Algebraic and geometric spread in finite frames

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ABSTRACT

When searching for finite unit norm tight frames (FUNTFs) of M vectors in $\mathbb{F}^N$ which yield robust representations, one is concerned with finding frames consisting of frame vectors which are in some sense as spread apart as possible. Algebraic spread and geometric spread are the two most commonly used measures of spread. A frame with optimal algebraic spread is called full spark and is such that any subcollection of N frame vectors is a basis for $\mathbb{F}^N$. A Grassmannian frame is a FUNTF which satisfies the Grassmannian packing problem; that is, the frame vectors are optimally geometrically spread given fixed M and N. A particular example of a Grassmannian frame is an equiangular frame, which is such that the absolute value of all inner products of distinct vectors is equal. The relationship between these two types of optimal spread is complicated. The folk knowledge for many years was that equiangular frames were full spark; however, this is now known not to hold for an infinite class of equiangular frames. The exact relationship between these types of spread will be further explored in this talk, as well as Plücker coordinates and coherence, which are measures of how much a frame misses being optimally algebraically or geometrically spread.

Keywords: finite frames, matroids, coherence, spark, Grassmannian frames, equiangular frames, Plücker coordinates

1. INTRODUCTION

Frames are a generalization of orthonormal bases which allows redundancy. They were first introduced in[1] and were brought to the forefront during the rise of wavelets. Due to their ability to analyze and synthesize data in a manner very similar to orthonormal bases while also encoding data in a way that protects against erasures, frames are used in a wide variety of applications like coding[2] and digital signal processing.[3] For an overview of the theory and applications of frames, see.[4,5]

Specifically, a collection $\Phi = \{\varphi_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* when there exist bounds $0 < A \leq B < \infty$ such that for all $x \in \mathcal{H}$,

$$A \|x\|^2 \leq \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \leq B \|x\|^2 . \tag{1}$$

When $\mathcal{H}$ is finite dimensional, like $\mathcal{H} = \mathbb{F}^N$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then we call Φ a *finite frame.* In this paper, we will only consider the finite case, where $I = [M] := \{1, 2, \ldots, M\}$ for some $M \geq N$. Equation 1 is a generalization of Parseval's equality, which all orthonormal bases satisfy. It may be surprising that even when $M > N$, Parseval's equality, or a scaled version of it, can still hold. Specifically, if there exists $A > 0$ such that for all $x \in \mathbb{F}^N$,

$$A \|x\|^2 = \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 , \tag{2}$$

then we call Φ a *tight frame* and A the *frame bound.* If Φ is a tight frame with frame bound 1, then we call it a *Parseval frame* since it satisfies Parseval's equality, Equation 2 is equivalent to the statement that for all $x \in \mathbb{F}^N$,

$$\sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i = Ax. \tag{3}$$

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Note that when $M < N$ the representation of a given x in Equation 3 is not unique. When each frame vector has unit length, we say that the frame is *unit norm*. We will mainly be concerned with finite unit norm tight frames, or *FUNTFs*. By scaling each vector of a FUNTF by $1/\sqrt{A}$, we may easily convert a FUNTF to a finite equal norm Parseval frame. So, we think of such classes of frames as being equivalent. If $\Phi = \{\varphi_i\}_{i=1}^M$ is a frame (resp., FUNTF) for $\mathbb{F}^N$, we will call it an (M, N) *frame* (resp., (M, N)-*FUNTF*). We will denote the set of (M, N)-FUNTFs as $\mathcal{F}(M, N)$. If the underlying field is important, then we denote it with a subscript $\mathcal{F}_{\mathbb{F}}(M, N)$. Frames have four important related operators, all of which are built from the *synthesis operator*. If Φ is an (M, N)-frame, then the synthesis operator is the matrix

$$\left(\begin{array}{c|c|c|c} \varphi_1 & \varphi_2 & \cdots & \varphi_M \end{array} \right).$$

By a slight abuse of notation, we will also use Φ to denote the synthesis operator. The *analysis operator* is the adjoint of the synthesis operator. For any $x \in \mathbb{F}^N$, we call $\Phi^T x \in \mathbb{F}^M$ the *analysis coefficients*. The *frame operator* is formed as $S = \Phi\Phi^*$. When Φ is a tight frame, then it follows from Equation 3 that

$$Sx = \Phi\Phi^* x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i = Ax.$$

The last operator of interest is the *Gram operator* $G = \Phi^*\Phi$, which has as entries the inner products of the frame vectors. Namely, $G_{i,j} = \langle \varphi_j, \varphi_i \rangle$. If Φ is a FUNTF, then

$$M = \sum_{i=1}^{M} \langle \varphi_i, \varphi_i \rangle = \operatorname{trace}(G) = \operatorname{trace}(\Phi^*\Phi) = \operatorname{trace}(\Phi\Phi^*) = \operatorname{trace}(S) = \operatorname{trace}(AI_N) = AN;$$

thus, $A = M/N$.

The question we pose is which frames are in some sense optimal for applications. Goyal *et al.* proved that an (M, N)-unit norm frame is optimally robust against noise and one erasure if the frame is a FUNTF.[6]

We consider a very simple example of wanting to transmit the vector $(2, 3)^T \in \mathbb{R}^2$ to someone else. Our first approach is to represent the data with respect to the canonical orthonormal basis of $\mathbb{R}^2$ and send 2 followed by 3. If both coefficients are received at the other end, then we have succeeded. However, if one of the coefficients is dropped along the way, say the 2, then we have lost an entire dimension of information and cannot recreate or even approximate our original vector in a meaningful manner. A very naive solution to this problem would be to send the sequence 2, 2, 3 instead. If we lose one of the first two coefficients, then we can still reconstruct the original vector, but if we lose the third coefficient, then we have again lost a dimension of information. Note that encoding $(2, 3)^T$ as $\{2, 2, 3\}$ is the same as mapping it to its analysis coefficients under

$$\Omega = \{\omega_1, \omega_2, \omega_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Ω is a finite unit norm frame but not a FUNTF; however, we will continue to use this example of a bad way to add redundancy as motivation throughout the paper. The different measures of algebraic spread will be presented in Section 2, followed by a discussion of geometric spread in Section 3. Finally, their relationship will be explored in Section 4.

2. ALGEBRAIC SPREAD

One way to generalize what went wrong with our toy example is that noting that since $\omega_1 = \omega_2$, Ω contains a set of two linearly dependent vectors. That is, Ω is a frame for $\mathbb{R}^2$, but not every subset of 2 vectors of Ω is a basis of $\mathbb{R}^2$. This leads us to the following definition, which was first introduced by.[7]

Definition 1. Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathbb{F}^N$. Then the *spark* of Φ is

$$\operatorname{spark}(\Phi) = \min \# \left\{ \{\varphi_{i_j}\}_j \subset \Phi : \{\varphi_{i_j}\}_j \text{ linearly dependent} \right\}.$$

If $\operatorname{spark}(\Phi) = N + 1$, we say that Φ is *full spark*.

Thus the spark of a frame is the size of the smallest set of linearly dependent vectors. If Φ is full spark, then then any subset of N vectors of Φ is a basis for $\mathbb{F}^N$. By thinking of Φ as the synthesis matrix, we can also speak of the spark of a matrix, which is the spark of the columns. The rank of a matrix is a best-case measure, where "good" here is linear independence, while the spark of a matrix is a worst-case measure. Since the largest set of linearly independent columns has size equal to the rank of Φ, $\operatorname{spark}\Phi \leq \operatorname{rank}\Phi + 1$, but we cannot tighten this relationship. To illustrate this, we will construct two different $N \times (N+1)$ matrices with rank N, one which has spark 1 and one which has spark $N+1$. We start with a unitary matrix U which has rank N. If we append the zero vector to U to obtain $(U|0)$, then the rank is still N, but the spark is 1, since a zero vector is dependent in itself. If instead, we append $\overline{u}$, the average of the (orthonormal) columns of U, then $(U|\overline{u})$ has rank N but spark $N+1$, since $\ker(U|\overline{u}) = \operatorname{span}(1,1,\ldots,1,-N)$. In other words, every non-zero element in the kernel of $(U|\overline{u})$ has $N+1$ non-zero elements. Spark was first introduced in the context of sparse coding and compressive sensing (for a discussion of the history, see[8, 9]). The base problem is an inverse problem of the form

$$\min \|x\|_0, \quad \text{s.t. } Ax = y,$$

where $\|x\|_0 = \{x_i \neq 0\}$. The spark of A tell us whether sparse x, that is, x with small $\|x\|_0$, lie in the kernel of A.

Spark actually has existed with a different name in matroid theory for a long time. The first publication in frame theory to make the connection to matroids is possibly,[10] although the author was at that time unaware of the term spark, calling full spark frames *generic frames*. Full spark frames are called *maximally robust to erasures* in[11] because a full spark (M,N)-frame has the property that for any $x \in \mathbb{F}^N$, if $M-N$ analysis coefficients $\Phi^* x$ are lost due to erasures, then x is still reconstructable.

Matroids abstractly codify certain traits arising in both linear algebra and graph theory. A standard reference for matroid theory is.[12] We include graph theory examples as motivation without defining the terms.

Definition 2. A *matroid* is an ordered pair $([M], \mathcal{B})$ where $\mathcal{B} \subset \mathcal{P}([M])$ and $\mathcal{P}([M])$ is the power set of $[M]$, which satisfies

- $\mathcal{B} \neq \emptyset$,
- $A, B \in \mathcal{B}$, $a \in A \backslash B \Rightarrow$ there exists $b \in B \backslash A$ such that $(A \backslash \{a\}) \cup \{b\} \in \mathcal{B}$.

This is the basis definition of matroids. Much like the same topological structure can be defined by presenting the open sets, the closed sets, or a base of open sets, there are other equivalent definitions of a matroid structure, by independent sets, circuits, and more. If we consider an $N \times M$ matrix Φ, then if we define $\mathcal{B} = \{\Lambda \subset [M] : \#\Lambda = N, \{\varphi_\lambda\}_{\lambda \in \Lambda} \text{ is linearly independent}\}$, $([M], \mathcal{B})$ is a (linear) matroid, which we label $\mathcal{M}(\Phi)$ and call the *matroid of the frame* Φ. Similarly, if $G = (V, E)$ is a graph with M edges labeled by $[M]$, then if we define $\mathcal{B} = \{\Lambda \subset [M] : (V, \{e_\lambda\}_{\lambda \in \Lambda}) \text{ is a spanning forest}\}$, $([M], \mathcal{B})$ is a graph matroid. The set of *independent sets* $\mathcal{I}$ of a matroid is the collection of subsets of bases. That is, $\mathcal{I} = \{I \in \mathcal{P}([M]) : I \subset B, B \in \mathcal{B}\}$. The *dependent sets* are the subsets of $[M]$ in $\mathcal{P}([M]) \backslash \mathcal{I}$. Now borrowing a few terms from graph theory, we define the *circuits* of a matroid to be minimal dependent sets. That is, C is a circuit if it is a dependent set, but $C \backslash \{c\} \in \mathcal{I}$ for all $c \in C$. And finally, the *girth* of a matroid is the size of the smallest circuit. If Φ is a frame, then the spark of Φ is precisely the girth of $\mathcal{M}(\Phi)$. Furthermore, if Φ is full spark, then $\mathcal{M}(\Phi)$ is the *uniform matroid of rank N on* $[M]$, $\mathcal{U}_{M,N}$; that is, $\mathcal{B}$ consists of all subsets of $[M]$ of size N.

There are other concepts in frame theory which exist in matroid theory.

Definition 3. Let Φ be an (M,N)-tight frame, then there exists an $(M, M-N)$-tight frame $\tilde{\Phi}$ called a *Naimark complement* such that

$$\begin{pmatrix} \frac{\sqrt{N}}{\sqrt{M}}\Phi \\ \frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi} \end{pmatrix}$$

is a unitary matrix. If Φ is an (M,N)-FUNTF, then $\tilde{\Phi}$ is an $(M, M-N)$-FUNTF.

Sometimes Naimark complements have slightly different scaling in the literature than in this definition. The Naimark complement exists because the rows of $\frac{\sqrt{N}}{\sqrt{M}}\Phi$ are N orthonormal vectors in $\mathbb{F}^M$. Thus, there are $M-N$ orthonormal vectors which, along with the other N vectors, form a complete orthonormal basis for $\mathbb{F}^M$. These are the rows of $\frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi}$. Since the rows are orthonormal, we have

$$\left(\frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi}^*\right)\frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi} = \frac{M-N}{M}\tilde{\Phi}^*\tilde{\Phi} = I_{M-N},$$

thus $\tilde{\Phi}$ satisfies Equation 3 and is a tight frame. If Φ is a FUNTF, then the columns of Φ have unit norm and the columns of $\left(\frac{\sqrt{N}}{\sqrt{M}}\Phi^* \;\middle|\; \frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi}^* \right)^*$ have unit norm. Thus, the columns of $\tilde{\Phi}$ have unit norm as well. Although when $M > N$, there is not a unique Naimark complement, we shall use the notation $\tilde{\Phi}$ unambiguously because the key properties hold for all Naimark complements. It is clear from construction that if $\tilde{\Phi}$ is a Naimark complement of Φ, then Φ is a Naimark complement of $\tilde{\Phi}$. Thus taking Naimark complements is reflexive. This sort of duality has an analog in matroid theory.

Definition 4. If $\mathcal{M} = ([M], \mathcal{B})$ is a matroid, then $\mathcal{M}^* = ([M], \mathcal{B}^*)$, where

$$\mathcal{B}^* = \{[M] \cap B^c : B \in \mathcal{B}\},$$

is a matroid, called the *dual matroid*.

Proposition 5. If Φ is an (M, N)-tight frame, then

$$(\mathcal{M}(\Phi))^* = \mathcal{M}(\tilde{\Phi}).$$

In[13] the dual of the matroid of a frame is implicitly used, and the fact that the dual of a matroid of a tight frame is the matroid of the Naimark complement is Proposition 4.8 in,[10] which was proven using results about Plücker coordinates.

Proof. We denote the vectors of $\tilde{\Phi}$ as $\{\tilde{\varphi}_i\}_{i=1}^M$ and the columns of

$$\begin{pmatrix} \frac{\sqrt{N}}{\sqrt{M}}\Phi \\ \frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi} \end{pmatrix}$$

as $\{u_i\}_{i=1}^M$. That is, each u_i is $\sqrt{N/M}\varphi_i$ stacked on top of $\sqrt{(M-N)/M}\tilde{\varphi}_i$. We prove that $I \subset [M]$ is an independent set in $\mathcal{M}(\Phi)$ if and only if $\{\tilde{\varphi}_i\}_{i\in[M]\setminus I}$ spans $\mathbb{F}^{M-N}$. By the reflexivity of Naimark complementation, this will show that $S \subset [M]$ is such that $\{\varphi_i\}_{i\in S}$ spans $\mathbb{F}^N$ if and only if $[M]\setminus S$ is an independent set in $\mathcal{M}(\tilde{\Phi})$. Thus, the bases of $\mathcal{M}(\tilde{\Phi})$ are precisely the complements in $[M]$ of the bases of $\mathcal{M}(\Phi)$, as desired.

We shall prove the contrapositive. Let $D \subset [M]$ be a dependent set in $\mathcal{M}(\Phi)$. Then $\{\varphi_i\}_{i\in D}$ is linearly dependent and there exist a_i, $i \in D$, not all zero, such that

$$0 = \sum_{i\in D} a_i\varphi_i.$$

Since the $\{u_i\}_{i=1}^M$ are orthonormal,

$$0 \neq \sum_{i\in D} a_i u_i.$$

We claim that the necessarily non-zero vector $\tilde{v} := \sum_{i\in D} a_i\tilde{\varphi}_i$ is not in the span of $\{\tilde{\varphi}_j\}_{j\in[M]\setminus D}$. Note that for $j \in [M]\setminus D$,

$$\langle \tilde{v}, \tilde{\varphi}_j\rangle = \left\langle \sum_{i\in D} a_i u_i, u_j \right\rangle - \left\langle \sum_{i\in D} a_i\varphi_i, \varphi_j \right\rangle = \sum_{i\in D} a_i \langle u_i, u_j\rangle - \langle 0, \varphi_j\rangle = 0.$$

Thus $\tilde{v}$ is a non-zero vector which is orthogonal to each $\tilde{\varphi}_j$ with $j \in [M]\backslash D$, and hence $\{\tilde{\varphi}_j\}_{j\in[M]\backslash D}$ does not span $\mathbb{F}^{M-N}$.

In the other direction, if $\tilde{D} \subset [M]$ is such that $\text{span}\{\tilde{\varphi}_i\}_{i\in\tilde{D}} \neq \mathbb{F}^{M-N}$. Then there exists a non-zero vector $\tilde{v} \in \mathbb{F}^{M-N}$ such that $\langle \tilde{v}, \tilde{\varphi}_i\rangle = 0$ for all $i \in \tilde{D}$. Further, since $\tilde{\Phi}$ is a frame, $\tilde{v} \in \text{span}\{\tilde{\varphi}_i\}_{i\in[M]}$. Thus, there exist scalars $\{a_i\}_{i\in[M]\backslash\tilde{D}}$, not all zero, such that $\tilde{v} = \sum_{i\in I} a_i\tilde{\varphi}_i$. Since the u_i form an orthonormal basis for $\mathbb{F}^M$, $u_v := \sum_{i\in[M]\backslash\tilde{D}} a_i u_i$ is a non-zero vector which is orthogonal to each u_i, $i \in \tilde{D}$. If we define $v = \sum_{i\in[M]\backslash D} a_i\varphi_i$, then for any $j \in \tilde{D}$,

$$\langle v, \varphi_j\rangle = \langle \tilde{v}, \tilde{\varphi}_j\rangle + \langle u_v, u_j\rangle = 0.$$

Thus, $v \in \text{span}\{\varphi_i; i \in [M]\backslash D\} \cap (\text{span}\{\varphi_i; i \in [M]\backslash D\})^{\perp}$. Hence it must be 0. Thus $\{\varphi_i; i \in [M]\backslash D\}$ is a linearly dependent set of vectors. □

Using matroid theory[12] or direct linear algebra computations,[14] one can show the following.

Proposition 6. A FUNTF Φ is full spark if and only if $\tilde{\Phi}$ is full spark.

It ends up that in some sense, full spark frames are very common. We have the following result from.[10,14,15]

Theorem 7. Fix $M \geq N$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then relative to the Euclidean topology, the set of full spark (M, N)-frames is open and dense in the set of (M, N)-frames. Further the set of full spark (M, N)-tight frames (resp., (M, N)-FUNTFs) is open and dense in the set of (M, N)-tight frames (resp., (M, N)-FUNTFs).

Spark is, however, a binary decision. Vectors are either linearly dependent or they are not. However, just as an invertible matrix may be poorly conditioned, there could be full spark frames which are "barely" full spark. A manner of quantifying the conditioning of full spark using a classical geometric measure first appeared in.[10]

Definition 8. If Φ is a real (M, N)-frame, then we define the *Plücker coordinates* $\text{Plu}(\Phi)$ to be the projection of

$$(\det(\varphi_{i_1}|\varphi_{i_2}|\cdots|\varphi_{i_N}))_{1\leq i_1<i_2<\cdots i_N<M}$$

onto the unit sphere of $\mathbb{R}^{\binom{M}{N}}$ with antipodal points identified.

Thus, the Plücker coordinates are all of the $N \times N$ minors of Φ. Since Φ spans $\mathbb{R}^N$, we know that there is at least one non-zero minor, so the projection onto the unit sphere is well-defined. Then Φ is full spark if and only if $\text{Plu}(\Phi)$ has no zero coordinates. We may further ask that the Plücker coordinates be as far away from zero as possible.

Definition 9. For real (M, N)-FUNTFs Φ and Ψ, define the *Plücker angle* between Φ and Ψ as

$$\Theta(\Phi, \Psi) = \cos^{-1}|\langle \text{Plu}(\Phi), \text{Plu}(\Psi)\rangle|.$$

For canonical orthonormal basis $\{e_i\}_{i=1}^M$, define $\mathcal{E}_{i_1,\cdots,i_N} = \text{span}\{e_{i_1}, \ldots, e_{i_N}\}$. We would like to solve the optimization problem

$$\min_{\Phi(M,N)\in\mathcal{F}_{\mathbb{R}}(M,N)} \max\{\Theta(\Phi, \mathcal{E}_{i_1,\cdots,i_N}) : 1 \leq i_1 < \cdots < i_N \leq M\}.$$

That is, we want to maximize the minimum Plücker coordinate. As of yet, not much has been done to further apply this idea. An explicit calculation of a real $(4, 2)$-FUNTF with optimal Plücker coordinates is found in.[10]

Theorem 10. Let $M = 4$ and $N = 2$, then

$$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}\right\} \in \underset{\Phi(M,N)\in\mathcal{F}_{\mathbb{R}}(M,N)}{\arg\min} \max\{\Theta(\Phi, \mathcal{E}_{i_1,\cdots,i_N}) : 1 \leq i_1 < \cdots < i_N \leq M\}$$

3. GEOMETRIC SPREAD

We now return to our example of redundancy not always improving the robustness of the representation. In the previous section, we said that

$$\Omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

was not desirable because ω_1 and ω_2 were linearly dependent, an algebraic condition. We can also generalize the equality of ω_1 and ω_2 geometrically. That is, we note that the angle between the vectors is 0. Then we might ask that the smallest angle between frame vectors is not too small.

Definition 11. Let Φ be an (M, N)-unit norm frame. Then the coherence μ is defined to be

$$\mu = \mu(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle| \,.$$

Coherence has also been called the *maximal frame correlation* (for example,[2,16]). Since for unit-norm vectors, $|\langle u, v\rangle| = |\cos(\theta_{u,v})|$, where $\theta_{u,v}$ is the angle between the vectors, μ is the cosine of the smallest angle between the one-dimensional subspaces spanned by each frame vector, that is, the *chordal distance* between the subspaces. Coherence has a lower bound called the Welsh bound.

Theorem 12. Let Φ be an (M, N)-unit-norm frame. Then

$$\mu \geq \sqrt{\frac{M-N}{N(M-1)}}. \tag{4}$$

[2] contains one proof of the bound as well as references to other methods. We may ask which frames have optimal coherence.

Definition 13. Fix M, N, $\mathbb{F}$. If

$$\Phi \in \underset{\Psi \in \mathcal{F}_{\mathbb{F}}(M,N)}{\arg\min} \; \mu(\Psi),$$

then we say that Φ is a *Grassmannian frame.*

A unit-norm frame is optimally robust against multiple erasures if it is Grassmannian.[2,16]

Clearly, if $\mu(\Phi) = \sqrt{\frac{M-N}{N(M-1)}}$, then it must be a Grassmannian frame. It ends up that such Φ have special structure.

Definition 14. If $\Phi \in \mathcal{F}(M, N)$ and there exists a constant C such that

$$|\langle \varphi_i, \varphi_j \rangle| = C, \quad i \neq j,$$

then we call Φ an *equiangular tight frame (ETF).*

Theorem 15. Let Φ be an (M, N)-unit-norm frame. Then

$$\mu = \sqrt{\frac{M-N}{N(M-1)}}$$

if and only if Φ is an ETF.

This really is a corollary to Theorem 12 as the two inequalities which appear in the standard proof are equal if and only if Φ is a tight frame and equiangular.

If $M = N$, then the ETFs are precisely the orthonormal bases. The first non-trivial ETF is the so-called Mercedes Benz frame of three vectors in $\mathbb{R}^2$, which is often presented as the cubic roots of unity. Namely,

$$\Phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}.$$

However, for all $M > 3$, the vectors that form the Mth roots of unity

$$\Phi = \begin{pmatrix} 1 & \cos(2\pi/M) & \cdots & \cos(2(M-1)\pi/M) \\ 0 & \sin(2\pi/M) & \cdots & \sin(2(M-1)\pi/M) \end{pmatrix}$$

are not an ETF, since

$$\left|\left\langle(1,0)^T, (\cos(2\pi/M), \sin(2\pi/M)^T\right\rangle\right| = |\cos(2\pi/M)| \neq |\cos(4\pi/M)| = \left|\left\langle(1,0)^T, (\cos(4\pi/M), \sin(4\pi/M)^T\right\rangle\right|.$$

On the other hand, roots of unity can still be used to create Grassmannian frames. In[16] all Grassmannian frames of M vectors in $\mathbb{R}^2$ were characterized. They are essentially the first M $2M$th roots of unity.

Theorem 16. Up to an equivalence relation, if $\Phi \in \mathcal{F}_{\mathbb{R}}(M,2)$ is Grassmannian, then

$$\Phi = \begin{pmatrix} 1 & \cos(\pi/M) & \cdots & \cos((M-1)\pi/M) \\ 0 & \sin(\pi/M) & \cdots & \sin((M-1)\pi/M) \end{pmatrix}.$$

By a compactness argument,[16] Grassmannian frames exist for all choices of M, N, $\mathbb{F}$. However, ETFs are quite rare and not completely characterized. In the paper[17] everything that is known about the existence or non-existence of ETFs for given M, N, and $\mathbb{F}$ is listed. There are infinite classes of constructions, as well as one-off examples. Similarly, there are infinite classes of $(M,N,\mathbb{F})$ for which no ETF exists, as well as specific $(M,N,\mathbb{F})$ which have been proven to not have an ETF. Although the conditions for an ETF to exist are not completely understood, there are numerous known restrictions. As an example, two such restrictions follow.[18]

Theorem 17. If $\Phi \in \mathcal{F}_{\mathbb{F}}(M,N)$ is an ETF, then

$$M \leq \frac{N(N+1)}{2}, \quad \mathbb{F} = \mathbb{R}, \text{ and} \tag{5}$$

$$M \leq N^2, \quad \mathbb{F} = \mathbb{C}. \tag{6}$$

The bounds essentially boil down to the dimension of the smallest subspace of $\mathbb{R}^{N\times N}$ (resp., $\mathbb{C}^{N\times N}$) which contains the symmetric (resp.. self-adjoint) matrices.

Theorem 18. Suppose that $1 < N < M-1$. If $M \neq 2N$, then a real (M,N)-ETF can exist only if

$$\sqrt{\frac{N(M-1)}{(M-N)}} \text{ and } \sqrt{\frac{(M-N)(M-1)}{N}}$$

are odd integers. A real $(2N,N)$-ETF can exist only if N is an odd integer and $2N-1$ is the sum of two squares.

The proof of this theorem makes use of results about algebraic integers. The simplest example of this being the fact that the square root of an integer is either an integer or irrational.

Just as the Naimark complement of a full spark frame is full spark, it is easy to see that the Naimark complement of an ETF is again an ETF. Namely, if u_i, $i \in [M]$ denote the columns of

$$\begin{pmatrix} \frac{\sqrt{N}}{\sqrt{M}}\Phi \\ \frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi} \end{pmatrix},$$

then for all $i, j \in [M]$,

$$\begin{aligned}\frac{M-N}{M}\left|\langle\tilde{\varphi}_i,\tilde{\varphi}_j\rangle\right| &= \left|\langle u_i,u_j\rangle - \frac{N}{M}\langle\varphi_i,\varphi_j\rangle\right| \\ &= \frac{N}{M}\left|\langle\varphi_i,\varphi_j\rangle\right| \\ &= \frac{N}{M}\mu(\Phi).\end{aligned}$$

Thus, $\tilde{\Phi}$ is an ETF with $\mu(\tilde{\Phi}) = \frac{N}{M-N}\mu(\Phi)$.

4. ALGEBRAIC AND GEOMETRIC SPREAD

We now move to the interaction of algebraic and geometric spread. From Theorems 10 and 16, we know that

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-\sqrt{2}/2\\\sqrt{2}/2\end{pmatrix}\right\}$$

is both a Grassmannian frame and has optimal Plücker coordinates. Also, both frames with bad algebraic spread (non-full spark) and with perfect geometric spread (ETF) are rare. Thus, one may hope that their intersection is empty and in general ETFs are also full spark. In fact, this has been claimed in various publications. Unfortunately, this is not the case. In fact, there are infinite classes of ETFs which have low spark.[19,20] These frames are constructed using a tensor-like combination of the incidence matrix of certain combinatorial designs with a regular simplex ETF (see below). In retrospect, numerous examples of non-full spark ETFs have existed in the literature for awhile. The following $(9,3)$-ETF with spark 3 appeared in:[21]

$$\Phi = \left(\ \varphi_1 \mid \cdots \mid \varphi_9\ \right) = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & i\sqrt{3} & -i\sqrt{3} & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2}\end{pmatrix}, \tag{7}$$

where $\zeta = e^{2\pi i/3}$. Then $\{\varphi_1,\varphi_2,\varphi_3\}$, $\{\varphi_4,\varphi_5,\varphi_6\}$, and $\{\varphi_7,\varphi_8,\varphi_9\}$ are all linearly dependent sets, but the dependencies are different. Namely, $\sum_{i=1}^{3}\varphi_i = 0$, while for $j \in \{4,7\}$, $\sum_{i=0}^{2}\zeta^i\varphi_{i+j} = 0$.

One might ask when ETFs are full spark. A number of examples are presented in.[14] There is also a very simple infinite class of full spark ETFs, the *regular simplex ETFs*.[22]

Proposition 19. For all $N \in \mathbb{N}$, there exists a $(N+1,N)$-ETF.

One construction[22] is to take the vertices of a regular simplex centered at the origin. One may also simply take an $(N, N+1)$ submatrix of the $(N+1)\times(N+1)$ discrete Fourier transform matrix and rescale the columns appropriately to get a complex valued $(N+1,N)$-ETF. These are full spark. In order to show that, we will use the following result from.[7,23]

Lemma 20. For any collection of unit norm vectors Φ,

$$\operatorname{spark}(\Phi) \geq 1 + \frac{1}{\mu(\Phi)}.$$

Proposition 21. An $(N+1,N)$-ETF is full spark.

Proof. Let Φ be an $(N+1,N)$-ETF. It follows from Theorem 12 that

$$\mu(\Phi) = \sqrt{\frac{(N+1)-N}{N(N+1-1)}} = \frac{1}{N}.$$

Then we use Lemma 20 to conclude that $\operatorname{spark}(\Phi) \geq N+1$ and Φ is full spark. □

Since the Welsh bound contains M and N in the denominator and numerator, it has values which bounce around. So in the case of $M = N + 1$, the lower bound in Lemma 20 is large enough to tell us that Φ must be full spark. Quite often that is not the case. However, there are numerous examples of non-full spark frames where $\text{spark}(\Phi) = 1 + \frac{1}{\mu(\Phi)}$, but this is not always the case, even when $\frac{1}{\mu(\Phi)} \in \mathbb{Z}$, for example when Φ is a real ETF. To see an example of both, we will consider the $(9,3)$-ETF Φ in Equation 7 and its Naimark complement, a $(9,6)$-ETF, $\tilde{\Phi}$. Note that $\mu(\Phi) = 1/2$ and $\text{spark}(\Phi) = 3 = 1 + 1/\mu(\Phi)$. Since $\tilde{\Phi}$ is also an ETF, it follows from Theorem 12 that

$$\mu(\tilde{\Phi}) = \sqrt{\frac{9-6}{6(9-1)}} = \frac{1}{4}.$$

We will follow the proof of Proposition 2 and find the size of the largest non-spanning set of Φ in order to find the spark of $\tilde{\Phi}$. Since the spark of Φ is equal to the dimension of the space, there is not "enough linear dependence" to build large non-spanning sets. In fact, the maximal non-spanning sets of Φ are precisely the minimal linearly dependent sets $\{\varphi_1, \varphi_2, \varphi_3\}$, $\{\varphi_4, \varphi_5, \varphi_6\}$, and $\{\varphi_7, \varphi_8, \varphi_9\}$. Since the largest non-spanning sets in Φ have size 3, the smallest linearly dependent sets in $\tilde{\Phi}$ have size $9 - 3$. However, $\text{spark}(\tilde{\Phi}) = 6 \neq 1 + 1/(1/4) = 1 + 1/\mu(\tilde{\Phi})$.

Beyond asking whether ETFs are full spark, there is another characteristic of finite frames which is both geometric and algebraic in nature. We have the following definition from.[24,25]

Definition 22. Let Φ be an (M,N) frame. If there exists a $K \subset [M]$ such that for all $i \in K$, $j \in [M]\backslash K$, $\langle \varphi_i, \varphi_j \rangle = 0$, then we say that Φ is *orthodecomposable*.

This trait is geometric in the sense that it is characterized by inner products, but it is also algebraic in the sense that the partition tells us about the composition of spanning sets and linearly independent sets. In,[24,25] these are characterized by a certain graph called the correlation network. A frame is orthodecomposable if and only if its correlation network is disconnected. We can also characterize orthodecomposable frames by connectivity of its associated matroid.

Definition 23. A matroid is called *connected* if and only if for every pair of distinct elements of $[M]$, there is a circuit containing both.

Theorem 24. $\Phi \in \mathcal{F}(M,N)$ is orthodecomposable if and only if $\mathcal{M}(\Phi)$ is disconnected.

Proof. We first assume that Φ is orthodecomposable. Then there exist non-empty sets $S_1, S_2, \in [M]$ such that

- $[M] = S_1 \sqcup S_2$ (that is, the disjoint union of S_1 and S_2) and
- $\langle \varphi_i, \varphi_j \rangle = 0$ for all $i \in S_1$, and $j \in S_2$.

Pick $\varphi_i, \varphi_j \in \Phi$ such that $i \in S_1$, $j \in S_2$. Assume that i and j are in a circuit $\mathcal{C}$, that is, that φ_i and φ_j are in a minimal dependent set. Since no φ_k with $k \in S_2$ can be linearly dependent on $\{\varphi_\ell\}$, $\ell \in S_1$ due to orthogonality and vice versa, $\mathcal{C} \cap \{\varphi_\ell; \ell \in S_1\}$ is a circuit of strictly smaller size, which contradicts the minimality of $\mathcal{C}$. Thus $\mathcal{M}(\Phi)$ is disconnected.

Now assume that $\mathcal{M}(\Phi)$ is disconnected. Let $\tilde{\Phi}$ denote the Naimark complement of Φ. By Proposition 4.2.6 of,[12] this holds if and only if there exists a non-empty $T \subsetneq [M]$ such that

$$r(T) + r^*(T) - |T| = 0,$$

where since $\mathcal{M}(\Phi)^* = \mathcal{M}(\tilde{\Phi})$ by Proposition 2, $r(T) = \dim \text{span}\{\varphi_t; t \in T\}$, $r^*(T) = \dim \text{span}\{\tilde{\varphi}_t; t \in T\}$, and $|T|$ is the size of T. Let u_i, $i \in [M]$ denote the orthonormal columns of

$$\begin{pmatrix} \frac{\sqrt{N}}{\sqrt{M}}\Phi \\ \frac{\sqrt{M-N}}{\sqrt{M}}\tilde{\Phi} \end{pmatrix}.$$

Thus we have

$$\begin{aligned}
|\{u_i : i \in T\}| &= \dim \operatorname{span}\{u_i : i \in T\} \\
&= |T| \\
&= r(T) + r^*(T) \\
&= \dim \operatorname{span}\{\varphi_t ; t \in T\} + \dim \operatorname{span}\{\tilde{\varphi}_t ; t \in T\}.
\end{aligned}$$

Let $T_1, T_2 \subset T$ be such that

$$\begin{aligned}
\dim \operatorname{span}\{\varphi_i ; i \in T_1\} &= \dim \operatorname{span}\{\varphi_t ; t \in T\} \text{ and} \\
\dim \operatorname{span}\{\tilde{\varphi}_j ; j \in T_2\} &= \dim \operatorname{span}\{\tilde{\varphi}_t ; t \in T\},
\end{aligned}$$

where necessarily $|T_1| + |T_2| = |T|$. Then

$$\operatorname{span}\{u_i ; i \in T\} = \operatorname{span}\left\{ \begin{pmatrix} \sqrt{\frac{N}{M}}\varphi_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\frac{M-N}{M}}\tilde{\varphi}_i \end{pmatrix} ; i \in T_1, j \in T_2 \right\}.$$

For any $k \in [M] \backslash T$ and $i \in T$,

$$\langle u_k, u_i \rangle = 0 \Rightarrow \left\langle \begin{pmatrix} \sqrt{\frac{N}{M}}\varphi_k \\ \sqrt{\frac{M-N}{M}}\tilde{\varphi}_k \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{N}{M}}\varphi_i \\ 0 \end{pmatrix} \right\rangle = 0 \Rightarrow \langle \varphi_k, \varphi_i \rangle = 0.$$

Thus, Φ is orthodecomposable. □

We note that although one can define a matroid from a graph and a graph matroid is disconnected precisely when the underlying graph is disconnected, the matroid of the correlation network frame is in general not the matroid of the frame. In fact, they do not usually even have the same base set $[M]$ as the vertices of the correlation network correspond to frame vectors while the edges of the matroid correspond to the frame vectors.

The balance between algebraic and geometric spread and their interplay with Naimark complementation is still not fully understood. In this short paper, we sought to explore this relationship some, but there are many open problems.

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