Application to ship maneuvering

Velocities and angles for a marine craft:

- \( u \): surge velocity
- \( v \): sway velocity
- \( w \): yaw velocity
- \( \Psi \): yaw angle
- \( \eta \): thruster angle

Proportional control: \( \eta(w, \psi) = \varepsilon_w w + \varepsilon_\psi \psi \)
- \( \varepsilon_w \): yaw damping control
- \( \varepsilon_\psi \): yaw restoring control

Equations of motion of our model:

\[
\begin{pmatrix}
\psi \\
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} =
\begin{pmatrix}
c_0 + c_1 u + c_2 u^2 + c_3 v w + \tau_1 \cos \eta \\
c_4 v - c_5 v + c_6 w + c_7 v w + f(v, w) + \tau_2 \sin \eta \\
c_8 w + c_9 w + c_{10} w + g(v, w) + \tau_3 \sin \eta
\end{pmatrix}
\]

where forces at high Reynolds number and large scale entail

\[
f(v, w) = a_1 v^2 + a_1^2 v|v| + a_2 v^2 |v| + a_2 v^2 |v| + a_2 vv|w|,
\]

\[
g(v, w) = b_1 v^2 + b_2 vv|w| + b_3 w|v| + b_4 vv|w|.
\]

Absolute value functions come from cross-flow drag.

Analysis of non-smooth bifurcations

Identification of characteristic parameters that determine whether a bifurcation of periodic states is ‘safe’ or ‘unsafe’ (super- or subcritical).

**Hopf bifurcations (HB)** in \((\varepsilon_w, \varepsilon_\psi)\)-plane:
- Radial normal form with criticality controlled by \( \sigma \):
  \[
  \dot{r} = -\mu r + \sigma r |r|
  \]

- Is the HB safe or unsafe for the straight motion of the ship?
- Smooth theory not applicable: new approach required!

Abstract setting: \( \dot{\mathbf{u}} = A(\mathbf{u}) + G(\mathbf{u}), \mathbf{u} \in \mathbb{R}^n, G(\mathbf{u}) = O(|\mathbf{u}|^2), \) piecewise smooth.

We rigorously derive explicit generalised Lyapunov coefficients \( \sigma, \sigma_2, \ldots, \) whose signs determine criticality and scaling.

**Theorem (sample):** Amplitude \( r_0 \) of periodic orbit:

- If \( \sigma_\varphi \neq 0 \): \( r_0 = -\frac{3\pi}{2\sigma_\varphi} + O(\mu^2) \)
- If \( \sigma_\varphi = 0 \): \( r_0 = \sqrt{\frac{2\varepsilon_w}{\sigma_2}} + O(\mu) \)

For the ‘Hamburg test case’ ship we prove: all bifurcations are safe!

Structure Identification for Hamiltonian Systems

**Simple Pendulum**

- Hamiltonian mechanics:
  \[ H(\varphi, \omega) = \frac{\omega^2}{2} + 5 \cdot \cos(\varphi) \]
- Differential equation system:
  \[ \dot{X} = \left( \begin{array}{c} \varphi \\ \omega \end{array} \right) = \left( \begin{array}{c} \partial H/\partial \omega \\ \partial H/\partial \varphi \end{array} \right) = \left( \begin{array}{c} -5 \cdot \sin(\varphi) \\ \cdot \end{array} \right) \]

**Data:** \([X, \dot{X}]\)

- Training: 50 random initial position round \( X = [2.0, 0.0] \), solved for 5s (RK4), added noise
- Test: Initial position \( X = [2.0, 0.0] \) solved for 30s (RK4)

Comparison of Different Identification Approaches

- **SINDy**
- **DMD**
- **Neural Networks**

- Solved with RK4: Black: Exact ode solved, Red: SINDy/DMD/Base model, Blue: HNN model

Sparse Identification of Nonlinear Dynamics (SINDy)

- Library: \( \Theta(X) = [1 \ X \ X^2 \cdots \sin(X) \ \cos(X) \ \cdots] \)
- Solve linear System \( \hat{X} = \Theta(X) \ \Xi \)
  \[ \Rightarrow \Xi \text{ shows which base-functions are used to build ODE} \]
- Application:
  - Polynomial degree: 3, use trigonometric functions
  - Learned model:
    \[
    \left( \begin{array}{c} \varphi \\ \omega \end{array} \right) = \left( \begin{array}{c} 3.0001 \cdot \omega \\ -4.99995 \cdot \sin(\varphi) \end{array} \right)
    \]

Dynamic Mode Decomposition

- Approximation of the modes of the Koopman operator
- Extracts temporal features
- Used for state estimation and prediction
- Application:
  - Losses energy during prediction

Neural Networks

- Base-Net learns \( X \mapsto \hat{X} \)
- Hamiltonian \( \hat{\mathbf{H}}(X) \)
  \[ \text{Loss: } \mathcal{L} = ||\hat{\mathbf{H}}/\partial \omega - \varphi||^2 + ||\hat{\mathbf{H}}/\partial \varphi - \dot{\varphi}||^2 \]
- Application:
  - Base-Net (red): Slowly losing energy
  - HNN (blue): Preserve’s Energy