UNIVERSITÄT BREMEN

Zentrum für Technomathematik
Fachbereich 3 – Mathematik und Informatik

Higher Order Real-Time Approximations of Pertubed Control Constrainted PDE Optimal Control Problems

Christof Büskens

Report 04–09

Berichte aus der Technomathematik

Report 04–09 **August 2004**

Higher Order Real–Time Approximations of Perturbed Control Constrainted PDE Optimal Control Problems

Christof Büskens
Universität Bremen
Fachbereich Mathematik und Informatik
AG Optimierung und Optimale Steuerung
D-28359 Bremen, Germany

May 19, 2004

Abstract

Numerous problems from natural sciences and engineering as well as from other disciplines lead to optimal control problems governed by systems of time-dependent partial differential equations (PDE). Control functions have to be determined such that a given performance index is optimized subject to additional constraints. These optimal control problems are solved numerically by time–consuming methods. Moreover, if the optimal controls are to be pursued in a real process, the possibility of data disturbances force recomputing the optimal controls in real–time to preserve constraints and optimality, at least approximately. For this purpose, a numerical method based on the parametric sensitivity analysis of nonlinear optimization problems is suggested to calculate higher order approximations for the optimal solution of the perturbed optimal control problems in real-time without solving the PDE system explicitely. By this method computing times can be reduced to a few nanoseconds on a typical one processor personal computer. The method is illustrated by the real–time optimal control of the nonlinear Burgers equation.

Keywords: perturbed optimal control problems; nonlinear programming methods; partial differential equations; parametric sensitivity analysis; real–time control

1 Introduction

This paper is concerned with the real–time approximation of perturbed optimal control problems governed by time-dependent partial differential equations and subject to control constraints. In practice, control problems are often subject to disturbances or perturbations in the system data. In mathematical terms, perturbations are expressed by means of parameter fluctuations that enter the dynamics, boundary conditions or control constraints. *Stability* and *sensitivity analysis* are concerned with the behavior of optimal solutions under parameter perturbations. The so-called parametric sensitivity derivatives are a helpful tool in real-time optimal control applications, see e.g. Pesch [19] and Büskens and Maurer [6]–[9] for optimal control problems governed by ordinary differential equations.

Although it is well–known that there exists a gap between the computational and theoretical aspects of optimal control problems with PDEs, the need on numerical methods capable to solve these problems is given today. Naturally this difficulty goes over to the computational and theoretical aspects of parametric sensitivity analysis of optimal control problems with PDEs, if numerical solution techniques are used. Numerical aspects of solving optimal control problems subject to partial differential equations by e.g. SQP methods are commonly discussed, e.g. by Heinkenschloss and Sachs [14], Casas, Tröltzsch and Unger [10, 11], Kunisch and Volkwein [15], Maurer and Mittelmann [17, 18] to name only some references.

In this paper the method of lines is used to discretize the partial differential equations, transforming the original system into a system of ordinary differential equations. To compute sensitivity differentials via nonlinear optimization methods a subsequent discretization of the ODE control problem is accomplished. This leads to so-called *direct optimization methods* which have proved to be powerful tools for solving ODE optimal control problems; cf., e.g., Büskens [3], Büskens and Maurer [6]–[9] and the references cited therein. These methods use only control and state variables as optimization variables and completely dispense with adjoint variables. Alt [1], Dontchev and Hager [12] and Malanowski, Büskens and Maurer [16] prove the convergence of the discretized problem to the continuous solution for ODE optimal control problems. For general optimal control problems involving partial differential equations, these results do not yet exist. Hence, we tacitly assume the convergence to the presumed unique continuous solution as the mesh size tends to zero.

In Büskens and Griesse [5] the *numerical computation* of sensitivity differentials for optimal control problems for PDEs is discussed. The proposed NLP-based method is capable of computing approximations to sensitivity differentials for the state, control and adjoint variables in parametric PDE optimal control problems. These sensitivity differentials allow an easy access to real–time optimal control approximations of perturbed problems. In case of deviations in the perturbation parameters the method presented in this paper is able to calculate higher order real–time approximations of the perturbed PDE optimal control problem without solving the PDE system again. To the authors knowledge this is this first time that approximations of perturbed PDE optimal control problems can be calculated in real–time.

The general mathematical structure of perturbed PDE optimal control problems is outlined in Section 2. Discretization details for the PDE can be found in Section 3. A short overview of basic results on sensitivity analysis and solution differentiability for perturbed finite-dimensional nonlinear optimization problems is offered in Section 4. By re-transforming the numerical solution and sensitivity quantities, information about the optimal solution and the sensitivity of the original PDE optimal control problem can be obtained. This will be covered in Section 5. Finally, Section 6 presents three numerical examples for PDE optimal control problems with control constraints.

2 Parametric PDE optimal control problems

We consider the following perturbed generally coupled time-dependent PDE optimal control problem **POCP(p)** with control constraints:

$$\begin{array}{rcll}
\text{Minimize} & & & \\
F(y,u,p) & = & \displaystyle\int_\Gamma f_1(y(x,t),u(x,t),x,t,p)\,dx\,dt & \\
& & \displaystyle +\int_\Omega f_2(y(x,t),u(x,t),x,t,p)\,dx\,dt & \\
\text{subject to} & & & \\
y_t(x,t) & = & f(y(x,t),y_x(x,t),y_{xx}(x,t),u(x,t),x,t,p), & \\
y(x,t_0) & = & y_0(x,p), & \text{for } \; x\in\Omega_x \\
y(x,t) & = & y_D(u(x,t),x,t,p), & \text{for } \; x\in\Gamma_x,\ t\in\Omega_t \\
\partial_\nu y(x,t) & = & y_N(y(x,t),u(x,t),x,t,p), & \text{for } \; x\in\Gamma_x,\ t\in\Omega_t \\
0 & \geq & C(u(x,t),x,t,p), & \text{for } \; (x,t)\in\Omega.
\end{array} \tag{1}$$

Herein $t\in[t_0,t_f]=\Omega_t\subset\mathbb{R}$ denotes the *time variable*, while $x\in\Omega_x\subset\mathbb{R}^{n_x}$ of dimension n_x denotes the *spatial variables*. Further, Ω_x is a bounded domain with piecewise smooth boundary $\Gamma_x=\partial\Omega_x$. Herewith $\Omega=\Omega_x\times\Omega_t$ and $\Gamma=\Gamma_x\times\Omega_t$ are defined. Moreover, let $y:\Omega\to\mathbb{R}^{n_y}$ be a vector function of dimension n_y of which $y_t(x,t)$ denotes the first derivative w.r.t. the time variable t. Likewise the first and second partial derivatives w.r.t. the spatial variables x are denoted by $y_x(x,t)$ and $y_{xx}(x,t)$. The control function $u:\Omega\to\mathbb{R}^{n_u}$ has components defined either on Ω *(distributed control)* or on the boundary Γ *(boundary control)*. *Perturbations*, which may appear in all functions of (1), are characterized by a parameter vector $p\in\Omega_p\subset\mathbb{R}^{n_p}$. A solution of the PDE system in (1) depends on the spatial variable x, the time variable t, the control function u and the fixed parameter vector p. While we mainly have a parabolic PDE system in mind, hyperbolic systems can be considered, too. Always providing that the PDE system in (1) is well-defined and uniquely solvable for given u and p, combinations of Dirichlet or Neumann conditions are defined by the functions $y_D:\mathbb{R}^{n_u}\times\Gamma_x\times\Omega_t\times\Omega_p\to\mathbb{R}^{n_y}$ or $y_N:\mathbb{R}^{n_y}\times\mathbb{R}^{n_u}\times\Gamma_x\times\Omega_t\times\Omega_p\to\mathbb{R}^{n_y}$. Herein derivatives in the direction of the outward unit normal ν of Γ_x are denoted by ∂_ν in (1). The vector function $C:\times\mathbb{R}^{n_u}\times\Omega_x\times\Omega_t\times\Omega_p\to\mathbb{R}^{n_c}$, $n_c\geq 0$, allows for additional inequality control constraints. We point out that in principle state constraints can be taken into account, too.

Nevertheless they might be to time consuming in view of the real–time approximations discussed later.

The problem is to determine a control vector — containing boundary or distributed control elements or both — that minimizes the functional F in (1) subject to the given restrictions. Note that (1) can be extended to contain more general terms, for example higher order derivatives or time–spatial derivatives.

3 Discretization of the PDE control problem

The main idea of discretization is to transform the PDE optimal control problem (1) into a finite dimensional nonlinear optimization problem (NLP). On the other hand for an efficient evaluation of the higher order real–time approximations discussed later on it is stringently necessary to find a discretization where the constraints are independent on the state $y(x,t)$ itself. To fulfill these requirements we proceed as follows:

First the method of lines is used to transform the partial differential equation into a system of ordinary differential equations by discretizing all functions with respect to the spatial variable x. Possible choices for the spatial discretization are finite differences, finite elements or others. The method of lines transforms the partial differential equation into a system of ordinary differential equations. Since the method is well known we will not discuss it in detail and tacitly assume that there exist a transcription to the perturbed optimal ODE control problem **(OCP(p))**:

$$\begin{array}{lrcl} \text{Minimize} & F(w,v,p) = \displaystyle\int_{t_0}^{t_f} f_0(w(t),v(t),t,p)\,dt & & \\ \text{subject to} & \dot{w}(t) & = & \tilde{f}(w(t),v(t),t,p), \\ & w(t_0) & = & w_0(p), \\ & \tilde{C}(v(t),t,p) & \leq & 0, \quad t \in [t_0,t_f]. \end{array} \tag{2}$$

We assumed, that the Dirichlet or Neumann conditions can be substituted directly into the ODE system in (2). Here, $w(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$ denote the state of the system and the control in a given time interval $[t_0,t_f] = \Omega_t$, respectively. Data perturbations are again modeled by a parameter $p \in \Omega_p$. The functions $f_0 : \mathbb{R}^{n+m+1} \times \Omega_p \to \mathbb{R}$, $\tilde{f} : \mathbb{R}^{n+m+1} \times \Omega_p \to \mathbb{R}^n$, $w_0 : \Omega_p \to \mathbb{R}^n$, and $\tilde{C} : \mathbb{R}^{n+m+1} \times \Omega_p \to \mathbb{R}^k$ are assumed to be sufficiently smooth on appropriate open sets. The admissible class of control functions is the class of piecewise continuous controls. The final time t_f can be either fixed or free. Note that the formulation in (2) contains only *pure control constraints* and that the former state variable $y(x,t)$ can be identified by the components of $w(t)$ depending on the discretization accomplished with the method of lines. Likewise the former control variable $u(x,t)$ can be identified by the components of $v(t)$.

As a second step a suitable *discretization* of the the control problem (2) is used by which it is transformed into a nonlinear optimization problem (NLP). These techniques are well developed and there exist a number of excellent methods. In principal these methods can be divided into two classes. The first type of methods is characterized by the fact that both the discretized state *and* control variables are taken as optimization variables. Unfortunately this approach leads to a high number of state depending constraints and hence are not suitable for the higher order real–time approximations presented in this paper. For the second class of NLP methods, only the discretized control variables are considered as optimization variables whereas the state variables are calculated as functions of the control variables using appropriate numerical integration methods. One obtains a NLP problem where the constraints are independent on the state of the system. We use one variation of the code NUDOCCCS of Büskens [2, 3] and reflect the main idea for the simple Euler method subsequently. Without loss of generality let $t_0 = 0$. For notational simplicity we choose equidistant mesh points $t_i := (i-1)h$, $i = 1,\dots,N_t$, $h := \frac{t_f}{N_t-1}$. Let $u^i \in \mathbb{R}^m$ denote approximations for $v(t_i)$. Then for given $z := (v^1,\dots,v^{N_t}) \in \mathbb{R}^{mN_t}$ state approximations $w^i \in \mathbb{R}^n$ of the values $w(t_i)$ can be archived recursively as functions of the control variables:

$$\begin{array}{rcl} w^1(z,p) & := & w_0(p), \\ w^{i+1}(z,p) & := & w^i(z,p) + h\tilde{f}(w^i(z,p),v^i,t_i,p), \quad i = 1,\dots,N_t-1. \end{array} \tag{3}$$

Hereby, the control problem (2) is replaced by:

$$\begin{array}{lll} \min\limits_{z} & h \cdot \sum\limits_{i=0}^{N-1} f_0(w^i(z,p), v^i, t_i, p) & \\ \text{subject to} & \tilde{C}(v^i, t_i, p) \leq 0, \quad i = 1, \ldots, N_t. \end{array} \tag{4}$$

Note that a free final time t_f can be handled as an additional variable in z and that due to the chosen discretization the state $w(z,p)$ appears in the objective of (4) but not in the constraints. Hence the numerical calculation of the objective is expensive due to the implicitly considered PDE problem while the calculation of the constraints is not. Problem (4) defines a perturbed NLP problem $\mathbf{NLP_1(p)}$ of form

$$\begin{array}{ll} \min\limits_{z} & H(z,p), \\ \text{subject to} & G_i(z,p) \leq 0, \quad i = 1, \ldots, N_c. \end{array} \tag{5}$$

which can be solved by standard techniques, e.g. SQP methods, if we use suitable definitions for N_c and the functions H and G_i.
All calculations described hereafter were performed by the code NUDOCCCS of Büskens [2, 3] in which also various higher order approximations of the state and control variables are implemented. The treatment of stiff ODEs, grid refinement techniques and a numerical check of second order sufficient optimality conditions can also be found in this code, see [3]. Recently, the convergence of solutions discretized via Euler's method to solutions of the continuous control problem has been proved in Malanowski, Büskens and Maurer [16].
By solving the NLP problem (5) we obtain an estimate of the *continuous* control and state variables (u,y) of (1) at appropriate $x_l \in \Omega_x$, $t_j \in \Omega_t$ depending on the applied discretization. Likewise, all other variables and functions of the continuous problem (1) can be determined approximately.

4 Parametric sensitivity analysis of perturbed NLP problems

In Section 3 a method to transform a perturbed control problem into a parametric NLP problem has been discussed. It should be mentioned that the results hereafter do not depend on the discretization technique used. After solving (5) we know the set and the number N_a of active constraints, i.e. those constraints in (5) with $G_i(t_i;z) = 0$. Let $G^a = (G_1^a, \ldots, G_{N_a}^a)^T$ denote the collection of these active constraints. Then the solution of (5) is the same as the solution of $\mathbf{NLP_2(p)}$

$$\begin{array}{ll} \min\limits_{z} & H(z,p) \\ \text{subject to} & G^a(z,p) = 0, \end{array} \tag{6}$$

since inactive constraints have no impact on the optimal solution. For this, we restrict the subsequent discussion to the formulation (6). The Lagrangian for (6) is defined as

$$L(z,\eta,p) = H(z,p) + \eta^\top G^a(z,p)$$

with Lagrange multiplier $\eta = (\eta_1, \ldots, \eta_{N_a})^\top$. The following theorem states sufficient conditions for the differentiability of an optimal solution $z(p)$ w.r.t. p.

Theorem 1: *Let H and G^a be twice continuously differentiable w.r.t. z and p. Let z_0 be a strong regular local solution of (6) for a fixed parameter p_0 with Lagrange multiplier η_0, i.e. $G^a(z_0,p_0) = 0$ and*

- *z_0 is regular in the sense $rg\,(\nabla_z G^a(z_0,p_0)) = N_a$, i.e., the gradients $\nabla_z G_i^a(z_0,p_0)$ are linearly independent,*
- *the first order necessary optimality conditions hold at z_0, i.e., $\nabla_z L(z_0,\eta_0,p_0) = 0, \eta_0^\top G^a(z_0,p_0) = 0$,*
- *the strict complementary condition holds at z_0, i.e., $(\eta_0)_i > 0$ for $i = 1, \ldots, N_a$,*

- *the second order sufficient conditions hold at* z_0*, i.e.,* $v^\top \nabla^2_{zz} L(z_0, \eta_0, p_0) v > 0$, $\forall v \in ker(\nabla_z G^a(z_0, p_0)(z_0, p_0))$, $v \neq 0$.

Then there exists a neighborhood $\mathcal{P}(p_0)$ *such that (6) possesses a unique strong regular local solution* $z(p)$ *and* $\eta(p)$ *for all* $p \in \mathcal{P}(p_0)$*. Furthermore,* $z(p)$ *and* $\eta(p)$ *are continuously differentiable functions of* p *in* $\mathcal{P}(p_0)$ *and it holds*

$$\begin{pmatrix} \nabla^2_{zz} L(z_0, \eta_0, p_0) & \nabla_z G^a(z_0, p_0)^\top \\ \nabla_z G^a(z_0, p_0)(z_0, p_0) & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{dz}{dp}(p_0) \\ \frac{d\eta}{dp}(p_0) \end{pmatrix} = - \begin{pmatrix} \nabla^2_{zp} L(z_0, \eta_0, p_0) \\ \nabla_p G^a(z_0, p_0) \end{pmatrix}. \tag{7}$$

Notice, that the left matrix in (7) is non-singular under the assumptions of Theorem 1. Hence, (7) points out a way to compute the sensitivity differentials dz/dp and $d\eta/dp$ at p_0 explicitly by solving the linear equation system. The proof of the theorem is based on the implicit function theorem and can be found in Fiacco [13] or Büskens [3]. The assumptions in Theorem 1 can be checked numerically by use of the projected or reduced Hessian, compare Büskens and Maurer [8] or Büskens [3]. In the following section these results are applied to real–time approximations.

5 Higher Order Admissible Real-Time Approximations

In the preceeding sections methods were devoted to calculate the nominal solution and the corresponding sensitivity differentials. In case of a deviation Δp in the parameter p a first order Taylor approximation for $z(p_0 + \Delta p)$ is given by

$$z(p) := z(p_0 + \Delta p) \approx \tilde{z}(p) := z(p_0) + \frac{dz}{dp}(p_0)\Delta p. \tag{8}$$

Since the quantities $z(p_0)$ and $\frac{dz}{dp}(p_0)$ in (8) can be computed *offline*, the benefit of (8) is that only a matrix-vector multiplication and a vector-vector addition has to be performed *online* to approximate $z(p_0 + \Delta p)$ extremely fast. Note that the calculation of $\tilde{z}(p)$ in (8) is independent on the PDE system, hence (8) is particularly suitable for time critical processes. We use (8) as a first *real-time approximation.* It holds, cf. Büskens [4]:

Theorem 2: *Let the assumptions of Theorem 1 hold and let the functions* H *and* G^a *in (6) be three times continuously differentiable w.r.t. to* z *and* p*. Then there exists a neighborhood* $U(p_0)$ *of* p_0 *with*

$$\begin{aligned} \|z(p) - \tilde{z}(p)\| &= \mathcal{O}(\|\Delta p\|^2), & (9) \\ \|H(z(p), p) - H(\tilde{z}(p), p)\| &= \mathcal{O}(\|\Delta p\|^2), & (10) \\ \|G^a(\tilde{z}(p), p)\| &= \mathcal{O}(\|\Delta p\|^2). & (11) \end{aligned}$$

In the unconstraint case, i.e., $N_a = 0$*, we have*

$$\|H(z(p), p) - H(\tilde{z}(p), p)\| = \mathcal{O}(\|\Delta p\|^3). \tag{12}$$

According to Theorem 2, Formula (8) yields, in view of optimality and admissibility, real–time approximations for small perturbations Δp for many problems. Nevertheless, the approximation $\tilde{z}(p)$ is generally not admissible, i.e., $\|G^a(\tilde{z}(p), p)\| = \mathcal{O}(\|\Delta p\|^2)$.

In particular, for larger perturbations Δp approximation $\tilde{z}(p)$ may be not acceptable in view of admissibility. To overcome this problem, we introduce an artifical perturbation q. Instead of (6) we treat the problem $\mathbf{NLP_3(p, q)}$

$$\begin{aligned} \min_{z} \quad & H(z, p) \\ \text{subject to} \quad & G^a(z, p) - q = 0. \end{aligned} \tag{13}$$

Obviously problem $\mathbf{NLP_2(p)}$ is equivalent to $\mathbf{NLP_3(p, q_0)}$ if the nominal perturbation is chosen to $q = q_0 = 0$. Moreover problem (13) fulfills the assumptions of Theorem 1, if (6) does, and hence we are able to calculate the sensitivities $\frac{dz}{dq}(0)$ and $\frac{d\eta}{dq}(0)$ similar to (7). Herewith we are able to formulate the following corrector iteration method to achieve admissibility for the constraints without loss of its optimality:

(i) Choose $\varepsilon^\infty > 0$ and initialize $\tilde{z}^{[1]}(p) := \tilde{z}(p)$. Set $k := 1$.

(ii) If $\|G^a(\tilde{z}^{[k]}(p), p)\| < \varepsilon^\infty$, then STOP.

(iii) Calculate

$$\tilde{z}^{[k+1]}(p) := \tilde{z}^{[k]} - \frac{dz}{dq}(0) G^a(\tilde{z}^{[k]}(p), p), \tag{14}$$

and set $k := k + 1$.

(iv) Go to (ii).

Iteration (14) achieves admissibility in the active constraints and additionally improves the optimality as shown by the following theorem, cf. Büskens [4].

Theorem 3: *Let the assumptions of Theorem 1 hold and let the functions H and G^a in (6) be three times continuously differentiable w.r.t. to z and p. Then there exists a neighborhood $U(p_0)$ of p_0 and a vector $v \in \mathbb{R}^{N_z}$ with $v \in ker(\nabla_z G^a(z_0, p_0))$ and $\|v\| = \mathcal{O}(\|\Delta p\|^2)$ such that for all $p \in U(p_0)$ the sequence $\tilde{z}^{[k]}(p)$ in (14) converges to a fixed point $\tilde{z}^{[\infty]}(p)$ with*

$$\|z(p) - \tilde{z}^{[\infty]}(p)\| = \|v\| + \mathcal{O}(\|\Delta p\|^3), \tag{15}$$

$$\|H(z(p), p) - H(\tilde{z}^{[\infty]}(p), p)\| = \mathcal{O}(\|\Delta p\|^3), \tag{16}$$

$$\|G^a(\tilde{z}^{[\infty]}(p), p)\| = 0. \tag{17}$$

Iteration (14) can be interpreted as follows: Approximation $\tilde{z}^{[k]}(p)$ causes a deviation in G^a which is identified as the perturbation q. For this perturbation an additional correction step of type (8) is performed. Of course, the new approximation will again cause a deviation in G^a and the corrector iteration method is born.
Note, that the fixed point in Theorem 3 is not unique, cf. Büskens [4]. Nevertheless, any fixed point of iteration (14) fulfills (15)–(17), especially the higher order of optimality in the objective. Now it becomes evident why the discretizations in (2)–(5) were chosen in such manner that the constraints are independent on the state: Iteration (14) needs no objective calculation and hence can be performed without solving the expensive underlying PDE system. Only the control constraints have to be computed, which is generally cheap.

6 Example: The Burgers Equation

In order to illustrate the performance of the algorithms presented in Sections 2–5 the nonlinear one-dimensional viscous Burgers equation is investigated subject to a tracking-type PDE optimal control problem with control constraints, cf. Volkwein [20]. All computations were performed on a 1GHz PIII personal computer using the code NUDOCCCS.

$$\begin{array}{rcl}
\text{Minimize} & & \\
F(y, u, p) & = & \dfrac{\alpha}{2}\left(\displaystyle\int_{\Omega} [y(x,t) - 0.035]^2 \, dx\, dt + (1 + p_2)\sigma \int_{t_0}^{t_f} [u_1(t)^2 + u_2(t)^2] \, dt \right), \\
\text{subject to} & & \\
y_t(x,t) & = & (1 + p_1)\nu y_{xx}(x,t) - y(x,t) y_x(x,t) + p_6 t, \\
y(x, t_0) & = & (1 + p_5) x^2 (1 - x)(1 - x), \\
y_x(0, t) & = & u_1(t), \\
y_x(1, t) & = & u_2(t), \\
0 & \geq & C(u_1(t), u_2(t), p_3, p_4),
\end{array} \tag{18}$$

with $n_x = n_y = 1$, $n_u = 2$, $n_p = 6$, $\Omega_t = [0, 1]$, $\Omega_x = (0, 1)$, $\sigma = 0.01$, $\nu = 0.1$, $\alpha = 1000$. For the method of lines, the first derivative $y_x(x_l, t)$ at a discretized spatial point x_l in the interior of Ω_x is approximated by the second order formula

$$y_x(x_l, t) \approx \frac{y(x_l + \triangle x, t) - y(x_l - \triangle x, t)}{2 \triangle x}, \tag{19}$$

while second order derivatives are approximated by the second order formula

$$y_{xx}(x_l,t) \approx \frac{y(x_l+\triangle x,t) - 2y(x_l,t) + y(x_l-\triangle x,t)}{\triangle x^2}. \tag{20}$$

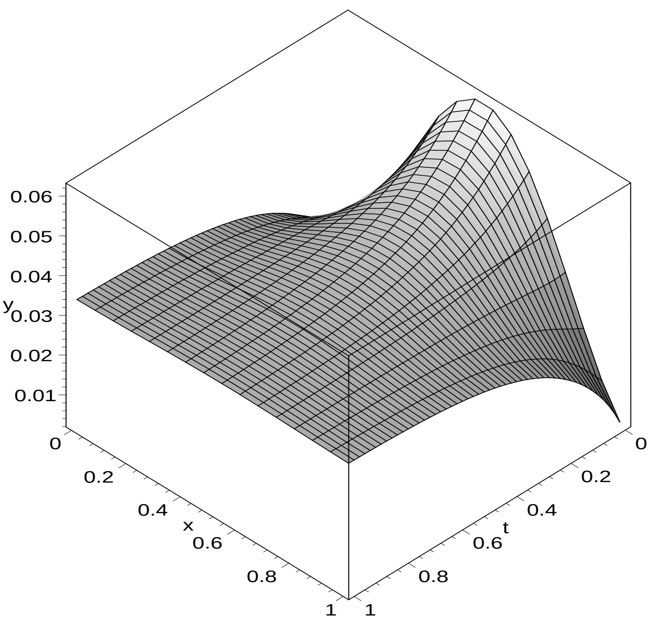
The state variables on the boundaries $y(0,t)$ and $y(1,t)$, entering the right hand side of the PDE when (18) approximating $y_x(x,t)$ and $y_{xx}(x,t)$, are calculated by the second order approximations

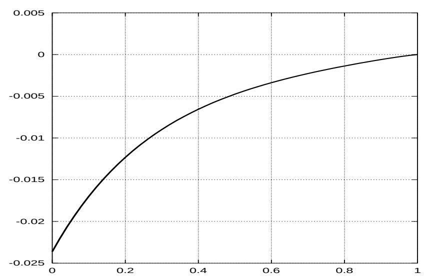
$$\begin{aligned} y(0,t) &\approx \frac{4y(\triangle x,t) - y(2\triangle x,t) - 2\triangle x u_1(t)}{3}, \\ y(1,t) &\approx \frac{4y(1-\triangle x,t) - y(1-2\triangle x,t) + 2\triangle x u_2(t)}{3}. \end{aligned} \tag{21}$$

All computations are performed with 18 lines for the spatial variable x which leads to $\triangle x = \frac{1}{16}$ and a system of 16 first order ordinary differential equations if the Neumann conditions are directly inserted into all functions. An explicit fourth order Runge–Kutta scheme and a linear interpolation of the control variable is used for the integration in time. In order to be able to use this explicit Runge–Kutta scheme, it is necessary to choose a sufficiently high number $N_t = 51$ of grid points for the discretized time interval. Similar results can be achieved by using an implicit Runge–Kutta method with fewer time steps, at the expense of additional numerical cost. The spatial integral in the objective is approximated by the trapezoidal method.
The perturbation parameter p_1 is connected to the viscosity coefficient ν while perturbations in the objective are concerned with p_2. The perturbations p_3 and p_4 will be used to simulate deviations in the constraints defined later on. The parameter p_5 allows for deviations in the initial values. Finally parameter p_6 produces a forcing term on the right hand side which grows with time.
The optimal solutions calculated hereafter are obtained from the discretized formulation (5) for the nominal perturbation parameter $p_0 = (0,0,0,0,0,0)^T$.

6.1 The unconstrained case

First we consider the unconstraint case with $n_c = 0$. After about 2.5 seconds of computational time, the optimal nominal solution is obtained with an objective value $F(y(p_0), u(p_0), p_0) \approx 2.176303 \cdot 10^{-2}$. The nominal state in the interior and the unperturbed optimal control functions are depicted in Figure 1.





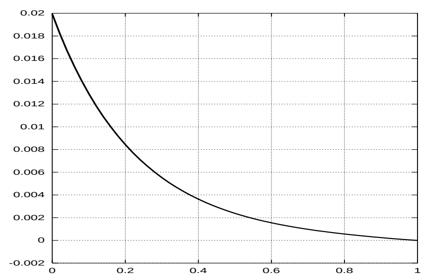


Figure 1: Nominal state $y(x,t)$ (left), nominal optimal controls $u_1(t)$, $u_2(t)$ (right) for the unconstrained case.

All assumptions of Theorem 1 have been carefully checked numerically for the discretized problem, and the Hessian of the Lagrangian has been found to be positive definite. Hence the sensitivity differentials of the control variables for p_1, p_2, p_5 and p_6, can be obtained from expression (7) and are depicted in Figure 2.

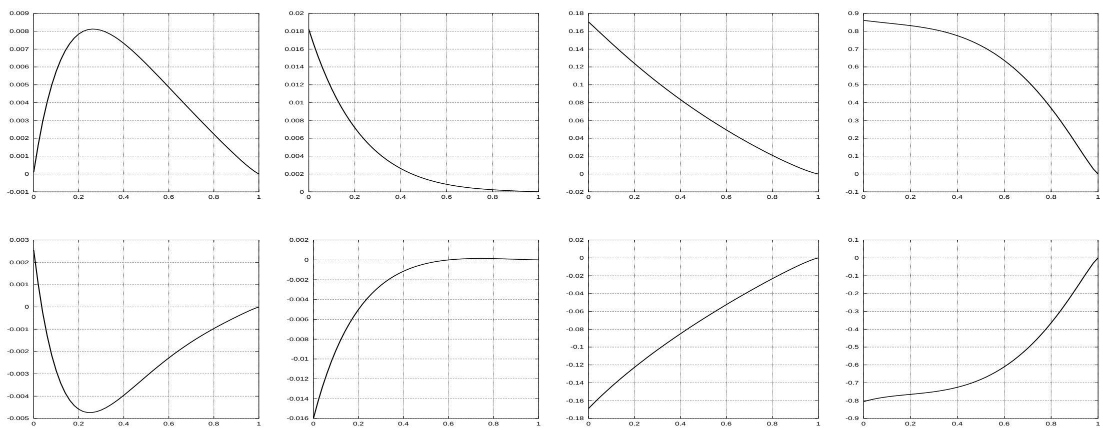


Figure 2: Sensitivity differentials $\frac{du_1(t)}{dp}(p_0)$ (top) and $\frac{du_2(t)}{dp}(p_0)$ (bottom) of the control variables for the unconstrained case.

In order to judge the quality of the real–time approximation (8) for the unconstrained Burgers problem, we set up the following Table 1 which lists the relative errors

$$\zeta_H^{[k]}(p) := \frac{H(\tilde{z}^{[k]}(p),p) - H(z(p),p)}{H(z(p),p)}, \qquad k = 0,1,2,\ldots \tag{22}$$

of the objective for different perturbations p. Herein and in the following $\zeta_H^{[0]}$ denotes the relative error of the objective obtained after an integration of the perturbed system using the nominal control variables.

	$\zeta_H^{[0]}(p)$	$\zeta_H^{[1]}(p)$
$p = (0,0,0,0,-0.1,-0.1)$	$4.38 \cdot 10^{-01}$	$3.18 \cdot 10^{-05}$
$p = (0,0,0,0,0.1,0.1)$	$4.27 \cdot 10^{-01}$	$3.05 \cdot 10^{-05}$
$p = (-0.1,-0.1,0,0,0,0)$	$5.77 \cdot 10^{-04}$	$8.22 \cdot 10^{-06}$
$p = (-0.02,0.02,0,0,-0.02,0.02)$	$7.95 \cdot 10^{-02}$	$2.81 \cdot 10^{-05}$
$p = (0.01,0.01,0,0,0.05,0.05)$	$3.29 \cdot 10^{-01}$	$1.05 \cdot 10^{-05}$
$p = (0.01,0.01,0,0,0.01,0.01)$	$3.91 \cdot 10^{-02}$	$4.33 \cdot 10^{-07}$
$p = (0.1,0.1,0,0,0.1,0.1)$	$4.94 \cdot 10^{-01}$	$5.56 \cdot 10^{-04}$

Table 1: Real–time approximations for different perturbations in the unconstrained case.

Although perturbations of up to 10% are considered to be large perturbations, the numerical results clearly indicate the real–time capability of the proposed method and show the higher order approximations of the objective for the unconstrained case as predicted by Theorem 2.
The computing time for calculating the real–time approximation of the complete controls varies between about $2 \cdot 10^{-7}$ seconds (only one perturbation holds $p_i \neq 0$, $i = 1,\ldots,6$) and about $8 \cdot 10^{-7}$ seconds (four perturbations hold $p_i \neq 0$, $i = 1,\ldots,6$) on a PIII 1GHz personal computer. Note that only a few floating point operations have to be performed. Inpractical implementations the computing time can be reduced further by a factor 51 (number of grid points in time), if the time, during the runtime of the real Burgers process, is used for computing the approximation (8).

6.2 Box constraints

Next we consider box constraints of form

$$-0.015(1+p_3) \leq u_i(t) \leq 0.015(1+p_3), \qquad i = 1,2, \tag{23}$$

with an additional perturbation p_3. The optimal nominal solution is now obtained with an objective value $F(y(p_0),u(p_0),p_0) \approx 2.178279 \cdot 10^{-2}$ after about 2.6 seconds of computing time. The

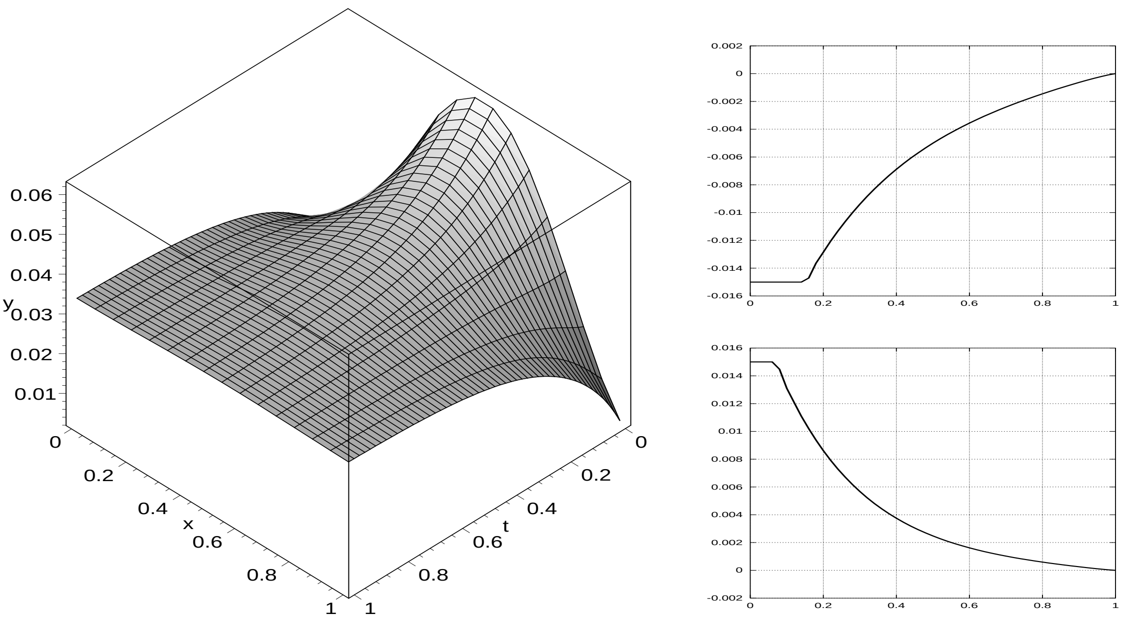


Figure 3: Nominal state $y(x,t)$ (left), nominal optimal controls $u_1(t)$, $u_2(t)$ (right) for the box constrained case.

nominal state in the interior and the unperturbed optimal controls are given in Figure 3.
Note that there are two boundary arcs for the controls.
The Hessian of the Lagrangian has been found to be positive definite on the kernel of the Jacobian of the active constraints and all other assumptions of Theorem 1 hold for the discretized problem, at least numerically. Hence expression (7) yields the sensitivity differentials of the control variables for p_1–p_3, p_5 and p_6. The sensitivity differentials of the control variables are shown in Figure 4.

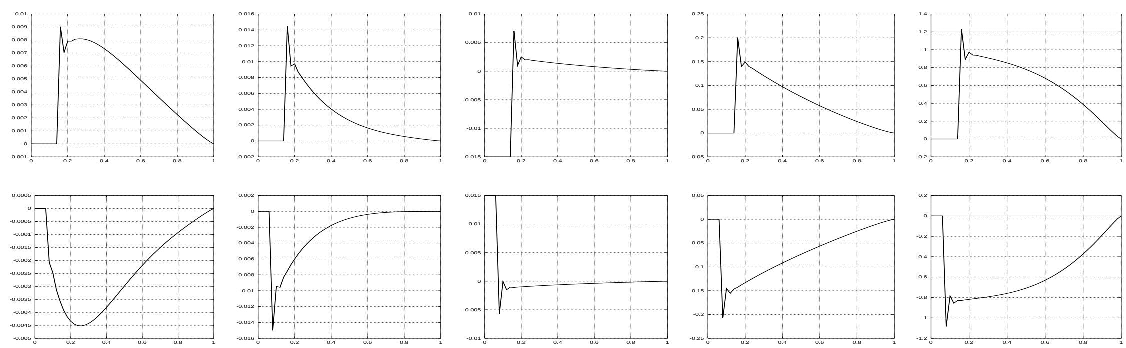


Figure 4: Sensitivity differentials $\frac{du_1(t)}{dp}(p_0)$ (top) and $\frac{du_2(t)}{dp}(p_0)$ (bottom) of the control variables for the box constrained case.

Note that the sensitivities of the controls in Figure 4 are zero on the boundary arcs for p_1, p_2, p_5 and p_6 and that the overshooting at each junction point of the control constraints results from the linear interpolation of the control variables. Table 2 lists the relative errors $\zeta_H^{[k]}(p)$ of the objective as defined in (22) for the box constrained Burgers problem.
Again the considered perturbations of up to 50% can be understood as large perturbations. Nevertheless the perturbed optimal solutions are well approximated. The crucial message from this example is however the admissibility of the approximated control variables after only one real–time correction step. This follows from Theorems 1–3 and is due to the fact that the treated constraints are linear in the control variables. Hence additional correction steps as proposed in (14) are not necessary for box constrained problems. Computing times are similar to the unconstrained case.

	$\zeta_H^{[0]}(p)$	$\zeta_H^{[1]}(p)$
$p = (0.1, 0.5, 0, 0, 0, 0)$	$3.22 \cdot 10^{-03}$	$8.75 \cdot 10^{-04}$
$p = (0, 0, 0.1, 0, 0, 0)$	$4.45 \cdot 10^{-04}$	$7.03 \cdot 10^{-06}$
$p = (0.01, 0.01, 0.01, 0, 0.001, 0)$	$3.98 \cdot 10^{-04}$	$1.99 \cdot 10^{-06}$
$p = (0.01, 0.01, 0.01, 0, 0, 0)$	$5.76 \cdot 10^{-05}$	$4.87 \cdot 10^{-10}$
$p = (-0.01, -0.01, -0.01, 0, 0, 0.01)$	$3.28 \cdot 10^{-04}$	$2.26 \cdot 10^{-06}$
$p = (-0.02, 0.02, -0.05, 0, 0, 0)$	$2.94 \cdot 10^{-04}$	$1.60 \cdot 10^{-06}$
$p = (0.05, -0.02, 0.05, 0, 0, 0)$	$3.23 \cdot 10^{-04}$	$2.83 \cdot 10^{-06}$
$p = (-0.01, 0.01, -0.01, 0, 0, 0)$	$5.53 \cdot 10^{-05}$	$5.62 \cdot 10^{-11}$

Table 2: Admissible real–time approximations for different perturbations in the box constrained case.

6.3 Nonlinear control constraints

Finally we consider a coupled nonlinear control constraint of the form

$$u_1(t)^2 + u_2(t)^2 \leq 0.000225(1 + p_4), \qquad i = 1, 2 \tag{24}$$

with an additional perturbation p_4. Since (24) is a nonlinear constraint we cannot expect, that in case of an active constraint the linear real–time approximation (8) ensures the admissibility of the solution in one step . Here, the nominal objective is calculated after about 3.5 seconds of computing time to $F(y(p_0), u(p_0), p_0) \approx 2.51 \cdot 10^{-2}$. The nominal state in the interior and the unperturbed optimal control functions are given in Figure 5.

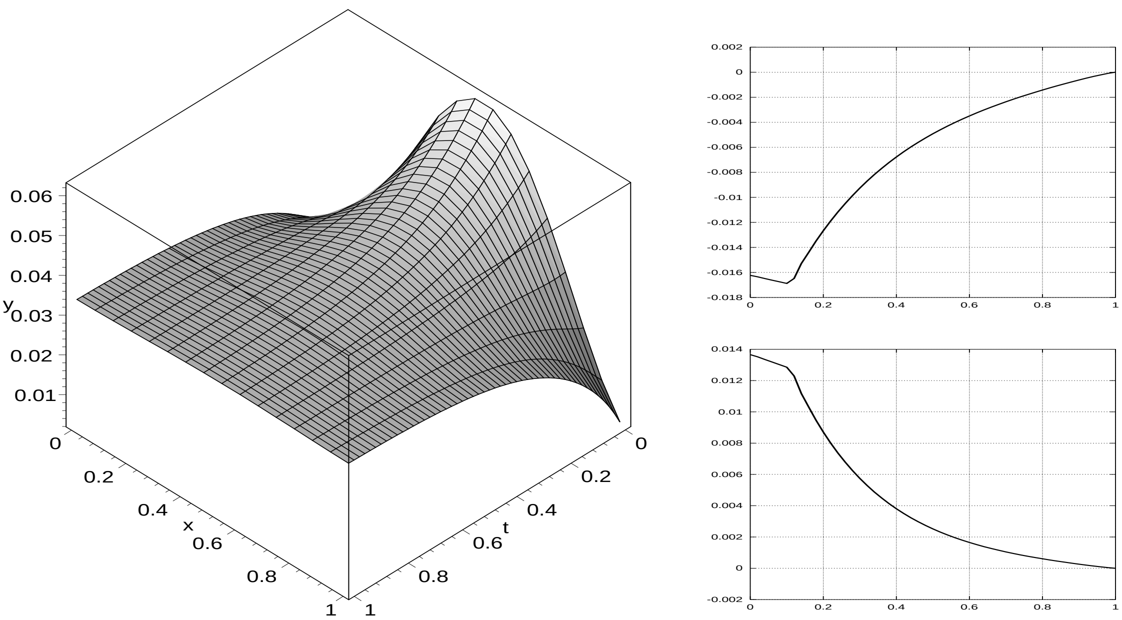


Figure 5: Nominal state $y(x,t)$ (left), nominal optimal controls $u_1(t)$, $u_2(t)$ (right) for the nonlinear control constrained case.

There is one boundary arc at the beginning of the time interval.
All assumptions of Theorem 1 have been checked numerically. The Hessian of the Lagrangian is positive definite on the kernel of the Jacobian of the active constraints. Hence expression (7) yields the sensitivity differentials $\frac{dz}{dp}(p_0)$ and $\frac{dz}{dq}(0)$ of the control variables for p_1, p_2 and p_4–p_6. The sensitivity differentials of the control variables w.r.t parameter p are shown in Figure 6. The sensitivity differentials of the control variables with respect to linear perturbations q in the constraints are neglected for the lack of space.
Note that the sensitivities of the controls in Figure 6 have jumps along the time axis at junction points and that the overshooting at each junction point of the control constraint results from the linear interpolation of the control variables. Table 3 lists the relative errors $\zeta_H^{[k]}(p)$ of the objective

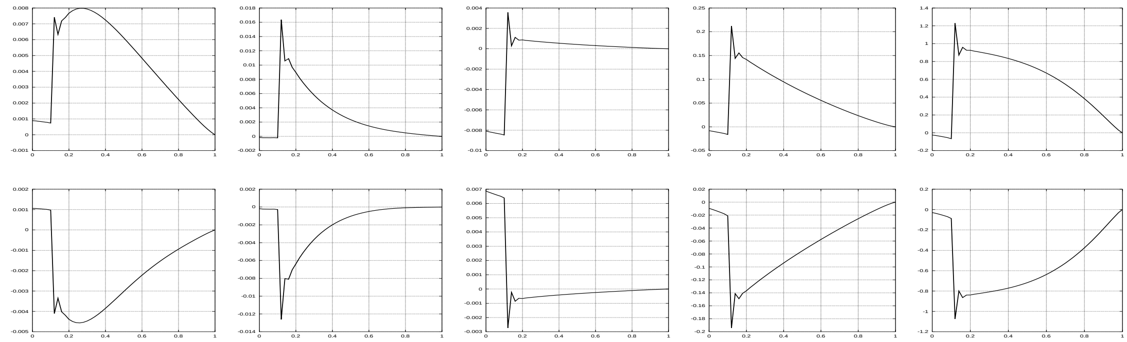


Figure 6: Sensitivity differentials $\frac{du_1(t)}{dp}(p_0)$ (top) and $\frac{du_2(t)}{dp}(p_0)$ (bottom) of the control variables for the nonlinear control constrained case.

as defined in (22) and the error $\zeta_G^{[k]}(p)$ in the nonlinear control constraint,

$$\zeta_G^{[k]}(p) := \max_i \frac{G_i^a(\tilde{z}^{[k]}(p), p)}{0.000225}, \qquad k = 0, 1, 2, \ldots, \tag{25}$$

for the first eight iterates, for perturbations $p_a = (0.05, 0, 0, -0.05, 0, 0)$, $p_b = (-0.1, 0.01, 0, 0.1, 0, 0)$, $p_c = (0.2, 0.05, 0, -0.5, 0.02, 0)$.

	$p = p_a$		$p = p_b$		$p = p_c$	
k	$\zeta_G^{[k]}(p)$	$\zeta_F^{[k]}(p)$	$\zeta_G^{[k]}(p)$	$\zeta_F^{[k]}(p)$	$\zeta_G^{[k]}(p)$	$\zeta_F^{[k]}(p)$
0	$5.00 \cdot 10^{-02}$	$1.08 \cdot 10^{-04}$	$1.00 \cdot 10^{-01}$	$3.01 \cdot 10^{-04}$	$5.00 \cdot 10^{-01}$	$5.57 \cdot 10^{-03}$
1	$6.35 \cdot 10^{-04}$	$1.88 \cdot 10^{-06}$	$2.54 \cdot 10^{-03}$	$3.96 \cdot 10^{-06}$	$6.27 \cdot 10^{-02}$	$2.36 \cdot 10^{-04}$
2	$1.60 \cdot 10^{-05}$	$1.14 \cdot 10^{-08}$	$1.25 \cdot 10^{-04}$	$5.83 \cdot 10^{-07}$	$1.66 \cdot 10^{-02}$	$5.99 \cdot 10^{-05}$
3	$4.05 \cdot 10^{-07}$	$3.55 \cdot 10^{-08}$	$6.12 \cdot 10^{-06}$	$3.58 \cdot 10^{-07}$	$4.75 \cdot 10^{-03}$	$8.14 \cdot 10^{-06}$
4	$1.02 \cdot 10^{-08}$	$3.67 \cdot 10^{-08}$	$2.99 \cdot 10^{-07}$	$3.69 \cdot 10^{-07}$	$1.38 \cdot 10^{-03}$	$7.04 \cdot 10^{-06}$
5	$2.60 \cdot 10^{-10}$	$3.68 \cdot 10^{-08}$	$1.46 \cdot 10^{-08}$	$3.68 \cdot 10^{-07}$	$4.04 \cdot 10^{-04}$	$1.15 \cdot 10^{-05}$
6	$6.58 \cdot 10^{-12}$	$3.68 \cdot 10^{-08}$	$7.12 \cdot 10^{-10}$	$3.69 \cdot 10^{-07}$	$1.18 \cdot 10^{-04}$	$1.28 \cdot 10^{-05}$
7	$1.67 \cdot 10^{-13}$	$3.68 \cdot 10^{-08}$	$3.47 \cdot 10^{-11}$	$3.69 \cdot 10^{-07}$	$3.46 \cdot 10^{-05}$	$1.32 \cdot 10^{-05}$
8	$4.28 \cdot 10^{-15}$	$3.68 \cdot 10^{-08}$	$1.70 \cdot 10^{-12}$	$3.69 \cdot 10^{-07}$	$9.95 \cdot 10^{-06}$	$1.33 \cdot 10^{-05}$

Table 3: Admissible real–time approximations for different perturbations in the nonlinear control constrained case.

Even in the case of a large perturbation p_c the method converges. Further iterations lead to admissible solutions within machine precision. Computing times for each of the eight iterates are similar to the two exaples discussed before since an evaluation of the nonlinear control constraint (24) at each discretized time step can be done by only six floating point operations.

7 Conclusion

Real–time methods based on the parametric sensitivity analysis of perturbed PDE optimal control problems with control constraints have been proposed. Under the assumption of convergence to the assumed unique continuous solution, the original control problem has been discretized via various stages to obtain a perturbed NLP problem for the discrete control variables. For this finite-dimensional problem, we are able to compute the sensitivity differentials of the optimal solution with respect to the perturbation parameters. These sensitivities allow for an approximation of the corresponding continuous variables of the original PDE optimal control problem and have been used for real–time optimal control strategies. The examples show that it is possible to calculate accurate solutions in a robust way.

References

[1] W. Alt Stability of solutions and the Lagrange-Newton method for nonlinear optimization and optimal control problems. Habilitationsschritft, Universität Bayreuth, 1990.

[2] C. Büskens. Direkte Optimierungsmethoden zur numerischen Berechnung optimaler Steuerungen. Diploma thesis, Institut für Numerische Mathematik, Universität Münster, Münster, Germany, 1993.

[3] C. Büskens. *Optimierungsmethoden und Sensitivitätsanalyse für optimale Steuerprozesse mit Steuer– und Zustands– Beschränkungen.* Dissertation, Institut für Numerische Mathematik, Universität Münster, Münster, Germany, 1998.

[4] C. Büskens. *Echtzeitoptimierung und Echtzeitoptimalsteuerung parametergestörter Probleme.* Habilitation, Fachbereich Mathematik und Physik, Universität Bayreuth, Bayreuth, Germany, 2002.

[5] C. Büskens and R. Griesse. Computational Parametric Sensitivity Analysis of Perturbed PDE Optimal Control Problems with State and Control Constraints. submitted.

[6] C. Büskens and H. Maurer. Sensitivity analysis and real-time control of nonlinear optimal control systems via nonlinear programming methods. In W.H. Schmidt, K. Heier, L. Bittner, and R. Bulirsch, editors, *Variational Calculus, Optimal Control and Applications*, volume 124 of *International Series of Numerical Mathematics*, pages 185–196. Birkhäuser Verlag, Basel, 1998.

[7] C. Büskens and H. Maurer. Real-time control of robots with initial value perturbations via nonlinear programming methods. *Optimization*, 47:383–405, 2000.

[8] C. Büskens and H. Maurer. SQP-methods for solving optimal control problems with control and state constraints: adjoint variables, sensitivity analysis and real-time control. *Journal of Computational and Applied Mathematics*, 120:85–108, 2000.

[9] C. Büskens and H. Maurer. Nonlinear programming methods for real–time control of an industrial robot. *Journal of Optimization Theory and Applications*, 107:505–527, 2000.

[10] E. Casas, F. Tröltzsch, and A. Unger. Second order sufficient optimality conditions for a class of elliptic control problems. *Journal for Analysis and its Applications*, 15:687–707, 1996.

[11] E. Casas, F. Tröltzsch, and A. Unger. Second order sufficient optimality conditions for some state–constrained control problems of semilinear elliptic equation. *SIAM Journal on Control and Optimization*, to appear.

[12] A.L. Dontchev and W.W. Hager Lipschitzian stability in nonlinear control and optimization *SIAM Journal on Control and Optimization*, 31, 569–603, 1993.

[13] A.V. Fiacco. *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, volume 165 of *Mathematics in Science and Engineering.* Academic Press, New York, 1983.

[14] M. Heinkenschloss and E.W. Sachs. Numerical solution of a constrained control problem for a phase field model. *Control and Estimation of Distributed Parameter Systems, Int. Ser. Num. Math*, 118:171–188, 1994.

[15] K. Kunisch and S. Volkwein. Augmented Lagrangian–SQP techniques and their approximations. *Contemporary Mathematics*, 209:147–159, 1997.

[16] K. Malanowski, C. Büskens and H. Maurer Convergence of approximations to nonlinear optimal control problems In A.V. Fiacco, editor, *Mathematical Programming with Data Perturbations*, Lecture Notes in Pure and Applied Mathematics, Volume 195, 253–284, Marcel Dekker Inc., 1998.

[17] H. Maurer and H.D. Mittelmann. Optimization techniques for solving elliptic control problems with control and state constraints. Part 1: Boundary control. *Computational Optimization and Applications*, 16:29–55, 2000.

[18] H. Maurer and H.D. Mittelmann. Optimization techniques for solving elliptic control problems with control and state constraints. Part 2: Distributed control. *Computational Optimization and Applications*, 18:141–160, 2001.

[19] H.J. Pesch. Real-time computation of feedback controls for constrained optimal control problems, Part 1: A correction method based on multiple shooting. *Optimal Control Applications & Methods*, pages 147–171,1989.

[20] S. Volkwein. Mesh-Independence of an Augmented Lagrangian-SQP Method in Hilbert Spaces and Control Problems for the Burgers Equation. *Dissertation Fachbereich Mathematik, Technische Universtät Berlin*, 1997.

Berichte aus der Technomathematik ISSN 1435-7968

http://www.math.uni-bremen.de/zetem/berichte.html

— Vertrieb durch den Autor —

Reports **Stand: 13. August 2004**

98–01. Peter Benner, Heike Faßbender:
An Implicitly Restarted Symplectic Lanczos Method for the Symplectic Eigenvalue Problem, Juli 1998.

98–02. Heike Faßbender:
Sliding Window Schemes for Discrete Least-Squares Approximation by Trigonometric Polynomials, Juli 1998.

98–03. Peter Benner, Maribel Castillo, Enrique S. Quintana-Ortí:
Parallel Partial Stabilizing Algorithms for Large Linear Control Systems, Juli 1998.

98–04. Peter Benner:
Computational Methods for Linear–Quadratic Optimization, August 1998.

98–05. Peter Benner, Ralph Byers, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Solving Algebraic Riccati Equations on Parallel Computers Using Newton's Method with Exact Line Search, August 1998.

98–06. Lars Grüne, Fabian Wirth:
On the rate of convergence of infinite horizon discounted optimal value functions, November 1998.

98–07. Peter Benner, Volker Mehrmann, Hongguo Xu:
A Note on the Numerical Solution of Complex Hamiltonian and Skew-Hamiltonian Eigenvalue Problems, November 1998.

98–08. Eberhard Bänsch, Burkhard Höhn:
Numerical simulation of a silicon floating zone with a free capillary surface, Dezember 1998.

99–01. Heike Faßbender:
The Parameterized SR Algorithm for Symplectic (Butterfly) Matrices, Februar 1999.

99–02. Heike Faßbender:
Error Analysis of the symplectic Lanczos Method for the symplectic Eigenvalue Problem, März 1999.

99–03. Eberhard Bänsch, Alfred Schmidt:
Simulation of dendritic crystal growth with thermal convection, März 1999.

99–04. Eberhard Bänsch:
Finite element discretization of the Navier-Stokes equations with a free capillary surface, März 1999.

99–05. Peter Benner:
Mathematik in der Berufspraxis, Juli 1999.

99–06. Andrew D.B. Paice, Fabian R. Wirth:
Robustness of nonlinear systems and their domains of attraction, August 1999.

99–07. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Balanced Truncation Model Reduction of Large-Scale Dense Systems on Parallel Computers, September 1999.

99–08. Ronald Stöver:
Collocation methods for solving linear differential-algebraic boundary value problems, September 1999.

99–09. Huseyin Akcay:
Modelling with Orthonormal Basis Functions, September 1999.

99–10. Heike Faßbender, D. Steven Mackey, Niloufer Mackey:
Hamilton and Jacobi come full circle: Jacobi algorithms for structured Hamiltonian eigenproblems, Oktober 1999.

99–11. Peter Benner, Vincente Hernández, Antonio Pastor:
On the Kleinman Iteration for Nonstabilizable System, Oktober 1999.

99–12. Peter Benner, Heike Faßbender:
A Hybrid Method for the Numerical Solution of Discrete-Time Algebraic Riccati Equations, November 1999.

99–13. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Numerical Solution of Schur Stable Linear Matrix Equations on Multicomputers, November 1999.

99–14. Eberhard Bänsch, Karol Mikula:
Adaptivity in 3D Image Processing, Dezember 1999.

00–01. Peter Benner, Volker Mehrmann, Hongguo Xu:
Perturbation Analysis for the Eigenvalue Problem of a Formal Product of Matrices, Januar 2000.

00–02. Ziping Huang:
Finite Element Method for Mixed Problems with Penalty, Januar 2000.

00–03. Gianfrancesco Martinico:
Recursive mesh refinement in 3D, Februar 2000.

00–04. Eberhard Bänsch, Christoph Egbers, Oliver Meincke, Nicoleta Scurtu:
Taylor-Couette System with Asymmetric Boundary Conditions, Februar 2000.

00–05. Peter Benner:
Symplectic Balancing of Hamiltonian Matrices, Februar 2000.

00–06. Fabio Camilli, Lars Grüne, Fabian Wirth:
A regularization of Zubov's equation for robust domains of attraction, März 2000.

00–07. Michael Wolff, Eberhard Bänsch, Michael Böhm, Dominic Davis:
Modellierung der Abkühlung von Stahlbrammen, März 2000.

00–08. Stephan Dahlke, Peter Maaß, Gerd Teschke:
Interpolating Scaling Functions with Duals, April 2000.

00–09. Jochen Behrens, Fabian Wirth:
A globalization procedure for locally stabilizing controllers, Mai 2000.

00–10. Peter Maaß, Gerd Teschke, Werner Willmann, Günter Wollmann:
Detection and Classification of Material Attributes – A Practical Application of Wavelet Analysis, Mai 2000.

00–11. Stefan Boschert, Alfred Schmidt, Kunibert G. Siebert, Eberhard Bänsch, Klaus-Werner Benz, Gerhard Dziuk, Thomas Kaiser:
Simulation of Industrial Crystal Growth by the Vertical Bridgman Method, Mai 2000.

00–12. Volker Lehmann, Gerd Teschke:
Wavelet Based Methods for Improved Wind Profiler Signal Processing, Mai 2000.

00–13. Stephan Dahlke, Peter Maass:
A Note on Interpolating Scaling Functions, August 2000.

00–14. Ronny Ramlau, Rolf Clackdoyle, Frédéric Noo, Girish Bal:
Accurate Attenuation Correction in SPECT Imaging using Optimization of Bilinear Functions and Assuming an Unknown Spatially-Varying Attenuation Distribution, September 2000.

00–15. Peter Kunkel, Ronald Stöver:
Symmetric collocation methods for linear differential-algebraic boundary value problems, September 2000.

00–16. Fabian Wirth:
The generalized spectral radius and extremal norms, Oktober 2000.

00–17. Frank Stenger, Ahmad Reza Naghsh-Nilchi, Jenny Niebsch, Ronny Ramlau:
A unified approach to the approximate solution of PDE, November 2000.

00–18. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Parallel algorithms for model reduction of discrete–time systems, Dezember 2000.

00–19. Ronny Ramlau:
A steepest descent algorithm for the global minimization of Tikhonov–Phillips functional, Dezember 2000.

01–01. Efficient methods in hyperthermia treatment planning:
Torsten Köhler, Peter Maass, Peter Wust, Martin Seebass, Januar 2001.

01–02. Parallel Algorithms for LQ Optimal Control of Discrete-Time Periodic Linear Systems:
Peter Benner, Ralph Byers, Rafael Mayo, Enrique S. Quintana-Ortí, Vicente Hernández, Februar 2001.

01–03. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:
Efficient Numerical Algorithms for Balanced Stochastic Truncation, März 2001.

01–04. Peter Benner, Maribel Castillo, Enrique S. Quintana-Ortí:
Partial Stabilization of Large-Scale Discrete-Time Linear Control Systems, März 2001.

01–05. Stephan Dahlke:
Besov Regularity for Edge Singularities in Polyhedral Domains, Mai 2001.

01–06. Fabian Wirth:
A linearization principle for robustness with respect to time-varying perturbations, Mai 2001.

01–07. Stephan Dahlke, Wolfgang Dahmen, Karsten Urban:
Adaptive Wavelet Methods for Saddle Point Problems - Optimal Convergence Rates, Juli 2001.

01–08. Ronny Ramlau:
Morozov's Discrepancy Principle for Tikhonov regularization of nonlinear operators, Juli 2001.

01–09. Michael Wolff:
Einführung des Drucks für die instationären Stokes–Gleichungen mittels der Methode von Kaplan, Juli 2001.

01–10. Stephan Dahlke, Peter Maaß, Gerd Teschke:
Reconstruction of Reflectivity Desities by Wavelet Transforms, August 2001.

01–11. Stephan Dahlke:
Besov Regularity for the Neumann Problem, August 2001.

01–12. Bernard Haasdonk, Mario Ohlberger, Martin Rumpf, Alfred Schmidt, Kunibert G. Siebert:
h–p–Multiresolution Visualization of Adaptive Finite Element Simulations, Oktober 2001.

01–13. Stephan Dahlke, Gabriele Steidl, Gerd Teschke:
Coorbit Spaces and Banach Frames on Homogeneous Spaces with Applications to Analyzing Functions on Spheres, August 2001.

02–01. Michael Wolff, Michael Böhm:
Zur Modellierung der Thermoelasto-Plastizität mit Phasenumwandlungen bei Stählen sowie der Umwandlungsplastizität, Februar 2002.

02–02. Stephan Dahlke, Peter Maaß:
An Outline of Adaptive Wavelet Galerkin Methods for Tikhonov Regularization of Inverse Parabolic Problems, April 2002.

02–03. Alfred Schmidt:
A Multi-Mesh Finite Element Method for Phase Field Simulations, April 2002.

02–04. Sergey N. Dachkovski, Michael Böhm:
A Note on Finite Thermoplasticity with Phase Changes, Juli 2002.

02–05. Michael Wolff, Michael Böhm:
Phasenumwandlungen und Umwandlungsplastizität bei Stählen im Konzept der Thermoelasto-Plastizität, Juli 2002.

02–06. Gerd Teschke:
Construction of Generalized Uncertainty Principles and Wavelets in Anisotropic Sobolev Spaces, August 2002.

02–07. Ronny Ramlau:
TIGRA – an iterative algorithm for regularizing nonlinear ill–posed problems, August 2002.

02–08. Michael Lukaschewitsch, Peter Maaß, Michael Pidcock:
Tikhonov regularization for Electrical Impedance Tomography on unbounded domains, Oktober 2002.

02–09. Volker Dicken, Peter Maaß, Ingo Menz, Jenny Niebsch, Ronny Ramlau:
Inverse Unwuchtidentifikation an Flugtriebwerken mit Quetschöldämpfern, Oktober 2002.

02–10. Torsten Köhler, Peter Maaß, Jan Kalden:
Time-series forecasting for total volume data and charge back data, November 2002.

02–11. Angelika Bunse-Gerstner:
A Short Introduction to Iterative Methods for Large Linear Systems, November 2002.

02–12. Peter Kunkel, Volker Mehrmann, Ronald Stöver:
Symmetric Collocation for Unstructured Nonlinear Differential-Algebraic Equations of Arbitrary Index, November 2002.

02–13. Michael Wolff:
Ringvorlesung: Distortion Engineering 2
Kontinuumsmechanische Modellierung des Materialverhaltens von Stahl unter Berücksichtigung von Phasenumwandlungen, Dezember 2002.

02–14. Michael Böhm, Martin Hunkel, Alfred Schmidt, Michael Wolff:
Evaluation of various phase-transition models for 100Cr6 for application in commercial FEM programs, Dezember 2002.

03–01. Michael Wolff, Michael Böhm, Serguei Dachkovski:
Volumenanteile versus Massenanteile - der Dilatometerversuch aus der Sicht der Kontinuumsmechanik, Januar 2003.

03–02. Daniel Kessler, Ricardo H. Nochetto, Alfred Schmidt:
A posteriori error control for the Allen-Cahn Problem: circumventing Gronwall's inequality, März 2003.

03–03. Michael Böhm, Jörg Kropp, Adrian Muntean:
On a Prediction Model for Concrete Carbonation based on Moving Interfaces - Interface concentrated Reactions, April 2003.

03–04. Michael Böhm, Jörg Kropp, Adrian Muntean:
A Two-Reaction-Zones Moving-Interface Model for Predicting $Ca(OH)_2$ *Carbonation in Concrete*, April 2003.

03–05. Vladimir L. Kharitonov, Diederich Hinrichsen:
Exponential estimates for time delay systems, May 2003.

03–06. Michael Wolff, Michael Böhm, Serguei Dachkovski, Günther Löwisch:
Zur makroskopischen Modellierung von spannungsabhängigem Umwandlungsverhalten und Umwandlungsplastizität bei Stählen und ihrer experimentellen Untersuchung in einfachen Versuchen, Juli 2003.

03–07. Serguei Dachkovski, Michael Böhm, Alfred Schmidt, Michael Wolff:
Comparison of several kinetic equations for pearlite transformation in 100Cr6 steel,
Juli 2003.

03–08. Volker Dicken, Peter Maass, Ingo Menz, Jenny Niebsch, Ronny Ramlau:
Nonlinear Inverse Unbalance Reconstruction in Rotor dynamics,
Juli 2003.

03–09. Michael Böhm, Serguei Dachkovski, Martin Hunkel, Thomas Lübben, Michael Wolff:
Übersicht über einige makroskopische Modelle für Phasenumwandlungen im Stahl,
Juli 2003.

03–10. Michael Wolff, Friedhelm Frerichs, Bettina Suhr:
Vorstudie für einen Bauteilversuch zur Umwandlungsplastizität bei der perlitischen Umwandlung des Stahls 100 Cr6,
August 2003.

03–11. Michael Wolff, Bettina Suhr:
Zum Vergleich von Massen- und Volumenanteilen bei der perlitischen Umwandlung der Stähle 100Cr6 und C80,
September 2003.

03–12. Rike Grotmaack, Adrian Muntean:
Stabilitätsanalyse eines Moving-Boundary-Modells der beschleunigten Karbonatisierung von Portlandzementen,
September 2003.

03–13. Alfred Schmidt, Michael Wolff, Michael Böhm:
Numerische Untersuchungen für ein Modell des Materialverhaltens mit Umwandlungsplastizität und Phasenumwandlungen beim Stahl 100Cr6 (Teil 1),
September 2003.

04–01. Liliana Cruz Martin, Gerd Teschke:
A new method to reconstruct radar reflectivities and Doppler information,
Januar 2004.

04–02. Ingrid Daubechies, Gerd Teschke:
Wavelet based image decomposition by variational functionals,
Januar 2004.

04–03. N. Guglielmi, F. Wirth, M. Zennaro:
Complex polytope extremality results for families of matrices,
März 2004.

04–04. I. Daubechies, G. Teschke:
Variational image restoration by means of wavelets: simultaneous decomposition, deblurring and denoising,
April 2004.

04–05. V.L. Kharitonov, E. Plischke:
Lyapunov matrices for time-delay systems,
April 2004.

04–06. Ronny Ramlau:
On the use of fixed point iterations for the regularization of nonlinear ill-posed problems,
Juni 2004.

04–07. Christof Büskens, Matthias Knauer:
Higher Order Real-Time Approximations In Optimal Control of Multibody-Systems For Industrial Robots,
August 2004.

04–08. Christof Büskens, Roland Griesse:
Computational Parametric Sensitivity Analysis of Perturbed PDE Optimal Control Problems with State and Control Constraints,
August 2004.

04–09. Christof Büskens:
Higher Order Real-Time Approximations of Perturbed Control Constrainted PDE Optimal Control Problems ,
August 2004.