# A remark on densities of hyperbolic dimensions for conformal iterated function systems with applications to conformal dynamics and fractal number theory

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**Abstract**

In this note we investigate radial limit sets of arbitrary regular conformal interated function systems. We show that for each of these systems there exists a variety of finite hyperbolic subsystems such that the spectrum made of the Hausdorff dimensions of the limit sets of these subsystems is dense in the interval between 0 and the Hausdorff dimension of the given conformal iterated function system. This result has interesting applications in conformal dynamics and elementary fractal number theory.

## Introduction and statement of main results

In this paper we consider *iterated function systems (IFS)* on a connected and compact subset $X$ of the $N$–dimensional Euclidean space $\mathbb{R}^N$, for $N \in \mathbb{N}$. We always assume that the boundary of $X$ is a Jordan curve. Recall that a IFS is generated by a family $\Phi$ of injective contractions $\phi_i : X \to X$, for $i$ in some given countable index set $I$. A IFS is called *conformal iterated function systems (CIFS)* if the following conditions are fulfilled. In here we use the notation $\phi_\omega := \phi_{i_1} \circ \phi_{i_2} \circ \ldots \circ \phi_{i_n}$ for $\omega = (i_1, i_2, \ldots i_n) \in I^n$.

*Open set condition.* There exists a nonempty open set $U \subset X$ in the topology of $X$ such that $\phi_i(U) \subset U$ for all $i \in I$, and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for all $i, j \in I, i \neq j$.

*Contraction.* There exists $0 < \theta < 1$ such that $|\phi_i'(x)| \leq \theta < 1$, for all $x \in X$ and $i \in I$.

*Conformality.* There exists an open connected set $X \subset V \subset \mathbb{R}^N$ such that each $\phi_i$ extends to a conformal diffeomorphism of $V$ into $V$.

*Bounded distortion property.* There exists $C \geq 1$ such that for all $n \in \mathbb{N}, \omega \in I^n$ and $x, y \in V$ we have

$$|\phi_\omega'(y)| \leq C \; |\phi_\omega'(x)|.$$

A CIFS is called *finite* if $I$ is finite, otherwise it is called *infinite.* Every CIFS gives rise to a unique *'radial limit set'*

$$\Lambda(\Phi) := \bigcap_{n \in \mathbb{N}} \bigcup_{\omega \in I^n} \phi_\omega(\mathrm{int}(X)).$$

Note that in general $\Lambda(\Phi)$ does not have to be a compact set. In this paper we exclusively consider *regular CIFS.* That is, the system $(X, \Phi)$ admits a *$h$–conformal measure* $\mu$, for $h := \dim_H(\Lambda(\Phi))$ referring to the Hausdorff dimension of $\Lambda(\Phi)$. This means that for each Borel set $B \subset X$ and $i \in I$, we have (cf. [14],[3],[4],[10])

$$\mu(\phi_i(B)) = \int_B |\phi_i|^h d\mu, \text{ and } \mu(\phi_i(X) \cap \phi_j(X)) = 0 \text{ for each } i, j \in I, i \neq j.$$

The following theorem represents the main result of this paper. In here, *hyperbolic subset* of $\Lambda(\Phi)$ refers to the limit set of some finite subsystem of the CIFS $(X, \Phi)$.

**Main Theorem.** *Let $(X, \Phi)$ be a regular conformal iterated function system. Then there exists a set $\mathcal{H}(\Phi)$ of hyperbolic subsets of $\Lambda(\Phi)$ such that*

$$\{dim_H(S) \mid S \in \mathcal{H}(\Phi))\} \text{ is dense in } [0, dim_H(\Lambda(\Phi))].$$

This result has various interesting applications in complex dynamics, and we shall now briefly comments on a few of these. For instance, for Julia sets of rational endomorphisms of the Riemann sphere $\bar{\mathbb{C}}$ it has the following immediate implication.

**Application 1.** *Let $T : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map of degree at least 2 such that the associated Julia set $J(T)$ does not contain critical points. Then there exists a set $\mathcal{H}(T)$ of hyperbolic subsets of $J(T)$ such that*

$$\{dim_H(S) \mid S \in \mathcal{H}(T))\} \text{ is dense in } [0, dim_H(J(T))].$$

Furthermore, for Kleinian groups acting on the $(N+1)$–dimensional hyperbolic space our Main Theorem gives rise to the following result. Note that in here we have restricted the application to the geometrically finite situation in which the action of the group admits a fundamental domain whose boundary consists of finitely many $N$–dimensional hyperbolic polygons. However, similar conclusions could also be drawn for radial limit sets of certain types of geometrically infinite Kleinian groups.

**Application 2.** *For each geometrically finite Kleinian group $G$ there exists a set $\mathcal{H}(G)$ of hyperbolic subsets of the limit set $L(G)$ such that*

$$\{dim_H(S) \mid S \in \mathcal{H}(G))\} \text{ is dense in } [0, dim_H(L(G))].$$

We remark that if $G$ has parabolic elements then the hyperbolic sets in $\mathcal{H}(G)$ can be described exclusively in terms of certain prescribed bounded cuspidal excursion patterns.

Finally, we also mention an application of our Main Theorem in elementary fractal number theory. For this let $[x_1, x_2, x_3, \ldots]$ refer to the regular continued fraction expansions of $x \in [0, 1]$. With $\mathcal{A}$ referring to the set of all finite words in the alphabet $\mathbb{N}$, we define for a finite subset $A$ of $\mathcal{A}$,

$$S(A) := \{[a_1, a_2, a_3, \ldots] \mid (a_1, a_2, a_3, \ldots) \in A^{\mathbb{N}}\}.$$

It is well known that every Markov map which is topologically conjugated to the full shift can be represented by a suitable IFS. Since the second forward iterate of the Gauss map is a uniformly expanding Markov map, it follows that the associated backward iterates give rise to a CIFS. Hence, our Main Theorem has the following immediate implication.

**Application 3.**

*The set $\{dim_H(S(A)) \mid A \subset \mathcal{A}$ finite$\}$ is dense in $[0, 1]$.*

Clearly, this application is closely related to the following so-called Texan conjecture, which was formulated independently by Hensley ([5]) and Mauldin & Urbański ([9]), and which has recently been shown to be true in the interval $[0, 1/2]$ (cf. [7]). We remark that the partial solution of this conjecture in [7] uses pertubation theory for a familiy of Ruelle–Frobenius operators.

**Texan conjecture.**

*The set $\{dim_H(S(A)) \mid A \subset \mathbb{N}$ finite$\}$ is dense in $[0, 1]$.*

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# 1 Density of hyperbolic dimensions

## Perforation spectra for hyperbolic sets

Let us recall first the notion of the $\nu$–dimension of a compact subset of $\Lambda$ of $\mathbb{R}^N$, for some non-atomic Borel probability measure $\nu$ on $\Lambda$. Namely, for a given $\epsilon > 0$, a covering $\{U_i\}_{i\in\mathbb{N}}$ of $\Lambda' \subset \Lambda$ is called a *$(\nu, \epsilon)$–covering of* $\Lambda'$ if $\nu(U_i) < \epsilon$ for all $i \in \mathbb{N}$. With $\mathcal{U}_\epsilon^\nu(\Lambda')$ referring to the set of all $(\nu, \epsilon)$–coverings of $\Lambda'$, the *$s$–dimensional $\nu$–Hausdorff measure* $\mathcal{H}_s^\nu(\Lambda')$ is given by, for $s \geq 0$,

$$\mathcal{H}_s^\nu(\Lambda') := \lim_{\epsilon\to 0} \inf \left\{ \sum_i (\nu(U_i))^s \mid \{U_i\}_{i\in\mathbb{N}} \in \mathcal{U}_\epsilon^\nu(\Lambda') \right\}.$$

Then the *$\nu$–dimension* $\dim_\nu(\Lambda')$ *of* $\Lambda'$ is defined as

$$\dim_\nu(\Lambda') := \sup\{s \ : \ \mathcal{H}_s^\nu(\Lambda') = \infty\} = \inf\{s \ : \ \mathcal{H}_s^\nu(\Lambda') = 0\}.$$

The following relates the $\nu$–dimension to the Hausdorff dimension. This result represents a folklore theorem in fractal geometry, which was implicitly obtained first in [1] (Theorem 14.1).

**Billingsley's Lemma.** *If there exists $t > 0$ such that for each $N$–ball $B(x, r)$ centred at $x \in \Lambda'$ of radius $r > 0$ we have*

$$\lim_{r\to 0} \frac{\log \nu(B(x, r))}{\log r} = t,$$

*then*

$$\dim_H(\Lambda') = t \dim_\nu(\Lambda').$$

Furthermore, we introduce the notion of the *$(\nu, \mathcal{H})$–perforation spectrum* of a CIFS $(X, \Phi)$ for a set $\mathcal{H}$ of hyperbolic subsets of $\Lambda(\Phi)$, which is given by

$$\{\dim_\nu(S) \mid S \in \mathcal{H}\}.$$

Finally, if $\nu$ is a $h$–conformal measure supported on $\Lambda(\Phi)$, then $\dim_H(\Lambda(\Phi')) = h \dim_\nu(\Lambda(\Phi'))$, for any finite subsystem $(X, \Phi')$ of $(X, \Phi)$. This follows since in this situation $\nu$ is *$h$–Ahlfors regular on* $\Lambda(\Phi')$, that is $\nu(B(x, r)) \asymp r^h$ for each $x \in \Lambda(\Phi')$ and $r > 0$. Hence, $\lim_{r\to 0}(\log \nu(B(x, r)))/\log r = h$ for all $x \in \Lambda(\Phi')$, and therefore Billingsley's Lemma is applicable.

**Proposition.** *Let $(X, \Phi)$ be a finite conformal iterated function system, and let $\nu$ be any $h$-conformal measure whose support contains $\Lambda(\Phi)$, for $h \geq h_\Phi := \dim_H(\Lambda(\Phi))$. Then there exists a set $\mathcal{H}$ of hyperbolic subsets of $\Lambda(\Phi)$ such that the $(\nu, \mathcal{H})$–perforation spectrum of $(X, \Phi)$ is dense in* $[0, \dim_\nu(\Lambda(\Phi))]$.

In the proof of this proposition the following lemma will turn out to be useful.

**Lemma.** *Let $(X, \Phi)$ be a finite conformal iterated function system , and let $\nu$ be any $h$–conformal measure whose support contains $\Lambda(\Phi)$, for $h \geq h_\Phi := \dim_H(\Lambda(\Phi))$. Furthermore, for each $n \in \mathbb{N}$ let $\tau(n) > 0$ be given by $\sum_{\omega\in I^n} \nu(\phi_\omega(X))^{\tau(n)} = 1$. We then have*

$$|\tau(n) - dim_\nu(\Lambda(\Phi))| \ll 1/n.$$

*Proof.* For $\omega$ in $I^n$, let

$$d_i(\omega) := \begin{cases} \text{ess inf } \frac{d\nu\circ\phi_\omega}{d\nu} & \text{for} \quad i = 1 \\ \text{ess sup } \frac{d\nu\circ\phi_\omega}{d\nu} & \text{for} \quad i = 2. \end{cases}$$

Note that $d_1(\omega) \leq d_2(\omega) \leq \theta < 1$ for all finite words $\omega$ in the alphabet $I$. This follows since $\nu$ is $h$–conformal, and therefore $d(\nu \circ \phi_\omega)/d\nu = |\phi'_\omega|^h$. Also, we define for $i = 1, 2$, $t \geq 0$ and $n \in \mathbb{N}$,

$$D_i(n, t) = \sum_{\omega\in I^n} (d_i(\omega))^t.$$

We then have that for each $n \in \mathbb{N}$ there exists $\delta_i(n)$ such that

$$D_i(n, \delta_i(n)) = 1 \text{ for } i = 1, 2,$$

and one immediately verifies that

$$\delta_1(n) \leq \tau(n) \leq \delta_2(n). \tag{1}$$

Note that by the bounded distortion property there exists a constant $1 \leq C_\nu < \infty$ such that $d_2(\omega) < C_\nu d_1(\omega)$. Then a straight forward adaptation of the methods of [8] shows that the following Hutchinson–type formula holds.

$$1 \leq D_2(n,t) \leq C_\nu \text{ for all } n \in \mathbb{N} \quad \text{if and only if} \quad \dim_\nu(\Lambda(\Phi)) = t. \tag{2}$$

Therefore, combining (1) and (2) is sufficient to show that $\epsilon(n) := \delta_2(n) - \delta_1(n) \ll 1/n$. For this we proceed as follows.

$$\begin{aligned} 1 &= \sum_{\omega \in I^n} (d_2(\omega))^{\delta_2(n)} \\ &\leq \sum_{\omega \in I^n} (C_\nu d_1(\omega))^{\delta_1(n)+\epsilon(n)} \\ &\leq C_\nu^{\delta_2(n)} \left(\sup\{d_1(\omega) \mid \omega \in I^n\}\right)^{\epsilon(n)}. \end{aligned}$$

This implies, using once again the bounded distortion property,

$$\epsilon(n) \leq \frac{\delta_2(n) \log C_\nu}{-\log\left(\sup\{d_1(\omega) \mid \omega \in I^n\}\right)} \ll \frac{1}{n}.$$

□

We are now in the position to give the proof of the Proposition.

*Proof of the Proposition.* Let $0 < s < \dim_\nu(\Lambda(\Phi))$ be given. It is sufficient to show that for each $n \in \mathbb{N}$ there exists a finite subsystem $(X, \Pi_n)$ of $(X, \Phi)$ such that $|s - \dim_\nu(\Lambda(\Pi_n))| \ll 1/n$. For this we fix some ordering

$$\{1, 2, \ldots |I|\}^n = \{\omega_1, \omega_2, \ldots \omega_{|I|^n}\}.$$

Then let $k_s(n) := \max\{k \mid \sum_{i=1}^{k} \nu(\phi_{\omega_i}(X))^s < 1\}$ and define

$$\Pi_n := (\phi_i)_{i=1,\ldots k_s(n)}.$$

Let $\kappa(n) \geq 0$ be the unique solution of the equation $\sum_{i=1}^{k_s(n)} \nu(\phi_{\omega_i}(X))^{s-\kappa(n)} = 1$. With $d_2(\omega)$ as defined in the proof of the previous lemma and $d_{\max} := \sup\{d_2(\omega) \mid \omega \in I\}$, we then have

$$\sum_{i=1}^{k_s(n)} \nu(\phi_{\omega_i}(X))^s \ll (d_{\max})^{n\,\kappa(n)},$$

which implies

$$\kappa(n) \leq \frac{\log \sum_{i=1}^{k_s(n)} \nu(\phi_{\omega_i}(X))^s}{n \log d_{\max}} \leq \frac{\log(1 - (d_{\max})^{ns})}{n \log d_{\max}} \ll \frac{1}{n}.$$

Combining this estimate with the Lemma above, the result follows. □

**Proof of the Main Theorem.**

Let $h := \dim_H(\Lambda(\Phi))$. If $(X, \Phi)$ is a finite CIFS then the assertion is an immediate consequence of the Proposition. Indeed, in this situation we can take the measure occuring in the above Proposition to be equal to the $h$–conformal measure $\mu$ associated with $(X, \Phi)$, and immediately obtain that there exists a set $\mathcal{H}$ of hyperbolic subsets of $\Lambda(\Phi)$ such that the $(\mu, \mathcal{H})$–perforation spectrum of $(X, \Phi)$ is dense in $[0, \dim_\mu(\Lambda(\Phi))]$. Since in this situation $\mu$ is $h$–Ahlfors regular on $\Lambda(\Phi)$, it follows that $\lim_{r\to 0}(\log \nu(B(x, r)))/ \log r = h$, and hence Billingsley's Lemma implies $\dim_\mu(\Lambda(\Phi)) = 1$. Combining this observation with the fact that for any finite subsystem $(X, \Phi')$ of $(X, \Phi)$ we have $\dim_H(\Lambda(\Phi')) = h \dim_\mu(\Lambda(\Phi'))$, the result follows.

For $I$ infinite, we proceed as follows. First, recall that there exists the following well known canonical exhaustion of the system $(X, \Phi)$ by hyperbolic sets (see e.g. [10],[6],[11],[12],[13]). Namely, for $\Phi_l := \{\phi_i \mid \sup_{x\in X} |\phi_i'(x)| > 1/l\}$ we have

$$\dim_H(\Lambda(\Phi_l)) \leq \dim_H(\Lambda(\Phi_{l+1})) \text{ for all } l \in \mathbb{N}, \text{ and } \lim_{l\to\infty} \dim_H(\Lambda(\Phi_l)) = h.$$

Next, note that the $h$–conformal measure associated with $(X, \Phi)$ is always $h$–Ahlfors regular on $\Lambda(\Phi_l)$, for each $l \in \mathbb{N}$ for which $\Lambda(\Phi_l) \neq \emptyset$. Therefore, it follows

$$\dim_\mu(\Lambda(\Phi_l)) \leq \dim_\mu(\Lambda(\Phi_{l+1})) \text{ for all } l \in \mathbb{N}, \text{ and } \lim_{l\to\infty} \dim_\mu(\Lambda(\Phi_l)) = 1.$$

Now, let $0 < t < 1$ be given. By the above, there then exists $l_t \in \mathbb{N}$ such that $\dim_\mu(\Lambda(\Phi_{l_t})) > t$. Note that by compactness of $X$, we have that $(X, \Phi_{l_t})$ is a finite CIFS. We can then apply the Proposition, which gives that there exists a set $\mathcal{H}_t$ of hyperbolic subsets of $\Lambda(\Phi_{l_t})$ such that the $(\mu, \mathcal{H}_t)$–perforation spectrum of $(X, \Phi_{l_t})$ is dense in $[0, \dim_\mu(\Lambda(\Phi_{l_t}))]$. From this we deduce that $\{h \dim_\mu(S) \mid S \in \mathcal{H}_t\}$ is dense in $[0, h \dim_\mu(\Lambda(\Phi_{l_t}))]$. By letting $t$ tend to 1 and using the fact that by Billingsley's Lemma we have $\dim_H(S) = h \dim_\mu(S)$ for each $S \in \mathcal{H}_t$, the result follows.

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