Entropy of interval maps via permutations

Christoph Bandt    Gerhard Keller
Bernd Pompe

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Abstract

For piecewise monotone interval maps we show that Kolmogorov-Sinai entropy can be obtained from order statistics of the values in a generic orbit. A similar statement holds for topological entropy.

1 Introduction

Entropies of interval maps. Entropies are basic invariants for dynamical systems. Let us define them for a piecewise monotone map $f$ from an interval $I$ into itself, cf. [5, section 4.3]. Let $\mathcal{P}$ be a partition of $I$ into the finitely many intervals on which $f$ is strictly monotone and denote by $\mathcal{P}_n$ the partition of $I$ into all sets of the form $P_1 \cap f^{-1}(P_2) \cap ... \cap f^{-(n-1)}(P_n)$ with $P_1, ..., P_n \in \mathcal{P}$. The elements of $\mathcal{P}_n$ are the maximal intervals on which $f^n$ is (strictly) monotone.

Let $\mu$ denote an invariant probability measure of $f$. We define

$$h_0(n) = \frac{1}{n} \log |\mathcal{P}_n| \quad \text{and} \quad h_\mu(n) = \frac{1}{n} H(\mathcal{P}_n) = \frac{1}{n} \sum_{P \in \mathcal{P}_n} \varphi(\mu(P)) \quad (1)$$

with $\varphi(p) = -p \log p$. Logarithms are taken here with base 2, and $|\mathcal{F}|$ denotes the cardinality of the set $\mathcal{F}$. For $n \to \infty$, the sequences in (1) have limits which coincide with the topological entropy $h_0$ of $f$ and the Kolmogorov-Sinai entropy $h_\mu$ of $(f, \mu)$, respectively. For the latter this may be seen from [7, Remark 6.2.6] in conjunction with the Kolmogorov-Sinai theorem (see e.g. Theorem 3.2.18, ibid.). For the topological entropy and continuous $f$ this is a result due to Misiurewicz and Szlenk [9], see also [8, Theorem 7.2]. If $f$ is discontinuous we will take $h_0 = \lim_{n \to \infty} h_0(n)$ simply as a definition.
Remark. Speaking about a partition of $I$ into intervals we did not specify to which interval a point separating two neighbouring intervals of the partition belongs to. Indeed, since this problem occurs for at most countably many points the KS entropy is independent of the choice we make. For the topological entropy we note that the result of Misiurewicz and Szlenk [9] is also not affected.

Contents of the paper. Here we show that these entropies can be determined in a quite different way, without explicitly using partitions. The idea is to compare the values $x_0 = x, x_1 = f(x), x_2 = f^2(x), \ldots$ of a typical orbit and see which order patterns appear among successive values. The permutation entropies which we define below can be directly estimated from data – from a sufficiently long orbit $x_0, x_1, x_2, \ldots$. For the definition, we do not need a formula of $f$ or any monotonicity assumptions.

Actually, permutation entropies for small order $n$ were introduced in [1] as a practical complexity measure for experimental time series, without providing justification from the theory of dynamical systems. In particular the existence of limits for $n \to \infty$ is open. Here we show that the limits exist at least for piecewise monotone interval maps, and coincide with topological and KS-entropy. Thus the viewpoint of permutations deserves further study. In fact, our arguments in section 3 can be extended to $d$-dimensional dynamical systems with $d > 1$, as will be shown in a subsequent paper.

We now define permutation entropies and then state our main result which is proved in sections 2 and 3. Before we start, let us mention some related work. A. Forrest recently investigated order types of complete trajectories $\{x_k|k \geq 0\}$ [6]. Permutations realized by periodic orbits of interval maps were studied for generalizations of the well-known Sharkovsky theorem [2, 3, 4].

Permutation entropies for interval maps. Even though the estimation of these entropies does not require explicit use of partitions [1], it is most convenient for our purpose to take partitions for their definition. They will not tell us whether the $f^i(x)$ belong to certain prescribed sets – they give the information about which of the iterates $f^i(x)$ are larger or smaller than others.

Let $f : I \to I$ be an interval map, and let $\pi = (k_1, k_2, \ldots, k_n)$ denote a permutation of the numbers $0, 1, \ldots, n - 1$. Writing $x_0 = x$ and $x_k = f^k(x)$ we define the set of points for which the orbit is ordered like $\pi$:

$$P_\pi = \{x \in I | x_{k_1} < x_{k_2} < \ldots < x_{k_n}\}$$

For $n \geq 2$, let the partition $\mathcal{P}_n^\pi$ consist of all non-empty sets $P_\pi$ where $\pi$ is a permutation of order $n$. We can also say that $\mathcal{P}_n^\pi$ is the partition generated by the
sets
\[ P_{ij} = \{ x | f^i(x) < f^j(x) \} \]
with \( 0 \leq i < j < n \). Using the term 'partition', we disregard the case \( f^i(x) = f^j(x) \) similarly as we have ignored in (1) critical points of \( f^n \) which may belong to two intervals of \( \mathcal{P}_n \). See example 1 for more details.

Similarly to (1), we now define an order \( n \) topological permutation entropy \( h_0(n) \) and – in the case when \( f \) has an invariant probability measure \( \mu \) – an order \( n \) Shannon permutation entropy \( h_\mu(n) \). We divide by \( n - 1 \) since comparisons start with the second value. The set of all \( n! \) permutations of order \( n \) is called \( S_n \).

\[
    h_0(n) = \frac{1}{n - 1} \log |P_0^n| = \frac{1}{n - 1} \log |\{ \pi \in S_n | P_\pi \neq \emptyset \}| ,
\]
\[
    h_\mu(n) = \frac{1}{n - 1} \sum_{P \in P_\mu^n} \varphi(\mu(P)) = \frac{1}{n - 1} \sum_{\pi \in S_n} \varphi(\mu(P_\pi)) .
\]

**Main result.** In the context of stationary time series [1], it is not clear that limits of \( h_0(n) \) and \( h_\mu(n) \) for \( n \to \infty \) exist. For our class of interval maps, however, this can be verified. The limits coincide with the classical topological entropy and the KS-entropy, respectively.

**Theorem.** Let \( f : I \to I \) be a piecewise monotone map on an interval \( I \). Then
\[
    h_0 = \lim_{n \to \infty} h_0(n) .
\]
For each \( f \)-invariant probability measure \( \mu \) we have
\[
    h_\mu = \lim_{n \to \infty} h_\mu(n) .
\]

Let us note that this not obvious. At first it seems that the permutation entropies are larger since \( P_\mu^n \) can contain \( n! \) sets while \( \mathcal{P}^n \) contains at most \( m^n \) partition sets, where \( m = |\mathcal{P}| \). This idea is supported by computer experiments for small \( n \).

**Example: the tent map.** Let \( I = [0, 1] \) and \( \mu \) Lebesgue measure on \( I \). Let \( f(x) = 2x \) for \( x \leq 1/2 \) and \( f(x) = 2(1 - x) \) for \( x \geq 1/2 \). The partitions \( \mathcal{P}_n \) consist of \( 2^n \) dyadic intervals \([k2^{-n}, (k + 1)2^{-n}]\) which gives \( h_0(n) = h_\mu(n) = 1 \) for all \( n \).

In contrast, the partition \( \mathcal{P}_2^* \) is determined by the fixed point \( 2/3 \), that is, \( P_{01} = (0, 2/3) \) and \( P_{10} = (2/3, 1) \). Thus \( h_0^*(2) = 1 \), but \( h_\mu^*(2) = \log 3 - \frac{2}{3} \).
Table 1: Permutation entropies $h^0_{\mu}(n)$ and $h^\mu_{\mu}(n)$ of the tent map up to order $n = 16$ (in bit). $m_n < |P^*_n|$ is the number of sets with two components in the partition $P^*_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>P^*_n</td>
<td>$</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>75</td>
<td>414</td>
<td>2137</td>
<td>10525</td>
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<tr>
<td>$m_n$</td>
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<td>0</td>
<td>3</td>
<td>36</td>
<td>261</td>
<td>1544</td>
<td>8222</td>
<td>41119</td>
<td>197444</td>
</tr>
<tr>
<td>$h^0_{\mu}(n)$</td>
<td>1</td>
<td>1.161</td>
<td>1.195</td>
<td>1.246</td>
<td>1.242</td>
<td>1.229</td>
<td>1.215</td>
<td>1.201</td>
<td>1.189</td>
</tr>
<tr>
<td>$h^\mu_{\mu}(n)$</td>
<td>0.918</td>
<td>1.075</td>
<td>1.130</td>
<td>1.195</td>
<td>1.200</td>
<td>1.192</td>
<td>1.182</td>
<td>1.171</td>
<td>1.161</td>
</tr>
</tbody>
</table>

For $n = 3$, the permutation 210 does not occur. So $P^*_3$ has 5 sets:

\[ P_{012} = (0, \frac{1}{3}), \ P_{021} = (\frac{1}{3}, \frac{2}{5}), \ P_{201} = (\frac{2}{5}, \frac{2}{3}), \ P_{012} = (\frac{2}{3}, \frac{4}{5}), \ P_{120} = (\frac{4}{5}, 1). \]

In $P^*_4$ we have 12 sets, but 3 of them are unions of two disjoint intervals, as for instance $P_{2031} = (\frac{2}{5}, \frac{4}{7}) \cup (\frac{4}{7}, \frac{3}{5})$. Table 1 indicates that for larger $n$, most sets of $P^*_n$ have this form. One can prove that no set of $P^*_n$ consists of more than two components. A slightly weaker estimate that suffices for our purposes is given at the end of this note.

In general, the points $x$ with $f^i(x) = f^j(x)$ for some $0 \leq i < j < n$ separate the sets of $P^*_n$. These points, and the lengths of corresponding intervals were processed by computer to obtain Table 1.

**Numerical study of a parametric family.** Our example belongs to the family of symmetric tent maps on $[0,1]$, parametrized by $r \in [1,2]$.

\[ f_r(x) = rx \text{ for } x \leq \frac{1}{2} \text{ and } f_r(x) = r(1-x) \text{ for } x \geq \frac{1}{2} \]

It is known that $f_r$ has topological entropy $\log r$ [8, p. 171]. Moreover, each $f_r$ has a unique absolutely continuous invariant measure $\mu_r$ with $h_{\mu_r} = \log r$ [7, Example 6.3.13]. So this is a suitable family for comparing permutation entropies with their limit. Figure 1 shows $h^0_{\mu}(n)$ and $h^\mu_{\mu}(n)$ with $n = 2 \ldots 10$ for 5000 parameter values $r$. For each $r$, an orbit with $10^6$ points was evaluated. It is obvious from the figure and the table that in contrast with the entropy of a partition, $h^0_{\mu}(n)$ and $h^\mu_{\mu}(n)$ need not decrease with $n$. For $r$ near to 2, $h^\mu_{\mu}(2)$ is even smaller than the limit $\log r$.

The topological permutation entropy has discontinuities at those parameter values where certain $P^*_n$ become non-empty, notably at the band-merging point $\sqrt{2}$ and
Figure 1: Permutation entropies of symmetric tent maps $f_r$ for order $n = 2 \ldots 10$ (from thick to thin lines). The lower thick curve represents the limit $\log r$. 
at \( r = (\sqrt{5} + 1)/2 \). At such places, \( h_\mu^*(n) \) can have local minima while \( h_0^*(n) \) seems to increase with \( r \) for each \( n \).

Figure 1 and the above table indicate that the convergence of permutation entropies is rather slow. Proposition 1 below says that \( h_0^*(n) - h_0 \) and \( h_\mu^*(n) - h_\mu \) are bounded by a term of size \( \log n/n \). The difference in Figure 1 is about half of the bound. So the upper bound seems to be of the right order of magnitude.

On the other hand, it is apparent that permutation entropies of order 10, or even 6, considered as a function of \( r \), are almost parallel to the limit function \( \log r \). Although we do not know why this is the case, this is an argument for the practical use of permutation entropies in time series. Changes in the parameter are reflected by permutation entropies of low order in the same way as by their limit.

2 The upper bound

Maps without stable periodic orbits. In this section we show that for piecewise monotone maps, the permutation entropies are not larger than the corresponding topological and Kolmogorov-Sinai entropy:

\[
\limsup_{n \to \infty} h_0^*(n) \leq h_0 \quad \text{and} \quad \limsup_{n \to \infty} h_\mu^*(n) \leq h_\mu.
\]

This follows directly by taking limits \( n \to \infty \) in proposition 1 below. We first assume that for each \( n \), there is at most one fixed point of \( f^n \) on each interval of monotonicity of \( f^n \). Later we shall remove that condition.

Note that there is at most one fixed point on intervals where \( f^n \) is decreasing. When \( f^n \) is increasing on the interval and has at least two fixed points, then one of these points must be stable, in the sense that all points of a subinterval are attracted to the point under the action of \( f^n \). Thus our condition holds when \( f \) has no stable periodic points (and no interval consisting entirely of periodic points).

For technical reasons, we slightly extend the assumption, allowing for \( K \) fixed points instead of one. Various sufficient conditions for this property (for example negative Schwarzian derivative with \( K = 3 \)) were given in [11, Proposition 5.4].

Let us study the points which generate permutation partitions. For \( n \geq 1 \), we define

\[
F_n = \{ x \in I \mid f^i(x) = f^j(x) \text{ for some } i, j \text{ with } 0 \leq i < j \leq n \}
\]

and denote by \( \tilde{P}_n \) the partition of \( I \) into intervals by the points from \( F_n \). Then \( \tilde{P}_n \) is a refinement of \( P_{n+1}^* \), so \( |P_{n+1}^*| \leq |\tilde{P}_n| = 1 + |F_n| \).
Proposition 1. Let $f : I \to I$ be a piecewise monotone map and $K \geq 1$ such that for each $n$, each interval of monotonicity of $f^n$ contains at most $K$ fixed points of $f^n$. Then

a) Each interval of monotonicity of $f^n$ contains at most $Kn$ points of $F_n$.

b) $h_0^*(n+1) \leq h_0(n) + \frac{\log Kn}{n}$ for $n \geq 2$.

c) If $\mu$ is an $f$-invariant probability measure then

\[ h_\mu^*(n+1) \leq h_\mu(n) + \frac{\log(Kn+1)}{n} \quad \text{for } n \geq 1. \]

Proof. a) In $F_n$, $k = j - i$ runs from 1 to $n$, and $i = 0, 1, ..., n - k$. Let $A_k$ denote the set of periodic points of $f$ with (minimal) period $k$. It is easy to see that $A_k \subseteq f^{-1}(A_k)$ and hence $f^{-m}(A_k) \subseteq f^{-m-1}(A_k)$. Thus

\[ F_n = \bigcup_{k=1}^{n} f^{-(n-k)}(A_k). \]  

Now consider an interval of monotonicity $J$ of $f^n$, and fix $k$. Then $J$ is mapped monotonously by $f^{n-k}$ into an interval of monotonicity of $f^k$, which by assumption contains at most $K$ points of $A_k$. This means that for $k = 1, ..., n$ the interval $J$ contains at most $K$ points of $f^{-(n-k)}(A_k)$.

b) We proved that $|F_n| \leq Kn \cdot |P_n|$. For $n \geq 2$, we can reduce this number by 1 since each fixed point of $f$ is also counted as a point of period $n$.

Thus $|P_n^*| \leq Kn \cdot |P_n|$ which with (1) and (2) implies the assertion.

c) We estimate the $\mu$-entropy of the partition $P_n \vee \bar{P}_n$ which is the common refinement of $P_n$ and $\bar{P}_n$. By a), each monotonicity interval $J$ of $f^n$ is the union of at most $Kn + 1$ sets from $P_n \vee \bar{P}_n$ with total measure $\mu(J)$. The sum of the $\varphi$–values of these sets is bounded by $\varphi(\mu(J)) + \mu(J) \cdot \log(Kn+1)$, the maximal value being assumed when all partition sets have measure $\mu(J)/(Kn+1)$. Summing over $J \in \mathcal{P}_n$ we get for the Shannon entropies

\[ H(P_{n+1}^*) \leq H(\bar{P}_n) \leq H(P_n \vee \bar{P}_n) \leq H(P_n) + \log(Kn + 1). \]
Study of stable periodic orbits. Now we assume that on an interval $J$ of monotonicity of $f^k$ there are two or more fixed points of $f^k$. The idea is that because each other fixed point is stable, this will not really create more sets in $\mathcal{P}_n^*$, it will only divide the existing sets into more components.

Let $x$ denote the smallest and $y$ the largest fixed point of $f^k$ in $J$, and let $J' = [x, y]$. Then $f^k$ is an increasing function from $J'$ onto itself. The period of a point $z$ is the smallest $m$ with $f^m(z) = z$. We shall prove now that all periodic points in $J'$ have period $k$ – with one possible exception.

**Proposition 2.** Let $J' = [x, y]$ be a maximal interval such that $f^k$ is an increasing function from $J'$ onto itself, and let $k$ be the smallest integer such that $f^k$ is increasing on $J'$ and $f^k(J') = J'$. Then

a) $x$ and $y$ have period $k$.

b) $J'$ can contain at most one periodic point $u$ of $f$ with period $m \neq k$. If such a point exists, then $k$ is even, $m = \frac{k}{2}$, and $f^m(x) = y, f^m(y) = x$.

**Proof.** a) We suppose $f^m(x) = x$ for some $m < k$. Since this implies $f^{k-m}(x) = x$, we can assume $m \leq \frac{k}{2}$.

If $f^m$ is increasing on $J'$, and $f^m(y) < y$, consider the map $h = (f^k)^m$. On one hand, $h(y) = y$. On the other hand, $h = (f^m)^k$, and $f^m$ maps $J'$ onto a proper subinterval, which leads to the contradiction $h(y) < y$. For $f^m(y) > y$ we use the same argument with $h^{-1}$.

Now consider the case that $f^m$ is decreasing on $J'$. If $m < \frac{k}{2}$, then $f^{2m}$ would be increasing with $2m < k$, which is not possible as we just proved. Thus $m = \frac{k}{2}$, $f^m(x) = x$ and $f^m(y) = y' < x$. Let $J'' = [y', x]$. Since $f^k = f^m \circ f^m$ maps $J'$ homeomorphically onto itself, and $f^m$ maps $J'$ homeomorphically onto $J''$, the map $f^m$ also maps $J''$ homeomorphically onto $J'$. But then $f^k$ is a homeomorphism on the set $J'' \cup J'$, so this set must be within an interval $J$ of monotonicity of $f^k$. Since $f^k(y') = y'$, this contradicts the minimum property of $x$. The assumption $f^m(x) = x$ for $m < k$ is disproved. The same holds for $y$.

b) Suppose $x < u < y$ and $u$ is a point of period $m$. By assumption on $f$, $x = f^k(x) < f^k(u) < f^k(y) = y$. Suppose $f^k(u) < u$. Then $u = f^{mk}(u) = (f^k)^m(u) < u$, a contradiction. Similarly, $f^k(u) > u$ is impossible. Hence $f^k(u) = u$ so that $m$ divides $k$, say $k = jm$ where $j > 1$ by assumption. If $f^m$ is increasing on $J'$ it follows that $x \leq f^m(x) \leq \ldots \leq f^{jm}(x) = x$ so that $f^m(x) = x$ in contradiction to the assumptions on $k$. So $f^m$ is decreasing on $J'$, and the previous argument applied to $f^{2m}$ yields $f^{2m}(x) = x$ which implies $k = 2m$. Also, since $f^m$ is decreasing on $J'$, $u$ is the only fixed point of $f^m$ in $J'$. 

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Removing the condition. Now we check the proof of proposition 1 a,b without assuming the condition on fixed points of $f$. We go through an interval $J$ of monotonicity of $f^n$ and count the elements of $F_n$ using (4). $A_k$ consists of the points with (minimal) period $k$ and $k$ runs from 1 to $n$. However, now we shall count only those points which really give rise to new sets in $P_{n+1}^*$. Let $H_k = J \cap f^{-(n-k)}(A_k)$ and $J_k$ the smallest interval containing $H_k$.

Proposition 2 says that any two intervals $J_k, J_m$ with $m < k$ are disjoint, except for the case that $k = 2m$ and $J_m$ consists of a single point which is mapped onto the midpoint of $J_k$ by some iterate of $f$.

Case 1. We consider $J_k$ where the midpoint is not of period $k/2$ (which is certainly true for odd $k$). We show that we need only count one change point in (4) since all the points $u \in H_k$ only separate the components of two partition sets of $P_{n+1}^*$. We assume $|J_k| > 1$ since otherwise there is nothing to prove. Let us consider the maximal number $p = p(u) \geq 1$ with $f^n(u) = f^{n-k}(u) = \ldots = f^{n-pk}(u)$. This number is the same for all $u \in H_k$ which are not endpoints of $J$.

To see this, assume $p(v) > p$ for some $v \in H_k$ so that $z = f^{n-(p+1)k}(u) \neq f^{n-pk}(u) = x$ and $f^{n-(p+1)k}(v) = f^{n-pk}(v) = y$. Let $x < y$, the other case is similar. Now $f^k$ is an increasing function from $J' = [x, y]$ onto itself, as in proposition 2. So $x < z < y$ is impossible, $z < x$ would imply $f^{n-(p+1)k}[u, v] = [z, y]$ where some point in $[u, v]$ maps to $x$ so that $f^{n-pk}$ is not monotone on $[u, v]$. Thus we can conclude $y < z$. In this case $f^{n-(p+1)k}[u, v] = [y, z]$ and $f^k[y, z] = [x, y]$ so that $y$ is a critical point of $f^k$. Since $f^n$ is monotonous on $J$ this means that $v$ is an endpoint of $J$. In this case, however, the change in $v$ will be counted for the neighboring monotonicity interval of $f^n$ where $p(v)$ is not larger than some other $p(u)$.

Since $p$ is constant on $H_k$ (except for an endpoint of $J$), we write $f^{n-kp}(J_k) = [x, y]$ and note that $f^k$ is an increasing function from $[x, y]$ to itself. This implies that for $z \in J_k \setminus H_k$ either $f^{n-kp}(z) < \ldots < f^n(z)$ or $f^{n-kp}(z) > \ldots > f^n(z)$ in both cases the order between $f^i(z)$ and $f^{i+k}(z)$ with $n - kp < i < n - k$ is determined by the fact that $f^k$ is increasing or decreasing on the sets $f^j[x, y]$. Moreover, the order relation between all other $f^i(z)$ is the same for all $z \in J_k$, by proposition 2 and the fact that $p$ is constant. Thus altogether we have only two different permutations of order $n$ for all points of $J_k \setminus H_k$, as well as for points between $J_k$ and the next $J_m$ (or the endpoint of $J$) left and right. Thus we need to count only one change point.

Case 2. Now let us assume $k = 2m$ and the midpoint $w$ of $f^{n-k}(J_k) = [x, y]$ has period $m$. Since we count change points for increasing $k$, and $w$ has period $m < k$,
the point $v = f^{-(n-k)}(w)$ has already been counted. $v$ divides $J_k$ into two parts for which we can proceed as in case 1. We find two permutation patterns left of $v$ and two right of $v$, thus three changes, but one change was already counted. So in this case we have to count two change points for $J_k$.

Since case 2 applies only to even $k$, summing up over $k = 1, \ldots, n$ gives at most $\frac{3}{2} \cdot n$ changes. The proof of proposition 1 now yields the following result.

**Corollary.** For all piecewise monotone interval maps, the relations in proposition 1, b) and c) are true with $K = \frac{3}{2}$.

### 3 The lower bound

**Lemma.** If $f$ has $m + 1$ intervals of monotonicity, we have for each $n$

$$(n + 1)^m \cdot |\mathcal{P}_n^*| \geq |\mathcal{P}_n \cup \mathcal{P}_n^*| \geq |\mathcal{P}_n|.$$ 

**Proof.** We consider a set $P_\pi$ in $\mathcal{P}_n^*$ and show that it can intersect at most $(n + 1)^m$ sets of $\mathcal{P}_n$. Let $c_1 < \ldots < c_m$ be the points that subdivide the interval $I$ into $m + 1$ intervals $P_0, P_1, \ldots, P_m$ of monotonicity of $f$. For $x \in P_\pi$ let $\Delta_n[x]$ denote the interval in $\mathcal{P}_n$ that contains $x$. (Again we neglect points $x$ that are endpoints of intervals from $\mathcal{P}_n$.) $\Delta_n[x]$ can be written as $P_{i_0} \cap f^{-1}P_{i_1} \cap \ldots \cap f^{-(n-1)}P_{i_{n-1}}$ so that it is specified by the $n$-tupel $i[x] = (i_0, \ldots, i_{n-1}) \in \{0, \ldots, m\}^n$.

Now $\pi$ is given by inequalities $x_{k_1} < \ldots < x_{k_m}$ with $\{k_1, \ldots, k_m\} = \{0, \ldots, n-1\}$ and $x_k = f^k(x)$. For each $x \in P_\pi$ we can extend these inequalities so that they give the common order of the $c_l$ and the $x_{k_l}$, where $l = 1, \ldots, m$ and $j = 0, \ldots, n-1$. Nevertheless, there are at most $(n + 1)^m$ possible extended orders since each $c_l$ has $n + 1$ possible places among the $x_{k_l}$.

Moreover, when we know the common order of the $c_l$ and $x_0, ..., x_{n-1}$ then $i[x]$ is uniquely determined. This completes the proof.

**Proof of the lower estimate.** The lemma implies $m \log(n + 1) + \log |\mathcal{P}_n^*| \geq \log |\mathcal{P}_n|$. Similarly, as in the proof of proposition 1c), we get $m \log(n + 1) + H(\mathcal{P}_n^*) \geq H(\mathcal{P}_n)$. Thus

$$h_0^*(n) \geq \frac{1}{n-1} (\log |\mathcal{P}_n| - m \cdot \log(n + 1)) > h_0(n) - \frac{m}{n-1} \cdot \log(n + 1),$$

and this inequality also holds for $h_\mu^*(n)$ and $h_\mu(n)$. We conclude that

$$\liminf_{n \to \infty} h_0^*(n) \geq h_0 \quad \text{and} \quad \liminf_{n \to \infty} h_\mu^*(n) \geq h_\mu.$$
The unimodal case. For the case when $f$ has only two intervals of monotonicity we can give a better estimate. There is only one point $c$, and we show that for a given order $x_{k_1} < \ldots < x_{k_n}$, there are at most two neighboring gaps where $c$ can possibly fit in.

Suppose that $f$ is increasing for $x < c$. Then $x_j < c$ if and only if $x_j < x_{j+1}$ which determines the relation of $x_j$ and $c$ for $j = 0, \ldots, n-2$. Only $x_{n-1}$ can be either right or left from $c$, and this can happen only for rather special permutations.

Corollary. For unimodal maps, we have $2 \cdot |P_n^*| \geq |P_n|$. Consequently,

$$h^*_0(n) \geq \frac{1}{n-1} \cdot (\log |P_n| - 1) = \frac{n \cdot h_0(n) - 1}{n-1},$$

and the same relation holds for $h^*_\mu(n)$ and $h_\mu(n)$.

References


Authors’ addresses:
C. Bandt, Institut für Mathematik und Informatik, Arndt-Universität, 17487 Greifswald, Germany
G. Keller, Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstr. 1 1/2, 91054 Erlangen, Germany
B. Pompe, Institut für Physik, Arndt-Universität, 17487 Greifswald, Germany
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