Learning Tikhonov Functionals

We consider classical inverse problems given by a linear or non-linear operator $A : X \rightarrow Y$, which maps $x \in X$ to $y \in Y$. The related inverse problem asks to determine an approximation to $x^\dagger$ from given noisy measurements $y^\delta = Ax^\dagger + \eta$.

This can be achieved by minimizing Tikhonov functionals of type

$$\mathcal{L}(x) = \|Ax - y^\delta\|^2 + \alpha R(x),$$

where $R : X \rightarrow \mathbb{R}$ is called regularization term and $\alpha \in \mathbb{R}^+$ is a scalar weighting factor.

- Traditional approaches use handcrafted regularizers to introduce certain properties to the solution
- In many cases much more advanced methods are needed on real data
- Learn a parameterized version $R_\theta$ directly from data\(^1,2\)
- Investigate the choice and influence of different network architectures on the regularizing properties
- Compare (learned) Tikhonov functional approaches against several other reconstruction methods
- Low-dose computed tomography reconstruction is used as real-world application for the comparison

### Learning Tikhonov Functionals

**DIP Approaches to Inverse Problems**

Recently, the Deep Image Prior (DIP)\(^4\) has been introduced as a novel image reconstruction technique. The idea combines the architecture of generative neural networks with model-based optimization by solving the problem

$$\hat{\theta} = \arg \min_\theta \mathcal{L}(A\varphi_\theta(x), y^\delta)$$

with a variant of stochastic gradient descent (SGD) and early stopping. This yields a reconstruction

$$\hat{x} = \varphi_\theta(z),$$

which is observed to be regularized in a beneficial way.

Interesting questions for further research are:

- What is essential for the regularization effect? Aspects of the architecture, the optimization strategy?
- Can the idea be improved for inverse problems involving a non-trivial forward operator (e.g. CT, MRI)?

Both of these questions involve theoretical and experimental aspects. Specific aims of the project are to

- prove mathematical properties of the method in order to provide guarantees and explain the regularizing behaviour, and to
- develop extensions of the DIP targeted at inverse problems.

### Application of Computed Tomography

- Highly relevant medical imaging application
- Linear, mildly ill-posed inverse problem
- Active field of research, including many neural network approaches

Current challenges include:

- Reconstruction from low-dose measurements
- Guarantees and explainability for network approaches
- 3D reconstruction