cdot 10^{-02}$	$2.81 \cdot 10^{-05}$
$p = (0.01, 0.01, 0, 0, 0.05, 0.05)$	$3.29 \cdot 10^{-01}$	$1.05 \cdot 10^{-05}$
$p = (0.01, 0.01, 0, 0, 0.01, 0.01)$	$3.91 \cdot 10^{-02}$	$4.33 \cdot 10^{-07}$
$p = (0.1, 0.1, 0, 0, 0.1, 0.1)$	$4.94 \cdot 10^{-01}$	$5.56 \cdot 10^{-04}$

Table 1: Real–time approximations for different perturbations in the unconstrained case.

Although perturbations of up to 10% are considered to be large perturbations, the numerical results clearly indicate the real–time capability of the proposed method and show the higher order approximations of the objective for the unconstrained case as predicted by Theorem 2.
The computing time for calculating the real–time approximation of the complete controls varies between about $2 \cdot 10^{-7}$ seconds (only one perturbation holds $p_i \neq 0$, $i = 1, \ldots, 6$) and about $8 \cdot 10^{-7}$ seconds (four perturbations hold $p_i \neq 0$, $i = 1, \ldots, 6$) on a PIII 1GHz personal computer. Note that only a few floating point operations have to be performed. Inpractical implementations the computing time can be reduced further by a factor 51 (number of grid points in time), if the time, during the runtime of the real Burgers process, is used for computing the approximation (8).

6.2 Box constraints

Next we consider box constraints of form

$$-0.015(1 + p_3) \leq u_i(t) \leq 0.015(1 + p_3), \qquad i = 1, 2, \tag{23}$$

with an additional perturbation p_3. The optimal nominal solution is now obtained with an objective value $F(y(p_0), u(p_0), p_0) \approx 2.178279 \cdot 10^{-2}$ after about 2.6 seconds of computing time. The

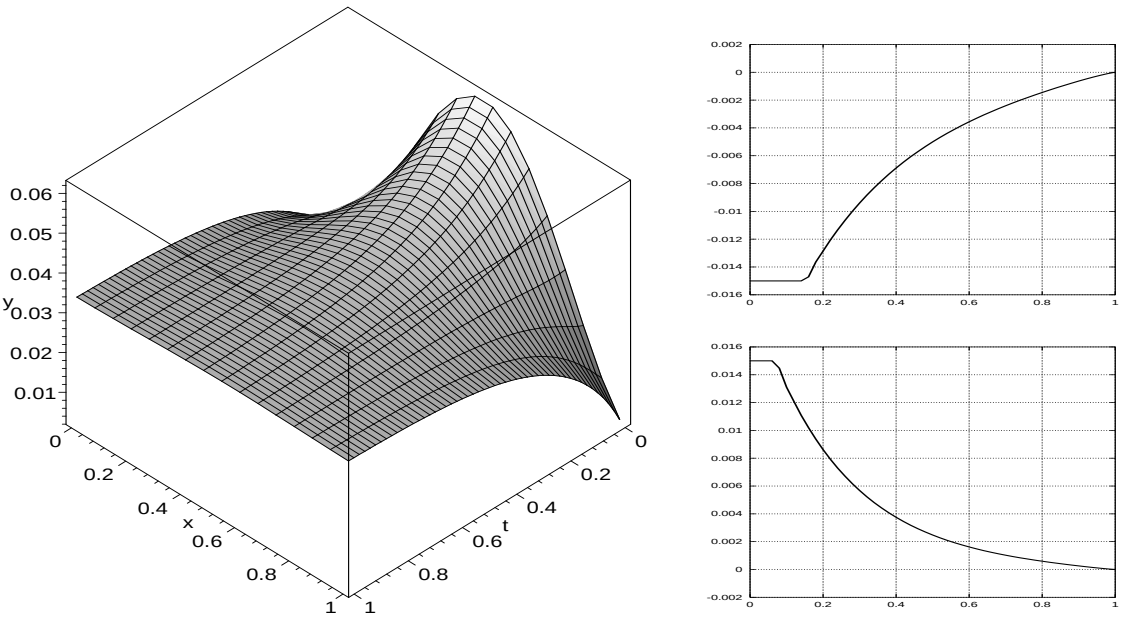

Figure 3: Nominal state $y(x,t)$ (left), nominal optimal controls $u_1(t)$, $u_2(t)$ (right) for the box constrained case.

nominal state in the interior and the unperturbed optimal controls are given in Figure 3.
Note that there are two boundary arcs for the controls.
The Hessian of the Lagrangian has been found to be positive definite on the kernel of the Jacobian of the active constraints and all other assumptions of Theorem 1 hold for the discretized problem, at least numerically. Hence expression (7) yields the sensitivity differentials of the control variables for p_1–p_3, p_5 and p_6. The sensitivity differentials of the control variables are shown in Figure 4.

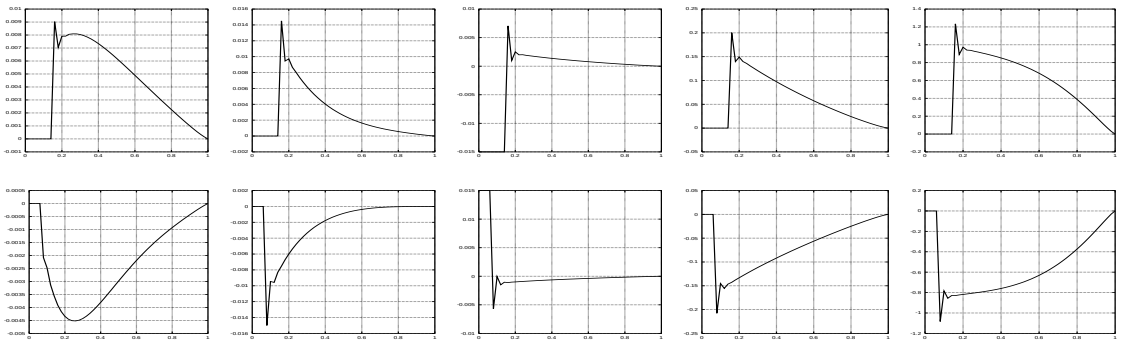

Figure 4: Sensitivity differentials $\frac{du_1(t)}{dp}(p_0)$ (top) and $\frac{du_2(t)}{dp}(p_0)$ (bottom) of the control variables for the box constrained case.

Note that the sensitivities of the controls in Figure 4 are zero on the boundary arcs for p_1, p_2, p_5 and p_6 and that the overshooting at each junction point of the control constraints results from the linear interpolation of the control variables. Table 2 lists the relative errors $\zeta_H^{[k]}(p)$ of the objective as defined in (22) for the box constrained Burgers problem.
Again the considered perturbations of up to 50% can be understood as large perturbations. Nevertheless the perturbed optimal solutions are well approximated. The crucial message from this example is however the admissibility of the approximated control variables after only one real–time correction step. This follows from Theorems 1–3 and is due to the fact that the treated constraints are linear in the control variables. Hence additional correction steps as proposed in (14) are not necessary for box constrained problems. Computing times are similar to the unconstrained case.

	$\zeta_H^{[0]}(p)$	$\zeta_H^{[1]}(p)$
$p = (0.1, 0.5, 0, 0, 0, 0)$	$3.22 \cdot 10^{-03}$	$8.75 \cdot 10^{-04}$
$p = (0, 0, 0.1, 0, 0, 0)$	$4.45 \cdot 10^{-04}$	$7.03 \cdot 10^{-06}$
$p = (0.01, 0.01, 0.01, 0, 0.001, 0)$	$3.98 \cdot 10^{-04}$	$1.99 \cdot 10^{-06}$
$p = (0.01, 0.01, 0.01, 0, 0, 0)$	$5.76 \cdot 10^{-05}$	$4.87 \cdot 10^{-10}$
$p = (-0.01, -0.01, -0.01, 0, 0, 0.01)$	$3.28 \cdot 10^{-04}$	$2.26 \cdot 10^{-06}$
$p = (-0.02, 0.02, -0.05, 0, 0, 0)$	$2.94 \cdot 10^{-04}$	$1.60 \cdot 10^{-06}$
$p = (0.05, -0.02, 0.05, 0, 0, 0)$	$3.23 \cdot 10^{-04}$	$2.83 \cdot 10^{-06}$
$p = (-0.01, 0.01, -0.01, 0, 0, 0)$	$5.53 \cdot 10^{-05}$	$5.62 \cdot 10^{-11}$

Table 2: Admissible real–time approximations for different perturbations in the box constrained case.

6.3 Nonlinear control constraints

Finally we consider a coupled nonlinear control constraint of the form

$$u_1(t)^2 + u_2(t)^2 \leq 0.000225(1 + p_4), \qquad i = 1, 2 \tag{24}$$

with an additional perturbation p_4. Since (24) is a nonlinear constraint we cannot expect, that in case of an active constraint the linear real–time approximation (8) ensures the admissibility of the solution in one step . Here, the nominal objective is calculated after about 3.5 seconds of computing time to $F(y(p_0), u(p_0), p_0) \approx 2.51 \cdot 10^{-2}$. The nominal state in the interior and the unperturbed optimal control functions are given in Figure 5.

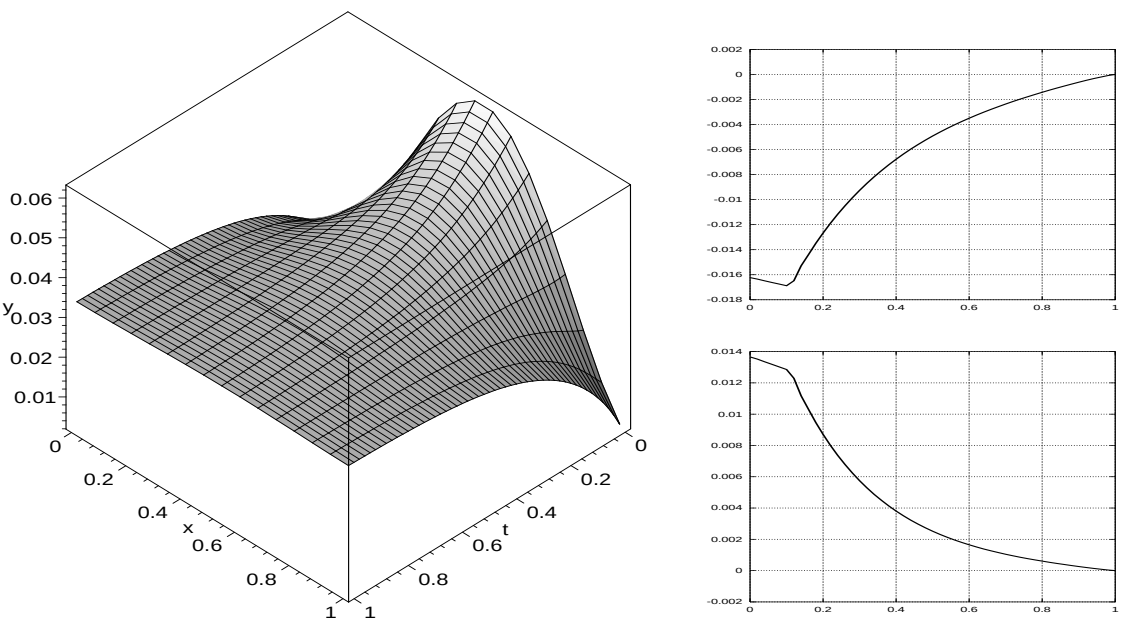

Figure 5: Nominal state $y(x, t)$ (left), nominal optimal controls $u_1(t)$, $u_2(t)$ (right) for the nonlinear control constrained case.

There is one boundary arc at the beginning of the time interval.
All assumptions of Theorem 1 have been checked numerically. The Hessian of the Lagrangian is positive definite on the kernel of the Jacobian of the active constraints. Hence expression (7) yields the sensitivity differentials $\frac{dz}{dp}(p_0)$ and $\frac{dz}{dq}(0)$ of the control variables for p_1, p_2 and p_4–p_6. The sensitivity differentials of the control variables w.r.t parameter p are shown in Figure 6. The sensitivity differentials of the control variables with respect to linear perturbations q in the constraints are neglected for the lack of space.
Note that the sensitivities of the controls in Figure 6 have jumps along the time axis at junction points and that the overshooting at each junction point of the control constraint results from the linear interpolation of the control variables. Table 3 lists the relative errors $\zeta_H^{[k]}(p)$ of the objective

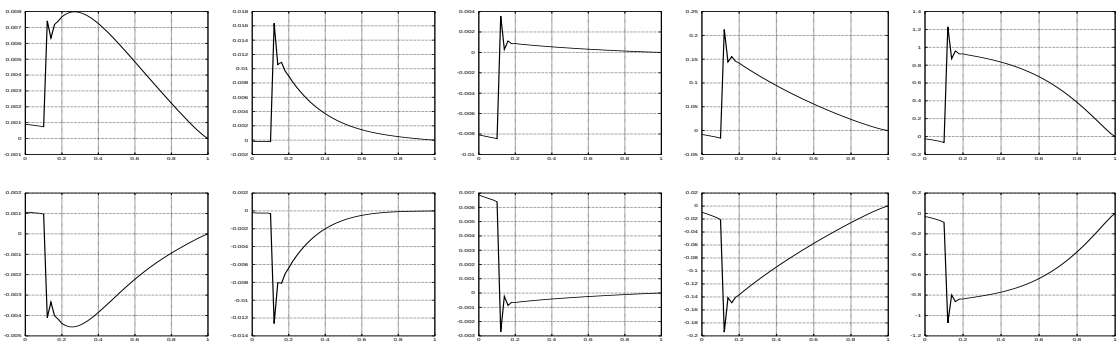

Figure 6: Sensitivity differentials $\frac{du_1(t)}{dp}(p_0)$ (top) and $\frac{du_2(t)}{dp}(p_0)$ (bottom) of the control variables for the nonlinear control constrained case.

as defined in (22) and the error $\zeta_G^{[k]}(p)$ in the nonlinear control constraint,

$$\zeta_G^{[k]}(p) := \max_i \frac{G_i^a(\tilde{z}^{[k]}(p), p)}{0.000225}, \qquad k = 0, 1, 2, \ldots, \tag{25}$$

for the first eight iterates, for perturbations $p_a = (0.05, 0, 0, -0.05, 0, 0)$, $p_b = (-0.1, 0.01, 0, 0.1, 0, 0)$, $p_c = (0.2, 0.05, 0, -0.5, 0.02, 0)$.

	$p = p_a$		$p = p_b$		$p = p_c$	
k	$\zeta_G^{[k]}(p)$	$\zeta_F^{[k]}(p)$	$\zeta_G^{[k]}(p)$	$\zeta_F^{[k]}(p)$	$\zeta_G^{[k]}(p)$	$\zeta_F^{[k]}(p)$
0	$5.00 \cdot 10^{-02}$	$1.08 \cdot 10^{-04}$	$1.00 \cdot 10^{-01}$	$3.01 \cdot 10^{-04}$	$5.00 \cdot 10^{-01}$	$5.57 \cdot 10^{-03}$
1	$6.35 \cdot 10^{-04}$	$1.88 \cdot 10^{-06}$	$2.54 \cdot 10^{-03}$	$3.96 \cdot 10^{-06}$	$6.27 \cdot 10^{-02}$	$2.36 \cdot 10^{-04}$
2	$1.60 \cdot 10^{-05}$	$1.14 \cdot 10^{-08}$	$1.25 \cdot 10^{-04}$	$5.83 \cdot 10^{-07}$	$1.66 \cdot 10^{-02}$	$5.99 \cdot 10^{-05}$
3	$4.05 \cdot 10^{-07}$	$3.55 \cdot 10^{-08}$	$6.12 \cdot 10^{-06}$	$3.58 \cdot 10^{-07}$	$4.75 \cdot 10^{-03}$	$8.14 \cdot 10^{-06}$
4	$1.02 \cdot 10^{-08}$	$3.67 \cdot 10^{-08}$	$2.99 \cdot 10^{-07}$	$3.69 \cdot 10^{-07}$	$1.38 \cdot 10^{-03}$	$7.04 \cdot 10^{-06}$
5	$2.60 \cdot 10^{-10}$	$3.68 \cdot 10^{-08}$	$1.46 \cdot 10^{-08}$	$3.68 \cdot 10^{-07}$	$4.04 \cdot 10^{-04}$	$1.15 \cdot 10^{-05}$
6	$6.58 \cdot 10^{-12}$	$3.68 \cdot 10^{-08}$	$7.12 \cdot 10^{-10}$	$3.69 \cdot 10^{-07}$	$1.18 \cdot 10^{-04}$	$1.28 \cdot 10^{-05}$
7	$1.67 \cdot 10^{-13}$	$3.68 \cdot 10^{-08}$	$3.47 \cdot 10^{-11}$	$3.69 \cdot 10^{-07}$	$3.46 \cdot 10^{-05}$	$1.32 \cdot 10^{-05}$
8	$4.28 \cdot 10^{-15}$	$3.68 \cdot 10^{-08}$	$1.70 \cdot 10^{-12}$	$3.69 \cdot 10^{-07}$	$9.95 \cdot 10^{-06}$	$1.33 \cdot 10^{-05}$

Table 3: Admissible real–time approximations for different perturbations in the nonlinear control constrained case.

Even in the case of a large perturbation p_c the method converges. Further iterations lead to admissible solutions within machine precision. Computing times for each of the eight iterates are similar to the two exaples discussed before since an evaluation of the nonlinear control constraint (24) at each discretized time step can be done by only six floating point operations.

7 Conclusion

Real–time methods based on the parametric sensitivity analysis of perturbed PDE optimal control problems with control constraints have been proposed. Under the assumption of convergence to the assumed unique continuous solution, the original control problem has been discretized via various stages to obtain a perturbed NLP problem for the discrete control variables. For this finite-dimensional problem, we are able to compute the sensitivity differentials of the optimal solution with respect to the perturbation parameters. These sensitivities allow for an approximation of the corresponding continuous variables of the original PDE optimal control problem and have been used for real–time optimal control strategies. The examples show that it is possible to calculate accurate solutions in a robust way.

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