Zentrum für Technomathematik
Fachbereich 3 – Mathematik und Informatik

Exponential estimates for time delay systems

Vladimir Kharitonov Diederich Hinrichsen

Report 03–05

Berichte aus der Technomathematik

Report 03–05 **April 2003**

Exponential estimates for time delay systems

V. L. Kharitonov
Control Automatico
CINVESTAV-IPN A.P. 14-740
07000 Mexico D.F., Mexico

E-mail: khar@ctrl.cinvestav.mx

D. Hinrichsen
Institut für Dynamische Systeme
Universität Bremen
D-28334 Bremen, Germany

E-mail: dh@math.uni-bremen.de

April 23, 2003

Abstract

In this paper we demonstrate how Lyapunov-Krasovskii functionals can be used to obtain exponential bounds for the solutions of time-invariant linear delay systems.

Keywords: time delay system, exponential estimate, Lyapunov-Krasovskii functional.

1 Introduction

The objective of this note is to describe a systematic procedure of constructing quadratic Lyapunov functionals for (exponentially) stable linear delay systems in order to obtain exponential estimates for their solutions.

The procedure we propose is a counterpart to the well known method of deriving exponential estimates for stable systems $\dot{x} = Ax$ by means of quadratic Lyapunov functions $V(x) = \langle x, Ux\rangle$. Here $U \succ 0$ is the solution of a Lyapunov equation $A^*U + UA = -W$ where $W \succ 0$ is any chosen positive definite matrix. If $2\omega U \preceq W$ for some $\omega > 0$ then

$$\|e^{At}\| \leq \kappa(U)^{1/2} e^{-\omega t}, \quad t \geq 0 \tag{1}$$

where $\|\cdot\|$ denotes the spectral norm and $\kappa(U) = \|U\|\,\|U^{-1}\|$ is the condition number of U, see [6]. Note that this estimate guarantees not only a uniform decay rate ω for all solutions of $\dot{x} = Ax$ but also a bound on the transients of the system.

It is surprising that a similar constructive method does not exist for delay systems. It is true, there exists an operator theoretic version of Lyapunov's equation in the abstract semigroup theory of infinite dimensional time-invariant linear systems, see [2], but this does not provide us with a constructive procedure. For constructive purposes more concrete Lyapunov functions must be considered. Since the fifties different types of Lyapunov functions have been proposed for the stability analysis of delay systems , see the pioneering works of Razumikhin [11] and Krasovskii [10]. Whereas Razumikhin [11] used Lyapunov type functions $V(x(t))$ depending on the current value $x(t)$ of the solution, Krasovskii [10] proposed to use functionals $V(x_t)$ depending on the whole solution segment x_t, i.e. the true state of

the delay system. These functionals, which are defined on the space of (continuous) initial functions, are called *Lyapunov-Krasovskii functionals*. For a brief discussion of these two approaches and some historic comments, see [5, §5.5].
The majority of results concerning Lyapunov's direct method for delay systems provides *sufficient criteria* for stability and asymptotic stability. These results assume that Lyapunov functionals with certain properties are given, and so the exponential estimates derived by means of these functionals are not obtained in a constructive manner. For a constructive procedure *converse results* are needed which show how Lyapunov functionals of a specific type can be constructed for a given class of stable delay equations. Such converse results are available for certain delay systems, see e.g. Halanay [4], but they do not abound as the sufficient stability criteria based on given Lyapunov functionals.
Quadratic Lyapunov functionals have been proposed for time-invariant linear delay equations by Repin [12], Datko [3], Infante and Castelan [8], Huang [7], Kharitonov and Zhabko [9]. However, with exception of the latter reference, these Lyapunov functionals cannot be used for deriving exponential estimates, if no additional a priori information is available. Infante and Castelan propose an interesting method of constructing exponential estimates whose decay rate comes arbitrarily close to the spectral abscissa of the delay system. But their method presupposes that a concrete exponential estimate is already available. The constants of the exponential estimate which they derive from a quadratic Lyapunov functional depend explicitely on the constants of the presupposed a priori estimate. (compare (3.6), (3.19) and (2.6) in [8]).
In order to overcome this dependence on an a priori estimate we use a modified Lyapunov-Krasovskii functional introduced in [9]. This functional is constructed in a similar way to that of Infante and Castelan but contains an additional integral term. As in [8] the construction is based on a solution of a matrix differential-difference equation on a finite time interval satisfying additional symmetry and boundary conditions. The matrix boundary value problem plays, roughly speaking, a similar role for linear delay equations as Lyapunov's equation in the delay free case. Our differential-difference equation is more complicated than that in [8] since Infante and Castelan only considered the *one* delay case. Since these matrix equations have only recently been discovered there is as yet no systematic solution theory available. This will be a subject for future research. However, first algorithms have been developed for the solution of the corresponding matrix boundary value problem.
The note is organized as follows. After some preliminaries on delay systems in the next section we describe the construction of the Lyapunov-Krasovskii functional according to [9] in section 3. Section 4 contains the main results of this note. Finally, section 5 presents an example illustrating the basic steps leading to an exponential estimate for a given exponentially stable system with two (commensurate) delays.

2 Preliminaries

In this paper we consider time delay systems of the following form

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - h_k). \tag{2}$$

where $A_0, A_1, ..., A_m \in \mathbb{R}^{n\times n}$ are given matrices and $0 < h_1 < ... < h_m = h$ are positive delays. For any continuous initial function $\varphi : [-h, 0] \to \mathbb{R}^n$ there exists the unique solution, $x(t, \varphi)$, of (2) satisfying the initial condition

$$x(\theta, \varphi) = \varphi(\theta), \quad \theta \in [-h, 0].$$

If $t \geq 0$ we denote by $x_t(\varphi)$ the trajectory segment

$$x_t(\varphi) : \theta \mapsto x(t + \theta, \varphi), \quad \theta \in [-h, 0].$$

Throughout this note we will use the Euclidean norm for vectors and the induced matrix norm for matrices. The space of continuous initial functions $\mathcal{C}([-h, 0], \mathbb{R}^n)$ is provided with the supremum norm $\|\varphi\|_\infty = \max_{\theta\in[-h,0]} \|\varphi(\theta)\|$.

Definition 1 *The system (2) is said to be* exponentially stable *if there exist $\sigma > 0$ and $\gamma \geq 1$ such that for every solution $x(t, \varphi)$, $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ the following exponential estimate holds*

$$\|x(t, \varphi)\| \leq \gamma e^{-\sigma t} \|\varphi\|_\infty, \quad t \geq 0. \tag{3}$$

For simplicity we will call an *exponentially stable* system just *stable*. The matrix-valued function $K : [-h, \infty] \to \mathbb{R}^{n\times n}$ which solves the matrix differential equation

$$\frac{d}{dt}K(t) = A_0K(t) + \sum_{j=1}^{m} A_jK(t - h_j), \ t \geq 0,$$

with initial condition

$$K(t) = 0 \text{ for } -h \leq t < 0, \qquad K(0) = I_n,$$

is called the *fundamental matrix* of the system (2),here I_n is the identity matrix. It is known that $K(t)$ also satisfies the differential equation [1]

$$\frac{d}{dt}K(t) = K(t)A_0 + \sum_{j=1}^{m} K(t - h_j)A_j, \quad t \geq 0. \tag{4}$$

The following result is known as the Cauchy formula for the solutions of system (2), see [1].

$$x(t, \varphi) = K(t)\varphi(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t - h_j - \theta)A_j\varphi(\theta)d\theta, \quad t \geq 0. \tag{5}$$

Every column of $K(t)$ is a solution of system (2), so if the system is exponentially stable, then the matrix satisfies an inequality of the form (3). As a consequence, the integral

$$U(\tau) = \int_0^\infty K^\top(t)WK(t + \tau)dt \tag{6}$$

is well defined for all $\tau \geq -h$ and any matrix $W \in \mathbb{R}^{n\times n}$.

3 Lyapunov-Krasovskii functionals

We will now construct quadratic Lyapunov functions for the system (2) in a similar way as for linear differential equations without delays. In this latter case, given any stable system $\dot{x} = Ax$, a quadratic Lyapunov function is determined in the following way. For an arbitrarily chosen quadratic function $w(x) = x^\top W x$ with positive definite $W \succ 0$ one constructs a quadratic function $v(x) = x^\top U x$, $U \succ 0$ such that

$$dv(x(t))/dt = -w(x(t)), \quad t \in \mathbb{R} \tag{7}$$

for every solution, $x(t)$, of $\dot{x} = Ax$. Function $v(x) = x^\top U x$ satisfies (7) if and only if U is the (uniquely determined, positive definite) solution of the *Lyapunov equation*

$$A^* U + UA = -W. \tag{8}$$

Analogously we choose for the delay system (2) positive definite $n \times n$ matrices W_0, W_1,..., W_{2m} and consider the following functional on $\mathcal{C}([-h, 0], \mathbb{R}^n)$

$$w(\varphi) = \varphi^\top(0) W_0 \varphi(0) + \sum_{k=1}^{m} \varphi^\top(-h_k) W_k \varphi(-h_k) + \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^\top(\theta) W_{m+k} \varphi(\theta) d\theta, \tag{9}$$

where $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ is arbitrary. If the system (2) is exponentially stable, then there exists a unique quadratic functional $v : \mathcal{C}([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ such that $t \mapsto v(x_t(\varphi))$ is differentiable on $\mathbb{R}_+$ and

$$\frac{dv(x_t(\varphi))}{dt} = -w(x_t(\varphi)), \quad t \geq 0 \tag{10}$$

for all solutions $x(t, \varphi)$ of (2), $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ [9]. Functional $v(\cdot)$ is called the *Lyapunov-Krasovskii functional* associated with (9). It has been shown in [9] that the functional is given by

$$\begin{aligned} v(\varphi) =& \varphi^\top(0) U(0) \varphi(0) + \sum_{k=1}^{m} 2\varphi^\top(0) \int_{-h_k}^{0} U(-h_k - \theta) A_k \varphi(\theta) d\theta + \\ &+ \sum_{k=1}^{m} \sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^\top(\theta_2) A_k^\top \left[\int_{-h_j}^{0} U(\theta_2 - \theta_1 + h_k - h_j) A_j \varphi(\theta_1) d\theta_1 \right] d\theta_2 + \\ &+ \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^\top(\theta) \left[W_k + (h_k + \theta) W_{m+k} \right] \varphi(\theta) d\theta \end{aligned} \tag{11}$$

where

$$U(\tau) = \int_0^\infty K^\top(t) \left[W_0 + \sum_{k=1}^{m} (W_k + h_k W_{m+k}) \right] K(t + \tau) dt, \quad \tau \geq -h. \tag{12}$$

Note that by exponential stability of (2) the matrix $U(\tau)$ is well defined for all $\tau \geq -h$ and is of the form (6) with

$$W = W_0 + \sum_{k=1}^{m} (W_k + h_k W_{m+k}). \tag{13}$$

We call $U(\tau)$ the *Lyapunov matrix function* for the system (2) associated with the functional $w(\cdot)$ (9). Note that the first $(1+m+m^2)$ terms in (11) are completely determined by U and

hence only depend upon the weighted sum W of the positive definite matrices W_k. However, the last m terms in (11) depends on the individual W_k's. We will see later that all these terms are needed in order to derive exponential estimates for (2) by means of the above quadratic functionals. For the case of a single delay in (2) a similar Lyapunov-Krasovskii functional has been considered by Infante and Castelan in [8]. However, in their paper the matrix $W_k + (h_k + \theta)W_{m+k}$ in each one of the last m terms of (11) is replaced by a constant positive definite matrix. This is due to the fact that Infante and Castelan did not include the integral term in their definition of the functional $w(\cdot)$, see [8, (3.4)]. However, we will see in Remark 5 that this integral term is an essential ingredient for deriving an exponential estimate of the form (3) for an exponentially stable delay system without any further a priori knowledge.

Remark 1 If (2) is without delays ($h_k = 0$, $k = 1, \ldots, m$) then the interval $[-h, 0]$ is reduced to $\{0\}$, $\mathcal{C}([-h, 0], \mathbb{R}^n)$ to $\mathcal{C}(\{0\}, \mathbb{R}^n) \cong \mathbb{R}^n$ and we have $K(t) = e^{At}$. Identifying $\varphi \in \mathcal{C}(\{0\}, \mathbb{R}^n)$ with $x = \varphi(0)$, the quadratic functional $w(\cdot)$ (9) is given by $w(x) = x^\top W x$ where W is defined by (13), and $v(\cdot)$ is given by $v(x) = x^\top U(0)x$, $x \in \mathbb{R}^n$. Note that in this case $U(0)$ is by definition equal to

$$U = \int_0^\infty e^{A^\top t} W e^{At} dt \tag{14}$$

so that $U = U(0)$ satisfies the Lyapunov equation (8). So the above construction of the Lyapunov-Krasovskii functional $v(\cdot)$ from $w(\cdot)$ generalizes the construction procedure via the Lyapunov equation (8) in the delay free case. □

Remark 2 In the delay free case the success of quadratic Lyapunov functions rely on the fact that for a given $w(x) = x^\top W x$ the corresponding $v(x) = x^\top U x$ is not obtained via the integral expression (14) but can be computed from the linear Lyapunov equation (8). Similarly, the above construction of the Lyapunov-Krasovskii functional (11) would not be practical if it required the evaluation of the integral (12) (and so, in particular, the knowledge of the fundamental matrix $K(t)$ on $\mathbb{R}_+$). But it is not difficult to show (see [9]) that $U(\tau)$, $\tau \in [-h, h]$ solves the following matrix delay differential equation

$$\frac{d}{d\tau} U(\tau) = U(\tau)A_0 + \sum_{k=1}^{m} U(\tau - h_k)A_k, \quad \tau \in [0, h] \tag{15}$$

and additionally the following conditions

- the symmetry condition

$$U(-\tau) = U^\top(\tau), \quad \tau \in [-h, h], \tag{16}$$

- the Lyapunov type linear matrix equation

$$U(0)A_0 + A_0^\top U(0) + \sum_{k=1}^{m} U^\top(h_k)A_k + A_k^\top U(h_k) + W = 0. \tag{17}$$

A systematic study of the equations (15), (16), (17) has not yet been accomplished. We conjecture that $U(\tau))$ is the *unique* solution of this set of equations, but a proof of this conjecture is an open problem. □

4 Main results

In this section we show how one can use functionals (9) and (11) in order to obtain exponential estimates for time delay systems.
We first specify two conditions under which a pair of (not necessarily quadratic) functionals $v, w : \mathcal{C}([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ satisfying (10) yields an exponential estimate of the form (3).

Proposition 3 *Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants and $v, w : \mathcal{C}([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ are continuous functionals such that $t \mapsto v(x_t(\varphi))$ is differentiable on $\mathbb{R}_+$. If the following conditions are satisfied for all $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$*

1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 w(\varphi)$,
2. $\beta_1 \|\varphi(0)\|^2 \leq w(\varphi) \leq \beta_2 \|\varphi\|_\infty^2$,
3. $\frac{d}{dt} v(x_t(\varphi)) \leq -w(x_t(\varphi))$ *for all* $t \geq 0$,

then for all $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$

$$\|x(t, \varphi)\| \leq \sqrt{\frac{\alpha_2 \beta_2}{\alpha_1}}\, e^{-\frac{1}{2\alpha_2} t}\, \|\varphi\|_\infty, \quad t \geq 0. \tag{18}$$

Proof: Given any $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$, conditions 1. and 3. imply that

$$\frac{d}{dt} v(x_t(\varphi)) \leq -\frac{1}{\alpha_2} v(x_t(\varphi)), \quad t \geq 0.$$

Integrating this inequality from 0 to t and applying Gronwall's Lemma we get

$$v(x_t(\varphi)) \leq v(\varphi) e^{-\frac{1}{\alpha_2} t}, \quad t \geq 0.$$

Then conditions 1. and 2. yield

$$\alpha_1 \|x(t, \varphi)\|^2 \leq v(x_t(\varphi)) \leq v(\varphi) e^{-\frac{1}{\alpha_2} t} \leq \alpha_2 \beta_2 \|\varphi\|_\infty^2 e^{-\frac{1}{\alpha_2} t}, \quad t \geq 0.$$

Comparing the left and the right hand sides, the exponential estimate (18) follows. □

We will now show that conversely, if the system (2) is exponentially stable, then positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be determined such that the quadratic functionals $v(\cdot), w(\cdot)$ constructed in the previous section satisfy the assumptions of Proposition 3. As a consequence we obtain an exponential estimate of the form (3) for every set of positive definite $n \times n$ matrices $W_0, \ldots, W_{2m}$.

Theorem 4 *If system (2) is exponentially stable and $W_0, W_1, \ldots, W_{2m}$ are positive definite real $n \times n$ matrices then there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that the quadratic Lyapunov-Krasovskii functionals $w(\cdot)$ and $v(\cdot)$ defined by (9) and (11), respectively, satisfy the assumptions of Proposition 3.*

Proof: We have seen in Section 3 that the functional $v(\cdot)$ is well defined by (11) and (12) since (2) is exponentially stable by assumption. Moreover it follows easily from the definitions (9) and (11) that the two functionals $v, w : \mathcal{C}([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ are continuous. Let $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$. We know from Section 3 that $w(\cdot)$ and $v(\cdot)$ are related by (10), i.e. $t \mapsto v(x_t(\varphi))$ is differentiable on $\mathbb{R}_+$ and $w(\cdot)$, $v(\cdot)$ satisfy the third condition of Proposition 3 with equality. It remains to show that there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that conditions 1. and 2. of Proposition 3 are satisfied. Let

$$\lambda_{\min} = \min_{k=0,\dots,2m} \lambda_{\min}(W_k), \quad \lambda_{\max} = \max_{k=0,\dots,2m} \lambda_{\max}(W_k),$$

where $\lambda_{\min}(W_k)$ and $\lambda_{\max}(W_k)$ denote the smallest and the largest eigenvalue of the positive definite matrix W_k, respectively. Then

$$\lambda_{\min} \left[\|\varphi(0)\|^2 + \sum_{k=1}^{m} \|\varphi(-h_k)\|^2 + \sum_{k=1}^{m} \int_{-h_k}^{0} \|\varphi(\theta)\|^2 \, d\theta \right] \leq$$
$$\leq w(\varphi) \leq \lambda_{\max} \left[\|\varphi(0)\|^2 + \sum_{k=1}^{m} \|\varphi(-h_k)\|^2 + \sum_{k=1}^{m} \int_{-h_k}^{0} \|\varphi(\theta)\|^2 \, d\theta \right].$$

From these inequalities we conclude that

$$\lambda_{\min} \|\varphi(0)\|^2 \leq w(\varphi) \leq \lambda_{\max} \left(1 + m + \sum_{k=1}^{m} h_k \right) \|\varphi\|_\infty^2 ,$$

i.e. the functional $w(\cdot)$ satisfies the second condition of proposition 3 with

$$\beta_1 = \lambda_{\min}, \quad \beta_2 = (1 + m + \sum_{k=1}^{m} h_k) \lambda_{\max}. \tag{19}$$

In order to check the first condition, let

$$\mu = \max_{\tau \in [0,h]} \{\|U(\tau)\|\}, \quad a = \max_{k=1,\dots,m} \|A_k\| .$$

Then one can easily verify the following inequalities

$$\varphi(0)^\top U(0)\varphi(0) \leq \mu \|\varphi(0)\|^2 ,$$
$$2\varphi(0)^\top \int_{-h_k}^{0} U(-h_k - \theta) A_k \varphi(\theta) d\theta \leq \mu a h_k \|\varphi(0)\|^2 + \mu a \int_{-h_k}^{0} \|\varphi(\theta)\|^2 \, d\theta,$$
$$\int_{-h_k}^{0} \varphi(\theta)^\top [W_k + (h_k + \theta) W_{m+k}] \varphi(\theta) d\theta \leq (1 + h_k) \lambda_{\max} \int_{-h_k}^{0} \|\varphi(\theta)\|^2 \, d\theta,$$

for $k = 1, ..., m$.
In order to find an upper estimate of the double integrals in (11) we make use of the fact that by the Cauchy-Schwartz inequality in $L^2(-h_i, 0; \mathbb{R}^n)$ we have

$$\left(\int_{-h_i}^{0} \|\varphi(\theta)\| \, d\theta \right)^2 \leq h_i \int_{-h_i}^{0} \|\varphi(\theta)\|^2 \, d\theta, \quad i = 1, \dots, m.$$

So

$$\int_{-h_k}^{0} \varphi(\theta_2)^\top A_k^\top \left[\int_{-h_j}^{0} U(\theta_1 - \theta_2 + h_k - h_j) A_j \varphi(\theta_1) d\theta_1 \right] d\theta_2 \leq$$
$$\leq \mu a^2 \left(\int_{-h_k}^{0} \|\varphi(\theta_2)\| \, d\theta_2 \right) \left(\int_{-h_j}^{0} \|\varphi(\theta_1)\| \, d\theta_1 \right) \leq$$
$$\leq \frac{1}{2} \mu a^2 \left[\left(\int_{-h_k}^{0} \|\varphi(\theta_2)\| \, d\theta_2 \right)^2 + \left(\int_{-h_j}^{0} \|\varphi(\theta_1)\| \, d\theta_1 \right)^2 \right] \leq$$
$$\leq \frac{1}{2} \mu a^2 h_k \int_{-h_k}^{0} \|\varphi(\theta_2)\|^2 \, d\theta_2 + \frac{1}{2} \mu a^2 h_j \int_{-h_j}^{0} \|\varphi(\theta_1)\|^2 \, d\theta_1,$$

for all $k, j = 1, ..., m$. As a consequence we obtain the following upper bound for $v(\varphi)$:

$$v(\varphi) \leq \mu \big(1 + a \sum_{k=1}^{m} h_k\big) \|\varphi(0)\|^2 + \sum_{k=1}^{m} \left(\mu a + (1 + h_k)\lambda_{\max} + \frac{m+1}{2} \mu h_k a^2 \right) \int_{-h_k}^{0} \|\varphi(\theta)\|^2 \, d\theta. \tag{20}$$

If we select $\alpha_2 > 0$ such that the following two conditions hold

- $\alpha_2 \lambda_{\min} \geq \mu \left(1 + a \sum_{k=1}^{m} h_k\right)$,
- $\alpha_2 \lambda_{\min} \geq \mu a + (1 + h)\lambda_{\max} + \frac{m+1}{2} \mu h a^2$,

then

$$v(\varphi) \leq \alpha_2 w(\varphi).$$

To obtain the required quadratic lower bound for $v(\varphi)$, we consider the modified functional

$$\widetilde{v}(\varphi) = v(\varphi) - \alpha \|\varphi(0)\|^2, \quad \varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$$

where $\alpha > 0$. Then $\frac{d}{dt}\widetilde{v}(x_t(\varphi)) = -\widetilde{w}(x_t(\varphi))$ where, for any $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$,

$$\widetilde{w}(\varphi) = -w(\varphi) - \alpha \varphi(0)^\top \left[A_0 \varphi(0) + \sum_{k=1}^{m} A_k \varphi(-h_k) \right] - \alpha \left[A_0 \varphi(0) + \sum_{k=1}^{m} A_k \varphi(-h_k) \right]^\top \varphi(0).$$

Omitting the last term in the definition of $w(\cdot)$ (see (9)) we obtain

$$\widetilde{w}(\varphi) \geq \left(\varphi(0)^\top, \varphi(-h_1)^\top, ..., \varphi(-h_m)^\top\right) W(\alpha) \begin{pmatrix} \varphi(0) \\ \varphi(-h_1) \\ \vdots \\ \varphi(-h_m) \end{pmatrix}, \quad \varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$$

where

$$W(\alpha) = \begin{pmatrix} W_0 & 0 & \cdots & 0 \\ 0 & W_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_m \end{pmatrix} + \alpha \begin{pmatrix} A_0 + A_0^\top & A_1 & \cdots & A_m \\ A_1^\top & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A_m^\top & 0 & \cdots & 0 \end{pmatrix}.$$

Using Schur complements we see that the matrix $W(\alpha)$ is positive definite if and only if

$$\widetilde{W}(\alpha) = W_0 + \alpha(A_0 + A_0^\top) - \alpha^2 \sum_{k=1}^{m} A_k W_k^{-1} A_k^\top \succ 0.$$

Since $W_0 \succ 0$ there exists $\alpha_1 > 0$ such that $\widetilde{W}(\alpha_1) \succ 0$. Integrating $\frac{d}{dt}\widetilde{v}(x_t(\varphi)) = -\widetilde{w}(x_t(\varphi))$ from 0 to ∞ we get for $\alpha = \alpha_1$,

$$v(\varphi) - \alpha_1 \|\varphi(0)\|^2 = \widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))dt \geq 0.$$

Altogether we have found $\alpha_1, \alpha_2 > 0$ such that the first condition in Proposition 3 is satisfied and this concludes the proof. □

Remark 5 The above proof shows that every quadratic functional $w(\cdot)$ of the form (9) with positive definite matrices $W_0, \ldots, W_{2m}$ satisfies the second condition of Proposition 3 with the constants $\beta_1, \beta_2 > 0$ given by (19). The first $(m+1)$ terms in (9) were needed to prove that the corresponding functional $v(\cdot)$ (11) satisfies the first inequality in the first condition of Proposition 3. The last m terms in (9), along with the first term, were used in the proof to derive the second inequality. So, $w(\cdot)$ defined by (9) can be viewed as a quadratic functional with a minimum number of quadratic terms to yield a Lyapunov-Krasovskii functional $v(\cdot)$ satisfying the first condition in Proposition 3. As mentioned above the integral terms in (9) are missing in the definition of $w(\cdot)$ in [8]. This is compensated by an additional exponential factor $e^{\delta t}$, $\delta > 0$ in the definition of $v(\cdot)$, see [8, (3.1)], where $-\delta$ is supposed to be strictly greater than the exponential growth rate of the delay system. Therefore the construction of Infante and Castelan requires some a priori knowledge about the spectral abscissa of the system. □

Remark 6 Clearly the exponential estimate obtained by Theorem 4 depends on the choice of the matrices $W_k \succ 0$, $k = 0, 1, ..., 2m$. These matrices may serve as free parameters in an optimization of the estimate. In this note we do not try to obtain tight estimates, we only wish to demonstrate that the above Lyapunov-Krasovskii approach yields a systematic procedure for determining exponential estimates for an exponentially stable delay system (2) without any additional a priori information. □

5 Illustration

In this section we illustrate the results of the previous section by an example.

Example 7 Consider the system

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.7 \\ 0.7 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} -0.49 & 0 \\ 0 & -0.49 \end{pmatrix} x(t-2). \quad (21)$$

The characteristic quasipolynomial of the system is

$$f(s) = s^2 + 3s + 2 + 0.98e^{-2s} + 0.98se^{-2s} + [0.49]^2 e^{-4s}.$$

All the roots of this quasipolynomial lie in the open left half complex plane. The roots closest to the imaginary axis are

$$s_{1,2} \simeq -0.363 \pm j1.388$$

so that the spectral abscissa of the system is -0.363. Since delay systems satisfy the spectrum determined growth assumption [2], there exists for every $\varepsilon > 0$ a constant $\gamma_\varepsilon \geq 1$ such that the solutions of (21) satisfy the inequality

$$\|x(t,\varphi)\| \leq \gamma_\varepsilon \|\varphi\|_\infty e^{-(0.363-\varepsilon)t}, \quad t \geq 0.$$

We will now apply Theorem 4 in order to obtain an exponential estimate for the system (21) without making use of any knowledge about its spectral abscissa. Let us choose $W_k = I_n$, $k = 0, 1, ..., 4$ so that the functional $w(\cdot)$ (9) is given by

$$w(\varphi) = \|\varphi(0)\|^2 + \|\varphi(-1)\|^2 + \|\varphi(-2)\|^2 + \int_{-1}^{0} \|\varphi(\theta)\|^2 d\theta + \int_{-2}^{0} \|\varphi(\theta)\|^2 d\theta.$$

for $\varphi \in \mathcal{C}([-2,0],\mathbb{R}^2)$. Obviously

$$\|\varphi(0)\|^2 \leq w(\varphi) \leq 6 \|\varphi\|_\infty^2,$$

so we may choose $\beta_1 = 1$ and $\beta_2 = 6$.
The matrix $\widetilde{W}(\alpha)$ used in the proof of Theorem 4 is given by

$$\widetilde{W}(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \alpha \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} - \alpha^2 \begin{pmatrix} 0.7301 & 0 \\ 0 & 0.7301 \end{pmatrix}.$$

$\widetilde{W}(\alpha)$ is positive definite for $\alpha = 0.23$, so we may choose $\alpha_1 = 0.23$.
The corresponding functional $v(x_t)$ is of the form

$$\begin{aligned} v(x_t) =& x^\top(t)U(0)x(t) + 1.4x^\top(t) \int_{-1}^{0} U(-1-\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t+\theta)d\theta - \\ &- 0.98x^\top(t) \int_{-2}^{0} U(-2-\theta)x(t+\theta)d\theta + \\ &+ 0.49 \int_{-1}^{0} x^\top(t+\theta_2) \left[\int_{-1}^{0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta_1-\theta_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t+\theta_1)d\theta_1 \right] d\theta_2 - \\ &- 0.686 \int_{-1}^{0} x^\top(t+\theta_2) \left[\int_{-2}^{0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta_1-\theta_2-1)x(t+\theta_1)d\theta_1 \right] d\theta_2 + \\ &+ 0.2401 \int_{-2}^{0} x^\top(t+\theta_2) \left[\int_{-2}^{0} U(\theta_1-\theta_2)x(t+\theta_1)d\theta_1 \right] d\theta_2 + \\ &+ \int_{-1}^{0} (2+\theta) \|x(t+\theta)\|^2 d\theta + \int_{-2}^{0} (3+\theta) \|x(t+\theta)\|^2 d\theta. \end{aligned}$$

Here the Lyapunov matrix function

$$U(\tau) = 6 \int_0^\infty K^\top(t)K(t+\tau)d\tau$$

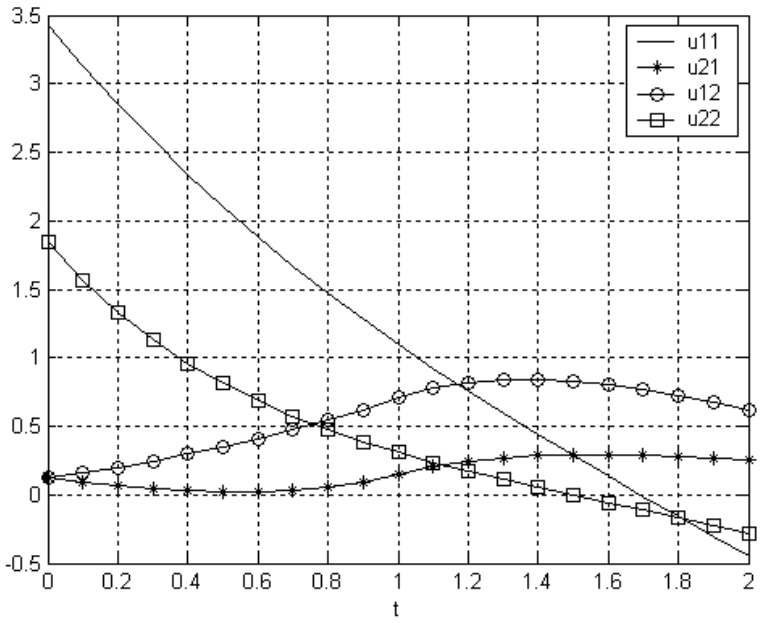

Figure 1: Matrix $U(\tau)$, piecewise linear approximation

satisfies the equation

$$\frac{d}{d\tau}U(\tau) = -U(\tau)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} + U(\tau-1)\begin{pmatrix}0 & 0.7\\ 0.7 & 0\end{pmatrix} - U(\tau-2)\begin{pmatrix}0.49 & 0\\ 0 & 0.49\end{pmatrix}. \qquad (22)$$

In Fig. 1 we plot the four components of a piecewise linear approximation of the Lyapunov matrix function. From the plot we get

$$\mu = 3.44 \text{ and } a = 0.7.$$

The upper bound (20) of $v(\cdot)$ has the following form

$$v(x_t) \leq \mu(1+3a)\|x(t)\|^2 + (\mu a + 2 + \frac{3}{2}\mu a^2)\int_{-1}^{0}\|x(t+\theta)\|^2\,d\theta + (\mu a + 3 + 3\mu a^2)\int_{-2}^{0}\|x(t+\theta)\|^2 d\theta.$$

The value α_2 should satisfy the conditions

$$\alpha_2 \geq \mu(1+3a) = 10.65, \quad \alpha_2 \geq \mu a + 3 + 3\mu a^2 = 10.46$$

and so we may choose $\alpha_2 = 10.65$.
As a result we obtain an exponential estimate (18) for the solutions of the system (21) with the following constants

$$\gamma = \sqrt{\frac{\alpha_2\beta_2}{\alpha_1}} \approx 16.69, \quad \sigma = \frac{1}{2\alpha_2} \approx 0.047.$$

In order to verify how well the piecewise linear approximation represents the Lyapunov matrix valued function we compute the solution of equation (22) with the initial condition generated by the linear piecewise approximation as initial function on $[-h, 0]$, see Remark (16). The four components of the corresponding solution are plotted on Fig. 2. A comparison Fig. 1 shows a good fit between the solution $U(\tau)$ and the piecewise linear approximation on $[0, h]$. □

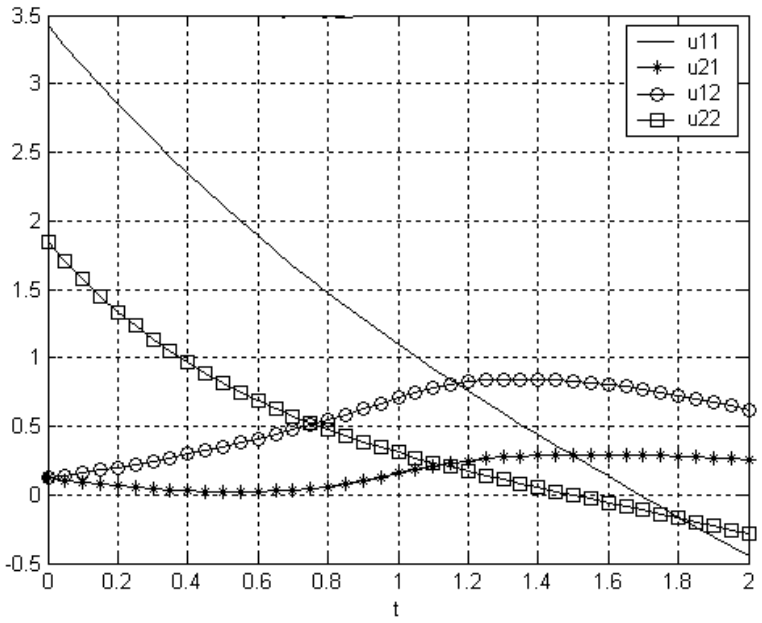

Figure 2: Matrix $U(\tau)$, numerical solution

References

[1] R. Bellman and K. L. Cooke. *Differential-Difference Equations*. Number 6 in Mathematics in Science and Engineering. Academic Press, New York, 1963.

[2] R. F. Curtain and H. J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Number 21 in Texts in Applied Mathematics. Springer-Verlag, New York, 1995.

[3] R. Datko. Uniform asymptotic stability of evolutionary processes in a Banach space. *SIAM J. Math. Anal.*, 3:428–445, 1972.

[4] A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags*. Academic Press, New York, 1966.

[5] J. K. Hale. *Theory of Functional Differential Equations*. Number 3 in Applied Mathematical Sciences. Springer-Verlag, New York, 2nd edition, 1977.

[6] D. Hinrichsen, E. Plischke, and A. J. Pritchard. Liapunov and Riccati equations for practical stability. In *Proc. 6th European Control Conf. 2001*, pages 2883–2888, 2001. Paper no. 8485 (CDROM).

[7] W. Z. Huang. Generalization of Liapunov's theorem in a linear delay system. *J. Math. Anal. Appl.*, 142(1):83–94, 1989.

[8] E. F. Infante and W. B. Castelan. A Liapunov functional for a matrix difference-differential equation. *J. Diff. Eqns.*, 29:439–451, 1978.

[9] V. L. Kharitonov and A. P. Zhabko. LLyapunov-Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39:15–20, 2003.

[10] N. N. Krasovskiĭ. On the application of the second method of Lyapunov for equations with time delays. *Prikl. Mat. Meh.*, 20:315–327, 1956.

[11] B. S. Razumikhin. On stability of systems with a delay. *Prikl. Mat. Meh.*, 20:500–512, 1956.

[12] I. M. Repin. Quadratic Liapunov functionals for systems with delay. *J. Appl. Math. Mech.*, 29:669–672, 1966. Translation of *Prikl. Mat. Meh.*, 29:564–566 (1965).

Berichte aus der Technomathematik ISSN 1435-7968

`http://www.math.uni-bremen.de/zetem/berichte.html`

— Vertrieb durch den Autor —

Reports **Stand: 22. April 2003**

98–01. Peter Benner, Heike Faßbender:
An Implicitly Restarted Symplectic Lanczos Method for the Symplectic Eigenvalue Problem, Juli 1998.

98–02. Heike Faßbender:
Sliding Window Schemes for Discrete Least-Squares Approximation by Trigonometric Polynomials, Juli 1998.

98–03. Peter Benner, Maribel Castillo, Enrique S. Quintana-Ortí:
Parallel Partial Stabilizing Algorithms for Large Linear Control Systems, Juli 1998.

98–04. Peter Benner:
Computational Methods for Linear–Quadratic Optimization, August 1998.

98–05. Peter Benner, Ralph Byers, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Solving Algebraic Riccati Equations on Parallel Computers Using Newton's Method with Exact Line Search, August 1998.

98–06. Lars Grüne, Fabian Wirth:
On the rate of convergence of infinite horizon discounted optimal value functions, November 1998.

98–07. Peter Benner, Volker Mehrmann, Hongguo Xu:
A Note on the Numerical Solution of Complex Hamiltonian and Skew-Hamiltonian Eigenvalue Problems, November 1998.

98–08. Eberhard Bänsch, Burkhard Höhn:
Numerical simulation of a silicon floating zone with a free capillary surface, Dezember 1998.

99–01. Heike Faßbender:
The Parameterized SR Algorithm for Symplectic (Butterfly) Matrices, Februar 1999.

99–02. Heike Faßbender:
Error Analysis of the symplectic Lanczos Method for the symplectic Eigenvalue Problem, März 1999.

99–03. Eberhard Bänsch, Alfred Schmidt:
Simulation of dendritic crystal growth with thermal convection, März 1999.

99–04. Eberhard Bänsch:
Finite element discretization of the Navier-Stokes equations with a free capillary surface, März 1999.

99–05. Peter Benner:
Mathematik in der Berufspraxis, Juli 1999.

99–06. Andrew D.B. Paice, Fabian R. Wirth:
Robustness of nonlinear systems and their domains of attraction, August 1999.

99–07. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Balanced Truncation Model Reduction of Large-Scale Dense Systems on Parallel Computers, September 1999.

99–08. Ronald Stöver:
Collocation methods for solving linear differential-algebraic boundary value problems, September 1999.

99–09. Huseyin Akcay:
Modelling with Orthonormal Basis Functions, September 1999.

99–10. Heike Faßbender, D. Steven Mackey, Niloufer Mackey:
Hamilton and Jacobi come full circle: Jacobi algorithms for structured Hamiltonian eigenproblems, Oktober 1999.

99–11. Peter Benner, Vincente Hernández, Antonio Pastor:
On the Kleinman Iteration for Nonstabilizable System, Oktober 1999.

99–12. Peter Benner, Heike Faßbender:
A Hybrid Method for the Numerical Solution of Discrete-Time Algebraic Riccati Equations, November 1999.

99–13. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Numerical Solution of Schur Stable Linear Matrix Equations on Multicomputers, November 1999.

99–14. Eberhard Bänsch, Karol Mikula:
Adaptivity in 3D Image Processing, Dezember 1999.

00–01. Peter Benner, Volker Mehrmann, Hongguo Xu:
Perturbation Analysis for the Eigenvalue Problem of a Formal Product of Matrices, Januar 2000.

00–02. Ziping Huang:
Finite Element Method for Mixed Problems with Penalty, Januar 2000.

00–03. Gianfrancesco Martinico:
Recursive mesh refinement in 3D, Februar 2000.

00–04. Eberhard Bänsch, Christoph Egbers, Oliver Meincke, Nicoleta Scurtu:
Taylor-Couette System with Asymmetric Boundary Conditions, Februar 2000.

00–05. Peter Benner:
Symplectic Balancing of Hamiltonian Matrices, Februar 2000.

00–06. Fabio Camilli, Lars Grüne, Fabian Wirth:
A regularization of Zubov's equation for robust domains of attraction, März 2000.

00–07. Michael Wolff, Eberhard Bänsch, Michael Böhm, Dominic Davis:
Modellierung der Abkühlung von Stahlbrammen, März 2000.

00–08. Stephan Dahlke, Peter Maaß, Gerd Teschke:
Interpolating Scaling Functions with Duals, April 2000.

00–09. Jochen Behrens, Fabian Wirth:
A globalization procedure for locally stabilizing controllers, Mai 2000.

00–10. Peter Maaß, Gerd Teschke, Werner Willmann, Günter Wollmann:
Detection and Classification of Material Attributes – A Practical Application of Wavelet Analysis, Mai 2000.

00–11. Stefan Boschert, Alfred Schmidt, Kunibert G. Siebert, Eberhard Bänsch, Klaus-Werner Benz, Gerhard Dziuk, Thomas Kaiser:
Simulation of Industrial Crystal Growth by the Vertical Bridgman Method, Mai 2000.

00–12. Volker Lehmann, Gerd Teschke:
Wavelet Based Methods for Improved Wind Profiler Signal Processing, Mai 2000.

00–13. Stephan Dahlke, Peter Maass:
A Note on Interpolating Scaling Functions, August 2000.

00–14. Ronny Ramlau, Rolf Clackdoyle, Frédéric Noo, Girish Bal:
Accurate Attenuation Correction in SPECT Imaging using Optimization of Bilinear Functions and Assuming an Unknown Spatially-Varying Attenuation Distribution, September 2000.

00–15. Peter Kunkel, Ronald Stöver:
Symmetric collocation methods for linear differential-algebraic boundary value problems, September 2000.

00–16. Fabian Wirth:
The generalized spectral radius and extremal norms, Oktober 2000.

00–17. Frank Stenger, Ahmad Reza Naghsh-Nilchi, Jenny Niebsch, Ronny Ramlau:
A unified approach to the approximate solution of PDE, November 2000.

00–18. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Parallel algorithms for model reduction of discrete–time systems, Dezember 2000.

00–19. Ronny Ramlau:
A steepest descent algorithm for the global minimization of Tikhonov–Phillips functional, Dezember 2000.

01–01. Efficient methods in hyperthermia treatment planning:
Torsten Köhler, Peter Maass, Peter Wust, Martin Seebass, Januar 2001.

01–02. Parallel Algorithms for LQ Optimal Control of Discrete-Time Periodic Linear Systems:
Peter Benner, Ralph Byers, Rafael Mayo, Enrique S. Quintana-Ortí, Vicente Hernández, Februar 2001.

01–03. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Efficient Numerical Algorithms for Balanced Stochastic Truncation, März 2001.

01–04. Peter Benner, Maribel Castillo, Enrique S. Quintana-Ortí:
Partial Stabilization of Large-Scale Discrete-Time Linear Control Systems, März 2001.

01–05. Stephan Dahlke:
Besov Regularity for Edge Singularities in Polyhedral Domains, Mai 2001.

01–06. Fabian Wirth:
A linearization principle for robustness with respect to time-varying perturbations, Mai 2001.

01–07. Stephan Dahlke, Wolfgang Dahmen, Karsten Urban:
Adaptive Wavelet Methods for Saddle Point Problems - Optimal Convergence Rates, Juli 2001.

01–08. Ronny Ramlau:
Morozov's Discrepancy Principle for Tikhonov regularization of nonlinear operators, Juli 2001.

01–09. Michael Wolff:
Einführung des Drucks für die instationären Stokes–Gleichungen mittels der Methode von Kaplan, Juli 2001.

01–10. Stephan Dahlke, Peter Maaß, Gerd Teschke:
Reconstruction of Reflectivity Desities by Wavelet Transforms, August 2001.

01–11. Stephan Dahlke:
Besov Regularity for the Neumann Problem, August 2001.

01–12. Bernard Haasdonk, Mario Ohlberger, Martin Rumpf, Alfred Schmidt, Kunibert G. Siebert:
h–p–Multiresolution Visualization of Adaptive Finite Element Simulations, Oktober 2001.

01–13. Stephan Dahlke, Gabriele Steidl, Gerd Teschke:
Coorbit Spaces and Banach Frames on Homogeneous Spaces with Applications to Analyzing Functions on Spheres, August 2001.

02–01. Michael Wolff, Michael Böhm:
Zur Modellierung der Thermoelasto-Plastizität mit Phasenumwandlungen bei Stählen sowie der Umwandlungsplastizität, Februar 2002.

02–02. Stephan Dahlke, Peter Maaß:
An Outline of Adaptive Wavelet Galerkin Methods for Tikhonov Regularization of Inverse Parabolic Problems, April 2002.

02–03. Alfred Schmidt:
A Multi-Mesh Finite Element Method for Phase Field Simulations, April 2002.

02–04. Sergey N. Dachkovski, Michael Böhm:
A Note on Finite Thermoplasticity with Phase Changes, Juli 2002.

02–05. Michael Wolff, Michael Böhm:
Phasenumwandlungen und Umwandlungsplastizität bei Stählen im Konzept der Thermoelasto-Plastizität, Juli 2002.

02–06. Gerd Teschke:
Construction of Generalized Uncertainty Principles and Wavelets in Anisotropic Sobolev Spaces, August 2002.

02–07. Ronny Ramlau:
TIGRA – an iterative algorithm for regularizing nonlinear ill–posed problems, August 2002.

02–08. Michael Lukaschewitsch, Peter Maaß, Michael Pidcock:
Tikhonov regularization for Electrical Impedance Tomography on unbounded domains, Oktober 2002.

02–09. Volker Dicken, Peter Maaß, Ingo Menz, Jenny Niebsch, Ronny Ramlau:
Inverse Unwuchtidentifikation an Flugtriebwerken mit Quetschöldämpfern, Oktober 2002.

02–10. Torsten Köhler, Peter Maaß, Jan Kalden:
Time-series forecasting for total volume data and charge back data, November 2002.

02–11. Angelika Bunse-Gerstner:
A Short Introduction to Iterative Methods for Large Linear Systems, November 2002.

02–12. Peter Kunkel, Volker Mehrmann, Ronald Stöver:
Symmetric Collocation for Unstructured Nonlinear Differential-Algebraic Equations of Arbitrary Index, November 2002.

02–13. Michael Wolff:
Ringvorlesung: Distortion Engineering 2
Kontinuumsmechanische Modellierung des Materialverhaltens von Stahl unter Berücksichtigung von Phasenumwandlungen, Dezember 2002.

02–14. Michael Böhm, Martin Hunkel, Alfred Schmidt, Michael Wolff:
Evaluation of various phase-transition models for 100Cr6 for application in commercial FEM programs, Dezember 2002.

03–01. Michael Wolff, Michael Böhm, Serguei Dachkovski:
Volumenanteile versus Massenanteile - der Dilatometerversuch aus der Sicht der Kontinuumsmechanik, Januar 2003.

03–02. Daniel Kessler, Ricardo H. Nochetto, Alfred Schmidt:
A posteriori error control for the Allen-Cahn Problem: circumventing Gronwall's inequality, März 2003.

03–03. Michael Böhm, Jörg Kropp, Adrian Muntean:
On a Prediction Model for Concrete Carbonation based on Moving Interfaces - Interface concentrated Reactions, April 2003.

03–04. Michael Böhm, Jörg Kropp, Adrian Muntean:
A Two-Reaction-Zones Moving-Interface Model for Predicting $Ca(OH)_2$ Carbonation in Concrete, April 2003.

03–05. Vladimir L. Kharitonov, Diederich Hinrichsen:
Exponential estimates for time delay systems, April 2003.