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Lyapunov matrices for time-delay systems

V.L. Kharitonov E. Plischke

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V. L. Kharitonov
Control Automatico
CINVESTAV-IPN A.P. 14-740
07000 Mexico D.F., Mexico

E-mail: khar@ctrl.cinvestav.mx

E. Plischke
Zentrum für Technomathematik
Universität Bremen
D-28334 Bremen, Germany

E-mail: elmar@math.uni-bremen.de

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Abstract

The construction of complete-type Lyapunov-Krasovkii functionals for a linear time-invariant delay system depends on so-called delay Lyapunov matrices which satisfy a matrix delay equation with additional boundary conditions. We study existence and uniqueness issues for these delay Lyapunov matrices.

1 Introduction

The use of Lyapunov methods for the stability analysis of time-delay systems has been an ever growing subject of interest starting with the pioneering works of Krasovskii [7] and Razumikhin [9]. Recently, Kharitonov and Zhabko [5] introduced modified Lyapunov-Krasovskii functionals for which the time derivative includes terms with not only depend on the present but also on the past states of the delay system. This modification allows to use the functionals for robustness analysis of time delay systems. The construction of these functionals is based on a solution of a linear matrix differential-difference equation on a finite time interval which satisfies additional symmetry and boundary conditions. This solution is called a *delay Lyapunov matrix* as it inherits properties of the classical quadratic Lyapunov functions for ordinary delay free differential equations. Delay Lyapunov matrices have also been used in [4] in order to derive exponential estimates for the solutions of exponentially stable time delay systems. In both papers the existence of these matrices was shown only to the case of exponentially stable systems. The uniqueness issue was not studied there. This paper closes this gap by showing that a unique delay Lyapunov matrix exists when the delay equation is exponentially stable. For the general case, however, there are currently no results available, but in the case of one delay systems we give here necessary and sufficient conditions for the existence of the delay Lyapunov matrices.

2 Preliminaries

We consider a linear time-invariant delay system of the form

$$\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - h_k), \quad t \geq 0 \tag{1}$$

where $A_0, \ldots, A_m \in \mathbb{R}^{n \times n}$ are given matrices, $0 = h_0 < h_1 < \cdots < h_m = H$ are given delays, and $m \geq 1$. To specify an initial value problem we prescribe a piece-wise continuous initial function $\varphi(\theta)$, $\theta \in [-H, 0]$ and call the associated unique solution $x(t; \varphi)$. A *trajectory segment* of $x(t, \varphi)$ is denoted by $x_t : [-H, 0] \to \mathbb{R}^n$. When it is necessary to indicate the initial condition the trajectory segment will be denoted as $x_t(\varphi)$. The set of all continuous function segments is given by $\mathcal{C} = \mathcal{C}[-H, 0]$.
The zeros of the *characteristic equation* $\Delta(s) = \det\left(sI - \sum_{k=0}^{m} A_k e^{-h_k s}\right)$ are the *eigenvalues* of (1), its *spectrum* is given by $\sigma((1)) = \{s \in \mathbb{C} \,|\, \Delta(s) = 0\}$. If s_0 is such an eigenvalue, then there exists an *eigenmotion* $x(t) = e^{s_0 t}\eta$ of (1) where $\eta \in \mathbb{C}^n$.
The system (1) is called exponentially stable if there exist constants $M \geq 1$, $\beta > 0$ such that for every solution $x(t; \varphi)$the following inequality holds

$$\|x(t; \varphi)\| \leq M e^{-\beta t} \|\varphi\|_H, \qquad \text{where} \quad \|\varphi\|_H := \sup_{\tau \in [-H, 0]} \|\varphi(\tau)\|.$$

A necessary and sufficient condition for the exponential stability of (1) is that all of its eigenvalues reside in the open left half-plane $\mathbb{C}_-$. As in the delay-free case one can check this stability property by using Lyapunov functions.

Definition 1. A functional $v : \mathcal{C} \to \mathbb{R}_+$ is called a *Lyapunov-Krasovskii functional* for (1) if it has the following properties: There exist $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \|x(t)\|^2 \leq v(x_t) \leq \alpha_2 \|x_t\|_H^2$, and there exists $\beta > 0$ with $\dot{v}(x_t) \leq -\beta \|x(t)\|^2$.

To construct a Lyapunov-Krasovskii functional we first choose a quadratic functional $w : \mathcal{C} \to \mathbb{R}_+$ and then determine a functional $v(\cdot)$ satisfying $\dot{v}(x_t) = -w(x_t)$. The following result has been shown in [5].

Proposition 2. *Given a quadratic functional of the form*

$$w(x_t) = \sum_{k=0}^{m} x^T(t - h_k) W_k x(t - h_k) + \sum_{k=1}^{m} \int_{-h_k}^{0} x^T(t + \theta) W_{m+k} x(t + \theta) d\theta \tag{2}$$

where $W_0, W_1, \ldots, W_{2m} \in \mathbb{R}^{n \times n}$ are positive definite weight matrices. If the system (1) is exponentially stable then there exists a unique quadratic functional v with $\dot{v}(x_t) = -w(x_t)$.

This functional is given by

$$v(x_t) = x^T(t)U(0)x(t) + \sum_{k=1}^{m} 2x^T(t) \int_{-h_k}^{0} U(-h_k - \theta)A_k x(t+\theta)d\theta +$$
$$+ \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0}\int_{-h_j}^{0} x^T(t+\theta_2)A_k^T U(\theta_2 - \theta_1 + h_k - h_j)A_j x(t+\theta_1)d\theta_1 d\theta_2 +$$
$$+ \sum_{k=1}^{m} \int_{-h_k}^{0} x^T(t+\theta)\left[W_k + (h_k+\theta)W_{m+k}\right] x(t+\theta)d\theta \quad (3)$$

where

$$U(\tau) = \int_0^\infty K^T(t)\left[\sum_{k=0}^{m}(W_k + h_k W_{m+k})\right] K(t+\tau)dt, \qquad \tau \in \mathbb{R}.$$

Here $K(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ *is the fundamental solution of* (1), *i.e.* $K(t) = 0_n$ *for* $t < 0$, $K(0) = I_n$ *and* $\dot{K}(t) = \sum_{k=0}^{m} A_k K(t-h_k)$ *for* $t \geq 0$.

As $U(\tau)$ in (3) takes over the role of a classical quadratic Lyapunov matrix for systems without delays we call it the *delay Lyapunov matrix* associated with (3). In the following we will study its properties. Note that $U(\tau)$ is of the form

$$U(\tau) = \int_0^\infty K^T(t)WK(t+\tau)dt, \qquad \tau \in \mathbb{R} \quad (4)$$

where $W \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Moreover, for the construction of the functional v of (3), $U(\tau)$ needs only to be known for $\tau \in [-H, H]$.
We have the following characterization of $U(\tau)$.

Problem 3. *For a given symmetric matrix* $W \in \mathbb{R}^{n\times n}$ *find a continuous solution* $U(\tau)$ *of the following matrix delay differential equation*

$$U'(\tau) = U(\tau)A_0 + \sum_{k=1}^{m} U(\tau - h_k)A_k, \quad \tau \geq 0 \quad (5)$$

which satisfies the conditions

$$U(-\tau) = U^T(\tau), \qquad \tau \geq 0, \quad (6)$$
$$U(0)A_0 + A_0^T U(0) + \sum_{k=1} U^T(h_k)A_k + A_k^T U(h_k) = -W. \quad (7)$$

The condition (6) is called the *symmetry condition*, while (7) is called the *algebraic condition.* Let us comment on the smoothness of solutions. If the initial function $U[-H, 0]$ is $\mathcal{C}^0$, then the solution of (5) is $\mathcal{C}^1$. But by symmetry (6), the initial function is then itself $\mathcal{C}^1$. Repeating this argument, we see that U is infinitely smooth, with a possible exception of

$\tau = 0$ where the delay equation (5) only describes the one-sided derivative $U'(+0)$. It is easily verified that the improper integral (4) gives a solution of Problem 3. It is well-defined for all $\tau \in \mathbb{R}$ because (1) is exponentially stable. For the choice $W = \sum_{k=0}^{m} [W_k + h_k W_{m+k}]$ we obtain a matrix $U(\tau)$ that can be used for the construction of Lyapunov-Krasovskii functionals in Proposition 2.

3 Uniqueness of the delay Lyapunov matrix

We will now show that equation (5) and conditions (6),(7) uniquely determine the delay Lyapunov matrix. We state the following uniqueness theorem.

Theorem 4. *Suppose that the delay equation* (1) *is exponentially stable. Then the matrix* $U(\tau)$ *given by* (4) *is the unique solution of* (5) *satisfying the conditions* (6) *and* (7).

Proof. Assume that for a given W, Problem 3 has two different solutions $U_1(\tau)$ and $U_2(\tau)$. We define two functionals of the form

$$v_i(x_t) = x^T(t)U_i(0)x(t) + \sum_{k=1}^{m} 2x^T(t) \int_{-h_k}^{0} U_i(-h_k - \theta)A_k x(t+\theta)d\theta + \\ + \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} x^T(t+\theta_2)A_k^T \left[U_i(\theta_2 - \theta_1 + h_k - h_j)A_j x(t+\theta_1)d\theta_1\right] d\theta_2 \quad (8)$$

corresponding to U_1 and U_2, respectively. Note that this choice mimics the construction presented in Proposition 2, where the weights are given by $W_0 = W, W_1 = \cdots = W_{2m} = 0$. By direct calculations one can check that

$$\dot{v}_i(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi) \qquad \text{for} \quad t \geq 0,\, i = 1,2.$$

Hence the difference $v(x_t) = v_2(x_t) - v_1(x_t)$ satisfies the equality $\dot{v}(x_t) = 0$, $t \geq 0$, which implies that for all initial conditions φ and all $t \geq 0$ we have $v(x_t(\varphi)) = v(\varphi)$ as v is constant along solutions. By exponential stability of (1), $\|x(t,\varphi)\| \to 0$ as $t \to \infty$, therefore it follows from Definition (1) that $v(x_t(\varphi)) \to 0$ for $t \to \infty$ which implies $v(\varphi) = 0$ for every initial segment φ. From Equation (8) we obtain

$$0 = v(\varphi) = v_2(\varphi) - v_1(\varphi) = \varphi^T(0)U(0)\varphi(0) + \sum_{k=1}^{m} 2\varphi^T(0) \int_{-h_k}^{0} U(-h_k - \theta)A_k\varphi(\theta)d\theta + \\ + \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_2)A_k^T \left(\int_{-h_j}^{0} U(\theta_2 - \theta_1 + h_k - h_j)A_j\varphi(\theta_1)d\theta_1\right) d\theta_2, \quad (9)$$

where $U(\tau) = U_2(\tau) - U_1(\tau)$ satisfies the conditions of Problem 3 with $W = 0$. Now for $\gamma \in \mathbb{R}^n$ consider the initial segment given by a piecewise continuous function,

$$\varphi(\theta) = \begin{cases} \gamma, & \theta = 0, \\ 0, & \theta \in [-H, 0). \end{cases} \quad (10)$$

For this initial segment φ, all integrals in (9) vanish and hence (9) reduces to $\gamma^T U(0)\gamma = 0$. Since γ is an arbitrary vector and $U(0)$ is a symmetric matrix, $U(0) = 0$ must hold. Now, fix an index $i \in \{1, 2 \ldots, m\}$ and choose $\tau \in [-h_i, -h_{i-1})$ and $\varepsilon > 0$ such that $\tau + \varepsilon < -h_{i-1}$. For any given vectors $\gamma, \eta \in \mathbb{R}^n$ consider now the initial function

$$\varphi(\theta) = \begin{cases} \gamma, & \theta = 0, \\ \eta, & \theta \in [\tau, \tau + \varepsilon], \\ 0, & \text{for all other} \quad \theta \in [-H, 0). \end{cases}$$

For this initial segment, (9) now reads

$$0 = \sum_{k=i}^{m} 2\gamma^T \left(\int_{\tau}^{\tau+\varepsilon} U(-h_k - \theta) A_k d\theta \right) \eta + \\ + \sum_{k=i}^{m} \sum_{j=i}^{m} \eta^T A_k^T \left(\int_{\tau}^{\tau+\varepsilon} \int_{\tau}^{\tau+\varepsilon} U(\theta_1 - \theta_2 - h_k + h_j) d\theta_1 d\theta_2 \right) A_j \eta. \quad (11)$$

If $\varepsilon > 0$ is small then the first integral is proportional to ε while the double integral is proportional to ε^2 so that (11) can be written as

$$0 = 2\varepsilon\gamma^T \left(\sum_{k=i}^{m} U(-h_k - \tau) A_k \right) \eta + o(\varepsilon),$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. The fact that γ and η are arbitrary vectors and that ε can be made arbitrarily small implies that

$$\sum_{k=i}^{m} U(\tau - h_k) A_k = 0 \quad \text{for} \quad \tau \in (h_{i-1}, h_i]. \quad (12)$$

Now, (12) holds for all $i = 1, 2, \ldots, m$. For $i = 1$ we therefore obtain the differential equation $U'(\tau) = U(\tau)A_0$, as $\sum_{k=1}^{m} U(\tau - h_k)A_k = 0$, $\tau \in (0, h_1]$. But we already know $U(0) = 0$, and hence $U(\tau) = 0$ for all $\tau \in [0, h_1]$. On the interval $(h_1, h_2]$ equations (5) and (12) for $i = 2$ now yield the delay equation $U'(\tau) = U(\tau)A_0 + U(\tau - h_1)A_1$. But on the interval $[0, h_1]$, $U(\tau)$ is constantly 0, therefore $U(\tau) = 0$ for $\tau \in (h_1, h_2]$. Continuing this process we conclude that $U(\tau) = 0$, $\tau \in [0, H]$, i.e. $U_1(\tau) = U_2(\tau)$ for all $\tau \in [-H, H]$. Hence every solution of Problem 3 when (1) is exponentially stable is given by the integral equation (4). □

Let us now investigate under which conditions equation (5) has no solution satisfying the conditions (6) and (7). Of course, by the previous Theorem 3 such a situation may only occur if system (1) is not exponentially stable. We need the following technical lemma.

Lemma 5. *For two nontrivial vectors $x, y \in \mathbb{C}^n$ there exists a real symmetric matrix $W \in \mathbb{R}^{n \times n}$ such that $x^\top W y \neq 0$.*

Proof. If there exists an index j such that $x_j y_j \neq 0$ then $W = e_j e_j^T$ satisfies $x^T W y \neq 0$. Here e_j denotes the j-th unit vector. If $x_j y_j = 0$ for all j then there exist indices i and k, $k \neq i$, such that $x_i \neq 0$ and $x_k = 0$ while $y_i = 0$ and $y_k \neq 0$. Hence setting $W = e_i e_k^T + e_k e_i^T$ gives $x^T W y = x_i y_k + x_k y_i = x_i y_k \neq 0$. $\square$

Proposition 6. *If the delay system* (1) *has two eigenvalues s_1 and s_2 with $s_1 + s_2 = 0$ then there exists a symmetric matrix W for which* (5) *has no solution satisfying the conditions* (6)–(7).

Proof. Assume by contradiction that for any symmetric matrix W equation (5) has a solution satisfying conditions (6)–(7). We can pick two eigenmotions of system (1) associated with the eigenvalues s_1 and s_2 which are given by

$$x^{(1)}(t) = e^{s_1 t} x, \qquad x^{(2)}(t) = e^{s_2 t} y, \qquad x, y \in \mathbb{C}^n.$$

By Lemma 5 there exists a symmetric matrix W such that $x^T W y \neq 0$. Now by assumption, (5) has a solution $U(\tau)$ which satisfies the conditions (6)–(7). Let us define the bilinear functional

$$\begin{aligned} z(\varphi, \psi) = \varphi^T(0) U(0) \psi(0) &+ \sum_{j=1}^{m} \varphi^T(0) \int_{-h_j}^{0} U(-h_j - \theta) A_j \psi(\theta) d\theta + \\ &+ \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta) A_k^T U(h_k + \theta) d\theta \psi(0) + \\ &+ \sum_{k=1}^{m} \sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_2) A_k^T \int_{-h_j}^{0} U(\theta_2 - \theta_1 + h_k - h_j) A_j \psi(\theta_1) d\theta_1 d\theta_2. \end{aligned}$$

Given two solutions of (1) one can verify by direct calculation (analogously to the calculation of $\dot{v}(x_t) = -w(x_t)$ where v is defined by (3)) that

$$\tfrac{d}{dt} z(x_t(\varphi), x_t(\psi)) = -x^T(t; \varphi) W x(t; \psi).$$

In particular, for the solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ we obtain

$$\tfrac{d}{dt} z(x_t^{(1)}, x_t^{(2)}) = - \left[x^{(1)}(t) \right]^T W x^{(2)}(t) = -e^{(s_1 + s_2)t} x^T W y = -x^T W y \neq 0. \tag{13}$$

On the other hand, direct substitution of these solutions into the bilinear functional yields

$$\begin{aligned} z(x_t^{(1)}, x_t^{(2)}) = e^{(s_1+s_2)t} x^T \Big[U(0) &+ \sum_{j=1}^{m} \int_{-h_j}^{0} \left(U(-h_j - \theta) A_j e^{s_2 \theta} + A_j^T U(h_j + \theta) e^{s_1 \theta} \right) d\theta + \\ &+ \sum_{k=1}^{m} \sum_{j=1}^{m} \int_{-h_k}^{0} \int_{-h_j}^{0} e^{s_2 \theta_1 + s_1 \theta_2} A_k^T U(\theta_2 - \theta_1 + h_k - h_j) A_j d\theta_1 d\theta_2 \Big] y. \end{aligned}$$

Observe that the matrix in square brackets does not depend on t. The condition $s_1+s_2=0$ therefore implies that

$$\tfrac{d}{dt} z(x_t^{(1)}, x_t^{(2)}) = 0.$$

But this is in contradiction to (13). Hence there exists no solution of (5) satisfying (6)–(7). ☐

Proposition 6 shows that delay Lyapunov matrices do not exist if there are two eigenvalues of (1) with sum 0. It is generally not known if these are the only critical conditions. We will investigate this question for systems with one delay ($m=1$) in the next section.

4 Existence and uniqueness issues for the one delay case

Let us now assume that system (1) has only one delay term,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \qquad h > 0. \tag{1'}$$

Then the symmetry condition (6) allows us to extract a delay-free ordinary differential matrix equation from the delay matrix equation (5). This case has been studied in [3]. A recent analysis of this approach may be found in [8] where this technique is used to locate those eigenvalues of (1') which lie on the imaginary axis. Consider the following

Problem 7. *For a given symmetric matrix $W \in \mathbb{R}^{n\times n}$ find a solution $U : [-h,h] \to \mathbb{R}^{n\times n}$ satisfying*

$$\begin{aligned}
&U'(\tau) = U(\tau)A_0 + U(\tau-h)A_1, && \tau \in [0,h], \qquad (14)\\
&U(\tau) = U^T(-\tau) && \textit{(symmetry condition)},\\
&U(0)A_0 + U^T(h)A_1 + A_0^T U(0) + A_1^T U(h) = -W && \textit{(algebraic condition)}.
\end{aligned}$$

As this problem is just a reformulation of Problem 3 for the one delay case, any solution of Problem 7 is called a delay Lyapunov matrix for (1'). Note that we do not assume exponential stability, so the integral representation (4) may be not defined. Therefore, not only uniqueness, but also existence of delay Lyapunov matrices must be checked. Now, consider the following boundary value problem for a delay-free system.

Problem 8. *For a given symmetric matrix $W \in \mathbb{R}^{n\times n}$ find solutions $U, V : [0,h] \to \mathbb{R}^{n\times n}$ of the ordinary differential system*

$$U'(\tau) = U(\tau)A_0 + V(\tau)A_1, \qquad V'(\tau) = -A_1^T U(\tau) - A_0^T V(\tau) \tag{15}$$

$$U'(0) - V'(h) = -W, \qquad U(0) - V(h) = 0. \tag{16}$$

Here $U'(0)$ and $V'(h)$ are a short-hand notation for the one-sided derivatives, $U'(0+0) = U(0)A_0 + V(0)A_1$ and $V'(h-0) = -A_1^T U(h) - A_0^T V(h)$.

Problems 7 and 8 are equivalent in the following sense.

Proposition 9. *If $U(\tau)$ is a solution of Problem 7 then the pair $(U(\tau), V(\tau)) = (U(\tau), U^T(h-\tau))$ solves Problem 8. If the pair $(U(\tau), V(\tau))$ solves Problem 8 then $\tilde{U}(\tau) = 1/2(U(\tau) + V^T(h-\tau))$ solves Problem 7 if we extend $\tilde{U} : [0, h] \to \mathbb{R}^{n\times n}$ to $[-h, h]$ by setting $\tilde{U}(\tau) = \tilde{U}^T(-\tau)$ for $\tau < 0$.*

Proof. Suppose that $U(\tau)$ solves Problem 7. Set $V(\tau) = U^T(h-\tau)$. By symmetry,

$$\begin{aligned} U'(\tau) &= U(\tau)A_0 + U(\tau-h)A_1 = U(\tau)A_0 + U^T(h-\tau)A_1 = U(\tau)A_0 + V(\tau)A_1, \\ V'(\tau) &= -A_0^T U^T(h-\tau) - A_1^T V^T(h-\tau) = -A_1^T U(\tau) - A_0^T V(\tau), \end{aligned}$$

moreover the symmetry and boundary conditions $U(0) = U^T(0)$ and $U(0) = V^T(h)$ give $U(0) = V(h)$. Applying this result and the condition $V(0) = U^T(h)$ to the algebraic condition yields

$$\begin{aligned} -W &= U(0)A_0 + U^T(h)A_1 + A_0^T U(0) + A_1^T U(h) = \\ &= U(0)A_0 + V(0)A_1 + A_0^T V(h) + A_1^T U(h) = U'(0) - V'(h) \end{aligned}$$

Now, suppose that the pair (U, V) solves Problem 8. Then the pair $(\hat{U}(\tau), \hat{V}(\tau)) = (V^T(h-\tau), U^T(h-\tau))$ also solves Problem 8 since

$$\begin{aligned} \hat{U}'(\tau) &= -\left[-A_1^T U(h-\tau) - A_0^T V(h-\tau)\right]^T = \hat{U}(\tau)A_0 + \hat{V}(\tau)A_1, \\ \hat{V}'(\tau) &= -\left[U(h-\tau)A_0 + V(h-\tau)A_1\right]^T = -A_1^T \hat{U}(\tau) - A_0^T \hat{V}(\tau). \end{aligned}$$

Furthermore we have $\hat{U}(0) - \hat{V}(h) = V^T(h) - U^T(0) = 0$ and by symmetry of W

$$\begin{aligned} \hat{U}'(0) - \hat{V}'(h) &= \hat{U}(0)A_0 + \hat{V}(0)A_1 + A_1^T \hat{U}(h) + A_0^T \hat{V}(h) = \\ &= V^T(h)A_0 + U^T(h)A_1 + A_1^T V^T(0) + A_0^T U^T(0) = \\ &= \left(A_0^T U(0) + A_1^T U(h) + V(0)A_1 + V(h)A_0\right)^T = (U'(0) - V'(h))^T = -W. \end{aligned}$$

From U and $\hat{U}$ we can construct the solution $\tilde{U}(\tau) = \frac{1}{2}\left(U(\tau) + \hat{U}(\tau)\right)$. It satisfies

$$\tilde{U}'(\tau) = \tfrac{1}{2}(U(\tau) + V^T(h-\tau))A_0 + \tfrac{1}{2}(V(\tau) + U^T(h-\tau))A_1 = \tilde{U}(\tau)A_0 + \tilde{U}^T(h-\tau)A_1. \quad (17)$$

As a final step we have to verify that $\tilde{U}$ satisfies the conditions of Problem 7. Since $\tilde{U}$ is defined on $\tau \in [-h, 0)$ by $\tilde{U}(\tau) = \tilde{U}^T(-\tau)$ we only need to check $\tilde{U}(0) = \tilde{U}^T(0)$. But the condition $U(0) = V(h)$ of (16) implies that

$$\tilde{U}(0) = \tfrac{1}{2}\left(U(0) + V^T(h)\right) = \tfrac{1}{2}\left(V(h) + U^T(0)\right) = \tilde{U}^T(0). \quad (18)$$

Since W is symmetric, we have by (16) that $-W = \frac{1}{2}\left[(U'(0) + V'(h)) + (U'(0) + V'(h))^T\right]$. From this equation we obtain using (18)

$$\begin{aligned} -W &= \tfrac{1}{2}\left((U(0) + V^T(h))A_0 + (V(0) + U^T(h))A_1\right) + \\ &+ \tfrac{1}{2}\left(A_1^T(U(h) + V^T(0)) + A_0^T(V(h) + U^T(0))\right) = \tilde{U}(0)A_0 + \tilde{U}^T(h)A_1 + A_1^T \tilde{U}(h) + A_0^T \tilde{U}(0) \end{aligned}$$

which is the algebraic condition of Problem 7. Hence $\tilde{U}$ is a solution of Problem 7, if we extend $\tilde{U}$ to $[-h, h]$ by $\tilde{U}(\tau) = \tilde{U}^T(-\tau)$ since then (17) is equivalent to (14). ∎

From the proof of Proposition 9 we get the following corollary.

Corollary 10. *Given a pair $(U(\tau), V(\tau))$ that solves Problem 8.*

1. *The pair $(\tilde{U}(\tau), \tilde{V}(\tau)) = {}^1\!/_2(U(\tau)+V^T(h-\tau), V(\tau)+U^T(h-\tau))$ also solves Problem 8. Additionally $\tilde{U}(\tau) = \tilde{V}^T(h-\tau)$.*
2. *If the solution pair is uniquely determined then $U(\tau) = V^T(h-\tau)$.*

The last item immediately rises the question of unique solutions, for which we present the following uniqueness theorem.

Theorem 11. *The solution pair $(U(\tau), V(\tau))$ of Problem 8 is uniquely determined if and only if the spectrum of* (1') *does not contain two eigenvalues with sum* 0*.*

For the proof we recall the following technical lemma, see e.g. [1].

Lemma 12 (Unique Representation of Quasi-Polynomials). *Given a quasi-polynomial $\varphi(\tau) = \sum_{i=1}^{\ell} e^{\lambda_i \tau} p_i(\tau)$ where $\lambda_i \in \mathbb{C}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, and $p_i \in \mathbb{C}[\tau]$ are polynomials. Then $\varphi \equiv 0$ implies $p_i \equiv 0$ for all $i = 1, \ldots, \ell$.*

Proof (of Theorem 11). Given a nontrivial solution pair $U, V : [0, h] \to \mathbb{R}^{n\times n}$ corresponding to $W = 0$. By Corollary 10 we can assume without loss of generality that $U(\tau) = V^T(h-\tau)$ holds for $\tau \in [0, h]$. By continuation of the solution we obtain $U, V : \mathbb{R} \to \mathbb{R}^{n\times n}$ which satisfy Problem 8 on $\mathbb{R}$, i.e. (15) and $U(\tau) = V^T(h-\tau)$ are satisfied for all $\tau \in \mathbb{R}$. The algebraic condition $U'(0) = V'(h)$ then holds with two-sided derivatives.
We now show that the symmetry condition $U(-\tau) = U^T(\tau)$ automatically holds for all $\tau \in \mathbb{R}$. For this we prove $U(\tau) = V(\tau + h)$. Consider the second order derivatives

$$\begin{aligned}
U''(\tau) &= U'(\tau)A_0 + V'(\tau)A_1 = U'(\tau)A_0 - \left[A_1^T U(\tau) + A_0^T V(\tau)\right] A_1 = \\
&= U'(\tau)A_0 - A_0^T U'(\tau) + A_0^T U(\tau)A_0 - A_1^T U(\tau)A_1, \\
V''(\tau) &= -A_1^T U'(\tau) - A_0^T V'(\tau) = -A_1^T \left[U(\tau)A_0 + V(\tau)A_1\right] - A_0^T V'(\tau) = \\
&= V'(\tau)A_0 - A_0^T V'(\tau) + A_0^T V(\tau)A_0 - A_1^T V(\tau)A_0.
\end{aligned}$$

Hence U and V are subject to the same second order differential equation

$$X''(\tau) = X'(\tau)A_0 - A_0^T X'(\tau) + A_0^T X(\tau)A_0 - A_1^T X(\tau)A_1. \tag{19}$$

Now, $U(0) = V(h)$ and $U'(0) = V'(h)$ which yields by time-invariance of (19) the symmetry result $U(\tau) = V(\tau+h) = U^T(-\tau)$. The solution $U(\tau)$ is given by a sum of eigenmotions of the finite-dimensional system (15). Namely, there exist $\lambda_i \in \mathbb{C}$ and matrices $Z_{ik} \in \mathbb{C}^{n\times n}$, $i = 1, \ldots, \ell,\ k = 0, \ldots, N_i$, such that $\left\{e^{\lambda_i \tau}\tau^k Z_{ik}\right\}$ forms a basis of the solution space for the U-component of (15) where λ_i are the associated eigenvalues and $Z_{ik} \in \mathbb{C}^{n\times n}$ are the U-components of generalized eigenvectors of (15). Therefore

$$U(\tau) = \sum_{i\in I} e^{\lambda_i \tau} \sum_{k\in K_i} \tau^k Z_{ik}, \qquad I \subset \{1, \ldots, \ell\}, K_i \subset \{0, \ldots, N_i\}, \quad \tau \in \mathbb{R},$$

(coefficients are incorporated in $Z_{ik} \neq 0$). Since $U(\tau) = V^T(h-\tau) = V(h+\tau)$, $U(\tau)$ satisfies $U'(\tau) = U(\tau)A_0 + U(\tau-h)A_1$. As the components of $U'(\tau)$ are formed by quasi-polynomials we obtain from Lemma 12 that

$$\lambda_i \left(\sum_{k\in K_i} \tau^k Z_{ik}\right) + \left(\sum_{k\in K_i\setminus\{0\}} k\tau^{k-1} Z_{ik}\right) = \left(\sum_{k\in K_i} \tau^k Z_{ik}\right) A_0 + e^{-\lambda_i h}\left(\sum_{k\in K_i} (\tau-h)^k Z_{ik}\right) A_1, \quad i \in I.$$

Now consider for a fixed index i the coefficient matrix of $\tau^{\hat{k}_i}$ belonging to the highest degree $\hat{k}_i = \max(K_i)$. Then $Z_{i\hat{k}_i}(\lambda I - A_0 - e^{-\lambda_i h}A_1) = 0$. As $Z_{i\hat{k}_i} \neq 0$ we conclude that $\det(\lambda_i I - A_0 - e^{-\lambda_i h}A_1) = 0$, i.e. $\lambda_i \in \sigma((1'))$. The symmetry property implies that $U(\tau) = \sum_{i\in I} e^{-\lambda_i\tau}\sum_{k\in K_i}(-\tau)^k Z_{ik}^T$. Hence, if (generalized) eigenmotions of λ_i contribute to $U(\tau)$ so do (generalized) eigenmotions of $-\lambda_i$. The same reasoning as above shows that $-\lambda_i$ is also contained in $\sigma((1'))$. Now let us suppose that there is $\lambda \in \sigma((1'))$ such that $-\lambda \in \sigma((1'))$. We can construct a non-trivial pair of solutions (U, V) of Problem 8 which satisfies $U'(0) = V'(h)$, hence breaking uniqueness of the trivial solution $(U, V) \equiv 0$. To see this we set $U(\tau) = e^{\lambda\tau}wv^T$ and $V(\tau) = e^{\lambda(\tau-h)}wv^T$ where $v, w \in \mathbb{C}^n$ satisfy $v^T(\lambda I - A_0 - e^{-\lambda h}A_1) = 0$, $(\lambda I + A_0^T + e^{\lambda h}A_1^T)w = 0$. Then

$$\begin{aligned} U'(\tau) &= \lambda e^{\lambda\tau}wv^T = e^{\lambda\tau}wv^T(A_0 + e^{-\lambda h}A_1) = U(\tau)A_0 + V(\tau)A_1, \\ V'(\tau) &= \lambda e^{\lambda(\tau-h)}wv^T = e^{\lambda(\tau-h)}(-A_0^T - e^{\lambda h}A_1^T)wv^T = -A_1^T U(\tau) - A_0^T V(\tau). \end{aligned}$$

and $U(0) = V(h)$, $U'(0) = \lambda wv^T = V'(h)$. Switching to the real parts if necessary, we have obtained a real non-trivial solution pair of Problem 8 which corresponds to $W = 0$. □

If the condition of Theorem 11 does not hold then there always exist non-trivial solutions of Problem 8 corresponding to $W = 0$. Moreover, Proposition 6 shows that under these conditions there exist matrices W for which there exists no solution at all. Applying Proposition 9 to the Theorem 11, we obtain the following conclusion for solution set of Problem 7.

Corollary 13. *A delay Lyapunov matrix U of Problem 7 is uniquely determined if and only if for all $\lambda, \mu \in \sigma((1'))$: $\lambda + \mu \neq 0$.*

With the help of Kronecker products Problem 8 can be vectorized and the resulting equations can be utilized in the numerical computation of solutions. The Kronecker product satisfies $\operatorname{vec} AXB = (B^T \otimes A)\operatorname{vec} X$, where $\operatorname{vec} X \in \mathbb{R}^{n^2}$ is obtained from $X \in \mathbb{R}^{n\times n}$ by stacking up its columns, see [2]. Problem 8 takes the following vectorized form.

Problem 14. *Given a symmetric matrix $W \in \mathbb{R}^{n\times n}$. Find a solution pair $u, v : [0, h] \to \mathbb{R}^{n^2}$ such that*

$$\begin{pmatrix} u'(\tau) \\ v'(\tau) \end{pmatrix} = A \begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix}, \qquad A = \begin{pmatrix} A_0^T \otimes I & A_1^T \otimes I \\ -I \otimes A_1^T & -I \otimes A_0^T \end{pmatrix} \tag{20}$$

$$M\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} + N\begin{pmatrix} u(h) \\ v(h) \end{pmatrix} = \begin{pmatrix} -w \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} A_0^T \otimes I & A_1^T \otimes I \\ I & 0 \end{pmatrix}, \ N = \begin{pmatrix} I \otimes A_1^T & I \otimes A_0^T \\ 0 & -I \end{pmatrix} \tag{21}$$

where $u = \operatorname{vec} U$, $v = \operatorname{vec} V$, *and* $w = \operatorname{vec} W$, $A, M, N \in \mathbb{R}^{2n^2 \times 2n^2}$.

Let us analyze the structure of the eigenvectors of the system matrix A in (20).

Proposition 15. *Suppose that* λ_0 *is an eigenvalue of the linear operator* $\mathcal{A}$ *given by*

$$\mathcal{A} : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}, \qquad \begin{pmatrix} U \\ V \end{pmatrix} \mapsto \begin{pmatrix} UA_0 + VA_1 \\ -A_1^T U - A_0^T V \end{pmatrix}$$

and $-\lambda_0$ *does not belong to* $\sigma(A_0)$, *then there exists an eigenvector corresponding to* λ_0 *of the operator* $\mathcal{A}$ *which is given by a pair of the form* $\binom{Y_0}{\zeta_0 Y_0}$ *where* $Y_0 \in \mathbb{C}^{n\times n}$ *and* $\zeta_0 \in \mathbb{C}$. *Moreover, if* $\binom{U_0}{V_0}$ *is an eigenvector of* $\mathcal{A}$ *corresponding to* λ_0, *then the pair* $\binom{V_0^T}{U_0^T}$ *forms an eigenvector of* $\mathcal{A}$ *corresponding to the eigenvalue* $-\lambda_0$.

Proof. Using the representation of Problem 14 we have that

$$\det(\lambda I - \mathcal{A}) = \det \begin{pmatrix} (\lambda I - A_0^T) \otimes I & -A_1^T \otimes I \\ I \otimes A_1^T & I \otimes (\lambda I + A_0^T) \end{pmatrix} \tag{22}$$

This determinant is equal to

$$\det(\lambda I - \mathcal{A}) = \det\left[(\lambda I - A_0^T) \otimes (\lambda I + A_0^T) + A_1^T \otimes A_1^T\right].$$

It vanishes if and only if there exists $U \in \mathbb{C}^{n\times n}$, $U \neq 0$, such that

$$\mathcal{L}(\lambda)U = (\lambda I + A_0^T)U(\lambda I - A_0) + A_1^T U A_1 = 0.$$

Now, let $\binom{U_0}{V_0}$ be an eigenvector of $\mathcal{A}$ corresponding to λ_0. Then

$$U_0(A_0 - \lambda_0 I) + V_0 A_1 = 0, \qquad A_1^T U_0 + (\lambda_0 I + A_0^T)V_0 = 0. \tag{23}$$

By pre-multiplying the first equation with $\lambda_0 I + A_0^T$ and post-multiplying the second one with $\lambda_0 I - A_0$, we get

$$\begin{aligned} (\lambda_0 I + A_0^T)U_0(A_0 - \lambda_0 I) + (\lambda_0 I + A_0^T)V_0 A_1 &= 0, \\ A_1^T U_0(\lambda_0 I - A_0) + (\lambda_0 I + A_0^T)V_0(\lambda_0 I - A_0) &= 0. \end{aligned}$$

Substitution of (23) into these equations gives $\mathcal{L}(\lambda_0)U_0 = 0$, and $\mathcal{L}(\lambda_0)V_0 = 0$. Hence both components of any eigenvector corresponding to λ_0 are contained in $\ker \mathcal{L}(\lambda_0)$. We therefore can define the following linear operator on the kernel of $\mathcal{L}(\lambda_0)$

$$\mathcal{M}(\lambda) : \ \ker \mathcal{L}(\lambda_0) \to \ker \mathcal{L}(\lambda_0), \ U \mapsto V = (\lambda_0 I + A_0^T)^{-1} A_1^T U,$$

so that the pair $\binom{U}{V}$ forms an eigenvector of $\mathcal{A}$ corresponding to λ_0. Now, this linear operator $\mathcal{M}(\lambda_0)$ posesses an eigenvector Y_0 with $\mathcal{M}(\lambda_0)Y_0 = \zeta_0 Y_0$. Hence there exists an eigenvector $\binom{Y_0}{\zeta_0 Y_0}$ of $\mathcal{A}$ corresponding to λ_0 which is formed from the eigenpair (ζ_0, Y_0) of

$\mathcal{M}(\lambda_0)$.

Finally, if $\left(\lambda_0, \binom{U_0}{V_0}\right)$ is an eigenpair of $\mathcal{A}$ then

$$\mathcal{A}\begin{pmatrix} V_0^T \\ U_0^T \end{pmatrix} = \begin{pmatrix} (A_0^T V_0 + A_1^T U_0)^T \\ (-V_0 A_1 - U_0 A_0)^T \end{pmatrix} = -\lambda_0 \begin{pmatrix} V_0^T \\ U_0^T \end{pmatrix}$$

i.e. $-\lambda_0$ is also an eigenvalue of $\mathcal{A}$, and the pair $\binom{V_0^T}{U_0^T}$ forms a corresponding eigenvector. $\square$

Remark 16.

1. If $\lambda_0 \in \sigma(\mathcal{A})$, but $\lambda_0 \notin \sigma(A_0)$ then there exists an eigenvector of $\mathcal{A}$ corresponding to λ_0 which is of the form $\binom{\zeta_0 Y_0}{Y_0}$, $\zeta_0 \in \mathbb{C}$.
2. If A_1 is a regular matrix, then the conditions $-\lambda_0 \notin \sigma(A_0)$ or $\lambda_0 \notin \sigma(A_0)$ can be dropped.
3. If A_1 is singular and $\lambda_0 \in \sigma(\mathcal{A}) \cap \sigma(A_0) \cap \sigma(-A_0)$ then eigenvectors corresponding to λ_0 can be constructed explicitely: they are formed by pairs $(uv^T, 0)$ and $(0, xy^T)$ where $A_1^T u = A_1^\top y = 0$, $v^T(A_0 - \lambda_0 I) = 0$, and $(\lambda_0 I + A_0^T)x = 0$.

We are now able to decide if the delay Lyapunov matrix for (1') is uniquely determined just by looking at the eigenvectors of the operator $\mathcal{A}$.

Corollary 17. *Under the conditions of Proposition 15, if the operator $\mathcal{A}$ only has eigenvalues of geometric multiplicity 1, then a delay Lyapunov matrix of* (1') *is uniquely determined if and only if $\zeta \neq e^{-\lambda_0 h}$ holds for all eigenpairs $(\lambda_0, \binom{U_0}{\zeta U_0})$.*

Proof. Assume that there exists an eigenpair such that $\zeta = e^{-\lambda_0 h}$, i.e. $U_0(A_0 + e^{-\lambda_0 h} A_1 - \lambda_0 I) = 0$. It means that $\lambda_0 \in \sigma((1')) \cap \sigma(\mathcal{A})$. But this implies $-\lambda_0 \in \sigma((1')) \cap \sigma(\mathcal{A})$ whence by Theorem 11 there is no uniquely determined delay Lyapunov matrix.

On the other hand, a nontrivial solution of Problem 8 corresponding to $W = 0$ satisfies $V(\tau + h) = U(\tau)$. Considering an eigenmotion $e^{\lambda_0 \tau}\binom{U_0}{\zeta U_0}$ with non-zero coefficients in the eigendecomposition of the solution pair $(U(\tau), V(\tau)) \not\equiv 0$ for Problem 8, we have $e^{\lambda_0(\tau+h)}\zeta U_0 = e^{\lambda_0 \tau} U_0$, therefore $\zeta = e^{-\lambda_0 h}$ and $\lambda_0 \in \sigma((1')) \cap \sigma(\mathcal{A})$. $\square$

Let us look at some examples.

Example 18. Consider the "hot shower problem" [6]

$$\dot{x}(t) = -\alpha x(t - h), \qquad \alpha > 0, h > 0.$$

Then the system matrix A and the matrices M, N for the left and right boundary condition in the Kronecker formulation of Problem 14 are given by

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\alpha \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad N = \begin{pmatrix} -\alpha & 0 \\ 0 & -1 \end{pmatrix}.$$

If the determinant of $M + Ne^{Ah}$ does not vanish, every $w = \operatorname{vec} W$ uniquely defines an initial value via $\binom{u}{v}(0) = (M + Ne^{Ah})^{-1}\binom{-w}{0}$, which then gives a unique solution. Now,

$$M + Ne^{Ah} = \begin{pmatrix} -\alpha\cos\alpha h & \alpha\sin\alpha h - \alpha \\ 1 - \sin\alpha h & -\cos\alpha h \end{pmatrix} \qquad \text{with determinant} \qquad 2\alpha(1 - \sin\alpha h).$$

The determinant vanishes for $\alpha h = 2\pi k + \frac{\pi}{2}$, $k \in \mathbb{N}$. In this case, $M + Ne^{Ah} = 0$ so every choice of initial values leads to a solution which corresponds to $w = 0$. Here any initial value $\binom{u}{v}$ yields the first component of a solution of Problem 8 given by $u(\tau) = u\cos(\alpha\tau) - v\sin(\alpha\tau)$, while a solution of Problem 7 has to be in the form $\tilde{u}(\tau) = u\cos(\alpha\tau)$ whenever $\alpha h = 2\pi k + \frac{\pi}{2}$, $k \in \mathbb{N}$. Note that $i\alpha - (-\alpha)e^{-i\alpha h} = i\alpha + \alpha e^{-i\frac{\pi}{2}} = 0$ if $\alpha h = 2\pi k + \frac{\pi}{2}$, hence $\pm i\alpha$ are common eigenvalues of the delay equation in Problem 7 and of the system matrix in Problem 8.

Example 19. Let us now look at the following second order system given by the data

$$A_0 = \begin{pmatrix} -1 & -7 \\ 0 & -4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & 3 \\ -2 & -2 \end{pmatrix}.$$

With the help of Corollary 17 we can now decide for which delay terms h the delay Lyapunov matrix associated with $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$ is not uniquely determined. The spectrum of the operator $\mathcal{A}$ is numerically given by

$$\sigma(\mathcal{A}) = \{-4.3 \pm 0.384i, 4.3 \pm 0.384i, \pm 1.67i, -0.341, 0.341\},$$

the matching scaling factors ζ with $V_0 = \zeta U_0$ for the eigenvectors $\binom{U_0}{V_0}$ are given by

$$\zeta \in \{-0.0989 \pm 0.182i, -2.31 \pm 4.24i, 0.0931 \pm 0.996i, -0.329, -3.04\}.$$

Now if $h > 0$ is a critical value for the delay, then $\log(\zeta) = -\lambda_0 h$ has to hold. For real eigenvalues, negative values of ζ are of no interest. The only critical delays h are given as a solution of $0.0931 - 0.996i = e^{-1.67ih}$ which has infinitely many positive solutions starting with $h_0 = 0.886$. As the system is stable for $h = 0$, $\sigma(A_0 + A_1) = \{-2.5 \pm 2.78i\}$, we see that the spectrum of the delay equation (1') hits the imaginary axis at h_0 for the first time when varying the delay term h. The boundary condition matrix $M + Ne^{Ah_0} \in \mathbb{R}^{8\times 8}$ has rank 6, hence there are some weights W for which there does not exist any delay Lyapunov function, while other weights lead to non-unique delay Lyapunov functions.

5 Conclusions

This paper provides useful steps towards a systematic analysis of delay Lyapunov matrices and answers the uniqueness question for exponentially stable delay systems and for systems with one delay term. Unfortunately, the ideas presented in Section 4 are not directly applicable to general non-stable and/or non-commensurable multi-delay systems.

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References

[1] V. I. Arnold. *Ordinary Differential Equations.* MIT Press, Cambridge, Massachusetts, 1978.

[2] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[3] E. F. Infante and W. B. Castelan. A Liapunov functional for a matrix difference-differential equation. *J. Diff. Eqns.*, 29:439–451, 1978.

[4] V. L. Kharitonov and D. Hinrichsen. Exponential estimates for time delay systems. Technical Report 03-05, Zentrum für Technomathematik, Universität Bremen, Germany, 2003.

[5] V. L. Kharitonov and A. P. Zhabko. Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39:15–20, 2003.

[6] V. Kolmanovskii and A. Myshkis. *Applied Theory of Functional Differential Equations*, volume 85 of *Mathematics and Its Applications (Soviet Series).* Kluwer Academic Publishers, Dordrecht, 1992.

[7] N. N. Krasovskiĭ. On the application of the second method of Lyapunov for equations with time delays. *Prikl. Mat. Meh.*, 20:315–327, 1956.

[8] J. Louisell. A matrix method for determining the imaginary axis eigenvalues of a delay system. *IEEE Trans. Automat. Contr.*, 46(12):2008–2012, 2001.

[9] B. S. Razumikhin. On stability of systems with a delay. *Prikl. Mat. Meh.*, 20:500–512, 1956.

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