Multivariate Wavelet Analysis

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Aims of the project

- Construction of wavelets and multiwavelets for general scalings
- Construction of wavelets on domains and manifolds
- Development of denoising strategies for general scalings
- Implementation
Bases of $L_2(\mathbb{R})$ obtained by dilating and translating one motherwavelet $\psi$

\[ \{ \psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \ j \in \mathbb{Z}, \ k \in \mathbb{Z} \} \]

Construction principle:

Mallat/Meyer (1986): multiresolution analysis

Basic tool:

scaling function $\phi$
Refinement equation

\[ \phi(x) = \sum_{k \in \mathbb{Z}} b_k \phi(2x - k), \quad (b_k)_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z}) \]

\[ \hat{\phi}(\omega) = \frac{1}{2} b \left( e^{-i\frac{\omega}{2}} \right) \hat{\phi} \left( \frac{\omega}{2} \right) \]

with the symbol

\[ b(z) := \sum_{k \in \mathbb{Z}} b_k z^k, \quad z = e^{-i\omega} \]

General scalings

\[ \phi(x) = \sum_{k \in \mathbb{Z}^d} b_k \phi(Mx - k), \quad (b_k)_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d) \]

\( M \in \mathbb{Z}^{d \times d} \) expanding, i.e. \(|\lambda_i| > 1 \ \forall i, \lambda_i \) eigenvalues of \( M \)
• Compact support

\[ b(z) \text{ Laurent polynomial} \]

• Interpolation

\[ \phi(k) = \delta_{0,k} \text{ for all } k \in \mathbb{Z} \]

\[ b(z) = 1 + zb^1(z^2) \]

Benefit

– Shannon-like sampling theorem: \( f \in \text{span}\{\phi(\cdot - k)\} \)

\[ \implies f(x) = \sum_{k \in \mathbb{Z}} f(k)\phi(x - k) \]

– Wavelet Galerkin methods, boundary conditions
• Regularity/Approximation order
  – Strang-Fix conditions

\[ b(z) = (1 + z)^m \eta(z) \]

• Orthonormality

\[ \{ \phi(\cdot - k), \ k \in \mathbb{Z} \} \text{ orthonormal} \]

\[ |b(z)|^2 + |b(-z)|^2 = 4 \]

\[ \Rightarrow \{ \psi_{j,k}(x), \ j, \ k \in \mathbb{Z} \} \text{ is orthonormal basis of } L_2(\mathbb{R}) \]
Lemma: Suppose $\phi$ is \{ orthonormal, compactly supported, interpolating \} \implies $\phi = \chi_{[0,1)}$
Multiwavelets

• **r-scaling vector**

\[ \Phi := (\phi_0, \ldots, \phi_{r-1})^T, \quad r > 0, \quad \phi_i \in L_2(\mathbb{R}) \]

• **matrix refinement equation**

\[ \Phi(x) = \sum_{k \in \mathbb{Z}} A_k \Phi(2x - k), \quad A_k \in \mathbb{R}^{r \times r} \]

\[ \hat{\Phi}(\omega) = \frac{1}{2} A \left( e^{-i\frac{\omega}{2}} \right) \hat{\Phi} \left( \frac{\omega}{2} \right) \]

• **symbol**

\[ A(z) := \sum_{k \in \mathbb{Z}} A_k z^k, \quad z = e^{-i\omega} \]
• Interpolation property for $r = 2$

$$\Phi \left( \frac{n}{2} \right) = \begin{pmatrix} \delta_{0,n} \\ \delta_{1,n} \end{pmatrix}, \quad n \in \mathbb{Z}$$

Shannon-like sampling theorem: $f \in \text{span}\{\phi_0(\cdot - k), \phi_1(\cdot - k), k \in \mathbb{Z}\}$

$$f(x) = \sum_{k \in \mathbb{Z}} f(k)\phi_0(x - k) + f(k + 1/2)\phi_1(x - k)$$
Examples

- $\phi$ scaling function with symbol $b(z) = b^0(z^2) + zb^1(z^2)$

$$\Phi(x) := \begin{pmatrix} \phi(2x) \\ \phi(2x - 1) \end{pmatrix}$$

$\Phi$ 2-scaling vector with symbol

$$A(z) = \begin{pmatrix} b^0(z) & b^1(z) \\ zb^0(z) & zb^1(z) \end{pmatrix}$$

This embedding preserves interpolation properties!

- Selesnick (1999): orthogonal and interpolating 2-scaling vectors with compact support
Aim:

Systematic and symbol-based construction of 2-scaling vectors, ensuring

- compact support
- interpolation
- high regularity/approximation order,

which provides enough freedom for further purposes such as orthogonality.
• Interpolation:

\[
A(z) = \begin{pmatrix} 1 & a_0(z) \\ z & a_1(z) \end{pmatrix}
\]

scalar case: \( b(z) = 1 + zb^1(z^2) \)

with Laurent polynomials \( a_0(z), a_1(z) \)

• \( L_2 \)-stability:

Plonka/Strela (1998): \( \lambda_1 = 2 \) and \( |\lambda_2| < 2 \), \( \lambda_i \) eigenvalues of \( A(1) \)

• Approximation order \( m \): scalar case: \( b(z) = (1 + z)^m \eta(z) \)

\[
A(z) = \frac{1}{2m-1} C_0(z^2) \cdots C_{m-1}(z^2) A^m(z) C_{m-1}^{-1}(z) \cdots C_0^{-1}(z)
\]

Plonka-factorization (1995) with a suitable matrix \( A^m(z) \) of Laurent polynomials
Φ interpolating:

\[ C_n(z) = \left( \frac{1}{2u} \right)^{\delta_{0,n}} \begin{pmatrix} 2 & -2 \\ -2z & 2 \end{pmatrix}, \quad u \neq 0 \]

**Lemma:** Each matrix \( C_n \) in the factorization yields 2 conditions for \( A \).
Explicit construction of $A(z)$:

1. Start with general symbol entries

$$a_j(z) := \sum_{k=\nu}^{\nu+m} a_j^{(k)} z^k, \quad j \in \{0, 1\}, \quad \nu = -\left\lfloor \frac{m}{2} \right\rfloor$$

2. Apply $L_2$-stability condition, choose $a_0(1) = a_1(1) = 1$

3. For $n = 0, \ldots, m - 1$ apply the factorization conditions

Redundancy in 2. and 3. $\implies$ one-parameter family $m \Phi_\alpha, m A_\alpha(z)$
Examples

- \( m = 2 \):

\[
2A_\alpha(z) = \begin{pmatrix}
1 & \left(\frac{1}{2} - \alpha\right)z^{-1} + \frac{1}{2} + \alpha z \\
z & \alpha z^{-1} + \frac{1}{2} + \left(\frac{1}{2} - \alpha\right)z
\end{pmatrix}
\]

\( \alpha = 0 \):

\[
2A_0(z) = \begin{pmatrix}
1 & \frac{1}{2}z^{-1} + \frac{1}{2} \\
z & \frac{1}{2} + \frac{1}{2}z
\end{pmatrix} \sim b(z) = \frac{1}{2}z^{-1}(1 + z)^2
\]

cardinal B-spline of order 2!

→ Animation ...
• $m = 3$:

$$3A_\alpha(z) = \begin{pmatrix}
1 & \left(\frac{3}{8} - 3\alpha\right)z^{-1} + \frac{3}{4} + 3\alpha - (\alpha + \frac{1}{8})z + \alpha z^2 \\
z & \alpha z^{-1} + \frac{3}{8} - \alpha + (\frac{3}{4} + 3\alpha)z - (\frac{1}{8} + 3\alpha)z^2
\end{pmatrix}$$

$\alpha = 0$:

$$3A_\alpha(z) = \begin{pmatrix}
1 & \frac{3}{8}z^{-1} + \frac{3}{4} - \frac{1}{8}z \\
z & \frac{3}{8} + \frac{3}{4}z - \frac{1}{8}z^2
\end{pmatrix} \sim b(z) = \frac{1}{8}(3z^{-1} + 8 + 6z - z^3)$$

Dahlke/Maass (1997)

→ Animation ...
Conclusion

For our purposes:

- **scalar** wavelets/scaling **functions** not flexible enough
- alternative: **multiwavelets/scaling vectors**
- provisional gain: higher regularity for a given support length
To do

• Inclusion of orthonormality conditions

\[ \delta_{i,j} \delta_{0,n} = \int \phi_i(x)\phi_j(x - n) \, dx, \quad n \in \mathbb{Z}, \, i, j \in \{0, \ldots, r - 1\} \]

\[ \Rightarrow \text{Bezout problem (cf. Daubechies 1992)} \]

• Extension to general scalings

  – Interpolation property \( \checkmark \)
  
  – Approximation order: Plonka-factorization?

  – Orthonormality ?