The generalized spectral radius and extremal norms

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Abstract

The generalized spectral radius, also known under the name of joint spectral radius, or (after taking logarithms) maximal Lyapunov exponent of a discrete inclusion is examined. We present a new proof for a result of Barabanov, which states that for irreducible sets of matrices an extremal norm always exists. This approach lends itself easily to the analysis of further properties of the generalized spectral radius. We prove that the generalized spectral radius is locally Lipschitz continuous on the space of compact irreducible sets of matrices and show a strict monotonicity property of the generalized spectral radius. Sufficient conditions for the existence of extremal norms are obtained.

1 Introduction

In recent years discrete inclusions have attracted the interest of researchers from quite distinct fields. They occur in the theory of wavelets, where discrete inclusions can be used to determine Hoelder exponents of compactly supported wavelets, see Daubechies and Lagarias [8], Heil and Strang [15], and references therein. For discussions of applications in the theory of Markov

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chains, iterated function systems, hysteresis nonlinearities we refer to references given in the papers [2, 14, 26]. For stability analysis of numerical algorithms using this framework we refer to Guglielmi and Zennaro [13]. And this list is, of course, far from complete.

Given a set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we are interested in the asymptotic behavior of solutions of the discrete inclusion

$$\begin{aligned} x(t+1) &\in \{Ax(t) \mid A \in \mathcal{M}\}\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{K}^n\,. \end{aligned} \tag{1}$$

This problem has been studied from an abstract point of view in [23, 1, 16, 2, 17, 14, 3, 10, 20, 24, 26, 12]. Infinite dimensional versions of this problem have been studied in [14, 28]. A more general spectral theory for a wide class of discrete inclusions can be found in [27], see also [7] for continuous time analogues.

This author was first interested in stability of discrete inclusions from a control theory point of view. A discrete inclusion of the form (1) may be interpreted as a model for time-varying uncertainty of a nominal system $x(t+1) = Ax(t)$. One problem area in this direction consists in the calculation of *stability radii*. Given an increasing family of sets $\mathcal{U} := \{\mathcal{M}_\gamma \mid \gamma \geq 0\}$ the problem is to determine the smallest $\gamma > 0$ such that (1) defined by $\mathcal{M}_\gamma$ is not exponentially stable, see also [29].

A recurrent problem is the question whether $\mathcal{M}$ has *left convergent products* or is *product bounded*. The first of these properties means that for any sequence $\{A(k)\}_{k\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}$ it holds that

$$A(k)A(k-1)\cdots A(0)$$

is convergent for $k \to \infty$. Product boundedness means that there is a constant $C > 0$ such that $\|A(k)A(k-1)\cdots A(0)\| < C$ for all possible products of matrices in $\mathcal{M}$. This property is also called *absolute stability* in [16] and *nondefectiveness* in [13].

The property of left convergent products has been studied in [8, 3, 10]. In particular, this property is characterized in a number of ways for finite sets of matrices by Vladimirov et al. [26], where also results on general sets of matrices are obtained, which are not quite as far-reaching.

One of the main tools in the study of discrete inclusions consists of the generalized (or joint) spectral radius. This approach originates with Rota and Strang [23], who defined the joint spectral radius and Daubechies and Lagarias [8], who did the same for the generalized spectral radius. We now define these two numbers. Associated to the set $\mathcal{M}$ we can consider the sets of products of length t

$$\mathcal{S}_t := \{A(t-1)\dots A(0) \mid A(s) \in \mathcal{M}\,, s = 0,\dots,t-1\}\,,$$

and the semigroup given by

$$\mathcal{S} := \bigcup_{t=1}^{\infty} \mathcal{S}_t\,.$$

Let $\|\cdot\|$ be some operator norm on $\mathbb{K}^{n\times n}$ and define for $t \in \mathbb{N}$

$$\overline{\rho}_t(\mathcal{M}) := \sup\{r(S_t)^{1/t} \mid S_t \in \mathcal{S}_t\}\,, \qquad \hat{\rho}_t(\mathcal{M}) := \sup\{\|S_t\|^{1/t} \mid S_t \in \mathcal{S}_t\}\,. \tag{2}$$

The *joint spectral radius*, respectively the *generalized spectral radius* are now defined as

$$\overline{\rho}(\mathcal{M}) := \limsup_{t\to\infty} \overline{\rho}_t(\mathcal{M})\,, \quad \hat{\rho}(\mathcal{M}) := \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M})\,.$$

However, there is no need to insist on different notation as Theorem 4 in Berger and Wang [2] states that for bounded $\mathcal{M}$ we have $\hat{\rho}(\mathcal{M}) = \overline{\rho}(\mathcal{M})$, so that we will simply use the notation $\rho(\mathcal{M})$. Alternative proofs for this equality can be found in [9, 24]. Note also that for all $t \geq 1$

$$\overline{\rho}_t(\mathcal{M}) \leq \rho(\mathcal{M}) \leq \hat{\rho}_t(\mathcal{M})\,. \tag{3}$$

In a paper by Lagarias and Wang [17] the by now famous "finiteness conjecture" was formulated, which states that for a finite set of matrices $\mathcal{M}$ there always exists a $t \geq 1$ such that

$$\rho(\mathcal{M}) = \overline{\rho}_t(\mathcal{M})\,.$$

It has recently been shown by Bousch and Mairesse [5], that this conjecture is false. But in special cases it can be shown to hold, see [14, 17].

The calculation of the generalized spectral radius has been treated using different approaches. While Gripenberg [11] and Maesumi [19] reduce the number of matrix products that have to be evaluated to obtain upper, respectively lower bounds given by $\hat{\rho}_t, \overline{\rho}_t$, an optimal control approach is used in [29]. Simple computational results cannot be really expected as Kozyakin [16] has shown that ρ is not an algebraic function on the vector space of k-tuples of $n\times n$ matrices and the determination of ρ is NP-hard by a result of Tsitsiklis and Blondel [25].

In this paper we show two further properties of the generalized spectral radius, namely local Lipschitz continuity on the set of irreducible compact sets of matrices and a monotonicity property. Our approach is based on a further important idea in the analysis of exponential stability of discrete inclusions that was introduced by Barabanov [1]. Recall that $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is called *irreducible* if only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are invariant under all matrices $A \in \mathcal{M}$. Otherwise $\mathcal{M}$ is called *reducible*.

An immediate consequence of irreducibility of $\mathcal{M}$ is that $\rho(\mathcal{M}) > -\infty$, because in this case the semigroup $\mathcal{S}$ is irreducible and does therefore not consist of nilpotent elements by the Levitzky theorem [18]. Note that this implies in particular, that we can always normalize an irreducible set of matrices $\mathcal{M}$ to $\rho(\mathcal{M})^{-1}\mathcal{M}$ which is a set with generalized spectral radius equal to 1.

The fundamental contribution of Barabanov consists of the following result.

Theorem 1.1 *If $\mathcal{M}$ is compact and irreducible, then there exists a norm v on $\mathbb{K}^n$ such that*

(i) for all $x \in \mathbb{K}^n, A \in \mathcal{M}$ it holds that

$$v(Ax) \leq \rho(\mathcal{M})v(x)\,,$$

(ii) for all $x \in \mathbb{K}^n$ there exists an $A \in \mathcal{M}$ such that

$$v(Ax) = \rho(\mathcal{M})v(x)\,.$$

We will in particular be interested in the existence of *extremal norms*, that is norms with the property that $\|A\| \leq \rho(\mathcal{M})$ for all $A \in \mathcal{M}$. It follows from the result in Kozyakin that an extremal norm exists for $\mathcal{M}$ if and only if $\rho(\mathcal{M})^{-1}\mathcal{M}$ is product bounded, [16, Theorem 3]. A further characterization is obtained in [12, Section 3]. As the question whether a pair of matrices is product bounded is undecidable by a recent result of Blondel and Tsitsiklis [4] we do not expect to obtain an easily checkable criterion and so our condition is just sufficient but not necessary.

The paper is organized as follows. In Section 2 we present the class of systems that is studied; as our methods work just as well for semigroups generated by continuous time systems we briefly introduce the necessary concepts. In Section 3 we introduce our main technical tool, which we call the *limit semigroup* and which is obtained as the ω-limit set of the semigroup normalized to a generalized spectral radius equal to 1.

In Section 4 we use the result of the previous section to show that ρ is locally Lipschitz continuous on the set of compact irreducible sets of matrices. In Section 5 we show that the generalized spectral radius is a strictly increasing function under a natural growth condition on a function with values in the compact sets of matrices. This result is motivated by the problem of calculating time-varying stability radii and its consequences will be discussed in a forthcoming paper.

Finally, in Section 6 we show the existence of extremal norms under a nondefectiveness condition, which generalizes the corresponding result for the spectral radius of a matrix. Note, that we found it useful to use a slightly different sense of the word nondefective than found in the literature. In [12] "nondefective" just means that an extremal norm exists.

2 Preliminaries

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Given a set $\emptyset \neq \mathcal{M} \subset \mathbb{K}^{n\times n}$ we consider the discrete inclusion

$$\begin{array}{rcll} x(t+1) & \in & \{Ax(t) \mid A \in \mathcal{M}\}\,, & t \in \mathbb{N} \\ x(0) & = & x_0 & \in \mathbb{K}^n\,. \end{array} \tag{4}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called a solution of (1) with initial condition x_0 if $x(0) = x_0$ and if for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$. We continue to use the notation introduced in Section 1.

As all our arguments are also valid in continuous time, we will just consider an irreducible semigroup $\mathcal{S} \subset \mathbb{K}^{n\times n}$ with an associated time scale $\mathbb{T} = \mathbb{N}, \mathbb{R}_+ := [0,\infty)$. To be concrete, in the case $\mathbb{T} = \mathbb{R}_+$ we assume that the semigroup is generated by a differential inclusion

$$\dot{x} \in \{Ax(t) \mid A \in \mathcal{M}\}\,, \tag{5}$$

where $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is compact. In the latter case the elements of $\mathcal{S}_t, t \in \mathbb{R}_+$ are the evolution operators $\Phi_{A(\cdot)}(t,0)$ corresponding to measurable functions $A : \mathbb{R}_+ \to \mathcal{M}$ and the time-varying differential equation

$$\dot{x}(t) = A(t)x(t),\ \text{a.e.}$$

For a semigroup defined by (5) the quantities $\overline{\rho}_t(\mathcal{S}), \hat{\rho}_t(\mathcal{S}), t \in \mathbb{R}_+$ can be defined analogously to (2) and make obviously sense.

We will denote the corresponding limit by $\rho(\mathcal{S})$. We call this quantity the maximal Lyapunov exponent if we consider differential inclusions (although in the literature this name is normally reserved for $\log \rho(\mathcal{S})$). There is abundant literature on the theory of Lyapunov exponents of differential inclusions, see e.g. [6, 7] and references therein.

If we fear that there is a chance of confusion we will denote the generalized spectral radius given by a set $\mathcal{M}$ via the discrete inclusion (1) by $\rho(\mathcal{M}, \mathbb{N})$ and the maximal Lyapunov exponent by $\rho(\mathcal{M}, \mathbb{R}_+)$.

Note that given a semigroup $(\mathcal{S}, \mathbb{R}_+)$ we can always associate a discrete inclusion by defining $\mathcal{M} := \mathcal{S}_1$. Under our assumptions it is an easy exercise to check that $\rho(\mathcal{S}, \mathbb{R}_+) = \rho(\mathcal{M}, \mathbb{N})$. In the sequel, we will always tacitly assume that $\mathcal{S}$ is generated by a discrete inclusions of the form (4) or a differential inclusion of the form (5), if we just speak of a semigroup $(\mathcal{S}, \mathbb{T})$.

Definition 2.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be a semigroup in $\mathbb{K}^{n\times n}$. A norm v on $\mathbb{K}^n$ is called* Barabanov norm *corresponding to $\mathcal{S}$ if*

(i) $v(Sx) \leq \rho(\mathcal{S})^t v(x)\,, \quad$ *for all* $x \in \mathbb{K}^n, t \in \mathbb{T}, S \in \mathcal{S}_t$,

(ii) for all $x \in \mathbb{K}^n$, $t \in \mathbb{T}$ there is an $S \in \operatorname{cl} \mathcal{S}_t$ such that

$$v(Sx) = \rho(\mathcal{S})^t v(x)\,.$$

A norm v on $\mathbb{K}^n$ is called extremal *for $\mathcal{S}$ if for the corresponding operator norm it holds that*

$$v(S) \leq \rho(\mathcal{S})^t\,, \ \textit{for all}\ t \in \mathbb{T}, S \in \mathcal{S}_t\,.$$

We will investigate further conditions guaranteeing the existence of extremal norms in Section 6.

We will also consider the behavior of the generalized spectral radius as a function of the set $\mathcal{M}$. As we only have to consider compact sets $\mathcal{M} \subset \mathbb{K}^{n\times n}$, we introduce

$$\mathcal{K}(\mathbb{K}^{n\times n}) := \{\mathcal{M} \subset \mathbb{K}^{n\times n} \mid \mathcal{M} \text{ compact, nonempty}\}\,.$$

The space $\mathcal{K}(\mathbb{K}^{n\times n})$ becomes a complete metric space if it is endowed with the usual Hausdorff metric defined by

$$H(\mathcal{M}, \mathcal{N}) := \max\{\max_{A\in\mathcal{M}} \operatorname{dist}(A, \mathcal{N}), \max_{B\in\mathcal{N}} \operatorname{dist}(B, \mathcal{M})\}\,.$$

Note that with respect to this topology the set

$$I(\mathbb{K}^{n\times n}) := \{\mathcal{M} \in \mathcal{K}(\mathbb{K}^{n\times n}) \mid \mathcal{M} \text{ irreducible}\}$$

is open and dense in $\mathcal{K}(\mathbb{K}^{n\times n})$.

3 The limit semigroup

In this section we present an alternative and we hope less intricate proof of Barabanov's result. We need the following property of irreducible semigroups.

Lemma 3.1 *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*,* $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ *and let* $(\mathcal{S}, \mathbb{T})$ *be an irreducible semigroup in* $\mathbb{K}^{n\times n}$*. Then there are* $\varepsilon > 0$ *and* $\tau \in \mathbb{T}$ *such that for all* $z \in \mathbb{K}^n, A \in \mathbb{K}^{n\times n}$ *there is an* $S \in \bigcup_{1\le t\le \tau} \mathcal{S}_t$ *with*

$$\|ASz\| \geq \varepsilon \|A\| \|z\| .$$

Proof: Assume the assertion is false, so that there are $\varepsilon_k \to 0$, $\tau_k \to \infty, \tau_k \in \mathbb{T}, z_k \in \mathbb{K}^n, A_k \in \mathbb{K}^{n\times n}$ such that for all $S \in \bigcup_{1\le t\le \tau_k} \mathcal{S}_t$ we have

$$\|A_k S z_k\| < \varepsilon_k \|A_k\| \|z_k\| . \tag{6}$$

Without loss of generality we may assume that $\|z_k\| = \|A_k\| = 1$. Thus we may assume $z_k \to z, A_k \to A$ with $\|z\| = \|A\| = 1$. Then irreducibility of $\mathcal{S}$ implies that there exists an $S^* \in \mathcal{S}$ with

$$\|AS^*z\| = \varepsilon^* > 0 ,$$

otherwise $\{Sz \mid S \in \mathcal{S}\}$ is contained in the kernel of A. This, however, contradicts irreducibility of $\mathcal{S}$ as $\mathbb{K}^n \neq \ker A$ due to $\|A\| = 1$. For all k large enough we have $S^* \in \bigcup_{1\le t\le \tau_k} \mathcal{S}_t$ and

$$\|A_k S^* z_k\| \geq \varepsilon^*/2 ,$$

which contradicts (6). This concludes the proof. □

Given our irreducible semigroup $(\mathcal{S}, \mathbb{T})$ we define the *limit semigroup* $\mathcal{S}_\infty$ by

$$\mathcal{S}_\infty := \{S \in \mathbb{K}^{n\times n} \mid \exists t_k \to \infty, S_{t_k} \in \mathcal{S}_{t_k} \text{ such that } \rho(\mathcal{S})^{-t_k} S_{t_k} \to S\} . \tag{7}$$

We note the following properties of $\mathcal{S}_\infty$.

Proposition 3.2 *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*,* $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ *and let* $(\mathcal{S}, \mathbb{T})$ *be an irreducible semigroup in* $\mathbb{K}^{n\times n}$*. The set* $\mathcal{S}_\infty$ *defined by* (7) *satisfies*

(i) $\mathcal{S}_\infty$ *is compact and nonempty,* $\mathcal{S}_\infty \neq \{0\}$*,*

(ii) $\mathcal{S}_\infty$ *is a semigroup,*

(iii) for $T \in \mathcal{S}_t, S \in \mathcal{S}_\infty$ we have

$$\rho(\mathcal{S})^{-t}TS\,, \quad \rho(\mathcal{S})^{-t}ST \in \mathcal{S}_\infty\,,$$

(iv) for all $t \in \mathbb{T}$, $S \in \mathcal{S}_\infty$ there exist $T \in \mathcal{S}_\infty, A \in \operatorname{cl}\mathcal{S}_t$ as well as $R \in \mathcal{S}_\infty, B \in \operatorname{cl}\mathcal{S}_t$ such that

$$S = \rho(\mathcal{S})^{-t}TA = \rho(\mathcal{S})^{-t}BR\,,$$

(v) for all $S \in \mathcal{S}_\infty$ there exist $R, T \in \mathcal{S}_\infty$ with

$$S = RT\,,$$

(vi) $\mathcal{S}_\infty$ is irreducible.

Proof: Without loss of generality we may assume $\rho(\mathcal{S}) = 1$ in this proof.

(i) For $A \in \mathcal{S}_t$ it holds that $r(A) \leq \rho(\mathcal{S})^t = 1$, hence $\{A^t\}$ is a bounded sequence which has an accumulation point S. By definition $S \in \mathcal{S}_\infty$. To see that $\mathcal{S}_\infty$ is closed it suffices to use a standard argument from the construction of ω-limit sets.

In order to show that $\mathcal{S}_\infty$ is bounded assume this is not the case and let $\varepsilon > 0$ and $\tau \in \mathbb{T}$ be the constants given by Lemma 3.1. Unboundedness of $\mathcal{S}_\infty$ implies that there exists some $t \in \mathbb{T}$, $S \in \mathcal{S}_t$ with $\|S\| > 2/\varepsilon$. Thus for $x_0, \|x_0\| = 1$ arbitrary, there is a $T \in \bigcup_{1 \leq t \leq \tau} \mathcal{S}_t$ with

$$\|STx_0\| > 2$$

and applying this argument repeatedly we obtain a sequence $\{T_k\}_{k \in \mathbb{N}} \subset \bigcup_{1 \leq t \leq \tau} \mathcal{S}_t$ such that

$$\|ST_k \dots ST_1 x_0\| > 2^k\,, \quad k \in \mathbb{N}\,.$$

This implies $\hat{\rho}_{kt+\tau_k}(\mathcal{S}) \geq 2^{1/(t+\tau)}$, where $k \leq \tau_k \leq k\tau$, a contradiction.

In particular, the last argument also shows, that $\mathcal{S}$ is bounded, on the other hand from (3) we have that each $\mathcal{S}_t$ contains an element of norm at least 1. Hence $\mathcal{S}_\infty$ contains a nonzero element.

(ii) Let $S, T \in \mathcal{S}_\infty$ and consider sequences $s_k, t_k \to \infty$, $S_k \in \mathcal{S}_{s_k}, S_k \to S$ and $T_k \in \mathcal{S}_{t_k}, T_k \to T$. Then

$$\|ST - S_kT_k\| \leq \|S - S_k\|\|T\| + \|S_k\|\|T - T_k\|\,,$$

which goes to zero as both terms go to zero for $k \to \infty$. Hence $ST \in \mathcal{S}_\infty$.

(iii) This is clear, as approximation of S by a sequence S_k implies approximation of TS and ST by TS_k, respectively S_kT.

(iv) Let $t_k \to \infty, S_k \in \mathcal{S}_{t_k}$ be sequences such that $S_k \to S$. We can write $S_k = T_k A_k$ with $T_k \in \mathcal{S}_{t_k - t}, A_k \in \mathcal{S}_t$. Without loss of generality $A_k \to A \in \operatorname{cl} \mathcal{S}_t$ and $T_k \to T \in \mathcal{S}_\infty$. This implies $S = TA$, as required. The argument for the left factorization is exactly the same.

(v) Let $t_k \to \infty$ be a sequence in $\mathbb{T}$. By (iv) we can factorize for each k $S = A_k T_k$ with $A_k \in \operatorname{cl} \mathcal{S}_{t_k}, T_k \in \mathcal{S}_\infty$. Now for suitable subsequences we have $A_k \to R \in \mathcal{S}_\infty$, as we may approximate A_k by elements in $\mathcal{S}_{t_k}$, and $T_k \to T$, as $\mathcal{S}_\infty$ is compact. This implies $S = RT$.

(vi) By (i), (ii) and (iii) we know that

$$\overline{\mathcal{S}} := \mathcal{S}_\infty \cup \bigcup_{t \in \mathbb{T}} \rho(\mathcal{S})^{-t} \mathcal{S}_t$$

is an irreducible semigroup of which $\mathcal{S}_\infty$ is a closed nonzero semigroup ideal. Now $\mathcal{S}_\infty$ is irreducible by [22, Lemma 1].

□

We give an easy example for the above construction, that will turn out to be of use in the remainder of the article.

Example 3.3 Consider the set

$$\mathcal{M} := \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

For $\mathbb{T} = \mathbb{N}$ it is easy to see that

$$\mathcal{S}_{2k} = \left\{ 0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

whereas $\mathcal{S}_{2k+1} = \mathcal{M} \cup \{0\}$. Hence $\mathcal{S}_\infty = \mathcal{M} \cup \mathcal{S}_2$.

Given our irreducible semigroup $(\mathcal{S}, \mathbb{T})$ and the associated limit semigroup $\mathcal{S}_\infty$ we now define the function

$$v(x) := \max_{S \in \mathcal{S}_\infty} \|Sx\| \tag{8}$$

and note that this defines the norm we are looking for.

Lemma 3.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroup in $\mathbb{K}^{n \times n}$. Then v is a Barabanov norm for $\mathcal{S}$.*

Proof:

(i) We first show that v is a norm. The properties $v(0) = 0, v(\lambda x) = |\lambda| v(x)$ are clear. If $x \neq 0$ then $v(x) \neq 0$ as otherwise span $\{x\}$ would be in the kernel of all $S \in \mathcal{S}_\infty$ contradicting irreducibility. The function $v(x)$ is finite as $\mathcal{S}_\infty$ is compact and finally

$$v(x+y) \leq \max_{S \in \mathcal{S}_\infty} \|Sx\| + \|Sy\| \leq \max_{S \in \mathcal{S}_\infty} \|Sx\| + \max_{S \in \mathcal{S}_\infty} \|Sy\| .$$

(ii) Without loss of generality let $\rho(\mathcal{S}) = 1$. Let $x \in \mathbb{K}^n, S \in \mathcal{S}$ be arbitrary, then

$$v(Sx) = \max_{T \in \mathcal{S}_\infty} \|TSx\| \leq \max_{T \in \mathcal{S}_\infty} \|Tx\| = v(x) , \tag{9}$$

as $TS \in \mathcal{S}_\infty$ for all $T \in \mathcal{S}_\infty$. To prove the second statement assume that $S_x \in \mathcal{S}_\infty$ is such that $v(x) = \|S_x x\|$, then by Proposition 3.2 (iv) S_x factors into $S_x = TA$ with $T \in \mathcal{S}_\infty, A \in \mathrm{cl}\, \mathcal{S}_t$. Hence

$$v(Ax) = \max_{S \in \mathcal{S}_\infty} \|SAx\| \geq \|TAx\| = v(x) ,$$

and so by (9) we have $v(Ax) = v(x)$.

□

The existence of a Barabanov norm has many consequences as already noted in [1]. For instance, it is immediate that $\rho(\mathcal{M}) = \rho(\mathrm{cl}\, \mathcal{M})$ and $\rho(\mathcal{M}) = \rho(\mathrm{conv}\, \mathcal{M})$. In particular, we cite the following continuity result from [1] which will be of use for us in the sequel. Alternatively, it has been noted by Heil and Strang [15] that the continuity of the generalized spectral radius is a direct consequence of the equality $\rho(\mathcal{M}) = \overline{\rho}(\mathcal{M}) = \hat{\rho}(\mathcal{M})$. (The argument is given for the case of pairs of matrices, but is easily seen to extend to general compact sets of matrices.)

Lemma 3.5 *The map $\mathcal{M} \to \rho(\mathcal{M})$ is continuous from $\mathcal{K}(\mathbb{K}^{n \times n})$ to $\mathbb{R}_+$.*

4 Lipschitz continuity of the generalized spectral radius

In this section we intend to show that the generalized spectral radius is locally Lipschitz continuous on the set of irreducible compact sets of matrices.

To this end we begin by an investigation of the variation of Barabanov norms under changes of $\mathcal{M}$. For irreducible $\mathcal{M}$ we will need to know how much the original norm is deformed under the definition (8). Therefore we introduce the quantities

$$c^-(\mathcal{M}) \;:=\; \min\{v(x) \mid \|x\| = 1\}\,, \tag{10}$$

$$c^+(\mathcal{M}) \;:=\; \max\{v(x) \mid \|x\| = 1\}\,. \tag{11}$$

Of course, these constant also depend on the choice $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{R}_+$, but we suppress this dependence. Note that for any $A \in \mathbb{K}^{n\times n}$ we have for the induced operator norm that

$$\frac{c^-(\mathcal{M})}{c^+(\mathcal{M})}\|A\| \le v(A) \le \frac{c^+(\mathcal{M})}{c^-(\mathcal{M})}\|A\|\,.$$

Theorem 4.1 *Let $P \subset I(\mathbb{K}^{n\times n})$ be compact and let $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{R}_+$. Then there is a constant $C > 0$ such that*

$$1 \le \frac{c^+(\mathcal{M})}{c^-(\mathcal{M})} \le C\,, \text{ for all } \mathcal{M} \in P\,.$$

Proof: Fix a timeset $\mathbb{T} \in \{\mathbb{N}, \mathbb{R}_+\}$ and consider the corresponding semigroups generated by the sets $\mathcal{M} \in P$. Assume to the contrary that there exists a sequence $\{\mathcal{M}_k\} \subset P$ such that

$$\frac{c^+(\mathcal{M}_k)}{c^-(\mathcal{M}_k)} \to \infty\,.$$

Without loss of generality we may assume that $\mathcal{M}_k \to \mathcal{M} \in P$.

For every k choose a $S_k \in \mathcal{S}_{\infty,k}$ (the limit semigroup corresponding to $(\mathcal{M}_k, \mathbb{T})$) such that $\|S_k\| = c^+(\mathcal{M}_k)$ and denote

$$\tilde{S}_k := \frac{S_k}{\|S_k\|}\,.$$

Then we may assume that $\tilde{S}_k \to \tilde{S}$ with $\|\tilde{S}\| = 1$.

Now let $x_0 \in \mathbb{K}^n, \|x_0\| = 1$ be arbitrary. We will show that $c^+(\mathcal{M}_k)/v_k(x_0)$ is bounded by a constant independent of x_0, which proves the assertion.

Let $\varepsilon > 0, \tau \in \mathbb{T}$ be the constants for $\mathcal{S}$ (the semigroup generated by $(\mathcal{M}, \mathbb{T})$) guaranteed by Lemma 3.1. Then by convergence of the sets $\mathcal{M}_k$ there exists a $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ and some $R_k \in \mathcal{S}_{t_k,k}, 1 \le t_k \le \tau$ we have

$$\|\tilde{S} R_k x_0\| \ge \frac{\varepsilon}{2}\,.$$

Note, that k_0 is chosen independently of x_0. For all $k \geq k_0$ we now define

$$T_k := \rho(\mathcal{S}_k)^{-t_k} S_k R_k \in \mathcal{S}_{\infty,k} \,,$$

and obtain for the norm v_k defined through $\mathcal{S}_{\infty,k}$ that

$$v_k(x_0) \geq \|T_k x_0\| = \rho(\mathcal{S}_k)^{-t_k} \|S_k R_k x_0\| = \frac{c^+(\mathcal{M}_k)}{\rho(\mathcal{S}_k)^{t_k}} \|\tilde{S}_k R_k x_0\|$$

$$\geq \frac{c^+(\mathcal{M}_k)}{\rho(\mathcal{S}_k)^{t_k}} \left(\|\tilde{S} R_k x_0\| - \|\tilde{S} - \tilde{S}_k\| \|R_k x_0\| \right) \geq \frac{c^+(\mathcal{M}_k)}{\rho(\mathcal{S}_k)^{t_k}} \left(\frac{\varepsilon}{2} - \|\tilde{S} - \tilde{S}_k\| \|R_k\| \right) .$$

The last term converges to zero be the definition of $\tilde{S}$ and as the set of all products of length at most τ is uniformly bounded over P. Furthermore, by continuity of ρ we have that $\rho(\mathcal{S}_k) \leq \rho(\mathcal{S}) + \varepsilon$ for $k \geq k_1 \geq k_0$, k_1 sufficiently large. This implies that for all k large enough we have

$$\frac{c^+(\mathcal{M}_k)}{v_k(x_0)} \leq \frac{4}{\varepsilon} \max\{1, \rho(\mathcal{S}) + \varepsilon\}^\tau \,.$$

This shows the assertion because again we have chosen k_1 independently of x_0. □

As an application of Theorem 4.1 we can sharpen Lemma 3.5. We first just treat the discrete time case.

Corollary 4.2 *The generalized spectral radius is locally Lipschitz continuous on $I(\mathbb{K}^{n\times n})$.*

Proof: Let $P \subset I(\mathbb{K}^{n\times n})$ be compact with respect to the Hausdorff metric. Fix $\mathcal{M}, \mathcal{N} \in P$ arbitrary and let v denote the Barabanov norm with respect to $\mathcal{M}$. In the Hausdorff metric induced by our original norm $\|\cdot\|$ we have

$$H(\mathcal{M}, \mathcal{N}) =: a \,,$$

which implies the in the Hausdorff metric H_v induced by v it holds that

$$H_v(\mathcal{M}, \mathcal{N}) \leq \frac{c^+(\mathcal{M})}{c^-(\mathcal{M})} a \leq Ca \,,$$

where C is a constant only depending on P which exists by Theorem 4.1. Hence for all $x \in \mathbb{K}^n, A \in \mathcal{N}$ it holds that there exists a $B \in \mathcal{M}$ with $v(A - B) \leq Ca$ and thus

$$v(Ax) \leq v(Bx) + v((A - B)x) \leq (\rho(\mathcal{M}) + Ca)\, v(x) \,.$$

Hence $\rho(\mathcal{N}) \leq \rho(\mathcal{M}) + Ca$ and by symmetry we obtain

$$|\rho(\mathcal{N}) - \rho(\mathcal{M})| \leq CH(\mathcal{M}, \mathcal{N}),$$

as desired. $\square$

We cannot expect that the generalized spectral radius $\rho(\cdot)$ is Lipschitz continuous on $\mathcal{K}(\mathbb{K}^{n\times n})$ as already standard perturbation theory of eigenvalues tells us that, generally, if an eigenvalue splitting occurs at an eigenvalue with modulus equal to the spectral radius then the spectral radius will behave like a Puiseux series, that is, not Lipschitzean at the splitting point. An example for this phenomenon is given by

$$A_\varepsilon := \begin{bmatrix} 1 & 1 \\ \varepsilon & 1 \end{bmatrix},$$

the spectral radius of which for $\varepsilon > 0$ is given by $r(A_\varepsilon) = 1 + \sqrt{\varepsilon}$.

We note that the result translates immediately to continuous time.

Corollary 4.3 *The maximal Lyapunov exponent is locally Lipschitz continuous on $I(\mathbb{K}^{n\times n})$.*

Proof: The map

$$\mathcal{M} \mapsto \mathcal{S}_1(\mathcal{M}, \mathbb{R}_+)$$

is locally Lipschitz continuous on $\mathbb{K}^{n\times n}$. We have already noted that

$$\rho(\mathcal{M}, \mathbb{R}_+) = \rho(\mathcal{S}_1(\mathcal{M}), \mathbb{N}).$$

Now the assertion is immediate from Corollary 4.2. $\square$

5 Strict monotonicity of the generalized spectral radius

In this section we will consider a further aspect of the generalized spectral radius under variation of the generating set $\mathcal{M}$. The methods we use here are restricted to the discrete time case, so that all results in this section are to understood with respect to the discrete inclusion (1). Whenever we treat different set of matrices $\mathcal{M}_1, \mathcal{M}_2$ in this section, we denote the semigroups generated by $\mathcal{M}_1$ and $\mathcal{M}_2$ by $\mathcal{S}(\mathcal{M}_i)$, $i = 1, 2$. On the other hand, the respective limit semigroups and Barabanov norms are denoted by $\mathcal{S}_{\infty,1}, \mathcal{S}_{\infty,2}$ and v_1, v_2 in order to avoid overloaded notation.

The results of this section are based on the following observation used by Radjavi [22], which we state for the sake of completeness and because it not formulated independently in Radjavi's paper. When we speak of a projection $P \in \mathbb{K}^{n\times n}$, we mean some matrix satisfying $P^2 = P$. Orthogonality is not required.

Lemma 5.1 *Let $\mathcal{S} \subset \mathbb{K}^{n\times n}$ be an irreducible semigroup. Then for every projection $P \in \mathbb{K}^{n\times n}$ with* $\operatorname{rank} P \geq 2$ *the set*

$$P\mathcal{S}P := \{PSP \mid S \in \mathcal{S}\}$$

is irreducible on $\operatorname{Im} P$.

Proof: Assume the assertion is false for some projection P with $\operatorname{rank} P \geq 2$ and let $X \subset \operatorname{Im} P$ be the nontrivial invariant subspace of $P\mathcal{S}P$ (with respect to $\operatorname{Im} P$). Then we have for $x \in X, S \in \mathcal{S}$ that

$$Sx = SPx = PSPx + (I-P)SPx \in X + \operatorname{Im}(I-P)\,.$$

The subspace on the right has dimension strictly less than n as X is a proper subspace of $\operatorname{Im} P$ and $\operatorname{Im} P$ and $\operatorname{Im}(I-P)$ are complementary subspaces. This shows that

$$Y := \operatorname{span}\{SX \mid S \in \mathcal{S}\}\,,$$

defines an invariant subspace of $\mathcal{S}$ of dimension less than n. Also $Y \neq \{0\}$ as otherwise X is in the kernel of every $S \in \mathcal{S}$ contradicting irreducibility. Thus Y is a nontrivial invariant subspace of $\mathcal{S}$, which contradicts our assumptions. $\square$

Note, that we cannot conclude that $P\mathcal{S}P$ is a semigroup unless $P \in \mathcal{S}$. Even in this case if we consider a semigroup $\mathcal{S}$ generated by $\mathcal{M}$ and assume that $\mathcal{M}$ thus also the semigroup $P\mathcal{S}P$ are irreducible, this does not imply that $P\mathcal{M}P$ is irreducible.

For the statement of the statement of the following lemma recall that a projection P is called *reducing* for A if $PA = AP$. A reducing eigenprojection corresponding to a subset $\Lambda \subset \sigma(A)$ is a reducing projection with the property that $\operatorname{Im} P$ is equal to the sum of the generalized eigenspaces corresponding to the eigenvalues $\lambda \in \Lambda$.

Lemma 5.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $\mathcal{M} \in I(\mathbb{K}^{n\times n})$ contain more than one point. Assume that for some $A \in \mathcal{M}$ we have $r(A) = \rho(\mathcal{M})$ and let P be the reducing eigenprojection of A corresponding to the eigenvalues with modulus $r(A)$. If one of the following properties is satisfied* *(i)* $P = I$,

(ii) $\mathbb{K} = \mathbb{R}$ *and* $\operatorname{rank} P \geq 3$,

(iii) $\mathbb{K} = \mathbb{C}$ *and* $\operatorname{rank} P \geq 2$

then for every $x \in \operatorname{Im} P$ *and every* $T \in \mathcal{S}_\infty$ *such that*

$$v_P(x) := \max_{S \in \mathcal{S}_\infty} \|PSx\| = \|PTx\|$$

there exists an $S \in \mathcal{S}_\infty$ *and a factorization* $S = \rho(\mathcal{M})^{-1}UAV$, $U, V \in \mathcal{S}_\infty$ *such that* $Sx = PTx$ *and*

$$\{PUBVx \mid B \in \mathcal{M}\}$$

contains more than one element.

Proof: Let $A \in \mathcal{M}$ and an eigenprojection P of A satisfy the assumptions. Assume $\rho(\mathcal{M}) = 1$ and fix $x \in \operatorname{Im} P$. Choose $T \in \mathcal{S}_\infty$ with $v_P(x) = \|PTx\|$.

As A restricted to $\operatorname{Im} P$ is an isomorphism, there exists an inverse of A^k on $\operatorname{Im} P$. We will denote by S_k the matrix whose kernel coincides with $\ker P$ and which satisfies $S_k A^k x = A^k S_k x = x$ for $x \in \operatorname{Im} P$. The assumptions guarantee that $P \in \mathcal{S}_\infty$ as a subsequence of the powers of A converges to P and thus also $S_k \in \mathcal{S}_\infty$, $k \geq 1$.

The idea is now to base the factorization on the equality $PTx = PPTx = S_k A^k PTx$. Assume for the moment that there exists an integer k such that the set

$$\{S_k P R_k P T x \mid R_k \in \mathcal{S}_k\} \tag{12}$$

contains more than one element and denote the smallest integer with this property by l. Then there exists $R_l \in \mathcal{S}_l$ such that

$$S_l P R_l P T x \neq S_l P A^l P T x = PTx\,.$$

If $l = 1$ then we are done by defining $U = S_1 P, V = PT$ and $S = S_1 PAPT = UAV$. Otherwise writing $R_l = BR_{l-1}$ with $R_{l-1} \in \mathcal{S}_{l-1}, B \in \mathcal{M}$ the assumption that l be minimal implies that

$$PTx = S_{l-1} P A^{l-1} PTx = S_{l-1} P R_{l-1} PTx = S_l A P R_{l-1} PTx = S_l P A R_{l-1} PTx\,,$$

so that the set

$$\{S_l P B R_{l-1} PTx \mid B \in \mathcal{M}\}$$

contains more than one element and the assertion is shown by defining $U = S_l P, V = R_{l-1} PT$ and $S = S_l PAR_{l-1} PT = UAV$. By construction and Lemma 3.2 it follows that $U, V \in \mathcal{S}_\infty$.

It remains to be shown that some k exists, such that the set in (12) contains more than one element. Assume this is false, so that for all $k \geq 1$ we have

$$S_k P A^k PTx = S_k P R_k PTx\,, \quad \text{for all } R_k \in \mathcal{S}_k\,.$$

Then, as S_k restricted to $\operatorname{Im} P$ is an isomorphism,

$$PA^k PTx = PR_k PTx\,, \quad \text{for all } R_k \in \mathcal{S}_k\,.$$

Consequently, PA^kP and PR_kP coincide on

$$Y := \operatorname{span}\{PTx, APTx, \dots, A^n PTx\} \subset \operatorname{Im} P$$

for all $R_k \in \mathcal{S}_k$. If $P = I$ this implies $A^k y = R_k y$ for $y \in Y, R_k \in \mathcal{S}_k$. By irreducibility $Y = \mathbb{K}^n$, but then $\mathcal{M}$ necessarily consists just of the matrix A in contradiction to the assumption.

If $P \neq I$ we can conclude that any A-invariant subspace of Y is invariant for the set $P\mathcal{S}P$. If $Y \neq \operatorname{Im} P$ then Y is a proper invariant subspace of $\operatorname{Im} P$ for $P\mathcal{S}P$. Otherwise, proper A-invariant subspaces of $Y = \operatorname{Im} P$ must exist in the case $\mathbb{K} = \mathbb{C}$ if $\operatorname{rank} P \geq 2$ and in the case $\mathbb{K} = \mathbb{R}$ if $\operatorname{rank} P \geq 3$. But the existence of proper invariant subspaces of $\operatorname{Im} P$ for the set $P\mathcal{S}P$ contradicts irreducibility of $\mathcal{S}$ by Lemma 5.1. This shows the assertion.

□

The main result of this section is the following proposition which states that the generalized spectral radius of a set of matrices $\mathcal{M}_2$ is strictly greater than that of a set of matrices $\mathcal{M}_1$, if $\mathcal{M}_1$ is contained in the interior of the convex hull of $\mathcal{M}_2$ where the interior is taken relative to the affine subspace generated by $\mathcal{M}_2$. Note that this result is a bit surprising because a similar statement for the maximum of the spectral radii is false, see for instance [21, Example 12].

In the following statement we denote by $\operatorname{aff} \mathcal{M}$ the affine subspace generated by $\mathcal{M}$, that is, the smallest affine subspace containing $\mathcal{M}$. The relative interior with respect to $\operatorname{aff} \mathcal{M}$ is denoted by $\operatorname{int}_{\operatorname{aff} \mathcal{M}}$.

Where were just able to show the assertion for the following (generic) case, although the natural conjecture is that it is always true. In the following we will denote by P_A the reducing projection of A corresponding to the eigenvalues of modulus $r(A)$.

Assumption 5.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and fix $\mathcal{M} \subset \mathcal{K}(\mathbb{K}^{n\times n})$. Let one of the following assumptions hold* *(i)* $K = \mathbb{C}$: *It holds that* $n = 1$ *or there exists an* $A \in \mathcal{M}$ *such that*

$$r(A) < \rho(\mathcal{M})\,, \quad or \quad \operatorname{rank} P_A \neq 1\,, \quad or \quad \sigma((I - P_A)A) \neq \{0\}\,,$$

(ii) $K = \mathbb{R}$: *It holds that* $n = 1, 2$ *or there exists an* $A \in \mathcal{M}$ *such*

$$r(A) < \rho(\mathcal{M})\,, \quad or \quad \operatorname{rank} P_A \neq 2\,, \quad or \quad \sigma((I - P_A)A) \neq \{0\}\,.$$

Proposition 5.4 *Let* $\mathcal{M}_1, \mathcal{M}_2 \in I(\mathbb{K}^{n\times n})$ *satisfy* $\mathcal{M}_1 \neq \mathcal{M}_2$ *and*

$$\mathcal{M}_1 \subset \operatorname{int}_{\operatorname{aff}\mathcal{M}_2} \operatorname{conv} \mathcal{M}_2\,. \tag{13}$$

Assume that $\mathcal{M}_1$ *satisfies Assumption 5.3 then*

$$\rho(\mathcal{M}_1) < \rho(\mathcal{M}_2)\,.$$

Remark 5.5 Note that in the extremal case that $\mathcal{M}_2$ is a singleton set, our assumption (13) does not guarantee that $\mathcal{M}_1 \neq \mathcal{M}_2$, so that an assumption forcing the two sets to differ is necessary. □

Proof: First note that the case $n = 1$ is trivial, as then $\rho(\mathcal{M}) = \max\{|a| \mid a \in \mathcal{M}\}$. So assume $n \geq 2$.

Assume the assertion is false, so that $\rho(\mathcal{M}_1) = \rho(\mathcal{M}_2) = 1$ can be assumed without loss of generality. Note that this assumption implies in particular, that $\mathcal{S}_{\infty,1} \subset \mathcal{S}_{\infty,2}$. Also we will assume, that $\mathcal{M}_1, \mathcal{M}_2$ are convex, which we may do without loss of generality as $\rho(\mathcal{M}) = \rho(\operatorname{conv}\mathcal{M})$.

The proof is carried out in two steps. First we show that the assertion is true if for some $S \in \mathcal{S}(\mathcal{M}_1)$ we have $r(S) < 1$. Then we prove the assertion for the case that $r(S) = 1$ for all $S \in \mathcal{S}(\mathcal{M}_1)$.

So assume that $r(S) < 1$ for some $S \in \mathcal{S}(\mathcal{M}_1)$. Fix $x \in \mathbb{K}^n$ with $v_1(x) = 1$. Let t be minimal such that $v_1(S^t x) < 1$ and $v_1(S^{t-1}x) = 1$. Such a t exists for all $x, v_1(x) = 1$ by our assumption on the spectral radius of S. Factorizing $S = A_k \cdots A_0$, $A_j \in \mathcal{M}_1, j = 0, \ldots k$ there is some $l = 0, \ldots, k$ such that

$$v_1\Big(\prod_{j=0}^{l-1} A_j S^{t-1}x\Big) = 1 \qquad \text{and } v_1\Big(\prod_{j=0}^{l} A_j S^{t-1}x\Big) < 1\,.$$

Denoting $y = A_{l-1}\cdots A_0 S^{t-1}x$ it follows that $v_1(y) = 1, v_1(A_l y) < 1$. By construction of the norm v_1, however, there is some $B \in \mathcal{M}_1$, such that $v_1(By) = 1$. Now for $Y = \{Ay \mid A \in \mathcal{M}_2\}$ we have

$$\{By, A_l y\} \subset \operatorname{int}_{\operatorname{aff} Y} Y\,.$$

As By lies on the boundary and $A_l y$ in the interior of the unit ball with respect to v_1, it follows that for some $C \in \mathcal{M}_2$ we have $v_1(Cy) > 1$, whence $v_1(CA_{l-1} \cdots A_0 S^{t-1} x) > 1$. Using a standard compactness argument it follows that there exists a constant $c > 1$ such that for every $x \in \mathbb{K}^n$ with $v_1(x) = 1$, there is an $S \in \mathcal{S}(\mathcal{M}_2)$ such that

$$v_1(Sx) > cv_1(x)\,.$$

By induction we obtain an unbounded solution of the discrete inclusion defined by $\mathcal{M}_2$, which contradicts $\rho(\mathcal{M}_2) = 1$. This completes the proof in the first case.

So assume now that $r(S) = 1$ for all $S \in \mathcal{S}(\mathcal{M}_1)$. This implies that $r(S) = v_1(S) = 1$ for all $S \in \mathcal{S}(\mathcal{M}_1)$ and it follows from [20, Theorem 2.5] that

$$\sigma(S) \subset \{0\} \cup \{z \in \mathbb{C} \mid |z| = 1\}\,, \quad \forall S \in \mathcal{S}(\mathcal{M}_1)\,. \tag{14}$$

In particular, this shows that we have already completed the proof in the case, that there is a matrix $A \in \mathcal{M}_1$ with $r(A) < \rho(\mathcal{M}_1)$ or $r(A) = \rho(\mathcal{M}_1)$ but $\sigma((I - P_A)A) \neq \{0\}$. For $S \in \mathcal{S}(\mathcal{M}_1)$ let P_S denote the reducing projection corresponding to the nonzero eigenvalues.

We now treat the case $\mathbb{K} = \mathbb{R}, \operatorname{rank} P_A = 1$ for some $A \in \mathcal{M}_1$. First note that $\operatorname{rank} P_A$ has to be constant on $\mathcal{M}_1$, because a drop in the rank of P_A means that an eigenvalue decreases in modulus under variation of $A \in \mathcal{M}_1$. This decrease has to be continuous as $\mathcal{M}_1$ is convex which produces an eigenvalue of modulus in the interval $(0,1)$ and this in contradiction to (14). Then it follows by induction that $\operatorname{rank} P_S$ is constant on $\mathcal{S}(\mathcal{M}_1)$ because each of the sets $\mathcal{S}_k(\mathcal{M}_1)$ is pathwise connected. Hence, $\operatorname{trace}(S) \equiv 1$ on $\mathcal{S}(\mathcal{M}_1)$ which implies reducibility of $\mathcal{S}_1$ by [22, Corollary 2], a contradiction.

Thus we have to treat the cases

(i) $\mathbb{K} = \mathbb{C}$, $\operatorname{rank} P_A \geq 2$ for all $A \in \mathcal{M}_1$,

(ii) $\mathbb{K} = \mathbb{R}$, $\operatorname{rank} P_A \geq 3$ for all $A \in \mathcal{M}_1$,

(iii) $\mathbb{K} = \mathbb{R}$, $n = 2$, $\operatorname{rank} P_A = 2$ (and hence $P_A = I$) for all $A \in \mathcal{M}_1$.

Note that in all these cases we can apply Lemma 5.2 to any of the reducing projections P_A, $A \in \mathcal{M}_1$.

We fix a strictly convex norm $\|\cdot\|$ on $\mathbb{K}^n$, $A \in \mathcal{M}_1$ and show that in the cases (i)-(iii) we have for $x \neq 0, x \in \operatorname{Im} P_A$

$$\|x\| \leq w_1(x) := \max_{S \in \mathcal{S}_{\infty,1}} \|P_A S P_A x\| < \max_{S \in \mathcal{S}_{\infty,2}} \|P_A S P_A x\| =: w_2(x)\,. \tag{15}$$

This implies for some $c > 1$ that $w_2(x) > c\|x\|, x \in \operatorname{Im} P_A, x \neq 0$ by a compactness argument. By compactness of $\mathcal{S}_{\infty,2}$ it follows in particular that for $x_0 \in \operatorname{Im} P_A, \|x_0\| = 1$ there exists an $S_1 \in \mathcal{S}_{\infty,2}$ with

$$\|P_A S_1 P_A x_0\| = c$$

and arguing inductively there are $S_1, \dots, S_k \in \mathcal{S}_{\infty,2}$ with

$$\|P_A S_k P_A \cdots P_A S_1 P_A x_0\| = c^k \,. \tag{16}$$

However, $P_A \in \mathcal{S}_{\infty,1} \subset \mathcal{S}_{\infty,2}$ and the latter set is a semigroup, so that for each k the matrix product in (16) is an element of $\mathcal{S}_{\infty,2}$. This implies that $\mathcal{S}_{\infty,2}$ is unbounded, a contradiction to Proposition 3.2 (i).

Thus it remains to show that (15) holds if Lemma 5.2 is applicable. First note, that because of $P_A \in \mathcal{S}_{\infty,1}$ we have $w_1(x) \geq \|P_A x\| = \|x\|$ for all $x \in \operatorname{Im} P_A$.

Also due to (13) it holds that whenever we have a set of the form

$$D := \{P_A U B x \mid B \in \mathcal{M}_2\} \,,$$

then

$$\max\{\|P_A U B x\| \mid B \in \mathcal{M}_2\} > \max\{\|P_A U B x\| \mid B \in \mathcal{M}_1\} \,,$$

unless D is a singleton set. The reason for this lies in assumption (13), the linearity of the map $B \mapsto P_A U B x$ and the strict convexity of our norm.

Fix $0 \neq x \in \operatorname{Im} P_A$ and let $S \in \mathcal{S}_{\infty,1}$ be such that

$$\|P_A S x\| = w_1(x) \,.$$

By Proposition 3.2 (iv) and Lemma 5.2 we can factorize $S = UAV$ with $U, V \in \mathcal{S}_{\infty,1}$ such that the set

$$\{P_A U B V x \mid B \in \mathcal{M}_2\} \tag{17}$$

consists of more than one element. Then it follows

$$w_2(x) \geq \max\{\|P_A U B V x\| \mid B \in \mathcal{M}_2\}$$

$$> \max\{\|P_A U B V x\| \mid B \in \mathcal{M}_1\} = w_1(x) \,.$$

This completes the proof. $\square$

Remark 5.6 It is worth pointing out, that the proof of the above result would be much simplified if we knew, that there exists a strictly convex Barabanov norm v_1 for $\mathcal{M}_1$. In this case (assuming $\rho(\mathcal{M}_1) = 1$) we would conclude immediately from (13) and strict convexity of v_1 that for each $x \neq 0$ there is some $A \in \mathcal{M}_2$ such that $v_1(Ax) > v_1(x)$, which implies $\rho(\mathcal{M}_1) < \rho(\mathcal{M}_2)$. To show that such an approach is not possible, let us demonstrate that for some irreducible sets of matrices no Barabanov norm is strictly convex.

In fact, we return to the set $\mathcal{M}$ introduced in Example 3.3. As we have already calculated $\mathcal{S}_\infty$, we see immediately, that for any norm w the corresponding Barabanov norm is given by

$$v\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \max\left\{ w\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right), w\left(\begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right)\right\}.$$

This norm is not strictly convex. $\square$

Before we note a consequence for strictly increasing function with values in $\mathcal{K}(\mathbb{K}^{n\times n})$ we need the following remark. If a bounded set $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is reducible, then after a suitable change of coordinates all matrices $A \in \mathcal{M}$ are of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1d} \\ 0 & A_{22} & A_{23} & \dots & A_{2d} \\ 0 & 0 & A_{33} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \\ 0 & & \dots & 0 & A_{dd} \end{bmatrix}, \tag{18}$$

where each of the sets $\mathcal{M}_{ii} := \{A_{ii}; A \in \mathcal{M}\}, i = 1 \dots d$ is irreducible. By Lemma 2 (c) in [2] it holds that

$$\rho(\mathcal{M}) = \max_{i=1,\dots,d} \rho(\mathcal{M}_{ii}). \tag{19}$$

Corollary 5.7 *Let $f : \mathbb{R}_+ \to \mathcal{K}(\mathbb{K}^{n\times n})$ be a function such that $f(\theta_1) \subset f(\theta_2)$ satisfy (13) for all $\theta_1 < \theta_2 \in \mathbb{R}_+$. Then*

(i) there exists a $\theta_0 \in \mathbb{R}_+$ such that $\rho \circ f$ is constant on $[0, \theta_0)$ and strictly increasing on $[\theta_0, \infty)$,

(ii) if $f(\theta) \in I(\mathbb{K}^{n\times n})$ satisfies Assumption 5.3 for some $\theta > 0$, then $\theta_0 \leq \theta$,

(iii) if f is continuous then $\rho \circ f$ is continuous,

(iv) if f is locally Lipschitz continuous then $\rho \circ f$ is locally Lipschitz continuous on $[0, \infty) \setminus F$, where F contains at most $n-1$ points.

Proof: (i) The interval $(0, \infty)$ can be partitioned into at most n intervals on which the invariant subspaces of $f(\theta)$ are constant. That is, there are numbers $0 = a_0 < a_1 < a_{k-1} < a_k = \infty$, $k \leq n$ such that on $(a_j, a_{j+1}), j = 0, \ldots, k-1$ all matrices $A \in \cup_{\theta \in (a_j, a_{j+1})} f(\theta)$ are of a fixed block-diagonal structure of the form (18), where for each $\theta \in (a_j, a_{j+1})$ and each $i = 1, \ldots, d$ the set $\mathcal{M}_i(\theta) := \{A_{ii} | A \in f(\theta)\}$ is irreducible. The assumptions do not guarantee that the family $\mathcal{U}_i := \{\mathcal{M}_i(\theta) \mid \theta \in (a_j, a_{j+1})\}$ is strictly increasing. Nevertheless, we know that

$$\operatorname{conv} \mathcal{M}_i(\theta_1) \subset \operatorname{int}_{\operatorname{aff} \mathcal{M}_i(\theta_2)} \operatorname{conv} \mathcal{M}_i(\theta_2) \,,$$

for $\theta_1 < \theta_2 \in (a_j, a_{j+1})$. This implies that the only possibility for $\mathcal{U}_i$ not to be increasing at $\theta_0 \in (a_j, a_{j+1})$ is that $\mathcal{U}_i$ is a singleton set. Furthermore, if $\rho(\mathcal{M}_i(\theta_1)) < \rho(\mathcal{M}_i(\theta_2))$ for some θ_1, θ_2, then for $B \in \mathcal{M}_i(\theta_1)$ we have $r(B) < \rho(\mathcal{M}_i(\theta))$ for all $\theta \geq \theta_2$, so that all $\mathcal{M}_i(\theta)$, $\theta \geq \theta_2$ satisfy Assumption 5.3, and so $\rho(\mathcal{M}_i(\theta))$ is strictly increasing on (θ_2, a_{j+1}).

Hence, for the map $\rho_i : \theta \mapsto \rho(\mathcal{M}_i(\theta))$, $\theta \in (a_j, a_{j+1})$ there are three possibilities

(i) ρ_i is constant on (a_j, a_{j+1}),

(ii) ρ_i is strictly increasing on (a_j, a_{j+1}),

(iii) there is a constant $\theta_0 \in (a_j, a_{j+1})$ such that ρ_i is constant on (a_j, θ_0) and strictly increasing on (θ_0, a_{j+1}).

Due to (19) the same is true for $\rho \circ f$ on (a_j, a_{j+1}). Now it follows that if there are $\theta_1 < \theta_2 \in \mathbb{R}_+$ with $\rho \circ f(\theta_1) < \rho \circ f(\theta_2)$ then $\rho \circ f$ is strictly increasing on $[\theta_2, \infty)$, because in θ_2 the maximum of the joint spectral radii is attained in one of the functions ρ_i. In this i-th block ρ_i is thus strictly increasing on (θ_2, a_{j+1}) and merging of blocks does not change the fact that Assumption 5.3 is fulfilled. As the assumptions guarantee that $\rho \circ f$ is increasing the only possibility for this function to be constant is on an interval of the form $[0, \theta_0)$. This shows the first assertion.

(ii) is an immediate consequence of Proposition 5.4, while (iii) follows from Lemma 3.5.

(iv) If f is Lipschitz continuous then by Corollary 4.2 ρ_i is locally Lipschitz continuous on the intervals (a_j, a_{j+1}) and thus also the maximum of these functions is locally Lipschitz continuous. Thus F contains at most the points $a_1, \dots, a_{k-1}$, and of these there are at most $n-1$.

$\square$

6 Extremal norms

We now investigate conditions for the existence of extremal norms. For this we need a notion of "defectiveness" of the generalized spectral radius in the case that $\mathcal{M}$ is reducible, which in some sense generalizes the notion of a defective eigenvalue with a modulus equal to the spectral radius. We intend to generalize the well known result that for a matrix A there exists an operator norm v with

$$v(A) = r(A)\,,$$

if and only if all eigenvalues $\lambda \in \sigma(A)$ with $|\lambda| = r(A)$ are nondefective. Unfortunately we are not able to recover the "only if" part of this statement.

For a set $\mathcal{M}$ of matrices of the form (18) let $J := \{1 \le i \le d \mid \rho(\mathcal{M}_{ii}) = \rho(\mathcal{M})\}$ denote the set of indices for which the generalized spectral radius is attained.

Definition 6.1 *A compact set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is said to have* non-defective generalized spectral radius *if there is a basis of $\mathbb{K}^n$ such that every matrix $A \in \mathcal{M}$ is of the from* (18) *and for all $i \in J$, $i < j \le \max J$ and all $A \in \mathcal{M}$ it holds that*

$$A_{ij} = 0\,.$$

Note that instead of requiring "zero rows" to the right of $A_{ii}, i \in J$ we could also have required "zero columns", that is for $i \in J$, $i < j \le \max J$, $A \in \mathcal{M}$ we have $A_{ji} = 0$. These two notions are equivalent, as one form is always similar to the other.

In particular, the above definition is satisfied if $\mathcal{M}$ is irreducible. Our proof is based on the following lemma, which follows from [12, Proposition 3.3].

Lemma 6.2 *Let $\mathbb{K}^n = \mathbb{K}^m \bigoplus \mathbb{K}^p$ and let $\mathcal{M} \in \mathcal{K}(\mathbb{K}^{n\times n})$ satisfy that every $A \in \mathcal{M}$ is of the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

with $A_{11} \in \mathbb{K}^{m\times m}, A_{12} \in \mathbb{K}^{m\times p}, A_{22} \in \mathbb{K}^{p\times p}$. *Denote*

$$\mathcal{M}_1 := \{A_{11} \mid A \in \mathcal{M}\} \subset \mathbb{K}^{m\times m}\,, \quad \mathcal{M}_2 := \{A_{22} \mid A \in \mathcal{M}\} \subset \mathbb{K}^{p\times p}\,.$$

(i) If $\rho(\mathcal{M}_1) < \rho(\mathcal{M}_2)$ *and there is an extremal norm* v_2 *on* $\mathbb{K}^p$ *corresponding to* $\mathcal{M}_2$ *then there exists an extremal norm* w *on* $\mathbb{K}^n$ *corresponding to* $\mathcal{M}$.

(ii) If $\rho(\mathcal{M}_1) > \rho(\mathcal{M}_2)$ *and there is an extremal norm* v_1 *on* $\mathbb{K}^m$ *corresponding to* $\mathcal{M}_1$ *then there exists an extremal norm* w *on* $\mathbb{K}^n$ *corresponding to* $\mathcal{M}$.

Now we are in a position to prove our main result on extremal norms.

Theorem 6.3 *Let* $\mathcal{M} \subset \mathbb{K}^{n\times n}$ *be compact with nondefective generalized spectral radius. Then there exists an extremal norm for* $\mathcal{M}$ *on* $\mathbb{K}^n$.

Proof: Assume that we have chosen a basis such that all matrices $A \in \mathcal{M}$ are in the form (18), with $A_{ii} \in \mathbb{K}^{n_i\times n_i}, i = 1,\dots,d$. If $d = 1$ the result is immediate from Theorem 1.1 so assume $d > 1$. Let $J = \{i_1 < \dots < i_k\} \subset \{1,\dots,d\}$ be the set of indices satisfying $\rho(\mathcal{M}_{ii}) = \rho(\mathcal{M})$. We will work inductively backwards on the set J. In the first step consider the matrices

$$\mathcal{M}_k := \left\{ \left[\begin{array}{ccccc} A_{i_{k-1}+1,i_{k-1}+1} & * & \dots & \dots & * \\ 0 & \ddots & * & \dots & * \\ 0 & 0 & \ddots & * & \vdots \\ \vdots & & \ddots & A_{i_k-1,i_k-1} & * \\ 0 & & \dots & 0 & A_{i_k,i_k} \end{array}\right] \;\middle|\; A \in \mathcal{M} \right\}.$$

Note that $\rho(\mathcal{M}_k) = \rho(\mathcal{M})$ and all blocks except for the one in the right lower corner have a generalized spectral radius strictly smaller than $\rho(\mathcal{M})$. Thus Lemma 6.2 (i) applies and there is an extremal norm w_k on

$$\bigoplus_{i=i_{k-1}+1}^{i_k} \mathbb{K}^{n_i}$$

corresponding to $\mathcal{M}_k$. Now on $\bigoplus_{i=i_{k-1}}^{i_k} \mathbb{K}^{n_i}$ all matrices are of the form

$$\left[\begin{array}{cc} A_{i_{k-1},i_{k-1}} & 0 \\ 0 & A_k \end{array}\right], \quad A_k \in \mathcal{M}_k\,.$$

Thus again applying Theorem 1.1 it is clear that there is an extremal norm on $\bigoplus_{i=i_{k-1}}^{i_k} \mathbb{K}^{n_i}$.

Now we may apply the same argument for the blocks corresponding to $\bigoplus_{i=i_{k-2}+1}^{i_k} \mathbb{K}^{n_i}$ to successively obtain extremal norms by repeatedly applying Lemma 6.2 (i). As a result we obtain an extremal norm on $\bigoplus_{i=i_1}^{i_k} \mathbb{K}^{n_i}$. Now the result follows after a further application of Lemma 6.2 (i) and (ii) to the remaining blocks with indices smaller than i_1, respectively larger than i_k. $\square$

Remark 6.4 Note that we cannot assume to be able to order the blocks in an order such that the generalized spectral radii are increasing or decreasing in (18) as this would imply properties of the invariant subspaces of $\mathcal{M}$. For instance for the set

$$\mathcal{M} := \left\{ \begin{bmatrix} \frac{1}{2} & a \\ 0 & 1 \end{bmatrix} \;\middle|\; a \in [0,1] \right\}$$

the only nontrivial invariant subspace is span $[1,0]'$ which is associated to the eigenvalue $1/2$. Hence no similarity transformation will transform $\mathcal{M}$ into a set of matrices of the form

$$\begin{bmatrix} 1 & * \\ 0 & \frac{1}{2} \end{bmatrix}.$$

This somewhat explains the awkward proof of Theorem 6.3. $\square$

A further interesting feature of extremal norms is that they allow to make the inequality in (3) more precise.

Lemma 6.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that $\mathcal{M} \subset \mathbb{K}^{n \times n}$ is bounded.*

(i) If there exists an extremal norm v for $\mathcal{M}$, then there exists a constant $M > 0$ such that for all $t \geq 1$

$$|\log \hat{\rho}_t(\mathcal{M}) - \log \rho(\mathcal{M})| < Mt^{-1}.$$

(ii) Otherwise there exists an $M > 0$ such that for all $t \geq 1$

$$|\log \hat{\rho}_t(\mathcal{M}) - \log \rho(\mathcal{M})| < M \frac{1 + \log t}{t}.$$

Proof: Let v be the extremal norm for $\mathcal{M}$. As all norms on finite dimensional vector spaces are equivalent it follows with (3) that

$$0 \leq \frac{1}{t} \log \sup_{S_t \in \mathcal{S}_t} \|S_t\| - \log \rho(\mathcal{M}) \leq \frac{1}{t} \log \sup_{S_t \in \mathcal{S}_t} cv(S_t) - \rho(\mathcal{M}) = \frac{1}{t} \log c. \quad (20)$$

This proves the assertion.
(ii) This follows from Lemma 2.3 in [29]. $\square$

Remark 6.6 Note that we cannot expect a similar statement for the lower bound $\overline{\rho}_t$. If we return to our Example 3.3 then we see that in this case $\overline{\rho}_{2k}(\mathcal{M}) = \rho(\mathcal{M}) = 1$ and $\overline{\rho}_{2k+1} = 0$ for all $k \in \mathbb{N}$. $\square$

We also note the following consequence of local uniform convergence of $\hat{\rho}(\mathcal{M})$ to $\rho(\mathcal{M})$.

Corollary 6.7 *Let $P \subset I(\mathbb{K}^{n\times n})$ be compact then there is a constant $M > 0$ such that for all $\mathcal{M} \in P$ and all $t \geq 1$ it holds that*

$$|\log \hat{\rho}_t(\mathcal{M}) - \log \rho(\mathcal{M})| < Mt^{-1},$$

i.e. $\hat{\rho}_t$ converges locally uniformly to $\rho(\mathcal{M})$ on $I(\mathbb{K}^{n\times n})$.

Proof: Just note that the constant c in the proof of Lemma 6.5 (i) can be chosen independently of $\mathcal{M} \in P$ by Theorem 4.1. $\square$

7 Conclusion

We have studied extremal norms for linear discrete and differential inclusions. For the special case of irreducible inclusions we give a constructive procedure for a special extremal norm. This approach yields Lipschitz continuity of the generalized spectral radius and a monotonicity property as a byproduct. A more general sufficient criterion guaranteeing the existence of an extremal norm has also been presented. Furthermore, we have pointed out that the convergence of $\hat{\rho}_t$ to the generalized spectral radius is linear if an extremal norm exists, in particular in the irreducible case.

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