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Exponential estimates for time delay systems

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Abstract

In this paper we demonstrate how Lyapunov-Krasovskii functionals can be used to obtain exponential bounds for the solutions of time-invariant linear delay systems.

Keywords: time delay system, exponential estimate, Lyapunov-Krasovskii functional.

1 Introduction

The objective of this note is to describe a systematic procedure of constructing quadratic Lyapunov functionals for (exponentially) stable linear delay systems in order to obtain exponential estimates for their solutions.
The procedure we propose is a counterpart to the well known method of deriving exponential estimates for stable systems $\dot{x} = Ax$ by means of quadratic Lyapunov functions $V(x) = \langle x, Ux \rangle$. Here $U \succ 0$ is the solution of a Lyapunov equation $A^*U + UA = -W$ where $W \succ 0$ is any chosen positive definite matrix. If $2\omega U \preceq W$ for some $\omega > 0$ then

$$\|e^{At}\| \leq \kappa(U)^{1/2} e^{-\omega t}, \quad t \geq 0 \tag{1}$$

where $\|\cdot\|$ denotes the spectral norm and $\kappa(U) = \|U\|\,\|U^{-1}\|$ is the condition number of U, see [6]. Note that this estimate guarantees not only a uniform decay rate ω for all solutions of $\dot{x} = Ax$ but also a bound on the transients of the system.
It is surprising that a similar constructive method does not exist for delay systems. It is true, there exists an operator theoretic version of Lyapunov's equation in the abstract semigroup theory of infinite dimensional time-invariant linear systems, see [2], but this does not provide us with a constructive procedure. For constructive purposes more concrete Lyapunov functions must be considered. Since the fifties different types of Lyapunov functions have been proposed for the stability analysis of delay systems , see the pioneering works of Razumikhin [11] and Krasovskii [10]. Whereas Razumikhin [11] used Lyapunov type functions $V(x(t))$ depending on the current value $x(t)$ of the solution, Krasovskii [10] proposed to use functionals $V(x_t)$ depending on the whole solution segment x_t, i.e. the true state of

the delay system. These functionals, which are defined on the space of (continuous) initial functions, are called *Lyapunov-Krasovskii functionals.* For a brief discussion of these two approaches and some historic comments, see [5, §5.5].

The majority of results concerning Lyapunov's direct method for delay systems provides *sufficient criteria* for stability and asymptotic stability. These results assume that Lyapunov functionals with certain properties are given, and so the exponential estimates derived by means of these functionals are not obtained in a constructive manner. For a constructive procedure *converse results* are needed which show how Lyapunov functionals of a specific type can be constructed for a given class of stable delay equations. Such converse results are available for certain delay systems, see e.g. Halanay [4], but they do not abound as the sufficient stability criteria based on given Lyapunov functionals.

Quadratic Lyapunov functionals have been proposed for time-invariant linear delay equations by Repin [12], Datko [3], Infante and Castelan [8], Huang [7], Kharitonov and Zhabko [9]. However, with exception of the latter reference, these Lyapunov functionals cannot be used for deriving exponential estimates, if no additional a priori information is available. Infante and Castelan propose an interesting method of constructing exponential estimates whose decay rate comes arbitrarily close to the spectral abscissa of the delay system. But their method presupposes that a concrete exponential estimate is already available. The constants of the exponential estimate which they derive from a quadratic Lyapunov functional depend explicitely on the constants of the presupposed a priori estimate. (compare (3.6), (3.19) and (2.6) in [8]).

In order to overcome this dependence on an a priori estimate we use a modified Lyapunov-Krasovskii functional introduced in [9]. This functional is constructed in a similar way to that of Infante and Castelan but contains an additional integral term. As in [8] the construction is based on a solution of a matrix differential-difference equation on a finite time interval satisfying additional symmetry and boundary conditions. The matrix boundary value problem plays, roughly speaking, a similar role for linear delay equations as Lyapunov's equation in the delay free case. Our differential-difference equation is more complicated than that in [8] since Infante and Castelan only considered the *one* delay case. Since these matrix equations have only recently been discovered there is as yet no systematic solution theory available. This will be a subject for future research. However, first algorithms have been developed for the solution of the corresponding matrix boundary value problem.

The note is organized as follows. After some preliminaries on delay systems in the next section we describe the construction of the Lyapunov-Krasovskii functional according to [9] in section 3. Section 4 contains the main results of this note. Finally, section 5 presents an example illustrating the basic steps leading to an exponential estimate for a given exponentially stable system with two (commensurate) delays.

2 Preliminaries

In this paper we consider time delay systems of the following form

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - h_k). \tag{2}$$

where $A_0, A_1, ..., A_m \in \mathbb{R}^{n\times n}$ are given matrices and $0 < h_1 < ... < h_m = h$ are positive delays. For any continuous initial function $\varphi : [-h, 0] \to \mathbb{R}^n$ there exists the unique solution, $x(t,\varphi)$, of (2) satisfying the initial condition

$$x(\theta, \varphi) = \varphi(\theta), \quad \theta \in [-h, 0].$$

If $t \geq 0$ we denote by $x_t(\varphi)$ the trajectory segment

$$x_t(\varphi) : \theta \mapsto x(t + \theta, \varphi), \quad \theta \in [-h, 0].$$

Throughout this note we will use the Euclidean norm for vectors and the induced matrix norm for matrices. The space of continuous initial functions $\mathcal{C}([-h, 0], \mathbb{R}^n)$ is provided with the supremum norm $\|\varphi\|_\infty = \max_{\theta\in[-h,0]} \|\varphi(\theta)\|$.

Definition 1 *The system (2) is said to be* exponentially stable *if there exist $\sigma > 0$ and $\gamma \geq 1$ such that for every solution $x(t, \varphi)$, $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ the following exponential estimate holds*

$$\|x(t, \varphi)\| \leq \gamma e^{-\sigma t} \|\varphi\|_\infty, \quad t \geq 0. \tag{3}$$

For simplicity we will call an *exponentially stable* system just *stable*. The matrix-valued function $K : [-h, \infty] \to \mathbb{R}^{n\times n}$ which solves the matrix differential equation

$$\frac{d}{dt}K(t) = A_0K(t) + \sum_{j=1}^{m} A_jK(t - h_j),\ t \geq 0,$$

with initial condition

$$K(t) = 0 \text{ for } -h \leq t < 0, \qquad K(0) = I_n,$$

is called the *fundamental matrix* of the system (2),here I_n is the identity matrix. It is known that $K(t)$ also satisfies the differential equation [1]

$$\frac{d}{dt}K(t) = K(t)A_0 + \sum_{j=1}^{m} K(t - h_j)A_j, \quad t \geq 0. \tag{4}$$

The following result is known as the Cauchy formula for the solutions of system (2), see [1].

$$x(t, \varphi) = K(t)\varphi(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t - h_j - \theta)A_j\varphi(\theta)d\theta, \quad t \geq 0. \tag{5}$$

Every column of $K(t)$ is a solution of system (2), so if the system is exponentially stable, then the matrix satisfies an inequality of the form (3). As a consequence, the integral

$$U(\tau) = \int_0^\infty K^\top(t)WK(t + \tau)dt \tag{6}$$

is well defined for all $\tau \geq -h$ and any matrix $W \in \mathbb{R}^{n\times n}$.

3 Lyapunov-Krasovskii functionals

We will now construct quadratic Lyapunov functions for the system (2) in a similar way as for linear differential equations without delays. In this latter case, given any stable system $\dot{x} = Ax$, a quadratic Lyapunov function is determined in the following way. For an arbitrarily chosen quadratic function $w(x) = x^\top W x$ with positive definite $W \succ 0$ one constructs a quadratic function $v(x) = x^\top U x$, $U \succ 0$ such that

$$dv(x(t))/dt = -w(x(t)), \quad t \in \mathbb{R} \tag{7}$$

for every solution, $x(t)$, of $\dot{x} = Ax$. Function $v(x) = x^\top U x$ satisfies (7) if and only if U is the (uniquely determined, positive definite) solution of the *Lyapunov equation*

$$A^* U + UA = -W. \tag{8}$$

Analogously we choose for the delay system (2) positive definite $n \times n$ matrices W_0, W_1,..., W_{2m} and consider the following functional on $\mathcal{C}([-h,0],\mathbb{R}^n)$

$$w(\varphi) = \varphi^\top(0) W_0 \varphi(0) + \sum_{k=1}^{m} \varphi^\top(-h_k) W_k \varphi(-h_k) + \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^\top(\theta) W_{m+k} \varphi(\theta) d\theta, \tag{9}$$

where $\varphi \in \mathcal{C}([-h,0],\mathbb{R}^n)$ is arbitrary. If the system (2) is exponentially stable, then there exists a unique quadratic functional $v : \mathcal{C}([-h,0],\mathbb{R}^n) \to \mathbb{R}$ such that $t \mapsto v(x_t(\varphi))$ is differentiable on $\mathbb{R}_+$ and

$$\frac{dv(x_t(\varphi))}{dt} = -w(x_t(\varphi)), \quad t \geq 0 \tag{10}$$

for all solutions $x(t,\varphi)$ of (2), $\varphi \in \mathcal{C}([-h,0],\mathbb{R}^n)$ [9]. Functional $v(\cdot)$ is called the *Lyapunov-Krasovskii functional* associated with (9). It has been shown in [9] that the functional is given by

$$\begin{aligned}
v(\varphi) =& \varphi^\top(0) U(0) \varphi(0) + \sum_{k=1}^{m} 2\varphi^\top(0) \int_{-h_k}^{0} U(-h_k - \theta) A_k \varphi(\theta) d\theta + \\
&+ \sum_{k=1}^{m} \sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^\top(\theta_2) A_k^\top \left[\int_{-h_j}^{0} U(\theta_2 - \theta_1 + h_k - h_j) A_j \varphi(\theta_1) d\theta_1 \right] d\theta_2 + \\
&+ \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^\top(\theta) \left[W_k + (h_k + \theta) W_{m+k} \right] \varphi(\theta) d\theta
\end{aligned} \tag{11}$$

where

$$U(\tau) = \int_0^\infty K^\top(t) \left[W_0 + \sum_{k=1}^{m} (W_k + h_k W_{m+k}) \right] K(t+\tau) dt, \quad \tau \geq -h. \tag{12}$$

Note that by exponential stability of (2) the matrix $U(\tau)$ is well defined for all $\tau \geq -h$ and is of the form (6) with

$$W = W_0 + \sum_{k=1}^{m} (W_k + h_k W_{m+k}). \tag{13}$$

We call $U(\tau)$ the *Lyapunov matrix function* for the system (2) associated with the functional $w(\cdot)$ (9). Note that the first $(1+m+m^2)$ terms in (11) are completely determined by U and hence only depend upon the weighted sum W of the positive definite matrices W_k. However, the last m terms in (11) depends on the individual W_k's. We will see later that all these terms are needed in order to derive exponential estimates for (2) by means of the above quadratic functionals. For the case of a single delay in (2) a similar Lyapunov-Krasovskii functional has been considered by Infante and Castelan in [8]. However, in their paper the matrix $W_k + (h_k + \theta)W_{m+k}$ in each one of the last m terms of (11) is replaced by a constant positive definite matrix. This is due to the fact that Infante and Castelan did not include the integral term in their definition of the functional $w(\cdot)$, see [8, (3.4)]. However, we will see in Remark 5 that this integral term is an essential ingredient for deriving an exponential estimate of the form (3) for an exponentially stable delay system without any further a priori knowledge.

Remark 1 If (2) is without delays ($h_k = 0$, $k = 1, \dots, m$) then the interval $[-h, 0]$ is reduced to $\{0\}$, $\mathcal{C}([-h, 0], \mathbb{R}^n)$ to $\mathcal{C}(\{0\}, \mathbb{R}^n) \cong \mathbb{R}^n$ and we have $K(t) = e^{At}$. Identifying $\varphi \in \mathcal{C}(\{0\}, \mathbb{R}^n)$ with $x = \varphi(0)$, the quadratic functional $w(\cdot)$ (9) is given by $w(x) = x^\top W x$ where W is defined by (13), and $v(\cdot)$ is given by $v(x) = x^\top U(0)x$, $x \in \mathbb{R}^n$. Note that in this case $U(0)$ is by definition equal to

$$U = \int_0^\infty e^{A^\top t} W e^{At} dt \tag{14}$$

so that $U = U(0)$ satisfies the Lyapunov equation (8). So the above construction of the Lyapunov-Krasovskii functional $v(\cdot)$ from $w(\cdot)$ generalizes the construction procedure via the Lyapunov equation (8) in the delay free case. □

Remark 2 In the delay free case the success of quadratic Lyapunov functions rely on the fact that for a given $w(x) = x^\top W x$ the corresponding $v(x) = x^\top U x$ is not obtained via the integral expression (14) but can be computed from the linear Lyapunov equation (8). Similarly, the above construction of the Lyapunov-Krasovskii functional (11) would not be practical if it required the evaluation of the integral (12) (and so, in particular, the knowledge of the fundamental matrix $K(t)$ on $\mathbb{R}_+$). But it is not difficult to show (see [9]) that $U(\tau)$, $\tau \in [-h, h]$ solves the following matrix delay differential equation

$$\frac{d}{d\tau} U(\tau) = U(\tau)A_0 + \sum_{k=1}^m U(\tau - h_k)A_k, \quad \tau \in [0, h] \tag{15}$$

and additionally the following conditions

- the symmetry condition

$$U(-\tau) = U^\top(\tau), \quad \tau \in [-h, h], \tag{16}$$

- the Lyapunov type linear matrix equation

$$U(0)A_0 + A_0^\top U(0) + \sum_{k=1}^m U^\top(h_k)A_k + A_k^\top U(h_k) + W = 0. \tag{17}$$

A systematic study of the equations (15), (16), (17) has not yet been accomplished. We conjecture that $U(\tau))$ is the *unique* solution of this set of equations, but a proof of this conjecture is an open problem. □

4 Main results

In this section we show how one can use functionals (9) and (11) in order to obtain exponential estimates for time delay systems.
We first specify two conditions under which a pair of (not necessarily quadratic) functionals $v, w : \mathcal{C}([-h,0],\mathbb{R}^n) \to \mathbb{R}$ satisfying (10) yields an exponential estimate of the form (3).

Proposition 3 *Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants and $v, w : \mathcal{C}([-h,0],\mathbb{R}^n) \to \mathbb{R}$ are continuous functionals such that $t \mapsto v(x_t(\varphi))$ is differentiable on $\mathbb{R}_+$. If the following conditions are satisfied for all $\varphi \in \mathcal{C}([-h,0],\mathbb{R}^n)$*

1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 w(\varphi)$,
2. $\beta_1 \|\varphi(0)\|^2 \leq w(\varphi) \leq \beta_2 \|\varphi\|_\infty^2$,
3. $\frac{d}{dt} v(x_t(\varphi)) \leq -w(x_t(\varphi))$ *for all* $t \geq 0$,

then for all $\varphi \in \mathcal{C}([-h,0],\mathbb{R}^n)$

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2\beta_2}{\alpha_1}}\, e^{-\frac{1}{2\alpha_2}t}\, \|\varphi\|_\infty, \quad t \geq 0. \tag{18}$$

Proof: Given any $\varphi \in \mathcal{C}([-h,0],\mathbb{R}^n)$, conditions 1. and 3. imply that

$$\frac{d}{dt} v(x_t(\varphi)) \leq -\frac{1}{\alpha_2} v(x_t(\varphi)), \quad t \geq 0.$$

Integrating this inequality from 0 to t and applying Gronwall's Lemma we get

$$v(x_t(\varphi)) \leq v(\varphi) e^{-\frac{1}{\alpha_2}t}, \quad t \geq 0.$$

Then conditions 1. and 2. yield

$$\alpha_1 \|x(t,\varphi)\|^2 \leq v(x_t(\varphi)) \leq v(\varphi) e^{-\frac{1}{\alpha_2}t} \leq \alpha_2\beta_2 \|\varphi\|_\infty^2 e^{-\frac{1}{\alpha_2}t}, \quad t \geq 0.$$

Comparing the left and the right hand sides, the exponential estimate (18) follows. □

We will now show that conversely, if the system (2) is exponentially stable, then positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be determined such that the quadratic functionals $v(\cdot), w(\cdot)$ constructed in the previous section satisfy the assumptions of Proposition 3. As a consequence we obtain an exponential estimate of the form (3) for every set of positive definite $n \times n$ matrices $W_0, \ldots, W_{2m}$.

Theorem 4 *If system (2) is exponentially stable and $W_0, W_1, \ldots, W_{2m}$ are positive definite real $n \times n$ matrices then there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that the quadratic Lyapunov-Krasovskii functionals $w(\cdot)$ and $v(\cdot)$ defined by (9) and (11), respectively, satisfy the assumptions of Proposition 3.*

Proof: We have seen in Section 3 that the functional $v(\cdot)$ is well defined by (11) and (12) since (2) is exponentially stable by assumption. Moreover it follows easily from the definitions (9) and (11) that the two functionals $v, w : \mathcal{C}([-h,0],\mathbb{R}^n) \to \mathbb{R}$ are continuous. Let $\varphi \in \mathcal{C}([-h,0],\mathbb{R}^n)$. We know from Section 3 that $w(\cdot)$ and $v(\cdot)$ are related by (10), i.e. $t \mapsto v(x_t(\varphi))$ is differentiable on $\mathbb{R}_+$ and $w(\cdot)$, $v(\cdot)$ satisfy the third condition of Proposition 3 with equality. It remains to show that there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that conditions 1. and 2. of Proposition 3 are satisfied. Let

$$\lambda_{\min} = \min_{k=0,\ldots,2m} \lambda_{\min}(W_k), \quad \lambda_{\max} = \max_{k=0,\ldots,2m} \lambda_{\max}(W_k),$$

where $\lambda_{\min}(W_k)$ and $\lambda_{\max}(W_k)$ denote the smallest and the largest eigenvalue of the positive definite matrix W_k, respectively. Then

$$\begin{aligned}\lambda_{\min}\left[\|\varphi(0)\|^2 + \sum_{k=1}^{m}\|\varphi(-h_k)\|^2 + \sum_{k=1}^{m}\int_{-h_k}^{0}\|\varphi(\theta)\|^2 d\theta\right] \leq \\ \leq w(\varphi) \leq \lambda_{\max}\left[\|\varphi(0)\|^2 + \sum_{k=1}^{m}\|\varphi(-h_k)\|^2 + \sum_{k=1}^{m}\int_{-h_k}^{0}\|\varphi(\theta)\|^2 d\theta\right].\end{aligned}$$

From these inequalities we conclude that

$$\lambda_{\min}\|\varphi(0)\|^2 \leq w(\varphi) \leq \lambda_{\max}\left(1 + m + \sum_{k=1}^{m} h_k\right)\|\varphi\|_\infty^2,$$

i.e. the functional $w(\cdot)$ satisfies the second condition of proposition 3 with

$$\beta_1 = \lambda_{\min}, \quad \beta_2 = (1 + m + \sum_{k=1}^{m} h_k)\lambda_{\max}. \tag{19}$$

In order to check the first condition, let

$$\mu = \max_{\tau\in[0,h]}\{\|U(\tau)\|\}, \quad a = \max_{k=1,\ldots,m}\|A_k\|.$$

Then one can easily verify the following inequalities

$$\begin{gathered}\varphi(0)^\top U(0)\varphi(0) \leq \mu\|\varphi(0)\|^2, \\ 2\varphi(0)^\top\int_{-h_k}^{0} U(-h_k-\theta)A_k\varphi(\theta)d\theta \leq \mu a h_k\|\varphi(0)\|^2 + \mu a\int_{-h_k}^{0}\|\varphi(\theta)\|^2 d\theta, \\ \int_{-h_k}^{0}\varphi(\theta)^\top[W_k + (h_k+\theta)W_{m+k}]\varphi(\theta)d\theta \leq (1+h_k)\lambda_{\max}\int_{-h_k}^{0}\|\varphi(\theta)\|^2 d\theta,\end{gathered}$$

for $k = 1, ..., m$.

In order to find an upper estimate of the double integrals in (11) we make use of the fact that by the Cauchy-Schwartz inequality in $L^2(-h_i,0;\mathbb{R}^n)$ we have

$$\left(\int_{-h_i}^{0}\|\varphi(\theta)\| d\theta\right)^2 \leq h_i\int_{-h_i}^{0}\|\varphi(\theta)\|^2 d\theta, \quad i = 1,\ldots,m.$$

So

$$\int_{-h_k}^{0}\varphi(\theta_2)^\top A_k^\top\left[\int_{-h_j}^{0}U(\theta_1-\theta_2+h_k-h_j)A_j\varphi(\theta_1)d\theta_1\right]d\theta_2 \leq$$

$$\leq \mu a^2\left(\int_{-h_k}^{0}\|\varphi(\theta_2)\|\,d\theta_2\right)\left(\int_{-h_j}^{0}\|\varphi(\theta_1)\|\,d\theta_1\right) \leq$$

$$\leq \frac{1}{2}\mu a^2\left[\left(\int_{-h_k}^{0}\|\varphi(\theta_2)\|\,d\theta_2\right)^2+\left(\int_{-h_j}^{0}\|\varphi(\theta_1)\|\,d\theta_1\right)^2\right] \leq$$

$$\leq \frac{1}{2}\mu a^2 h_k\int_{-h_k}^{0}\|\varphi(\theta_2)\|^2\,d\theta_2+\frac{1}{2}\mu a^2 h_j\int_{-h_j}^{0}\|\varphi(\theta_1)\|^2\,d\theta_1,$$

for all $k,j=1,...,m$. As a consequence we obtain the following upper bound for $v(\varphi)$:

$$v(\varphi)\leq\mu\big(1+a\sum_{k=1}^{m}h_k\big)\,\|\varphi(0)\|^2+\sum_{k=1}^{m}\left(\mu a+(1+h_k)\lambda_{\max}+\frac{m+1}{2}\mu h_k a^2\right)\int_{-h_k}^{0}\|\varphi(\theta)\|^2\,d\theta. \tag{20}$$

If we select $\alpha_2>0$ such that the following two conditions hold

- $\alpha_2\lambda_{\min}\geq\mu\left(1+a\sum_{k=1}^{m}h_k\right)$,
- $\alpha_2\lambda_{\min}\geq\mu a+(1+h)\lambda_{\max}+\frac{m+1}{2}\mu h a^2$,

then

$$v(\varphi)\leq\alpha_2 w(\varphi).$$

To obtain the required quadratic lower bound for $v(\varphi)$, we consider the modified functional

$$\widetilde{v}(\varphi)=v(\varphi)-\alpha\,\|\varphi(0)\|^2,\quad \varphi\in\mathcal{C}([-h,0],\mathbb{R}^n)$$

where $\alpha>0$. Then $\frac{d}{dt}\widetilde{v}(x_t(\varphi))=-\widetilde{w}(x_t(\varphi))$ where, for any $\varphi\in\mathcal{C}([-h,0],\mathbb{R}^n)$,

$$\widetilde{w}(\varphi)=-w(\varphi)-\alpha\varphi(0)^\top\left[A_0\varphi(0)+\sum_{k=1}^{m}A_k\varphi(-h_k)\right]-\alpha\left[A_0\varphi(0)+\sum_{k=1}^{m}A_k\varphi(-h_k)\right]^\top\varphi(0).$$

Omitting the last term in the definition of $w(\cdot)$ (see (9)) we obtain

$$\widetilde{w}(\varphi)\geq\big(\varphi(0)^\top,\varphi(-h_1)^\top,...,\varphi(-h_m)^\top\big)\,W(\alpha)\begin{pmatrix}\varphi(0)\\ \varphi(-h_1)\\ \vdots\\ \varphi(-h_m)\end{pmatrix},\quad \varphi\in\mathcal{C}([-h,0],\mathbb{R}^n)$$

where

$$W(\alpha)=\begin{pmatrix}W_0 & 0 & \cdots & 0\\ 0 & W_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & W_m\end{pmatrix}+\alpha\begin{pmatrix}A_0+A_0^\top & A_1 & \cdots & A_m\\ A_1^\top & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ A_m^\top & 0 & \cdots & 0\end{pmatrix}.$$

Using Schur complements we see that the matrix $W(\alpha)$ is positive definite if and only if

$$\widetilde{W}(\alpha) = W_0 + \alpha(A_0 + A_0^\top) - \alpha^2 \sum_{k=1}^{m} A_k W_k^{-1} A_k^\top \succ 0.$$

Since $W_0 \succ 0$ there exists $\alpha_1 > 0$ such that $\widetilde{W}(\alpha_1) \succ 0$. Integrating $\frac{d}{dt}\widetilde{v}(x_t(\varphi)) = -\widetilde{w}(x_t(\varphi))$ from 0 to ∞ we get for $\alpha = \alpha_1$,

$$v(\varphi) - \alpha_1 \|\varphi(0)\|^2 = \widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))dt \geq 0.$$

Altogether we have found $\alpha_1, \alpha_2 > 0$ such that the first condition in Proposition 3 is satisfied and this concludes the proof. $\square$

Remark 5 The above proof shows that every quadratic functional $w(\cdot)$ of the form (9) with positive definite matrices $W_0, \ldots, W_{2m}$ satisfies the second condition of Proposition 3 with the constants $\beta_1, \beta_2 > 0$ given by (19). The first $(m+1)$ terms in (9) were needed to prove that the corresponding functional $v(\cdot)$ (11) satisfies the first inequality in the first condition of Proposition 3. The last m terms in (9), along with the first term, were used in the proof to derive the second inequality. So, $w(\cdot)$ defined by (9) can be viewed as a quadratic functional with a minimum number of quadratic terms to yield a Lyapunov-Krasovskii functional $v(\cdot)$ satisfying the first condition in Proposition 3. As mentioned above the integral terms in (9) are missing in the definition of $w(\cdot)$ in [8]. This is compensated by an additional exponential factor $e^{\delta t}$, $\delta > 0$ in the definition of $v(\cdot)$, see [8, (3.1)], where $-\delta$ is supposed to be strictly greater than the exponential growth rate of the delay system. Therefore the construction of Infante and Castelan requires some a priori knowledge about the spectral abscissa of the system. $\square$

Remark 6 Clearly the exponential estimate obtained by Theorem 4 depends on the choice of the matrices $W_k \succ 0$, $k = 0, 1, ..., 2m$. These matrices may serve as free parameters in an optimization of the estimate. In this note we do not try to obtain tight estimates, we only wish to demonstrate that the above Lyapunov-Krasovskii approach yields a systematic procedure for determining exponential estimates for an exponentially stable delay system (2) without any additional a priori information. $\square$

5 Illustration

In this section we illustrate the results of the previous section by an example.

Example 7 Consider the system

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.7 \\ 0.7 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} -0.49 & 0 \\ 0 & -0.49 \end{pmatrix} x(t-2). \qquad (21)$$

The characteristic quasipolynomial of the system is

$$f(s) = s^2 + 3s + 2 + 0.98e^{-2s} + 0.98se^{-2s} + [0.49]^2 e^{-4s}.$$

All the roots of this quasipolynomial lie in the open left half complex plane. The roots closest to the imaginary axis are

$$s_{1,2} \simeq -0.363 \pm j1.388$$

so that the spectral abscissa of the system is -0.363. Since delay systems satisfy the spectrum determined growth assumption [2], there exists for every $\varepsilon > 0$ a constant $\gamma_\varepsilon \geq 1$ such that the solutions of (21) satisfy the inequality

$$\|x(t,\varphi)\| \leq \gamma_\varepsilon \|\varphi\|_\infty e^{-(0.363-\varepsilon)t}, \quad t \geq 0.$$

We will now apply Theorem 4 in order to obtain an exponential estimate for the system (21) without making use of any knowledge about its spectral abscissa. Let us choose $W_k = I_n$, $k = 0, 1, ..., 4$ so that the functional $w(\cdot)$ (9) is given by

$$w(\varphi) = \|\varphi(0)\|^2 + \|\varphi(-1)\|^2 + \|\varphi(-2)\|^2 + \int_{-1}^{0} \|\varphi(\theta)\|^2 d\theta + \int_{-2}^{0} \|\varphi(\theta)\|^2 d\theta.$$

for $\varphi \in \mathcal{C}([-2,0], \mathbb{R}^2)$. Obviously

$$\|\varphi(0)\|^2 \leq w(\varphi) \leq 6 \|\varphi\|_\infty^2,$$

so we may choose $\beta_1 = 1$ and $\beta_2 = 6$.
The matrix $\widetilde{W}(\alpha)$ used in the proof of Theorem 4 is given by

$$\widetilde{W}(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \alpha \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} - \alpha^2 \begin{pmatrix} 0.7301 & 0 \\ 0 & 0.7301 \end{pmatrix}.$$

$\widetilde{W}(\alpha)$ is positive definite for $\alpha = 0.23$, so we may choose $\alpha_1 = 0.23$.
The corresponding functional $v(x_t)$ is of the form

$$\begin{aligned} v(x_t) =& x^\top(t)U(0)x(t) + 1.4x^\top(t) \int_{-1}^{0} U(-1-\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t+\theta)d\theta - \\ &- 0.98x^\top(t) \int_{-2}^{0} U(-2-\theta)x(t+\theta)d\theta + \\ &+ 0.49 \int_{-1}^{0} x^\top(t+\theta_2) \left[\int_{-1}^{0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta_1-\theta_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t+\theta_1)d\theta_1 \right] d\theta_2 - \\ &- 0.686 \int_{-1}^{0} x^\top(t+\theta_2) \left[\int_{-2}^{0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta_1-\theta_2-1)x(t+\theta_1)d\theta_1 \right] d\theta_2 + \\ &+ 0.2401 \int_{-2}^{0} x^\top(t+\theta_2) \left[\int_{-2}^{0} U(\theta_1-\theta_2)x(t+\theta_1)d\theta_1 \right] d\theta_2 + \\ &+ \int_{-1}^{0} (2+\theta) \|x(t+\theta)\|^2 d\theta + \int_{-2}^{0} (3+\theta) \|x(t+\theta)\|^2 d\theta. \end{aligned}$$

Here the Lyapunov matrix function

$$U(\tau) = 6 \int_{0}^{\infty} K^\top(t)K(t+\tau)d\tau$$

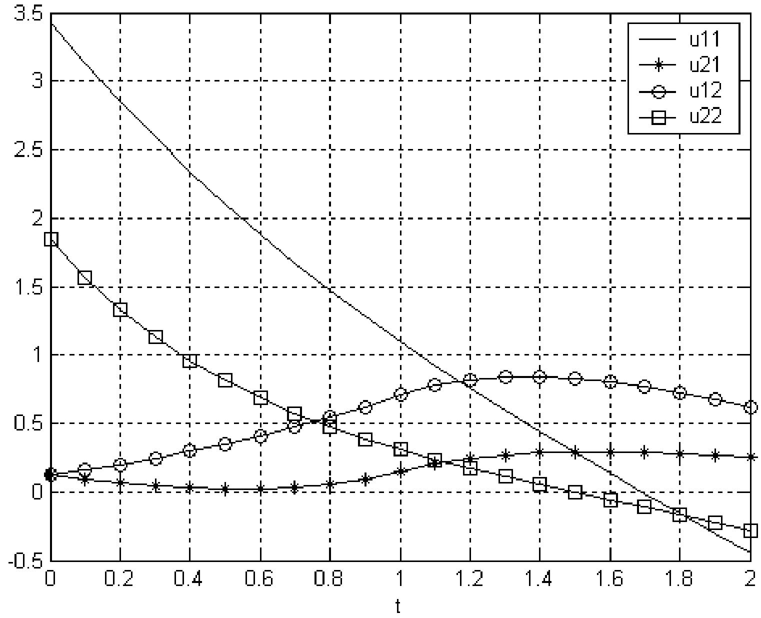

Figure 1: Matrix $U(\tau)$, piecewise linear approximation

satisfies the equation

$$\frac{d}{d\tau}U(\tau) = -U(\tau)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + U(\tau-1)\begin{pmatrix} 0 & 0.7 \\ 0.7 & 0 \end{pmatrix} - U(\tau-2)\begin{pmatrix} 0.49 & 0 \\ 0 & 0.49 \end{pmatrix}. \tag{22}$$

In Fig. 1 we plot the four components of a piecewise linear approximation of the Lyapunov matrix function. From the plot we get

$$\mu = 3.44 \text{ and } a = 0.7.$$

The upper bound (20) of $v(\cdot)$ has the following form

$$v(x_t) \leq \mu(1+3a)\|x(t)\|^2 + (\mu a + 2 + \frac{3}{2}\mu a^2)\int_{-1}^{0}\|x(t+\theta)\|^2\, d\theta + (\mu a + 3 + 3\mu a^2)\int_{-2}^{0}\|x(t+\theta)\|^2 d\theta.$$

The value α_2 should satisfy the conditions

$$\alpha_2 \geq \mu(1+3a) = 10.65, \quad \alpha_2 \geq \mu a + 3 + 3\mu a^2 = 10.46$$

and so we may choose $\alpha_2 = 10.65$.
As a result we obtain an exponential estimate (18) for the solutions of the system (21) with the following constants

$$\gamma = \sqrt{\frac{\alpha_2\beta_2}{\alpha_1}} \approx 16.69, \quad \sigma = \frac{1}{2\alpha_2} \approx 0.047.$$

In order to verify how well the piecewise linear approximation represents the Lyapunov matrix valued function we compute the solution of equation (22) with the initial condition generated by the linear piecewise approximation as initial function on $[-h, 0]$, see Remark (16). The four components of the corresponding solution are plotted on Fig. 2. A comparison Fig. 1 shows a good fit between the solution $U(\tau)$ and the piecewise linear approximation on $[0, h]$. $\square$

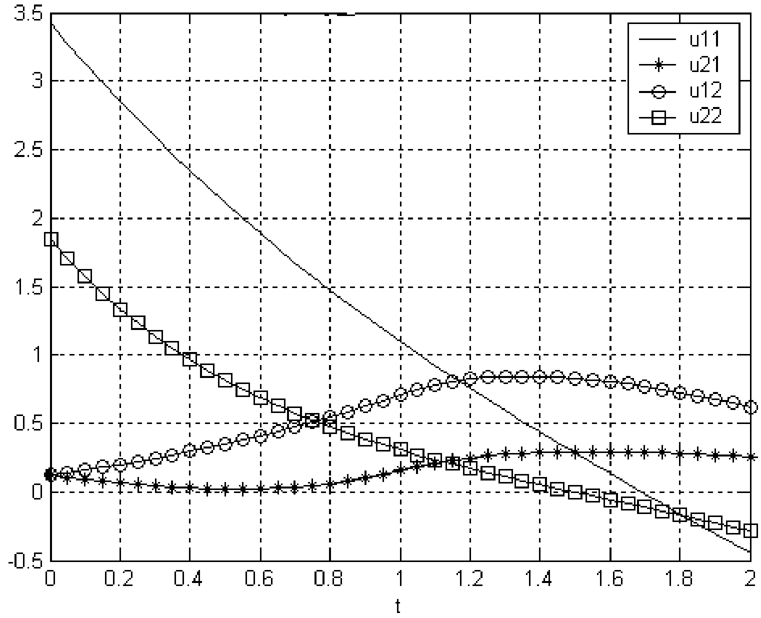

Figure 2: Matrix $U(\tau)$, numerical solution

References

[1] R. Bellman and K. L. Cooke. *Differential-Difference Equations.* Number 6 in Mathematics in Science and Engineering. Academic Press, New York, 1963.

[2] R. F. Curtain and H. J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory.* Number 21 in Texts in Applied Mathematics. Springer-Verlag, New York, 1995.

[3] R. Datko. Uniform asymptotic stability of evolutionary processes in a Banach space. *SIAM J. Math. Anal.*, 3:428–445, 1972.

[4] A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags.* Academic Press, New York, 1966.

[5] J. K. Hale. *Theory of Functional Differential Equations.* Number 3 in Applied Mathematical Sciences. Springer-Verlag, New York, 2nd edition, 1977.

[6] D. Hinrichsen, E. Plischke, and A. J. Pritchard. Liapunov and Riccati equations for practical stability. In *Proc. 6th European Control Conf. 2001*, pages 2883–2888, 2001. Paper no. 8485 (CDROM).

[7] W. Z. Huang. Generalization of Liapunov's theorem in a linear delay system. *J. Math. Anal. Appl.*, 142(1):83–94, 1989.

[8] E. F. Infante and W. B. Castelan. A Liapunov functional for a matrix difference-differential equation. *J. Diff. Eqns.*, 29:439–451, 1978.

[9] V. L. Kharitonov and A. P. Zhabko. LLyapunov-Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39:15–20, 2003.

[10] N. N. Krasovskiĭ. On the application of the second method of Lyapunov for equations with time delays. *Prikl. Mat. Meh.*, 20:315–327, 1956.

[11] B. S. Razumikhin. On stability of systems with a delay. *Prikl. Mat. Meh.*, 20:500–512, 1956.

[12] I. M. Repin. Quadratic Liapunov functionals for systems with delay. *J. Appl. Math. Mech.*, 29:669–672, 1966. Translation of *Prikl. Mat. Meh.*, 29:564–566 (1965).

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