Electromagnetic inverse scattering problem
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Scattering on nano-structured surfaces

- Non-destructive testing method for nano structures.
- Electromagnetic wave propagation modeled by Maxwell’s equations
  \[ \nabla \times (\mu^{-1} \nabla \times E) - \omega^2 \left( \varepsilon + i \frac{\sigma}{\omega} \right) E = F \quad \text{in } \mathbb{R}^3, \]
  where permeability \( \mu \) is assumed to be periodic and permittivity \( \varepsilon \) is also periodic but locally perturbed.
- Electromagnetic wave propagation in TE mode modeled by Helmholtz equation
  \[ \Delta u + k^2 n^2 u = f \quad \text{in } \mathbb{R}^3, \]
  where refractive index \( n^2 \) is assumed to be periodic but locally perturbed.
- Main tool: Bloch-Floquet transform defined by
  \[ (J \phi)_{(\alpha, x_1, x_2, x_3)} = \sum_{j \in \mathbb{Z}^2} \phi_{(x_1 + j_1, x_2 + j_2, x_3)} e^{i \alpha \cdot j}. \]
- Theorem: Under some assumptions there exists a unique solution to the Maxwell’s equations and the Helmholtz equation.

Discretization

- Bloch-Floquet transform gives a family of quasi-periodic scattering problems with coupling.
  \[ \Rightarrow \text{ suits perfectly for discretization and parallelizes greatly} \]
- Finite-element space: locally constant functions in \( \alpha \), and Nédelc or Lagrange elements in space. Solve large linear equation system by GMRES combined with incomplete LU decomposition.

Inverse Problem

- Goal: detect perturbation in periodic structure having measurements of the scattered wave.
- Measurement operators: \( \Lambda \) measures full wave, \( S \) measures near-field in one period and \( T \) measure far field of scattered wave.
- Theorem: Under some assumptions the measurement operators are injective, ill-posed and Frechet differentible.
- Newton method gives nice results.

Figure 1: \( L^2 \)-error related to discretization in space (Helmholtz and Maxwell).

Factorization method

- The Factorization method is a fast imaging method for reconstructing the support of the perturbation.
- Let \( F \) be the far field operator, \( F_\#: = |\text{Re} F| + |\text{Im} F| \).
- Theorem: Under some assumptions the operator \( F_\# \) is strictly positive and
  \[ z \in \text{ support of perturbation } \iff \sum_{j=1}^\infty \frac{|\langle \phi_\infty^z, \psi_j \rangle_{L^2(S)}|^2}{\lambda_j} < \infty, \]
  where \( \phi_\infty^z \) is the far field of fundamental solution and \( \{\lambda_j, \psi_j\}_{j=1}^\infty \) the eigen system of \( F_\# \).
- Numerical results clearly show the perturbed part:

Figure 2: Middle: Helmholtz problem, right: Maxwell problem.