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Ziping Huang

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HUANG Ziping

Department of Applied Mathematics, Tongji-University,
Shanghai 200092, People's Republic of China

Abstract

In this paper we discribe and analyse two finite element methods for mixed problems with penalty. In each case the bilinear form corresponding to mixed problems with penalty is modified to become coercive over the finite element space. Error estimates are derived for each procedure.

1. Introduction

The Mindlin-Reissner model is often used by engineers for the study of plate and shell problems. It is known that many numerical schemes for this model are satisfactory only when the thickness parameter t is "not too small". For a small t some bad behavior (such as the "looking" phenomenon) may occur. Brezzi and Fortin [4] have transfered this model into solving two elliptic problems and one mixed problems with penalty. In this paper we present two finite element methods for mixed problems with penalty. It is proved that the methods converge with optimal order uniformly with respect to the penalty parameter.

The mixed problems with penalty we consider is as follows:

Problem (S). Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $t \in (0,1]$, find $[\phi, p] \in \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$ such that

$$(\mathbf{grad}\phi, \mathbf{grad}\psi) - (\mathrm{div}\psi, p) = (\mathbf{f}, \psi) \qquad \forall \psi \in \mathbf{H_0^1(\Omega)}, \tag{1}$$

$$-(\mathrm{div}\phi, q) - t^2(\nabla p, \nabla q) = 0 \qquad \forall q \in \hat{H}^1(\Omega). \tag{2}$$

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For simplicity of the notation we set

$$D := \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$$

and we rewrite (1) and (2) as

$$\begin{cases} a(\phi,\psi) + b(\psi,p) = (\mathbf{f},\psi) \\ b(\phi,q) - t^2 d(p,q) = 0 \end{cases} \qquad \forall [\psi,q] \in D$$

with the bilinear forms

$$a(\phi,\psi) := (\mathbf{grad}\phi, \mathbf{grad}\psi), \quad b(\psi,q) := -(\mathrm{div}\psi, q), \quad d(p,q) := (\nabla p, \nabla q)$$

on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$, $\mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$, $\hat{H}^1(\Omega) \times \hat{H}^1(\Omega)$, resp. Here $(\cdot,\cdot)$ is the inner product in L^2. Moreover, $H^k(\Omega)$, $k \in N$, and $L^2(\Omega) = H^0(\Omega)$ are the usual Sobolev and Lebesgue spaces equipped with the norms (cf. Adams [1])

$$\|v\|_k := \left\{ \sum_{|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^2 dx \right\}^{\frac{1}{2}}.$$

Furthermore, $\mathbf{H}_0^1(\Omega) = [H_0^1(\Omega)]^2$. We use a circumflex "ˆ" above a function space to denote the subspace of the elements with mean value zero. We use two standard differential operators:

$$\mathbf{grad}\psi = \begin{bmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{bmatrix}, \qquad \mathrm{div}\psi = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}.$$

We assume that the domain Ω is a convex polygon.

A standard mixed method for approximating the solution of Problem (S) would depend on choosing a pair of spaces $\mathbf{V_h} \subset \mathbf{H_0^1(\Omega)}$ and $W_h \subset \hat{H}^1(\Omega)$ such that "inf-sup" condition

$$\inf_{\psi \in \mathbf{V_h}} \sup_{q \in W_h} \ge \frac{b(\psi,q)}{\|\psi\|_1 \|q\|_0} \ge \beta > 0,$$

β independent of h, holds (cf. Arnold and Falk [2], Huang [7]). As can be seen in the book of Girault and Raviart [6], there are quite a few such spaces known for this problem, however, most of these combinations employ some basis functions that are not found in many of the engineering code packages that are most commonly used. Hughes et al. [8] and Brezzi and Douglas [3] proposed to modify Stokes problem so that the associated bilinear form is coercive over $\mathbf{V}_h \times W_h$ and almost any pair of spaces can be choosen for $\mathbf{V}_h \times W_h$, and the resulting method can be implemented easily and rapidly within the framework of many existing engineering codes. In this paper we shall modify the mixed problems with penalty as Hughes et al. [8] and Brezzi and Douglas [3]. It follows that the associated bilinear form corresponding mixed problems with penalty is coercive over $\mathbf{V}_h \times W_h$, and almost any pair of spaces can be also choosen for $\mathbf{V}_h \times W_h$. This method will be illustrated and analysed in Section 2. The method of Section 3 is a penalty procedure. We point out that this method introduces a penalty error of order $O(h)$ independent of the choise of the discretization space; consequently, though the use of higher order spaces is feasible for the method without stabily problem, such usage is not recommended.

2. A variant Method of Hughes, Balestra and Franca

Let T_h be a triangulation of Ω and h be the longest side of the triangles of T_h. We suppose that the triangles of T_h satisfying the usual regularity assumptions for finite elements (cf. Ciarlet [5]) and that

$$\begin{aligned} & h_K \leq c\rho_K, \qquad \text{for all } K \in T_h, \qquad (3) \\ & h_K := \mathrm{diam}(K), \\ & \rho_K := \sup\{\mathrm{diam}(B) \mid B \text{ is a ball contained in } K\}. \end{aligned}$$

Here and in the rest of the paper, c, c_1, c_2, $\cdots$ will be positive constants independent of h and t. They may have different values in different formulas.

Let S_h be the space of continuous, piecewise linear finite elements corresponding to T_h. We define the following finite element spaces:

$$\mathbf{X}_h := [S_h \cap H_0^1(\Omega)]^2, \qquad M_h := S_h \cap \hat{H}^1(\Omega).$$

Forthermore, we set

$$D_h := \mathbf{X}_h \times M_h.$$

We consider a variant method of Hughes, Balestra and Franca for Problem (S):

Problem ($\mathbf{S}_h$). Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $t \in (0,1]$, find $[\phi_h, p_h] \in D_h$ such that for all $[\psi, q] \in D_h$

$$a(\phi_h, \psi) + b(\psi, p_h) = (\mathbf{f}, \psi), \qquad (4)$$

$$b(\phi_h, q) - t^2 d(p_h, q) - \alpha \sum_{K \in T_h} h_K^2[(\nabla p_h, \nabla q)_K - (\Delta \phi_h, \nabla q)_K] = -\alpha \sum_{K \in T_h} h_K^2 (\mathbf{f}, \nabla q)_K, \qquad (5)$$

where α is a constant which is independent of t, h and its value will be determined in Lemma 2.1. We rewrite (4) and (5) as

$$A([\phi_h, p_h], [\psi, q]) = (\mathbf{f}, \psi) + \alpha \sum_{K \in T_h} h_K^2 (\mathbf{f}, \nabla q)_K$$

with

$$A([\phi_h, p_h],[\psi, q]) = a(\phi_h, \psi) + b(\psi, p_h) - b(\phi_h, q) + t^2 d(p_h, q) + \alpha \sum_{K\in T_h} h_K^2[(\nabla p_h, \nabla q)_K - (\Delta\phi_h, \nabla q)_K].$$

Next lemma prove that the operator $A(\cdot,\cdot) : D_h \times D_h \longrightarrow R$ is coercive over the space D_h, if D_h is equipped with the norm

$$\|[\psi, q]\|_{D_h} := \{\|\psi\|_1^2 + t^2\|\nabla q\|_0^2 + \sum_{K\in T_h} h_K^2\|\nabla q\|_{0,K}^2\}^{\frac{1}{2}}, \qquad \forall[\psi, q] \in D_h.$$

Lemma 2.1. Suppose that T_h is a regularity triangulation of Ω satisfying the assumption (3). Then,

$$A([\psi, q]; [\psi, q]) \geq c\|[\psi, q]\|_{D_h}^2,$$

where c is a constant which is independent of $\mathbf{f}$, t and h.
Proof. From the inverse estimate (cf. Ciarlet [5]) we have for all $[\psi, q] \in D_h$

$$\begin{aligned} & A([\psi, q]; [\psi, q]) \\ = \ & a(\psi, \psi) + t^2 d(q, q) + \alpha \sum_{K\in T_h} h_K^2[(\nabla q, \nabla q)_K - (\Delta\psi, \nabla q)_K] \\ \geq \ & \|\nabla\psi\|_0^2 + t^2\|\nabla q\|_0^2 + \alpha \sum_{K\in T_h} h_K^2(\|\nabla q\|_{0,K}^2 - c_1 h_K^{-1}\|\nabla\psi\|_{0,K}\|\nabla q\|_{0,K}) \\ \geq \ & \|\nabla\psi\|_0^2 + t^2\|\nabla q\|_0^2 + \alpha \sum_{K\in T_h} h_K^2\|\nabla q\|_{0,K}^2 - \frac{\alpha}{2}\sum_{K\in T_h}(c_1^2\|\nabla\psi\|_{0,K}^2 + h_K^2\|\nabla q\|_{0,K}) \\ \geq \ & \min\left\{1 - \frac{\alpha}{2}c_1^2, 1, \frac{\alpha}{2}\right\}(\|\nabla\psi\|_0^2 + t^2\|\nabla q\|_0^2 + \sum_{K\in T_h} h_K^2\|\nabla q\|_{0,K}^2). \end{aligned}$$

Choosing $\alpha = 2/(1+c_1^2)$ and referring the definition of the norm $\|\cdot\|_{D_h}$ we obtain

$$\begin{aligned} A([\psi, q],\ [\psi, q]) \ & \geq \ \frac{1}{1+c_1^2}(\|\nabla\psi\|_0^2 + t^2\|\nabla q\|_0^2 + \sum_{K\in T_h} h_K^2\|\nabla q\|_{0,K}^2) \\ & \geq \ c\|[\psi, q]\|_{D_h}^2. \qquad \square \end{aligned}$$

For the error estimate we need a regularity property:
Lemma 2.2 (Regularity). Suppose that Ω is a convex polygon and that $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$ and $t \in (0, 1]$. Then there is a unique pair $[\phi, p] \in \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$ to solving

$$\begin{aligned} (\mathbf{grad}\phi,\ \mathbf{grad}\psi) - (\mathrm{div}\psi,\ p) &= (\mathbf{f}, \psi) \qquad && \forall\psi \in \mathbf{H_0^1(\Omega)}, \\ -(\mathrm{div}\phi,\ q) - t^2(\nabla p, \nabla q) &= (g, q) \qquad && \forall q \in \hat{H}^1(\Omega). \end{aligned}$$

Moreover, $\phi \in \mathbf{H}_0^1(\Omega) \bigcap \mathbf{H}^2(\Omega)$, $p \in \hat{H}^1(\Omega) \bigcap H^2(\Omega)$ and

$$\|\phi\|_2 + \|p\|_1 + t\|p\|_2 \leq d(\|\mathbf{f}\|_0 + t^{-1}\|g\|_0),$$

where c is a constant which is independent of $\mathbf{f}$, g and t .
Proof. See Huang [7].

In the next we give the erroe estimate in D_h-norm between the solutions of Problem (S) and (S$'$:
Theorem 2.1. Suppose that T_h be a regularity triangulation of Ω satisfying the assumption (3). Let $[\phi, p] \in D$ and $[\phi_h, p_h] \in D_h$ be the solutions of Problem (S) and (S$_h$), resp. Then,

$$\|[\phi - \phi_h, p - p_h]\|_{D_h} \leq ch\|\mathbf{f}\|_0,$$

where c is a constant which is independent of $\mathbf{f}$, t and h.

Proof. From Problem (S) and (S$_h$) we have for all $[\psi, q] \in D_h$

$$\begin{aligned} &a(\phi - \phi_h, \psi) + b(\psi, p - p_h) &= 0, \\ &b(\phi - \phi_h,\ q) - t^2 d(p - p_h,\ q) - \alpha \sum_{K \in T_h} h_K^2 (\bigtriangledown(p - p_h) - \Delta(\phi - \phi_h), \bigtriangledown q)_K &= 0. \end{aligned}$$

Let $[\chi, r] \in D_h$ be an optimal order correct interpolation of $[\phi, p] \in D$ and then

$$\begin{aligned} &a(\chi - \phi_h, \psi) + b(\psi, r - p_h) = a(\chi - \phi, \psi) + b(\psi, r - p), \\ &b(\chi - \phi_h, q) - t^2 d(r - p_h, q) - \alpha \sum_{K \in T_h} h_K^2 (\bigtriangledown(r - p_h) - \Delta(\chi - \phi_h), \bigtriangledown q)_K \\ &\qquad = b(\chi - \phi, q) - t^2 d(r - p, q) - \alpha \sum_{K \in T_h} h_K^2 (\bigtriangledown(r - p) - \Delta(\chi - \phi), \bigtriangledown q)_K. \end{aligned}$$

Choosing $\psi = \chi - \phi_h$, $q = r - p_h$ and referring Lemma 2.1 it follows that

$$\begin{aligned} & c\|[\chi - \phi_h, r - p_h]\|_{D_h}^2 \\ \leq\ & A([\chi - \phi_h, r - p_h];\ [\chi - \phi_h, r - p_h]) \\ =\ & [a(\chi - \phi_h, \chi - \phi_h) + b(\chi - \phi_h, r - p_h)] - [b(\chi - \phi_h, r - p_h) \\ & -t^2 d(r - p_h,\ r - p_h) - \alpha \sum_{K \in T_h} h_K^2 (\bigtriangledown(r - p_h) - \Delta(\chi - \phi_h), \bigtriangledown(r - p_h))_K] \\ =\ & [a(\chi - \phi, \chi - \phi_h) + b(\chi - \phi_h, r - p)] - [b(\chi - \phi, r - p_h) \\ & -t^2 d(r - p, r - p_h) - \alpha \sum_{K \in T_h} h_K^2 (\bigtriangledown(r - p) - \Delta(\chi - \phi), \bigtriangledown(r - p_h))_K]. \end{aligned}$$

This yields

$$\begin{aligned}
&\|[\chi-\phi_h, r-p_h]\|_{D_h}^2\\
\le\ & c\Big\{\|\chi-\phi\|_1^2+\|r-p\|_0^2+\sum_{K\in T_h} h_K^{-2}\|\chi-\phi\|_{0,K}^2+t^2\|\bigtriangledown(r-p)\|_0^2\\
&+\sum_{K\in T_h} h_K^2\|\bigtriangledown(r-p)\|_0^2+\sum_{K\in T_h} h_K^2\|\chi-\phi\|_{2,K}^2\Big\}.
\end{aligned}$$

From this and Lemma 2.2 we conclude

$$\begin{aligned}
\|[\phi-\phi_h,\ p-p_h]\|_{D_h} &\le \|[\phi-\chi,\ p-r]\|_{D_h}+\|[\chi-\phi_h,\ r-p_h]\|_{D_h}\\
&\le ch(\|\phi\|_2+\|p\|_1+t\|p\|_2)\\
&\le ch\|\mathbf{f}\|_0. \quad \Box
\end{aligned}$$

We change spaces S_h, $\mathbf{X}_h$, M_h, D_h as

$$\begin{aligned}
&S_h := \{v\in C^0(\overline{\Omega}) \mid v|_K\in P_m(K),\ K\in T_h\},\quad m\ge j,\\
&\mathbf{X}_h' := [S_h\textstyle\bigcap H_0^1(\Omega)]^2,\\
&M_h' := S_h\textstyle\bigcap \hat{H}^1(\Omega),\\
&D_h' := \mathbf{X}_h'\times M_h'
\end{aligned}$$

with $P_m(K)$ the space of polynomials, of degree less or equal to m, on K and consider the finite element approximation over D_h':

Problem ($\mathbf{S}_h'$). Given $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $t\in(0,1]$, find $[\phi_h,p_h]\in D_h'$ such that for all $[\psi,q]\in D_h'$

$$\begin{aligned}
&a(\phi_h,\psi)+b(\psi,p_h) &&= (\mathbf{f},\psi),\\
&b(\phi_h,q)-t^2d(p_h,q)-\alpha\sum_{K\in T_h} h_K^2[(\bigtriangledown p_h,\bigtriangledown q)_K-(\Delta\phi_h,\bigtriangledown q)_K] &&= -\alpha\sum_{K\in T_h} h_K^2(\mathbf{f},\bigtriangledown q)_K.
\end{aligned}$$

Analogously, we can prove the following error estimate for Problem (S$_h'$):

Theorem 2.2. Suppose that T_h be a regular triangulation of Ω satisfying the assumption (3). Let $[\phi,p]\in D$ and $[\phi_h,p_h]\in D_h$ be the solutions of Problem (S) and (S$_h'$), resp. Then,

$$\|[\phi-\phi_h,\ p-p_h]\|_{D_h}\le ch^j\{\|\phi\|_{j+1}+\|p\|_j+t\|p\|_{j+1}\},$$

where c is a constant which is independent of $\mathbf{f}$, t and h.

Now we give the L^2-estimate:

Theorem 2.3 (L_2-estimate). Suppose that T_h be a regular triangulation of Ω satisfying the assumption (3). Let $[\phi, p] \in D$ and $[\phi_h, p_h] \in D_h$ be the solutions of Problem (S) and (S'_h), resp. Then,

$$\|\phi - \phi_h\|_0 \le c_1 h^{j+1}\{\|\phi\|_{j+1} + \|p\|_j + t\|p\|_{j+1}\},$$
$$\|p - p_h\|_0 \le c_2 h^{j}\{\|\phi\|_{j+1} + \|p\|_j + t\|p\|_{j+1}\},$$

where c_1, c_2 are constants which are independent of $\mathbf{f}$, t and h.

Proof. We use a duality argument to derive the L^2-estimate. Let $[\rho, s] \in D$ be the solution of

$$\begin{cases} a(\rho,\psi) + b(\psi, s) = (\phi - \phi_h,\ \psi) \\ b(\rho, q) - t^2 d(s,q) = 0 \end{cases} \qquad \forall[\psi, q] \in D. \tag{6}$$

From this we have for all $[\psi, q] \in D_h$

$$\begin{aligned} \|\phi - \phi_h\|_0^2 &= a(\rho, \phi - \phi_h) + b(\phi - \phi_h, s) \\ &= [a(\rho - \psi, \phi - \phi_h) + b(\phi - \phi_h, s - q)] \\ &\quad + [a(\psi, \phi - \phi_h) + b(\phi - \phi_h, q)] \\ &= I + II. \end{aligned} \tag{7}$$

Choosing $[\psi, q] \in D_h$ as an optimal order correct interpolation of $[\rho, s] \in D$ and by Lemma 2.2 we obtain

$$\begin{aligned} I &= a(\rho - \psi, \phi - \phi_h) + b(\phi - \phi_h, s - q) \\ &\le c\|\phi - \phi_h\|_1 \ (\|\rho - \psi\|_1 + \|s - q\|_0) \\ &\le ch\|\phi - \phi_h\|_1 \ (\|\rho\|_2 + \|s\|_1) \\ &\le ch\|\phi - \phi_h\|_1 \ \|\phi - \phi_h\|_0. \end{aligned} \tag{8}$$

From Problem (S), (S'_h) and (6) we have

$$\begin{aligned} II &= a(\psi, \phi - \phi_h) + b(\phi - \phi_h, q) \\ &= -b(\psi, p - p_h) + t^2 d(p - p_h, q) \\ &\quad + \alpha \sum_{K \in T_h} h_K^2 (\nabla(p - p_h) - \Delta(\phi - \phi_h), \nabla q)_K \\ &= -b(\psi - \rho, p - p_h) + t^2 d(p - p_h, q - s) \\ &\quad + \alpha \sum_{K \in T_h} h_K^2 (\nabla(p - p_h) - \Delta(\phi - \phi_h), \nabla q)_K \\ &= II_1 + II_2. \end{aligned} \tag{9}$$

From this and Lemma 2.2 we get

$$
\begin{aligned}
II_1 &= -b(\psi - \rho, p - p_h) + t^2 d(p - p_h, q - s) \\
&\leq \|\rho - \psi\|_0 \| \bigtriangledown (p - p_h)\|_0 + t^2 \| \bigtriangledown (p - p_h)\|_0 \| \bigtriangledown (s - q)\|_0 \\
&\leq ch(\|\rho\|_2 + t\|s\|_2) \left\{ \sum_{K \in T_h} h_K^2 \| \bigtriangledown (p - p_h)\|_{0,K}^2 + t^2 \| \bigtriangledown (p - p_h)\|_0^2 \right\}^{\frac{1}{2}} \\
&\leq ch\|\phi - \phi_h\|_0 \left\{ \sum_{K \in T_h} h_K^2 \| \bigtriangledown (p - p_h)\|_{0,K}^2 + t^2 \| \bigtriangledown (p - p_h)\|_0^2 \right\}^{\frac{1}{2}} . \qquad (10)
\end{aligned}
$$

Choosing $\chi \in \mathbf{X}_h$ as an optimal order correct interpolation of ϕ we conclude that

$$
\begin{aligned}
II_2 &= \alpha \sum_{K \in T_h} h_K^2 (\nabla(p - p_h) - \Delta(\phi - \phi_h), \nabla q)_K \\
&= c \sum_{K \in T_h} h_K^2 [(\nabla(p - p_h), \nabla s + \nabla(q - s))_K - (\Delta(\phi - \chi) + \Delta(\chi - \phi_h), \nabla q)_K] \\
&\leq ch\|\phi - \phi_h\|_0 \left\{ \left[\sum_{K \in T_h} h_K^2 (\| \bigtriangledown (p - p_h)\|_{0,K}^2 + \|\phi - \chi\|_{2,K}^2) \right]^{\frac{1}{2}} + \|\chi - \phi_h\|_1 \right\} (11)
\end{aligned}
$$

From (7-11), (3) and Theorem 2.2 it is follows that

$$\|\phi - \phi_h\|_0 \leq ch^{j+1} \{\|\phi\|_{j+1} + \|p\|_j + t\|p\|_{j+1}\}.$$

Let the dual problem be changed to find $[\rho, s] \in D$ such that

$$
\begin{cases}
a(\rho, \psi) + b(\psi, s) &= 0 \\
b(\rho, q) - t^2 d(s, q) &= (p - p_h, q)
\end{cases}
\qquad \forall [\psi, q] \in D.
$$

so that $\|\rho\|_1 + \|s\|_0 + t\|s\|_1 \leq c\|p - p_h\|_0$ (see: Huang [7]). Then, with $\psi \in D_h$,

$$
\begin{aligned}
\|p - p_h\|_0^2 &= b(\rho, p - p_h) - t^2 d(p - p_h, s) \\
&= b(\psi, p - p_h) + b(\rho - \psi, p - p_h) - t^2 d(p - p_h, s) \\
&= -a(\phi - \phi_h, \psi) + b(\rho - \psi, p - p_h) - t^2 d(p - p_h, s) \\
&\leq c(\|\phi - \phi_h\|_1 \|\psi\|_1 + \|\rho - \psi\|_0 \| \bigtriangledown (p - p_h)\|_0 + t^2 \|s\|_1 \| \bigtriangledown (p - p_h)\|_0) \\
&\leq c\|p - p_h\|_0 (\|\phi - \phi_h\|_1 + h\| \bigtriangledown (p - p_h)\|_0 + t\| \bigtriangledown (p - p_h)\|_0).
\end{aligned}
$$

From this, (5) and Theorem 2.2 it follows that

$$
\begin{aligned}
\|p - p_h\|_0 &\leq c(\|\phi - \phi_h\|_1 + h\| \bigtriangledown (p - p_h)\|_0 + t\| \bigtriangledown (p - p_h)\|_0) \\
&\leq ch^j \{\|\phi\|_{j+1} + \|p\|_j + t\|p\|_{j+1}\}. \quad \square
\end{aligned}
$$

3. A penalty stabilization

Consider the penalty version of Problem (P):

Problem ($\mathbf{P}^h$). Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $t \in (0,1]$, find $[\phi^h, p^h] \in D$ such that

$$\left\{ \begin{array}{ll} a(\phi^h, \psi) + b(\psi, p^h) & = (\mathbf{f}, \psi), \\ b(\phi^h, q) - (t^2 + h^2) d(p^h, q) = 0, & \end{array} \right. \qquad \forall [\psi, q] \in D.$$

Then the finite element approximation of Problem (P^h) over D_h is as follows:

Problem ($\mathbf{P}_h$). Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $t \in (0,1]$, find $[\phi_h, p_h] \in D_h$ such that

$$\left\{ \begin{array}{ll} a(\phi_h, \psi) + b(\psi, p_h) & = (\mathbf{f}, \psi), \\ b(\phi_h, q) - (t^2 + h^2) d(p_h, q) = 0, & \end{array} \right. \qquad \forall [\psi, q] \in D_h.$$

First let us analyse the difference between the solution of Problem (S) and (P^h):

Lemma 3.1. Let $[\phi, p]$ and $[\phi^h, p^h]$ be the solutions of Problem (S) and (P^h), resp. Then,

$$\begin{array}{l} \|\phi - \phi^h\|_1 + (t+h) \| \bigtriangledown (p - p^h)\|_0 \le ch \|\mathbf{f}\|_0, \\ \|\phi - \phi^h\|_0 \le ch^2 \|\mathbf{f}\|_0, \\ \|p - p^h\|_0 \le ch \|\mathbf{f}\|_0. \end{array}$$

where c is a constant which is independent of $\mathbf{f}$, t and h.

Proof. From Problem (S) and (P^h) we have

$$\left\{ \begin{array}{ll} a(\phi - \phi^h, \psi) + b(\psi, p - p^h) & = 0 \\ b(\phi - \phi^h, q) - (t^2 + h^2) d(p - p^h, q) = -h^2 d(p, q) & \end{array} \right. \qquad \forall [\psi, q] \in D. \tag{12}$$

Choosing $\psi = \phi - \phi^h$ and $q = p - p^h$ in above we have

$$\|\phi - \phi^h\|_1^2 + (t^2 + h^2) \| \bigtriangledown (p - p^h)\|_0^2 \le ch^2 \| \bigtriangledown p\|_0 \cdot \| \bigtriangledown (p - p^h)\|_0.$$

From this and Lemme 2.2 it follows that

$$\|\phi - \phi^h\|_1 + (t+h) \| \bigtriangledown (p - p^h)\|_0 \le ch \|p\|_1 \le ch \|\mathbf{f}\|_0.$$

We use a duality argument to estimate $\|p - p^h\|_0$. Let $[\rho, s]$ be the solution of

$$\left\{ \begin{array}{l} a(\rho, \psi) + b(\psi, s) = 0 \\ b(\rho, q) - t^2 d(p, q) = (p - p^h, q) \end{array} \right. \qquad \forall [\psi, q] \in D.$$

Choosing $q = p - p^h$ in above and referring (12) and Huang [7] we conclude that

$$\begin{aligned}
\|p - p^h\|_0^2 &= b(\rho, p - p^h) - t^2 d(s, p - p^h) \\
&= -a(\phi - \phi^h, \rho) - t^2 d(s, p - p^h) \\
&\leq c(\|\phi - \phi^h\|_1 + t\| \bigtriangledown (p - p^h)\|_0)(\|\rho\|_1 + t\| \bigtriangledown s\|_0) \\
&= ch\|\mathbf{f}\|_0 \|p - p^h\|_0.
\end{aligned}$$

Then,

$$\|p - p^h\|_0 \leq ch\|\mathbf{f}\|_0.$$

Let the dual problem be changed to find $[\rho, w] \in D$ such that

$$\left\{ \begin{array}{l} a(\rho, \psi) + b(\psi, s) = (\phi - \phi^h, \psi) \\ b(\rho, q) - t^2 d(s, q) = 0 \end{array} \right. \qquad \forall [\psi, q] \in D. \tag{13}$$

Choosing $\psi = \phi - \phi^h$ in above and referring (12-13) and Lemma 2.2 it follows that

$$\begin{aligned}
\|\phi - \phi^h\|_0^2 &= a(\rho, \phi - \phi^h) + b(\phi - \phi^h, s) \\
&= -b(\rho, p - p^h) + t^2 d(p - p^h, s) - h^2 d(p^h, s) \\
&= -h^2 d(p^h, s) \\
&\leq ch^2 \| \bigtriangledown p^h\|_0 \cdot \| \bigtriangledown s\|_0 \\
&\leq ch^2 \|\mathbf{f}\|_0 \cdot \|\phi - \phi^h\|_0, \quad \square
\end{aligned}$$

i.e.,

$$\|\phi - \phi^h\|_0 \leq ch^2 \|\mathbf{f}\|_0. \quad \square$$

Next we give the error estimate between the solutions of Problem (P^h) and (P_h):
Lemma 3.2. Let $[\phi^h, p^h]$ and $[\phi_h, p_h]$ be the solutions of Problem (P^h) and (P_h), resp. Then,

$$\begin{aligned}
&\|\phi^h - \phi_h\|_1 + (t + h)\| \bigtriangledown (p^h - p_h)\|_0 \leq ch\|\mathbf{f}\|_0, \\
&\|\phi^h - \phi_h\|_0 \leq ch^2 \|\mathbf{f}\|_0, \\
&\|p^h - p_h\|_0 \leq ch\|\mathbf{f}\|_0.
\end{aligned}$$

where c is a constant which is independent of $\mathbf{f}$, t and h.
Proof. Set

$$\overline{A}([\phi, p]; [\psi, q]) := a(\phi, \psi) + b(\psi, p) - b(\phi, q) + (t^2 + h^2) d(p, q).$$

From this we obtain

$$\begin{aligned}
&\|\phi^h - \phi_h\|_1^2 + (t^2+h^2)\| \bigtriangledown (p^h - p_h)\|_0^2 \\
\leq\ & c\overline{A}([\phi^h - \phi_h, p^h - p_h], [\phi^h - \phi_h, p^h - p_h]) \\
\leq\ & c \inf_{[\psi,q]\in D_h} \overline{A}([\phi^h - \psi, p^h - q], [\phi^h - \psi, p^h - q]) \\
\leq\ & ch^2\{\|\phi^h\|_2^2 + \|p^h\|_1^2 + t^2\|p^h\|_2^2\}.
\end{aligned}$$

From this and Lemma 2.2 it follows that

$$\|\phi^h - \phi_h\|_1 + (t+h)\| \bigtriangledown (p^h - p_h)\|_0 \leq ch\{\|\phi^h\|_2 + \|p^h\|_1 + t\|p^h\|_2\} \leq ch\|\mathbf{f}\|_0. \tag{14}$$

We use the duality argument to estimate $\|\phi^h - \phi_h\|_0$. Let $[\rho, s]$ be the solution of

$$\begin{cases} a(\rho,\psi) + b(\psi, s) = (\phi^h - \phi_h, \psi) \\ b(\rho, q) - (t^2+h^2)d(s,q) = 0 \end{cases} \quad \forall[\psi, q] \in D. \tag{15}$$

Choosing $\psi = \phi^h - \phi_h$ in above we have for all $[\chi, \eta] \in D_h$

$$\begin{aligned}
\|\phi^h - \phi_h\|_0^2 &= a(\rho, \phi^h - \phi_h) + b(\phi^h - \phi_h, s) \\
&= a(\rho - \chi, \phi^h - \phi_h) + a(\chi, \phi^h - \phi_h) \\
&\quad + b(\phi^h - \phi_h, s - \eta) + b(\phi^h - \phi_h, \eta) \\
&= I + II + III + IV.
\end{aligned} \tag{16}$$

Choosing $[\chi, \eta] \in D_h$ as an optimal order correct interpolation of $[\rho, s] \in D$ and referring Lemma 2.2 it follows that

$$\begin{aligned}
I + III &= a(\rho - \chi, \phi^h - \phi_h) + b(\phi^h - \phi_h, s - \eta) \\
&\leq ch\|\rho\|_2\|\phi^h - \phi_h\|_1 + ch\|s\|_1\|\phi^h - \phi_h\|_1 \\
&\leq ch\|\phi^h - \phi_h\|_0\|\phi^h - \phi_h\|_1.
\end{aligned} \tag{17}$$

From Problem (P^h), (P_h), Lemma 2.2 and (15) we have

$$\begin{aligned}
II + IV &= a(\chi, \phi^h - \phi_h) + b(\phi^h - \phi_h, \eta) \\
&= -b(\chi, p^h - p_h) + (t^2+h^2)d(p^h - p_h, \eta) \\
&\quad + b(\rho, p^h - p_h) - (t^2+h^2)d(s, p^h - p_h) \\
&= b(\rho - \chi, p^h - p_h) - (t^2+h^2)d(p^h - p_h, s - \eta) \\
&\leq c\|\rho - \chi\|_0\| \bigtriangledown (p^h - p_h)\|_0 + (t^2+h^2)\| \bigtriangledown (s-\eta)\|_0\| \bigtriangledown (p^h - p_h)\|_0 \\
&\leq c(h^2\|\rho\|_2 + h(t^2+h^2)\|s\|_2)\| \bigtriangledown (p^h - p_h)\|_0 \\
&\leq ch(h + \sqrt{t^2+h^2})\|\phi^h - \phi_h\|_0 \cdot \| \bigtriangledown (p^h - p_h)\|_0.
\end{aligned} \tag{18}$$

Equations (14), (16-18) imply

$$\|\phi^h - \phi_h\|_0 \le ch(\|\phi^h - \phi_h\|_1 + (t+h)\| \bigtriangledown (p^h - p_h)\|_0) \le ch^2\|\mathbf{f}\|_0.$$

Let the dual problem be changed to find $[\rho, s] \in D$ such that

$$\begin{cases} a(\rho,\psi) + b(\psi,s) &= 0 \\ b(\rho,q) - (t^2+h^2)d(s,q) &= (p^h - p_h, q) \end{cases} \qquad \forall[\psi,q] \in D.$$

Choosing $q = p^h - p_h$ in above from Problem (P^h) and (P_h) we have for all $\chi \in \mathbf{X}_h$

$$\begin{aligned} \|p^h - p_h\|_0^2 &= b(\rho, p^h - p_h) - (t^2+h^2)d(s, p^h - p_h) \\ &= b(\rho - \chi, p^h - p_h) + b(\chi, p^h - p_h) - (t^2+h^2)d(s, p^h - p_h) \\ &= b(\rho - \chi, p^h - p_h) - a(\phi^h - \phi_h, \chi) - (t^2+h^2)d(s, p^h - p_h) \\ &\le c(h\|\rho\|_1\| \bigtriangledown (p^h - p_h)\|_0 + \|\phi^h - \phi_h\|_1\|\chi\|_1 \\ &\quad + (t^2+h^2)\|s\|_1\| \bigtriangledown (p^h - p_h)\|_0). \end{aligned}$$

Choosing $\chi \in \mathbf{X}_h$ as an optimal order correct interpolation of $\rho \in \mathbf{H}_0^1(\Omega)$ and referring Huang [7] it follows that

$$\begin{aligned} \|p^h - p_h\|_0^2 &\le c\|p^h - p_h\|_0(h\| \bigtriangledown (p^h - p_h)\|_0 + \|\phi^h - \phi_h\|_1 \\ &\quad + \sqrt{t^2+h^2}\| \bigtriangledown (p^h - p_h)\|_0), \end{aligned}$$

i.e.,

$$\|p^h - p_h\|_0 \le c((t+h)\| \bigtriangledown (p^h - p_h)\|_0 + \|\phi^h - \phi_h\|_1) \le ch\|\mathbf{f}\|_0. \quad \square$$

From Lemma 3.3 and 3.4 we can immediately obtain the following result:

Theorem 3.1. Let $[\phi, p]$ and $[\phi_h, p_h]$ be the solutions of Problem (S) and (P_h), resp. Then,

$$\begin{aligned} &\|\phi - \phi_h\|_1 + (t+h)\| \bigtriangledown (p - p_h)\|_0 \le ch\|\mathbf{f}\|_0, \\ &\|\phi - \phi_h\|_0 \le ch^2\|\mathbf{f}\|_0, \\ &\|p - p_h\|_0 \le ch\|\mathbf{f}\|_0. \quad \square \end{aligned}$$

where c is a constant which is independent of $\mathbf{f}$, t and h.

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