Proof. Suppose z_1 and z_2 are on the same side of z_3z_4. The four points are concyclic if the counter clockwise angles of rotation from z_1z_3 to z_1z_4 and from z_2z_3 to z_2z_4 are equal. In this case, the ratio

$$\frac{z_4 - z_1}{z_3 - z_1} \Big/ \frac{z_4 - z_2}{z_3 - z_2}$$

of the complex numbers is real, (and indeed positive).

On the other hand, if z_1, z_2 are on opposite sides of z_3z_4, the two angles differ by π, and the cross ratio is a negative real number.

6.6 Construction of the regular 17-gon

6.6.1 Gauss' analysis

Suppose a regular 17−gon has center $0 \in \mathbb{C}$ and one vertex represented by the complex number 1. Then the remaining 16 vertices are the roots of the equation

$$\frac{x^{17} - 1}{x - 1} = x^{16} + x^{15} + \cdots + x + 1 = 0.$$

If ω is one of these 16 roots, then these 16 roots are precisely $\omega, \omega^2, \ldots, \omega^{15}, \omega^{16}$. (Note that $\omega^{17} = 1$.) Geometrically, if A_0, A_1 are two distinct vertices of a regular 17−gon, then successively marking vertices $A_2, A_3, \ldots, A_{16}$ with

$$A_0A_1 = A_1A_2 = \ldots = A_{14}A_{15} = A_{15}A_{16},$$

we obtain all 17 vertices. If we write $\omega = \cos\theta + i\sin\theta$, then $\omega + \omega^{16} = 2\cos\theta$. It follows that the regular 17−gon can be constructed if one can construct the number $\omega+\omega^{16}$. Gauss observed that the 16 complex numbers $\omega^k, k = 1, 2, \ldots, 16$, can be separated into two "groups" of eight, each with a sum constructible using only ruler and compass. This is decisively the hardest step. But once this is done, two more applications of the same idea eventually isolate $\omega+\omega^{16}$ as a constructible number, thereby completing the task of construction. The key idea involves the very simple fact that if the coefficients a and b of a quadratic equation $x^2 - ax + b = 0$ are constructible, then so are its roots x_1 and x_2. Note that $x_1 + x_2 = a$ and $x_1x_2 = b$.

Gauss observed that, modulo 17, the first 16 powers of 3 form a permutation of the numbers 1, 2, ..., 16:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3^k	1	3	9	10	13	5	15	11	16	14	8	7	4	12	2	6

Let

$$\begin{aligned} y_1 &= \omega+\omega^9+\omega^{13}+\omega^{15}+\omega^{16}+\omega^8+\omega^4+\omega^2, \\ y_2 &= \omega^3+\omega^{10}+\omega^5+\omega^{11}+\omega^{14}+\omega^7+\omega^{12}+\omega^6. \end{aligned}$$

Note that

$$y_1+y_2=\omega+\omega^2+\cdots+\omega^{16}=-1.$$

Most crucial, however, is the fact that the product y_1y_2 does *not* depend on the choice of ω. We multiply these directly, but adopt a convenient bookkeeping below. Below each power ω^k, we enter a number j (from 1 to 8 meaning that ω^k can be obtained by multiplying the jth term of y_1 by an appropriate term of y_2 (unspecified in the table but easy to determine):

ω	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}	ω^{15}	ω^{16}
3	2	2	1	4	1	1	1	4	3	1	1	1	2	1	2
4	3	3	2	5	2	3	3	5	4	5	2	5	6	2	3
6	5	4	4	6	3	7	4	7	5	6	4	6	7	6	7
7	6	6	5	8	5	8	8	8	7	7	8	8	8	7	8

From this it is clear that

$$y_1y_2=4(\omega+\omega^2+\cdots+\omega^{16})=-4.$$

It follows that y_1 and y_2 are the roots of the quadratic equation

$$y^2+y-4=0,$$

and are constructible. We may take

$$y_1=\frac{-1+\sqrt{17}}{2}, \qquad y_2=\frac{-1-\sqrt{17}}{2}.$$

Now separate the terms of y_1 into two "groups" of four, namely,

$$z_1=\omega+\omega^{13}+\omega^{16}+\omega^4, \qquad z_2=\omega^9+\omega^{15}+\omega^8+\omega^2.$$

Clearly, $z_1+z_2=y_1$. Also,

$$z_1z_2=(\omega+\omega^{13}+\omega^{16}+\omega^4)(\omega^9+\omega^{15}+\omega^8+\omega^2)=\omega+\omega^2+\cdots+\omega^{16}=-1.$$

It follows that z_1 and z_2 are the roots of the quadratic equation

$$z^2-y_1z-1=0,$$

and are constructible, since y_1 is constructible. Similarly, if we write

$$z_3 = \omega^3 + \omega^5 + \omega^{14} + \omega^{12}, \qquad z_4 = \omega^{10} + \omega^{11} + \omega^7 + \omega^6,$$

we find that $z_3 + z_4 = y_2$, and $z_3 z_4 = \omega + \omega^2 + \cdots + \omega^{16} = -1$, so that z_3 and z_4 are the roots of the quadratic equation

$$z^2 - y_2 z - 1 = 0$$

and are also constructible.

Finally, further separating the terms of z_1 into two pairs, by putting

$$t_1 = \omega + \omega^{16}, \qquad t_2 = \omega^{13} + \omega^4,$$

we obtain

$$\begin{aligned} t_1 + t_2 &= z_1, \\ t_1 t_2 &= (\omega + \omega^{16})(\omega^{13} + \omega^4) = \omega^{14} + \omega^5 + \omega^{12} + \omega^3 = z_3. \end{aligned}$$

It follows that t_1 and t_2 are the roots of the quadratic equation

$$t^2 - z_1 t + z_3 = 0,$$

and are constructible, since z_1 and z_3 are constructible.

6.6.2 Explicit construction of a regular 17-gon [4]

To construct two vertices of the regular 17-gon inscribed in a given circle $O(A)$.

1. On the radius OB perpendicular to OA, mark a point J such that $OJ = \frac{1}{4}OA$.

2. Mark a point E on the segment OA such that $\angle OJE = \frac{1}{4}\angle OJA$.

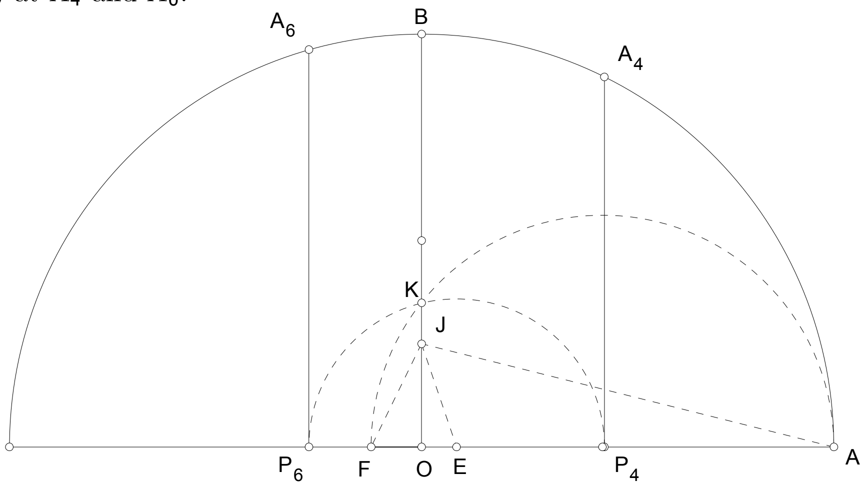
3. Mark a point F on the diameter through A such that O is between E and F and $\angle EJF = 45^\circ$.

4. With AF as diameter, construct a circle intersecting the radius OB at K.

[4] H.S.M.Coxeter, Introduction to Geometry, 2nd ed. p.27.

5. Mark the intersections of the circle $E(K)$ with the diameter of $O(A)$ through A. Label the one between O and A points P_4, and the other and P_6.

6. Construct the perpendicular through P_4 and P_6 to intersect the circle $O(A)$ at A_4 and A_6. [5]



Then A_4, A_6 are two vertices of a regular 17-gon inscribed in $O(A)$. The polygon can be completed by successively laying off arcs equal to A_4A_6, leading to A_8, A_{10}, $\dots A_{16}$, $A_1 = A$, A_3, A_5, $\dots$, A_{15}, A_{17}, A_2.

[5]Note that P_4 is not the midpoint of AF.