A DISTRIBUTIONAL LIMIT LAW FOR THE CONTINUED FRACTION DIGIT SUM

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ABSTRACT. We consider the continued fraction digits as random variables measured with respect to Lebesgue measure. The logarithmically scaled and normalized fluctuation process of the digit sums converges strongly distributional to a random variable uniformly distributed on the unit interval. For this process normalized linearly we determine a large deviation asymptotic.

1. INTRODUCTION AND STATEMENT OF MAIN Results

Any number \( x \in \mathbb{I} = [0, 1] \setminus \mathbb{Q} \) has a simple infinite continued fraction expansion

\[
x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots}},
\]

where the unique continued fraction digits \( a_n(x) \) are from the positive integers \( \mathbb{N} \). The Gauss transformation \( G : \mathbb{I} \to \mathbb{I} \) is given by

\[
G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \in \mathbb{R} \). Write \( G^n \) for the \( n \)-th iterate of \( G \), \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \) with \( G^0 = \text{id} \). It is then well known that for all \( n \in \mathbb{N} \), we have

\[
a_n(x) = \left\lfloor \frac{1}{G^{n-1}x} \right\rfloor.
\]

Clearly, the \( a_n, n \in \mathbb{N} \), define random variables on the measure space \((\mathbb{I}, \mathcal{B}, \mathbb{P})\), where \( \mathcal{B} \) denotes the Borel \( \sigma \)-algebra of \( \mathbb{I} \) and \( \mathbb{P} \) some probability measure on \( \mathcal{B} \). Then each \( a_n \) has infinite expectation with respect to the Lebesgue measure on \([0, 1]\), which we will denote by \( \lambda \). By the ergodicity of the Gauss transformation with respect to the Gauss measure \( d\mu(x) := \frac{1}{\log 2} \frac{1}{1+x} d\lambda(x) \) we readily reproduce Khinchin’s result on the geometric mean of the continued fraction digits, i.e.

\[
\sqrt[n]{a_1 \cdots a_n} \to K, \ \lambda\text{-a.e.},
\]

where \( K = 2, 685\ldots \) denotes the Khinchin constant. A similar result holds for the harmonic mean. Also by a classical result of Khinchin (cf. [Khi64]), we know that for \( \lambda \)-almost every \( x \in [0, 1] \) we have for infinitely many \( n \in \mathbb{N} \)

\[
a_n(x) > n \log n.
\]
From this Khinchin deduced that the arithmetic mean of the continued fraction digits is divergent, i.e.

\[
\lim_{n \to \infty} \frac{S_n}{n} = \infty \quad \text{\(\lambda\)-a.e.}
\]

where \(S_n(x) := a_1(x) + \cdots + a_n(x), \ x \in \mathbb{I}\). It was again Khinchin who showed that nevertheless for a suitable normalising sequence a weak law of large numbers holds. That is \(\frac{S_n}{n \log n}\) converges in measure to \(1/\log 2\) with respect to \(\mathcal{L}\). However, according to [Phi88] there is no (reasonable) normalising sequence \((n_k)\) with \((n_k/k)\) non-decreasing such that a strong law of large numbers is satisfied. More precisely, we either have

\[
\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{S_k}{n_k} = 0 \quad \text{\(\lambda\)-a.e., or} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{S_k}{n_k} = \infty \quad \text{\(\lambda\)-a.e.}
\]

In contrast to this Diamond and Vaaler showed in [DV86] that for the trimmed sum \(S^0_n := S_n - \max_{1 \leq \ell \leq n} a_\ell\) a strong law of large numbers holds in the sense that

\[
\lim_{n \to \infty} \frac{S^0_n}{n \log n} = \frac{1}{\log 2} \quad \text{\(\lambda\)-a.e.}
\]

This shows that the intricate stochastic properties of \(S_n\) arise from the occurrences of rare but exceptionally large continued fraction digits. Lévy was the first to derive non-degenerated limit laws in the context of continued fractions (cf. [Lév52]) – namely stable laws. Actually, we have that \((a_n)\) belongs to the domain of attraction to a stable law with characteristic exponent 1. More precisely we have the following convergence in distribution with respect to any absolutely continuous probability measure \(\mu \ll \lambda\)

\[
\frac{S_k}{k/\log 2} - \log k \underset{\mu}{\overset{d}{\to}} F,
\]

where \(F\) has a stable distribution (cf. [Hei87] and [Phi88], and for related results see also [Hen00]). Instead of taking a global centralising sequence as in the above situation we will focus on the pointwise behaviour of the fluctuation of \(S_n\). For this we define

\[
X_n(x) := \max \{S_k(x) : S_k(x) \leq n, \ k \in \mathbb{N} \}, \quad x \in \mathbb{I},
\]

and investigate the process \(n - X_n\) of pointwise fluctuations. Employing infinite ergodic theory we are able to derive non-degenerated results describing new aspects of the stochastic structure of \(S_n\). The underlying dynamical system will be given by the Farey map and the connection to metrical number theory is established via a certain return time process for this system. This process turns out to be conceptually related to renewal theory which guided us to find the following main results of this paper.

- **Uniform Law:** For \(U\) uniformly distributed random variable on \([0, 1]\) we have the following convergence in distribution with respect to \(\lambda\)

\[
\frac{\log(n - X_n)}{\log(n)} \overset{\lambda}{\to} U.
\]

- **Large Deviation Law:** For all \(\varepsilon \in (0, 1)\) we have

\[
\lambda \left( \frac{n - X_n}{n} > \varepsilon \right) \sim -\frac{\log \varepsilon}{\log n},
\]

where \(a_n \sim b_n\) means that \(\lim_{n \to \infty} \frac{a_n}{b_n} = 1\).
These statements are implied by the slightly more general Theorem 3.4 stated and proved in Subsection 3.2.

Note that the uniform law gives the following convergence in probability
\[ n - X_n \xrightarrow{\mathbb{P}} \infty, \quad (1.2) \]
whereas the large deviation law in particular means
\[ \frac{n - X_n}{n} \xrightarrow{\mathbb{P}} 0. \quad (1.3) \]

For further interesting result in the context of continued fraction digit sums we would like to refer the reader to [GLJ93, GLJ96], wherein alternating sums of continued fraction digits are considered.

The paper consists of two major parts. In the first part (Section 2) we recall and develop some expects from infinite ergodic theory allowing us to apply a theorem from [KS09] in the proof of the uniform law within this number theoretical context. In the second part we study the Farey map as an infinite measure preserving transformation and use its connections to the fluctuation process in question to finally give the proof of both the uniform law and the large deviation asymptotic.

2. INFINITE ERGODIC THEORY.

In the first subsection of this section we recall some basics definitions and facts from infinite ergodic theory and give new sufficient conditions for a set to be uniformly returning. In the second subsection we state the general limit law which we will apply in the context of continued fractions. Also, as an example we consider a certain interval map.

2.1. Preliminaries. With \((X, T, \mathcal{A}, \mu)\) we always denote a conservative ergodic measure preserving dynamical systems where \(\mu\) is an infinite \(\sigma\)-finite measure. For a good overview, further definitions and details we refer the reader to [Aar97].

Let
\[ \mathcal{P}_\mu := \{ \nu : \nu \text{ probability measure on } \mathcal{A} \text{ with } \nu \ll \mu \} \]
denote the set of probability measures on \(\mathcal{A}\) which are absolutely continuous with respect to \(\mu\). The measures from \(\mathcal{P}_\mu\) represent the admissible initial distributions for the processes associated with the iteration of \(T\). The symbol \(\mathcal{P}_\mu\) will also be used for the set of the corresponding densities.

Note that for such dynamical systems the mean return time to sets of finite positive measure is infinite leading to the notion of wandering rate. For a fixed set \(A \in \mathcal{A}\) with \(0 < \mu (A) < \infty\) we set
\[ K_n := \bigcup_{k=0}^{n} T^{-k} A \quad \text{and} \quad W_n := W_n (A) := \mu (K_n), \quad n \geq 0, \]
and call the sequence \((W_n(A))\) the \textit{wandering rate} of \(A\). Note that for the wandering rate the following identity holds

\[ W_n(A) = \sum_{k=0}^{n} \mu(A \cap \{ \varphi > k \}), \]

where

\[ \varphi(x) := \inf\{n \geq 1 : T^n(x) \in A\}, \quad x \in X, \]

denotes the first return time to the set \(A\).

Since \(T\) is conservative and ergodic, for all \(\nu \in \mathcal{P}_\mu\),

\[ \lim_{n \to \infty} \nu(K_n) = 1 \quad \text{and} \quad \nu(\{\varphi < \infty\}) = 1. \]

To understand the stochastic properties of a nonsingular transformation of a \(\sigma\)-finite measure space one often has to study the long-term behaviour of the iterates of its transfer operator

\[ \hat{T} : L_1(\mu) \longrightarrow L_1(\mu), \quad f \mapsto \hat{T}(f) := \frac{d(\nu_f \circ T^{-1})}{d\mu}, \]

where \(\nu_f\) denotes the measure with density \(f\) with respect to \(\mu\). Clearly, \(\hat{T}\) is a positive linear operator characterised by

\[ \int_B \hat{T}(f) \, d\mu = \int_{T^{-1}(B)} f \, d\mu, \quad f \in L_1(\mu), \quad B \in \mathcal{A}. \]

The ergodic properties of \((X, T, A, \mu)\) can be characterised in terms of the transfer operator in the following way (cf. [Aar97, Proposition 1.3.2]). A system is conservative and ergodic if and only if for all \(f \in L_1^+(\mu) := \{f \in L_1(\mu) : f \geq 0 \text{ and } \int_X f \, d\mu > 0\}\) we have \(\mu\)-a.e.

\[ \sum_{n \geq 0} \hat{T}^n(f) = \infty. \]

Invariance of \(\mu\) under \(T\) means \(\hat{T}(1) = 1\).

The following two definitions are in many situation crucial within infinite ergodic theory.

- A set \(A \in \mathcal{A}\) with \(0 < \mu(A) < \infty\) is called \textit{uniform} for \(f \in P_\mu\) if there exists a sequence \((b_n)\) of positive reals such that

\[ \frac{1}{b_n} \sum_{k=0}^{n-1} \hat{T}^k(f) \longrightarrow 1 \quad \mu - \text{a.e. uniformly on } A \]

(i.e. uniform convergence in \(L_\infty(\mu|_{A \cap A})\)).
- The set \(A\) is called a \textit{uniform} set if it is uniform for some \(f \in P_\mu\).
The concept of regularly varying functions and sequences plays a central role in infinite ergodic theory (see also [BGT89] for a comprehensive account).

A measurable function $R : \mathbb{R}^+ \to \mathbb{R}$ with $R > 0$ on $(a, \infty)$ for some $a > 0$ is called regularly varying at $\infty$ with exponent $\rho \in \mathbb{R}$ if

$$\lim_{t \to \infty} \frac{R(\lambda t)}{R(t)} = \lambda^\rho \text{ for all } \lambda > 0.$$ 

A regularly varying function $L$ with exponent $\rho = 0$ is called slowly varying at $\infty$, i.e.

$$\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1 \text{ for all } \lambda > 0.$$ 

Clearly, a function $R : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying at $1$ with exponent $2 \rho$ if and only if $R(t) = t^\rho L(t)$, $t \in \mathbb{R}^+$,

for some slowly varying function $L$.

A sequence $(u_n)$ is regularly varying with exponent $\rho$ if $u_n = R(n)$, $n \geq 1$, for some $R : \mathbb{R}^+ \to \mathbb{R}$ regularly varying at $\infty$ with exponent $\rho$.

**Remark 2.1.** From [Aar97, Proposition 3.8.7] we know, that $(b_n)$ is regularly varying with exponent $\alpha$ if and only if $(W_n)$ is regularly varying with exponent $(1 - \alpha)$. In this case $\alpha$ lies in the interval $[0, 1]$ and

$$b_nW_n \sim \frac{n}{\Gamma(1 + \alpha)\Gamma(2 - \alpha)}.$$  

(2.1)

In the following example we introduce interval maps giving rise to an infinite measure preserving transformation.

**Example 2.2.** ([Tha80], [Tha83]) Let $\xi_1 = \{B(k) : k \in I\}$ be a finite or infinite family of pairwise disjoint subintervals of $[0, 1]$ such that $\lambda(\bigcup_{k \in I} B(k)) = 1$. We consider transformations $T : [0, 1] \to [0, 1]$, satisfying the following Thaler Conditions.

1. $T|_{B(k)}$ is twice differentiable and $TB(k) = [0, 1]$ for all $k \in I$.
2. There exists a non-empty finite set $J \subseteq I$ such that each $B(j)$, $j \in J$, contains a unique fixed point $x_j$ with $T''(x_j) = 1$ (indifferent fixed point).
3. $|T'| \geq \rho(\varepsilon) > 1$ on $\bigcup_{k \in I} B(k) \setminus \bigcup_{j \in J} (x_j - \varepsilon, x_j + \varepsilon)$ for each $\varepsilon > 0$.
4. There exists $\eta > 0$ such that for all $j \in J$, $T'$ is decreasing on $(x_j - \eta, x_j) \cap B(j)$ and increasing on $(x_j, x_j + \eta) \cap B(j)$.
5. $\frac{\mu}{\lambda}$ is bounded on $\bigcup_{k \in I} B(k)$ (Adler’s Condition).

As proved in [Tha80], [Tha83] $T$ is conservative and ergodic with respect to $\lambda$, and admits an infinite $\sigma$-finite invariant measure $\mu$ equivalent to $\lambda$. The density $\frac{\mu}{\lambda}$ has a version $h$ of the form

$$h(x) = h_0(x) \prod_{j \in J} \frac{x - x_j}{x - u_j(x)}, \quad x \in [0, 1] \setminus \{x_j : j \in J\},$$

where \( u_j = (T_{\beta(j)})^{-1}, j \in J, \) and \( h_0 \) is continuous and positive on \([0, 1] \). By [Tha95] we know that there exists a sequence \((b_n)\) of positive numbers such that for all \( f \in L_1(\mu)\) with \( fh \) Riemann-integrable on \([0, 1] \) we have

\[
\frac{1}{b_n} \sum_{k=0}^{n-1} \hat{T}^k(f) \longrightarrow \int f \, d\mu,
\]

uniformly on compact subsets of \([0, 1] \setminus \{x_j : j \in J\} \). Hence, any such subset is uniform for any \( f \in P_\mu \). In particular, if

\[
T(x) = x \mp a_j |x - x_j|^{p_j+1} + o \left( |x - x_j|^{p_j+1} \right) \quad (x \to x_j)
\]

with \( a_j > 0, \ p_j \in \mathbb{N} \ (j \in J), \) and \( p = \max \{p_j : j \in J\} \), then we have

\[
b_n \sim \text{const} \cdot \begin{cases} \frac{n}{\log(n)}, & p = 1, \\ n^{1/p}, & p > 1. \end{cases}
\]

To state the crucial condition for the uniform law (see (UL) in Subsection 2.2) we need the notion of uniform returning sets introduced in [KS09].

**Definition 2.3.** A set \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \) is called **uniformly returning for** \( f \in P_\mu \) if there exists a positive increasing sequence \((b_n)\) such that

\[
b_n \hat{T}^n(f) \longrightarrow 1 \quad \mu \text{-a.e. uniformly on } A.
\]

The set \( A \) is called **uniformly returning** if it is uniformly returning for some \( f \in P_\mu \).

**Remark 2.4.** By [KS09, Proposition 1.1] we know that \((b_n)\) is regularly varying with exponent \( \beta \in [0, 1) \) if and only if \((W_n)\) is regularly varying with the same exponent. In this case,

\[
b_n \sim W_n \Gamma(1 - \beta) \Gamma(1 + \beta) \quad (n \to \infty).
\]

**Example 2.5.** Let \( T : [0, 1] \to [0, 1] \) be an interval map with two increasing full branches and an indifferent fixed point at 0 such that there exists an absolutely continuous invariant infinite measure for \( T \) like for instance in Example 2.2. In [Tha00] Thaler introduced for this type of maps some extra conditions (like the convexity of \( T \) in a neighbourhood of 0 and some regularity of the density) which in our context guarantee that any set \( A \in B_{[0,1]} \) with positive distance from the indifferent fixed point 0 and \( \lambda(A) > 0 \) is uniformly returning.

Whenever a map allows an absolutely continuous invariant infinite measure but does not satisfy Thaler’s condition in [Tha00] the following proposition will prove to be helpful (cf. Section 3).

**Proposition 2.6.** Let \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \) be a uniform set for \( f \). If the wandering rate \((W_n)\) is regularly varying with exponent \( 1 - \alpha \) for \( \alpha \in (0, 1] \) and the sequence \((\hat{T}^n(f)|_A)\) is decreasing, then \( A \) is a uniformly returning set for \( f \). In this case,

\[
W_n \hat{T}^n(f) \longrightarrow \frac{1}{\Gamma(\alpha) \Gamma(2 - \alpha)} \quad \mu \text{-a.e. uniformly on } A.
\]
Proof. Let $\lambda, \eta \in \mathbb{R}$ be fixed but arbitrary with $0 < \lambda < \eta < \infty$. Putting

$$V_n := \sum_{k=0}^{n} \hat{T}^k (f),$$

we have by monotonicity of $(\hat{T}^n (f) \mid A)$

$$\frac{\hat{T}^{[n \eta]} (f)}{V_n} \cdot ([n \eta] - [n \lambda]) \leq \frac{V_{[n \eta]} - V_{[n \lambda]}}{V_n} \leq \frac{\hat{T}^{[n \lambda]} (f)}{V_n} \cdot ([n \eta] - [n \lambda]).$$

Since $[n \eta] - [n \lambda] \sim n (\eta - \lambda)$ as $n \to \infty$, we have for fixed $\varepsilon \in (0, 1)$ and all $n$ sufficiently large

$$n (1 - \varepsilon) (\eta - \lambda) \leq [n \eta] - [n \lambda] \leq n (1 + \varepsilon) (\eta - \lambda).$$

This implies for all $n$ sufficiently large

$$\frac{n \hat{T}^{[n \eta]} (f)}{V_n} \cdot (1 - \varepsilon) (\eta - \lambda) \leq \frac{V_{[n \eta]} - V_{[n \lambda]}}{V_n} \leq \frac{n \hat{T}^{[n \lambda]} (f)}{V_n} \cdot (1 + \varepsilon) (\eta - \lambda).$$

Since

$$\frac{V_{[n \eta]} - V_{[n \lambda]}}{V_n} \to \eta^\alpha - \lambda^\alpha \quad \text{as} \quad n \to \infty \quad \mu\text{-a.e. uniformly on } A,$$

we obtain on the one hand

$$\frac{1}{1 + \varepsilon} \cdot \frac{\eta^\alpha - \lambda^\alpha}{\eta - \lambda} \leq \liminf \frac{n \hat{T}^{[n \lambda]} (f)}{V_n} \quad \mu\text{-a.e. uniformly on } A.$$

Letting $\eta \to \lambda$ and $\varepsilon \to 0$, it follows that

$$\alpha \lambda^{\alpha - 1} \leq \liminf \frac{n \hat{T}^{[n \lambda]} (f)}{V_n} \quad \mu\text{-a.e. uniformly on } A.$$

On the other hand, we obtain similarly

$$\limsup \frac{n \hat{T}^{[n \eta]} (f)}{V_n} \leq \alpha \eta^{\alpha - 1} \quad \mu\text{-a.e. uniformly on } A.$$

Since $\lambda$ and $\eta$ are arbitrary, we have for arbitrary $c > 0$

$$\frac{n \hat{T}^{[nc]} (f)}{V_n} \to \alpha c^{\alpha - 1} \quad \mu\text{-a.e. uniformly on } A.$$

Finally using $V_{[nc]} \sim c^{\alpha} V_n \quad \mu\text{-a.e. uniformly on } A$ and $[nc] \sim cn$, we obtain for $m = [nc]$

$$\frac{n \hat{T}^{m} (f)}{V_m} = \frac{[nc]}{n} \cdot \frac{\hat{T}^{[nc]} (f)}{V_{[nc]}} \cdot \frac{V_n}{V_{[nc]}} \to \alpha \quad \mu\text{-a.e. uniformly on } A.$$

From this and (2.1) the assertion follows. \qed
2.2. Limit laws. An important question when studying convergence in distribution for processes defined in terms of a non-singular transformation is to what extent the limiting behaviour depends on the initial distribution. This is formalised as follows.

Let \((R_n)_{n \geq 1}\) be a sequence of real valued random variables on the \(\sigma\)-finite measure space \((X, A, \mu)\) and \(R\) some random variable with values in \([-\infty, \infty]\). Then strong distributional convergence of \((R_n)_{n \geq 1}\) to \(R\) abbreviated by \(R_n \xrightarrow{\mathcal{L}(\mu)} R\) means that \(R_n \xrightarrow{\nu} R\) holds for all \(\nu \in \mathcal{P}_{\mu}\). In particular for \(c \in [-\infty, \infty]\),

\[
R_n \xrightarrow{\mathcal{L}(\mu)} c \iff R_n \xrightarrow{\mu} c.
\]

Now we are in the position to connect a certain renewal process with the processes we are investigating in this paper and state the corresponding limit laws.

Let \(A \subseteq A\) be a set with \(0 < \mu(A) < \infty\), and let \((\tau_n)_{n \in \mathbb{N}}\) be the sequence of return times, i.e. integer valued positive random variables defined recursively by

\[
\begin{align*}
\tau_1(x) &:= \varphi(x) = \inf \{ p \geq 1 : T^p(x) \in A \}, \quad x \in X, \\
\tau_n(x) &:= \inf \{ p \geq 1 : T^{p+\sum_{k=1}^{n-1} \tau_k(x)}(x) \in A \}, \quad x \in X.
\end{align*}
\]

The renewal process is then given by

\[
N_n(x) := \begin{cases} 
\max \{ k \leq n : S_k(x) \leq n \}, & x \in K_n = \bigcup_{k=0}^n T^{-k}A, \\
0, & \text{else},
\end{cases}
\]

where

\[
S_0 := 0, \quad S_n := \sum_{k=1}^{n} \tau_k, \quad n \in \mathbb{N}.
\]

Now we consider the so-called spent time process \(\sigma_n\) given by

\[
\sigma_n(x) := \begin{cases} 
\frac{n - S_{N_n(x)}(x)}{n}, & x \in K_n, \\
0, & \text{else},
\end{cases}
\]

and the normalised spent time Kac process

\[
\Psi_n := \frac{\sum_{k=0}^{\sigma_n} \mu(A \cap \{ \varphi > k \})}{\mu(K_n)} = \frac{W_{\sigma_n}}{W_n}.
\]

Note that

\[
S_{N_n(x)}(x) = Z_n(x) := \begin{cases} 
\max \{ k \leq n : T^k(x) \in A \}, & x \in K_n, \\
0, & \text{else}.
\end{cases}
\] (2.2)

We will make use of the following result from [KS09].

\textbf{(UL) Uniform law:} Let \(A \subseteq A\) with \(0 < \mu(A) < \infty\) be a uniformly returning set. If the wandering rate \((W_n)\) is slowly varying, then we have

\[
\Psi_n \xrightarrow{\mathcal{L}(\mu)} U,
\]

where the random variable \(U\) is distributed uniformly on \([0, 1]\).
Example 2.7. Let us consider the Lasota–Yorke map \( T : [0, 1] \to [0, 1] \), defined by
\[
T(x) := \begin{cases} \frac{1}{2x}, & x \in [0, \frac{1}{2}], \\ 2x - 1, & x \in \left(\frac{1}{2}, 1\right]. \end{cases}
\]
This map satisfies Thaler’s conditions (i)–(iv) in [Tha00]. Any compact subset \( A \) of \( (0, 1] \) with \( \lambda (A) > 0 \) is a uniformly returning set and we have
\[
W_n \sim \log (n) \quad \text{as} \quad n \to \infty.
\]
Hence,
\[
\frac{\log (\sigma_n)}{\log (n)} \xrightarrow{\mathcal{L}(\mu)} U.
\]

3. Application to continued fractions

In this section we will make use of the fact that one can receive the Gauss map from the Farey map by inducing. This allows us to connect the renewal process for the Farey map and the fluctuation process for the continued fraction digit sum. Unfortunately the Farey map – unlike the Lasota–Yorke map – does not satisfy Thaler’s condition (i)–(iv) in [Tha00] forcing us to study this map in some detail in order to show that the interval \( (1/2, 1] \) is uniformly returning.

3.1. The Farey and Gauss map. We consider the Farey map \( T : [0, 1] \to [0, 1] \), defined by
\[
T(x) := \begin{cases} T_0(x), & x \in [0, \frac{1}{2}], \\ T_1(x), & x \in \left[\frac{1}{2}, 1\right]. \end{cases}
\]
where
\[
T_0(x) := \frac{x}{1-x} \quad \text{and} \quad T_1(x) := \frac{1}{x - 1}.
\]

With \( B(0) = \left[0, \frac{1}{2}\right], B(1) = \left(\frac{1}{2}, 1\right], \) and \( J = \{0\}, \) it is not difficult to see that Thaler’s condition from Example 2.2 are fulfilled. It is easy to verify that with \( h(x) := \frac{\partial T}{\partial x}(x) = \frac{1}{x} \) we have \( T(1) = 1 \) and hence \((0, 1], T, B, \mu)\) defines a conservative ergodic measure preserving dynamical system. Also any Borel set \( A \in B \) with \( \lambda (A) > 0 \) which is bounded away from the indifferent fixed point 0 is a uniform set. Furthermore, we have
\[
W_n = \int_{x=0}^{1} \frac{1}{x} \, dx = \log (n + 2) \sim \log (n) \quad (n \to \infty).
\]
The inverse branches of the Farey map are
\[
u_0(x) := (T_0)^{-1}(x) = \frac{x}{1 + x},
\]
\[
u_1(x) := (T_1)^{-1}(x) = \frac{1}{1 + x}.
\]
For \( x \neq 0 \) the map \( \nu_0(x) \) is conjugated to the right translation \( x \mapsto F(x) := x + 1, \) i.e.
\[
u_0 = J \circ F \circ J \quad \text{with} \quad J(x) = J^{-1}(x) = \frac{1}{x}.
\]
This shows that for the $n$-th iterate we have
\[ u_0^n(x) = J \circ F^n \circ J(x) = \frac{x}{1 + nx}. \] (3.1)

Moreover, we have $u_1(x) = J \circ F(x)$.

Let $\mathcal{F} = \{ A_n \}_{n \geq 1}$ be the countable collection of pairwise disjoint subintervals of $[0, 1]$ given by $A_n = \left( \frac{1}{n+1}, \frac{1}{n} \right]$. Setting $A_0 = [0, 1)$, it is easy to check that $T(A_n) = A_{n-1}$ for all $n \geq 1$. The first entry time $e : \mathbb{I} \to \mathbb{N}$ in the interval $A_1$ is defined as
\[ e(x) := \min \{ k \geq 0 : T^k(x) \in A_1 \}. \]

Then the first entry time is connected to the first digit in the continued fraction expansion by
\[ a_1(x) = 1 + e(x) \quad \text{and} \quad \varphi(x) = a_1 \circ T(x), \quad x \in \mathbb{I}. \]

We now consider the induced map $S : \mathbb{I} \to \mathbb{I}$ defined by
\[ S(x) := T^{e(x)+1}(x). \]

Since for all $n \geq 1$
\[ \{ x \in \mathbb{I} : e(x) = n - 1 \} = A_n \cap \mathbb{I}, \]
we have by (3.1) for any $x \in A_n \cap \mathbb{I}$
\[ S(x) = T^n(x) = T_1 \circ T_0^{n-1}(x) = \frac{1}{x} - n = \frac{1}{x} - a_1(x). \]

This implies that the induced transformation $S$ coincides with Gauss map $G$ on $\mathbb{I}$.

In the next lemma we connect the number theoretical process $X_n$ defined in (1.1) with the renewal process $Z_n$ with respect to the Farey map defined in (2.2).
Lemma 3.1. Let $A_1 := \left( \frac{1}{2}, 1 \right]$ and $K_n := \bigcup_{k=0}^{n} T^{-k} A_1$. Then for the process $X_n$ defined in (1.1) we have for all $x \in \mathbb{I}$

$$X_n (x) = \begin{cases} 
1 + Z_{n-1}, & x \in K_{n-1}, \\
0, & \text{else.}
\end{cases}$$

Proof. From the above discussion and the definition of the renewal theoretic process $S_N n$ in Subsection 2.2 we deduce

- for $x \in \mathbb{I} \cap K_{n-1}^C$ we have $a_1 (x) > n$ implying $X_n (x) = 0$.
- for $x \in \mathbb{I} \cap K_{n-1}$ we distinguish two cases. Either the process starts in $x \in A_1$, then we have $a_1 (x) = 1$ and inductively for $n \geq 2$

$$a_n (x) = \tau_{n-1} (x),$$

or the process starts in $x \in A_1^C$, then we have $a_1 (x) = 1 + \tau_1 (x)$ and inductively for $n \geq 2$

$$a_n (x) = \tau_n (x).$$

From this the assertion follows.

To show that $A_1$ is uniformly returning we need the following lemma.

Lemma 3.2. Let

$$\mathcal{D} := \{ f \in \mathcal{P}_\mu : f \in C^2 ((0, 1)) \text{ with } f' > 0 \text{ and } f'' \leq 0 \}.$$ Then we have for all $n \in \mathbb{N}$

$$f \in \mathcal{D} \implies \hat{T}^n (f) \in \mathcal{D}.$$

Proof. First note, that it suffices to show the assertion only for $n = 1$ since the lemma then follows by induction. For $f \in \mathcal{D}$ we have

$$\hat{T} (f) = \frac{1}{h} \cdot \hat{P} (h \cdot f),$$

where $\hat{P}$ denotes the Perron-Frobenius operator of $T$ restricted to $L_1 (\lambda)$. Using the inverse branches of $T$ this operator is given by

$$\hat{P} (g) = g \circ u_0 \cdot |u'_0| + g \circ u_1 \cdot |u'_1|, \quad \text{for all } g \in L_1 (\lambda).$$

It follows that

$$\hat{T} (f) (x) = \frac{f \left( \frac{x}{x+1} \right) + xf \left( \frac{1}{x+1} \right)}{x + 1}, \quad \text{for all } x \in [0, 1].$$

Hence, also $\hat{T} (f)$ is differentiable on $(0, 1)$ and by the monotonicity of $f$ and $f'$ we have

$$\hat{T} (f)' (x) = \frac{f' \left( \frac{x}{x+1} \right) - xf' \left( \frac{1}{x+1} \right)}{(x + 1)^3} + \frac{f \left( \frac{1}{x+1} \right) - f \left( \frac{x}{x+1} \right)}{(x + 1)^2} \geq 0,$$

implying $\hat{T} (f)' > 0$. 

Furthermore, an easy calculation shows

\[
\begin{align*}
\hat{T}(f)(x) &= f''\left(\frac{x}{x+1}\right) + xf''\left(\frac{1}{x+1}\right) + \frac{2}{x+1} \left(f\left(\frac{x}{x+1}\right) - f\left(\frac{1}{x+1}\right)\right) \\
&\quad + \frac{2(x+1)}{(x+1)^3} \left((x-1)f'\left(\frac{1}{x+1}\right) - 2f'\left(\frac{x}{x+1}\right)\right) \leq 0.
\end{align*}
\]

This finishes the proof.

We remark that \(D \neq \emptyset\), since \(\text{id}_{[1,0]} \in D\).

**Lemma 3.3.** The set \(A_1 = \left(\frac{1}{2}, 1\right]\) is uniformly returning for any \(f \in D\).

**Proof.** Since \(A_1\) is uniform for any \(f \in P_\alpha\) (cf. Example 2.2) we have in view of Proposition 2.6 only to verify that \(\hat{T}^n(f)\mid_{A_1}\) is decreasing. For all \(x \in A_1, n \in \mathbb{N}_0\), we have

\[
\begin{align*}
\hat{T}^{n+1}(f)(x) &= \hat{T}^n(f)\left(\frac{x}{x+1}\right) + x\hat{T}^n(f)\left(\frac{1}{x+1}\right) \\
&= \frac{1}{x+1} \hat{T}^n(f)\left(\frac{x}{x+1}\right) + \frac{x}{x+1} \hat{T}^n(f)\left(\frac{1}{x+1}\right)
\end{align*}
\]

Since by Lemma 3.2 each \(\hat{T}^n(f)\) is concave and increasing on \([0, 1]\) and \(\frac{1}{x+1} + \frac{x}{x+1} = 1\), we have for all \(x \geq \sqrt{2} - 1\),

\[
\hat{T}^{n+1}(f)(x) \leq \hat{T}^n(f)\left(\frac{2x}{x+1}\right) \leq \hat{T}^n(f)(x).
\]

This finishes the proof.

### 3.2. Distributional limit laws for digit sums.

With the above preparations we can now state the main results.

**Theorem 3.4.** Let \(X_n\) be the process given in (1.1). Then the following holds.

1. We have

\[
\frac{\log(n - X_n)}{\log(n)} \xrightarrow{\mathcal{L}(\mu)} U,
\]

where the random variable \(U\) is distributed uniformly on \([0, 1]\).

2. For \(f \in D\) set \(d\nu := f\,d\mu\). Then for any \(a \in (0, 1)\) we have

\[
\nu\left(\frac{n - X_n}{n} > a\right) \sim -\log\left(\frac{a}{\log(n)}\right) \quad \text{as} \quad n \to \infty.
\]

**Proof.** Using Lemma 3.3 and the fact that \(W_n \sim \log(n)\) we have by (UL)

\[
\frac{W_{n-X_n}}{\log n} \xrightarrow{\mathcal{L}(\mu)} U.
\]
Consequently $n - Z_n \xrightarrow{\mathcal{L}(\mu)} \infty$ and hence
\[ \frac{W_{n-Z_n}}{\log (n-Z_n)} \rightarrow 1. \]
Thus by Lemma 3.1, the convergence in (3.2) holds.

For the second part of the theorem let $f \in \mathcal{D}$ and $a \in (0, 1)$ be fixed and set $d\nu := f \, d\mu$. It is not difficult to verify that
\[ \nu \left( \frac{n-Z_n}{n} > a \right) \sim \nu \left( \frac{n-X_n}{n} > a \right) \quad \text{as} \quad n \to \infty. \]

Therefore, to prove (3.3) it suffices to show
\[ \nu \left( \frac{n-Z_n}{n} > a \right) \sim \frac{-\log (a)}{\log (n)} \quad \text{as} \quad n \to \infty. \]

In fact, we have
\[ \nu \left( \frac{n-Z_n}{n} > a \right) = \sum_{k=0}^{\lfloor n(1-a) \rfloor} \nu \left( K_n \cap \{ Z_n = k \} \right) \]
\[ = \sum_{k=0}^{\lfloor n(1-a) \rfloor} \nu \left( T^{-k} \left( A_1 \cap \{ \varphi > n-k \} \right) \right) \]
\[ = \sum_{k=0}^{\lfloor n(1-a) \rfloor} \int_{A_1} 1_{A_1 \cap \{ \varphi > n-k \}} \hat{T}^k (f) \, d\mu. \]

Let $\delta \in (0, 1-a)$ and $\varepsilon \in (0, 1)$ be fixed but arbitrary and divide the above sum into two parts as follows.
\[ \nu \left( \frac{n-Z_n}{n} > a \right) = \sum_{k=0}^{\lfloor n\delta \rfloor - 1} \cdots + \sum_{k=\lfloor n\delta \rfloor}^{\lfloor n(1-a) \rfloor} \cdots =: I (n) + J (n). \]

We first note, that since $A_1 \cap \{ \varphi > n \} = \left[ \frac{n+\varepsilon}{n+\varepsilon+1}, 1 \right]$ we have
\[ \mu \left( A_1 \cap \{ \varphi > n \} \right) = \int_{n+\varepsilon+1}^{n+\varepsilon+2} \frac{1}{x} \, dx \sim \frac{1}{n} \quad \text{as} \quad n \to \infty. \quad (3.4) \]

Also, by monotonicity of $\left( 1_{A_1 \cap \{ \varphi > n \}} \right)$ we have
\[ I (n) \leq \int_{A_1} 1_{A_1 \cap \{ \varphi > n+1-[n\delta] \}} \sum_{k=0}^{[n\delta]-1} \hat{T}^k (f) \, d\mu. \]

Combining both observation and using the fact that $A_1$ is uniform for $f$ (cf. Example 2.2) and that (2.1) holds, we obtain for sufficiently large $n$
\[ I (n) \leq (1 + \varepsilon)^2 \left[ \frac{[n\delta]-1}{n-[n\delta]+1} \cdot \frac{1}{\log ([n\delta]-1)} \right. \]
\[ \sim (1 + \varepsilon)^2 \left[ \frac{\delta}{1-\delta} \cdot \frac{1}{\log (n)} \right. \quad \text{as} \quad n \to \infty. \]

Thus,
\[ \limsup_{n \to \infty} \log (n) \cdot I (n) \leq (1 + \varepsilon)^3 \frac{\delta}{1-\delta}. \]
Letting $\delta \to 0$, we conclude
\[ I(n) = o \left( \frac{1}{\log(n)} \right), \quad n \to \infty. \quad (3.5) \]

For the second part of the sum we have to show that
\[ J(n) \sim -\frac{\log(a)}{\log(n)} \quad \text{as} \quad n \to \infty. \quad (3.6) \]

A similarly argument as in [KS09, Lemma 2.3] shows that for all $n$ sufficiently large and $k \in ([n\delta], [n(1-a)])$ we have uniformly on $A_1$
\[ (1 + \varepsilon) \frac{1}{\log(n)} \leq \tilde{T}^k(f) \leq (1 + \varepsilon)^2 \frac{1}{\log(n)}. \quad (3.7) \]

Hence, using the right-hand side of (3.7) and the asymptotic formula (3.4), we obtain for $n$ sufficiently large
\[ J(n) \leq \frac{(1 + \varepsilon)^2}{\log(n)} \sum_{k=n^{-\lfloor n\delta \rfloor}}^{n-\lfloor n\delta \rfloor} \mu(A_1 \cap \{ \varphi > k \}) \]
\[ \sim \frac{(1 + \varepsilon)^3}{\log(n)} \cdot \log \left( \frac{1 - \delta}{a} \right) \quad \text{as} \quad n \to \infty, \]
which implies
\[ \limsup_{n \to \infty} \log(n) \cdot J(n) \leq (1 + \varepsilon)^4 \log \left( \frac{1 - \delta}{a} \right). \]

Similarly, using the left-hand side of (3.7), we get
\[ \liminf_{n \to \infty} \log(n) \cdot J(n) \geq (1 + \varepsilon)^3 \log \left( \frac{1 - \delta}{a} \right). \]

Since $\varepsilon$ and $\delta$ were arbitrary, (3.6) holds.

Combining (3.5) and (3.6) proves the second part of the theorem. \[ \square \]

Finally, since $\text{id}_{[0,1]} \in D$, Theorem 3.4 in particular implies the claims stated in the introduction. Also note that since $\lambda \sim \mu$, by (1.3) we have
\[ \frac{n - X_n}{n} \xrightarrow{\mathcal{L}(\mu)} 0, \]
and it follows directly from (3.2) that
\[ n - X_n \xrightarrow{\mathcal{L}(\mu)} \infty. \]

REFERENCES


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