Ergod. Th. & Dynam. Sys. (2001), **XX**, 1–28
Printed in the United Kingdom

A multifractal formalism for growth rates and applications to geometrically finite Kleinian groups

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(*Received* 25 *July* 2001)

Abstract. We elaborate thermodynamical and multifractal formalisms for general classes of potential functions and their average growth rates. We then apply these formalisms to certain geometrically finite Kleinian groups which may have parabolic elements of different ranks. We show that for these groups our revised formalisms give access to a description of the spectrum of 'homological growth rates' in terms of Hausdorff dimension. Furthermore, we derive necessary and sufficient conditions for the existence of 'thermodynamical phase transitions'.

1. *Introduction and statement of results*

In this paper we elaborate a multifractal formalism for general classes of potential functions and their average growth rates. For this we set up a thermodynamical formalism which generalizes the so far existing one. We then show that these so revised formalisms are applicable to geometrically finite Kleinian groups, where they reveal intricate fractal aspects of the 'homological growth rates'.

More precisely, we consider tuples $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$ of certain sequences of continuous real–valued functions $\{\Delta_n\}$ and $\{A_n\}$, defined over a subshift of finite type (X, σ). Here the family of functions Δ_n fulfils some mild growth properties with respect to the family of 'cocycle functions' A_n, and we refer to these as to the 'average growth property'. Our first goal is to establish a thermodynamical formalism for this general class of functions. We introduce a generalized notion of pressure $\mathcal{P}$ for which we then verify some variational principles. This then permits to study non–additive ensembles for which the geometry is not strictly compatible with the dynamics.

In the second part of the paper we restrict the investigations to so-called families of distance functions $\boldsymbol{\Delta}$, that is functions which are positive, increasing and pointwise

† Research supported by EPSRC (Ref.: GR/N01392)

unbounded. This allows to introduce a metric on X and hence the notion of Hausdorff dimension $\dim_{\boldsymbol{\Delta}}$. Here our goal is to determine $\dim_{\boldsymbol{\Delta}}$ for the level sets

$$\mathcal{F}_{\boldsymbol{\Delta}}(a) = \left\{ x \in X : \lim_{n\to\infty} \frac{\Delta_n(x)}{n} = a \right\}.$$

For this we establish a type of Young formula for non–ergodic measures, and by combining this formula with our modified thermodynamical formalism we obtain the Hausdorff dimensions of these level sets. A good reference for dimension theory in the context of dynamical systems can be found in [**Pes97**].

The main results of this analysis is summarized in the following theorem. In here the function P is given by $P(s) := \mathcal{P}(-s\boldsymbol{\Delta})$.

THEOREM 1.1. *For $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$, where $\boldsymbol{\Delta}$ is a family of distance functions and $\boldsymbol{A}$ is a family of positive functions, we have for each $s \in \mathbb{R}$ and positive $a \in [-P^+(s), -P^-(s)]$ that*

$$\dim_\Delta(\mathcal{F}_\Delta(a)) = \frac{as + P(s)}{a},$$

where $P^\pm$ refers to the left and right derivative of the pressure P.

In this general setting our proof requires a modification of an intricate argument of Cajar, which then allows to approximate the dimension with respect to Δ by measure theoretical dimensions derived from certain non–atomic weak Gibbs measures. Here we find that this argument can be avoided for the very special case in which there is a cocycle I associated with $\boldsymbol{\Delta}$ which has the property that there exists a non–atomic weak Gibbs measure for the potential $-\delta I$ (with δ denoting the least zero of P).

The third goal of the paper is to apply our general formalisms to geometrically finite Kleinian groups G, which are allowed to have parabolic fixed points of different ranks. We derive a multifractal analysis of the limit sets $L(G)$ of these groups. For simplicity we restrict the discussion to 'essentially free' Kleinian groups, that is geometrically finite Kleinian groups which are free groups apart from the possibility that they might have parabolic fixed points of rank k (with stabilizer a finite extension of a free abelian group on k generators). Also, for ease of exposition, we restrict the discussion to Kleinian groups acting on hyperbolic 3–space $\mathbb{D}^3$, where k can be equal to either 1 or 2. Some aspects of the methods used here, were already developed in [**Fri00a, Fri00b**].

More precisely, for these applications we consider the special family of distance functions

$$\Delta_n := \sum_{[x_0,\ldots,x_{n-1}]\in\mathfrak{Z}_n} d_h(0, x_0 \cdots x_{n-1}(0))\, \mathbf{1}_{[x_0,\ldots,x_{n-1}]}.$$

Here $\mathfrak{Z}_n$ refers to the set of n-cylinders built on the generators of G, $x_0 \cdots x_{n-1} \in G$ coding the limit point $x \in L(G)$ in the usual way, d_h denotes the hyperbolic distance, and $\mathbf{1}_A$ refers to the indicator function of the set A. We show that the analysis of the previous section is applicable under these circumstances, and in this way we derive a complete description of the multifractal spectrum for limit sets of the type of Kleinian groups under consideration.

The main results are summarized in the following theorem, which in terms of the pressure gives a fractal–dimensional characterization of the level sets

$$\mathcal{F}_G(a) = \left\{ \pi(x_0, x_1, \cdots) \in L(G) : \lim_{n\to\infty} \frac{d_h(0, x_0 \cdots x_{n-1}(0))}{n} = a \right\},$$

where π denotes the canonical projection of X onto $L(G)$

THEOREM 1.2. *Let $G < Con(N)$ be a non–elementary geometrically finite, 'essentially free' Kleinian group with exponent of convergence δ. If P is not linear, then there exists an interval (α_-, α_+) such that for each $a \in (\alpha_-, \alpha_+)$ we have*

$$\dim_H (\mathcal{F}_G(a)) = \frac{\hat{P}(-a)}{-a}, \tag{1}$$

where $\hat{P}$ refers to the Legendre transform of the pressure P.

- *If G is convex cocompact, then (1) holds for $a \in [\alpha_-, \alpha_+]$ and P is real–analytic everywhere. Furthermore, with $(P')^{-1}$ denoting the inverse of the derivative of P, we have for $a \in (\alpha_-, \alpha_+)$ that*

$$\hat{P}(-a) = -a\,(P')^{-1}(-a) - P\left((P')^{-1}(-a)\right).$$

- *If G has parabolic elements, then P is real–analytic on $(-\infty, \delta)$ and equal to 0 otherwise. Additionally, the following cases can be distinguished. Here $k_{\max}$ denotes the maximal possible rank of the parabolic elements in G.*
  - *For $\delta \le (k_{\max} + 1)/2$ we have that P is differentiable everywhere.*
  - *For $\delta > (k_{\max} + 1)/2$ there exists a 'thermodynamical phase transition'. More precisely, the right derivative of P at δ vanishes, whereas the left derivative of P at δ is strictly negative.*

We remark that in the parabolic case the natural choice for the relevant cocycle gives a function which is no longer Hölder continuous. In order to overcome this problem we introduce a coding of the radial limit set which uses an infinite alphabet (based on the so called 'top representation', previously introduced in [**SV95**] in connection with investigations of the Patterson measure), and then study the 'infinite ergodic theory' of the associated jump–transformation. In particular, in these investigations we employ general principles such as the 'Sarig variational principle', the 'Mauldin–Urbański construction' which guarantees the existence of unique invariant Gibbs measures for the infinite alphabet, and 'Abramov's formula' which relates the measure theoretical entropies of the finite and the infinite alphabet. Subsequently, using a formula of Kac as well as strong analytic properties of the pressure function for the infinite alphabet, we derive the necessary and sufficient conditions for the existence of 'thermodynamical phase transitions'.

We end this introduction by giving some conclusions which can be drawn from our analysis in this paper. In particular, we briefly discuss in which way the type of phase transition obtained in this paper relates to the phenomena of 'weak multifractal phase

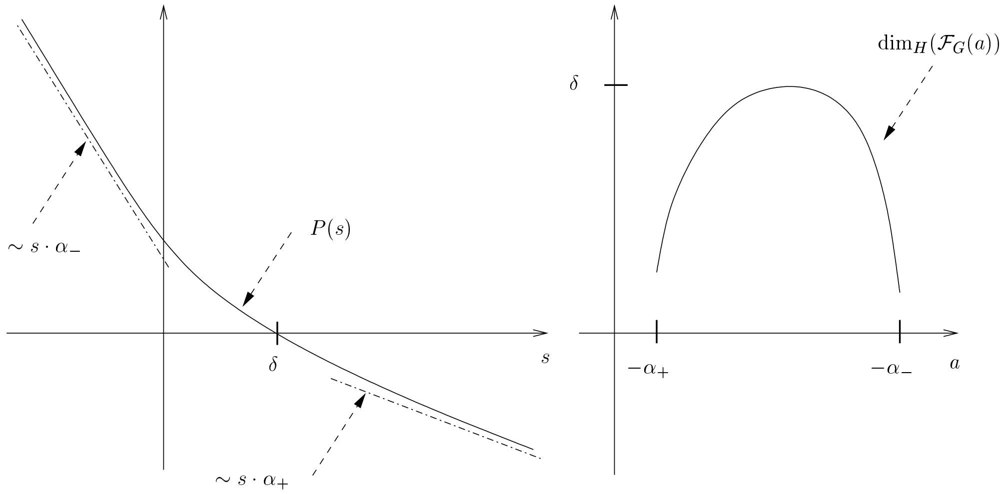


FIGURE 1. Typical shape of P and the dimension spectrum in the convex cocompact case. Here, $\alpha_\pm := \lim_{s \to \pm\infty} P(s)/s$ are the 'asymptotic directions' of P. The unique zero of P is the Hausdorff dimension of the limit set $L(G)$.

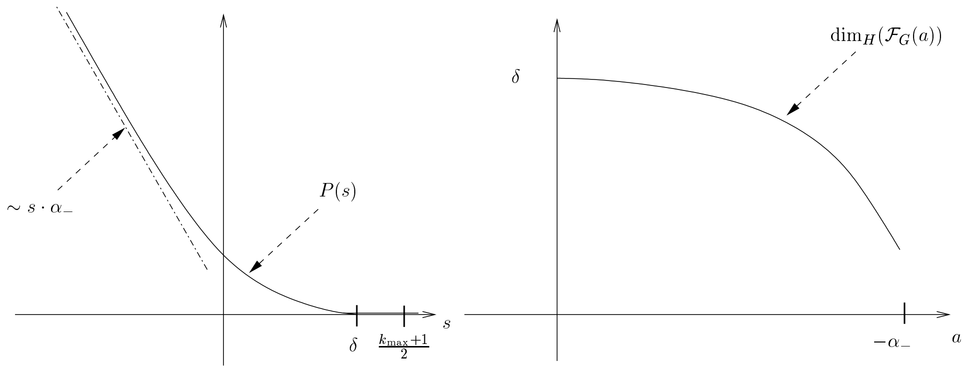


FIGURE 2. Typical shape of P and the dimension spectrum in the parabolic case without thermodynamical phase transition; that is, $\dim_H(L(G)) \leq (k_{\max}+1)/2$ and P is differentiable everywhere.

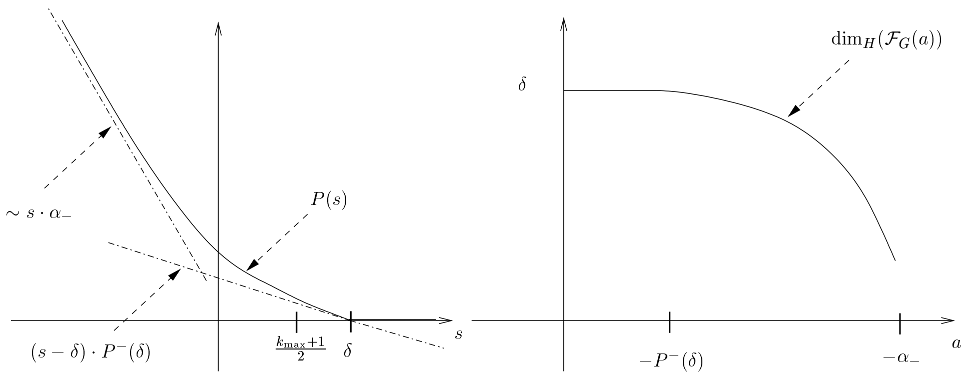


FIGURE 3. Typical shape of P and the dimension spectrum in the parabolic case with thermodynamical phase transition; that is, $\dim_H(L(G)) > (k_{\max}+1)/2$ and P is not differentiable at δ.

transition' for Kleinian groups obtained in [**Str95a**], [**Str99**]. In the following, if not stated otherwise, let G be a geometrically finite, essentially free Kleinian group with parabolic elements, and let δ denote the associated exponent of convergence.

Interpretation in terms of homological increments. For the first immediate interpretation, recall that every point in $L(G)$ admits a canonical coding by elements of G. Namely roughly speaking, to each $\xi \in L(G)$ we may associate a sequence $(g_n^{(\xi)})$ of elements in G such that the ray s_ξ from the origin to ξ intersects $g_n^{(\xi)}(F)$ in succession. Here, F denotes the Dirichlet fundamental domain at the origin. (Note that $g_1^{(\xi)} = \mathrm{id.}$ for each $\xi \in L(G)$).

For $\xi \in L(G)$ we define the *average homological increment* $\kappa(\xi)$ at ξ by (if the limit exists)

$$\kappa(\xi) := \lim_{n\to\infty} \frac{d(0, g_n^{(\xi)}(0))}{n}.$$

In this terminology, the level sets considered in Theorem 1.2 have the interpretation

$$\mathcal{F}_G(a) = \{\xi \in L(G) : \kappa(\xi) = a\}.$$

Hence our analysis gives precise estimates for the Hausdorff dimension of the set of limit points with a fixed prescribed average geometric increment, where the estimates are in terms of the Legendre transformation of the pressure function (as stated in Theorem 1.2).

Interpretation in terms of geodesic flow. Our analysis in section 3, which uses the method of cocycles, admits in particular the following. Let m be some fixed element of the hyperbolic manifold $M := \mathbb{D}^{N+1}/G$ associated with G. Also, let $\mathcal{UT}_m(M)$ denote the unit tangent bundle at m, and let $\mathcal{L}_m(M)$ be the set of generating geodesic loops at m (corresponding to the generators of G as well as the elements in the stabilizers of parabolic fixed points of G). Clearly, there is a 1-1-correspondence between elements $\xi \in \mathcal{UT}_m(M)$ and geodesic rays $\sigma_\xi \subset M$ initiating from m. In particular, each such ray admits a unique coding by a sequence $(l_1(\xi), l_2(\xi), \ldots)$ of elements in $\mathcal{L}_m(M)$. That is, roughly speaking, σ_ξ has a closest quasi-geodesic ray which is determined by performing in succession the homological movements $l_1(\xi), l_2(\xi), \ldots$. For fixed $\xi \in \mathcal{UT}_m(M)$, we consider the sequence $(\xi_1, \xi_2, \xi_3, ...)$ given by the property that $\xi_i \in \mathcal{UT}_m(M)$ is coded by $(l_i(\xi), l_{i+1}(\xi), \ldots)$, for each $i \in \mathbb{N}$. Note that the set of rays $\{\sigma_{\xi_i} : i \in \mathbb{N}\}$ represents a set of asymptotic geodesic movements in M. Let $W_i := \{\eta \in \mathcal{UT}(M) : \lim_{t\to\infty} d(\pi(\phi_t \eta), \pi(\phi_t \xi_i)) = 0\}$ denote the strong stable manifold at ξ_i (where $\mathcal{UT}(M)$ refers to the unit tangent space of M, ϕ_t to the geodesic flow on $\mathcal{UT}(M)$, and π to the canonical projection of $\mathcal{UT}(M)$ onto M). We then have for each $\xi \in \mathcal{UT}_m(M)$ that there exists a sequence $(t_i(\xi))$ of real numbers such that $\phi_{t_i(\xi)}(W_i) = W_{i+1}$. In this terminology, the level sets considered in Theorem 1.2 have the interpretation

$$\mathcal{F}_G(a) = \{\xi \in \mathcal{UT}_m(M) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} t_i(\xi) = a\}.$$

In this way Theorem 1.2 determines the Hausdorff dimensions of sets of elements with a fixed prescribed average growth of the sum of the distances of successive crossings of strong stable manifolds arising from the associated quasi-geodesics.

1.1. *Aspects of thermodynamical phase transition.* Recall that if for a dynamical system the left and right derivatives of the pressure function do not coincide at some value s, then the system is said to have a thermodynamical phase transition at s. First examples of dynamical systems exhibiting phase transition were obtained in [**Hof77**], [**Wal78**], and [**PS92**]. For any hyperbolic system arising from some expanding map T, Makarov [**Mak99**] gave the following dynamical characterization of the existence of this type of phase transition.

> An expanding map T exhibits a thermodynamical phase transition if and only T has a 'dominant periodic point' z_0. That is, if there exists $\kappa > 0$ such that for the multiplier $\lambda(w)$ of any periodic point $w \neq z_0$ we have that $\lambda(w) \leq \lambda(z_0) - \kappa$. (Here, the multiplier $\lambda(z)$ of a periodic point z of period l is defined by $\lambda(z) := |(T^l)'(z)|^{1/l}$).

For expansive dynamical systems the results in this paper admit the following characterization of the existence of a thermodynamical phase transition, which is in spirit of Makarov's observation for the expanding case. Note that our characterization is given in terms of G, and we remark that an analogous statement can be given for instance for parabolic rational maps, using the results in [**ADU93**] and [**SU00**] (see also [**Byr98**] and [**Smi96**]). For p an element of the set $\mathcal{P}$ of parabolic fixed points of G, let γ_p denote the generator of the stabilizer of p in G. We then define

$$\tau(G) := \inf_{p\in\mathcal{P}} \lim_{n\to\infty} \left| \frac{\log || \left(\gamma_p^n\right)' ||}{\log n} \right|,$$

where $|| \cdot ||$ refers to the supremum-norm. It is well-known that for any Kleinian group with parabolic elements we have that $\tau(G) = 2$ (cf. e.g. [**Str95b**], [**Flo80**]).

Also, for arbitrary $\xi \in L(G)$ and positive t, and for $d_\xi(t)$ denoting the length of the current cusp excursion at time t (while travelling from the origin towards ξ), we define †

$$\sigma(\xi) := \limsup_{t\to\infty} \frac{d_\xi(t)}{\log t}.$$

As an immediate consequence of the Khintchine law for the δ-conformal measure μ (also referred to as logarithmic law of geodesics; cf. [**SV95**], [**Sul82**]) we have that

$$\sigma(\xi) = \sigma(G) := \left(\delta - \frac{k_{\max}}{2}\right)^{-1} \quad \text{for } \mu\text{-almost every } \xi \in L(G).$$

Now, a combination of these results and Theorem 1.2 gives the following.

> *A geometrically finite, essentially free Kleinian group G with parabolic elements exhibits a thermodynamical phase transition if and only if $\sigma(G) < \tau(G)$.*

† Note that for the purposes in this paper the 'inf' would not be necessary. We give the definition in a way that it is also immediately applicable to parabolic rational maps, where the distortion of the map close to a parabolic fixed points very much depends on the number of petals associated with the parabolic point.

1.2. *Comparing thermodynamical and weak multifractal phase transitions.* Finally, we comment on the relationship between thermodynamical and weak multifractal phase transition. Again, we only consider the case of a Kleinian group G with parabolic elements, and remark that analogous statements can be derived for parabolic rational maps (combining the results of [**ADU93**] and [**SU99**]). One of the original motivations which lead to the work in this paper was to clarify the relationship between these two types of phase transition. The outcome here is that we found there is no such relation between these two phenomena. Roughly speaking, thermodynamical phase transition is in essence primarily an "ergodic phenomena", whereas weak multifractal phase transition is a purely "geometric phenomena". A first indication for this outcome is that the level sets $\mathcal{F}_G(a)$ have no immediate canonical description in pure terms of cusp excursions. Unlike the lower lim-sup spectra, which can be described exclusively by means of cusp excursions (cf. [**Str95a**], [**Str99**]), the sets $\mathcal{F}_G(a)$ represent a subtle intertwining of 'word lengths' of elements of the group and their 'geometrical sizes'. In order to give a clearer indication, we add to the above characterization of the existence of thermodynamical phase transition a corresponding characterization for weak multifractal case.

Recall that the 'lower lim-sup spectrum' of the δ-conformal measure μ is given by

$$D_\theta(\mu) := \dim_H\{\xi \in L(G) : \limsup_{r\searrow 0} \frac{\log \mu(B(\xi, r))}{\log r} \geq \theta\}.$$

For s in the interior of the range of possible scaling exponents of μ, the group G is said to exhibit a weak multifractal phase transition at s if and only if $D_\theta(\mu)$ is not differentiable at $\theta = s$.

In this terminology, the main result of [**Str95a**] can be stated as follows.

> A geometrically finite Kleinian group G with parabolic elements exhibits a weak multifractal phase transition if and only if $\delta > 1$ and if additionally G has a rank 1 parabolic fixed point.

Hence, this clearly shows that the existence of these two types of phase transitions is governed by completely different rules. For instance if G has exclusively rang two cusps and $\delta > 3/2$ then there exists a thermodynamical but no weak multifractal phase transition. On the other hand if G has rang one cusps as well as rang two cusps and $1 < \delta < 3/2$, then there exists a weak multifractal but no thermodynamical phase transition.

2. *General Formalisms for averages*

Throughout, let (X, σ, d) denote a transitive subshift of finite type over a finite alphabet containing at least two elements (we refer to [**DGS76**] for the details concerning subshifts of finite type). Here we have equipped X with the metric d which is given for $x, y \in X$ by $d(x, y) := 2^{-\sup\{i\in\mathbb{N}: x_i = y_i\}}$. For $n \in \mathbb{N}$, we let $[x_0, \ldots, x_{n-1}] := \{y = (y_0, y_1, \ldots) \in X : y_i = x_i;\ \text{for all } 0 \leq i < n\}$ denote a *cylinder set of length* n, or simply an n*–cylinder*, with the convention that the cylinder of length 0 is equal to the space X. Let $\mathfrak{Z}_n$ denote the set of all n-cylinders. The set of all cylinder sets generates the Borel σ-algebra $\mathcal{B}$ of X. Note that for every $x \in X$ and $n \in \mathbb{N}$, there exists a unique n-cylinder

$C_n(x)$ which contains x. Furthermore, we let $\mathcal{M}(X)$ denote the set of Borel probability measures on X, and we refer to the set of shift invariant measures as

$$\mathcal{M}(X,\sigma) := \left\{\mu \in \mathcal{M}(X) : \forall B \in \mathcal{B} \;\; \mu(B) = \mu(\sigma^{-1}B)\right\}.$$

2.1. *Generalized thermodynamical formalism.* In this section we elaborate a thermodynamical formalism for certain families $\mathbf{\Delta}$ of continuous functions. First we define the notion of pressure $\mathcal{P}(\mathbf{\Delta})$ in this context, for which we then derive a variational principle. Subsequently, we investigate properties of the map $P : s \mapsto \mathcal{P}(-s\Delta)$, such as its continuity and convexity. Furthermore, we show that P may be derived alternatively as the limit of a sequence of real–analytic functions. By convexity, P may fail to be differentiable on a set which can be at most countable. However, we show that at each point in the domain of P there exist explicit variational formulae for the left and right derivatives of P. — We shall use the standard notation for Birkhoff sums of a real–valued function f, that is we let $S_0 f := 0$ and $S_n f := \sum_{i=0}^{n-1} f \circ \sigma^i$ for $n \geq 1$.

Definition 2.1. A tuple $(\mathbf{\Delta}, \boldsymbol{A})$ of sequences of continuous functions $\mathbf{\Delta} := \{\Delta_n : X \to \mathbb{R}\}$ and $\boldsymbol{A} := \{A_n : X \to \mathbb{R}\}$ is said to be an element of the class $\mathcal{D}$ if the following holds.

(a) For each $n \in \mathbb{N}$ we have that Δ_n depends only on the first n coordinates. That is, we have $\Delta_n(x) = \Delta_n(y)$ for all $x, y \in C$ and all $C \in \mathfrak{Z}_n$.

(b) $\mathbf{\Delta}$ has the *average growth property (with respect to $\boldsymbol{A}$).* That is, for each $n, k \in \mathbb{N}$ there exists a positive number $c_{n,k}$ such that, with $c_k := \limsup_{n\to\infty} c_{n,k}/n < \infty$,

$$\|S_n A_k - \Delta_n\| \leq c_{n,k} \;\; \text{and} \;\; \lim_{k\to\infty} c_k = 0.$$

Since $\Delta_n(x)$ is independent of the particular choice of $x \in C \in \mathfrak{Z}_n$, we shall often use the notation $\mathbf{\Delta}(C) := \Delta_n(x)$, where we make the convention that $\mathbf{\Delta}(X) = 0$. Also, we remark that by a straight forward calculation, the average growth property implies that $\|\Delta_n\| \ll n$. Here we have used the common notation $a_n \ll b_n$, which refers to that a_n/b_n is uniformly bounded from above. (Also, we shall use the notation $a_n \asymp b_n$ if a_n/b_n and b_n/a_n are both uniformly bounded from above.)

Definition. For a family $\mathbf{\Delta} := \{\Delta_n : X \to \mathbb{R}\}$ of continuous functions which fulfils the property (a) in Definition 2.1, we define the sequence

$$\mathcal{P}_n(\mathbf{\Delta}) := \frac{1}{n} \log \sum_{C\in\mathfrak{Z}_n} \exp\left(\mathbf{\Delta}(C)\right).$$

If this sequence converges, then we shall write $\mathcal{P}(\mathbf{\Delta}) := \lim_{n\to\infty} \mathcal{P}_n(\mathbf{\Delta})$ and refer to this limit as the *pressure* of $\mathbf{\Delta}$.

Remark. For ease of notation, if φ is an element of the set $\mathcal{C}(X)$ of real–valued, continuous functions on X, then we shall use the notation $\mathcal{P}(\varphi)$ to denote the pressure of the family $\left\{\sum_{C\in\mathfrak{Z}_n} \sup_{x\in C} S_n\varphi(x)\, \mathbf{1}_C\right\}$. Note that in this case our definition coincides with the classical definition of the pressure for continuous functions (cf [**Wal82**]). Hence, we have for the topological entropy h_{top} that $h_{\text{top}} = \mathcal{P}(0)$, which necessarily has to be positive (since (X, σ, d) is transitive) (cf [**DGS76**]).

PROPOSITION 2.1. *Let $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$. There exists a map*

$$P : \mathbb{R} \to \mathbb{R} \quad \textit{given by} \quad s \mapsto \mathcal{P}(-s\,\boldsymbol{\Delta}).$$

Furthermore, for each $k \in \mathbb{N}$ there is a real–analytic map

$$P_k : \mathbb{R} \to \mathbb{R} \quad \textit{given by} \quad s \mapsto \mathcal{P}(-s\,A_k),$$

such that P_k converges to P uniformly on compact subset of the real line, for k tending to infinity. In particular, the functions P_k and P are continuous, convex and decreasing.

Proof. Let $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$ be fixed. For each $s \in \mathbb{R}$, we then have that also $(s\,\boldsymbol{\Delta}, s\,\boldsymbol{A}) \in \mathcal{D}$. The stated properties of P_k are well-known, and we refer to [**KG92**] for their derivation.

For $n, k, \ell \in \mathbb{N}$ the inequality in Definition 2.1 (b) implies

$$\|S_n A_k - S_n A_\ell\| \leq c_{n,k} + c_{n,\ell}.$$

A straight forward calculation then gives that

$$|\mathcal{P}(-sA_k) - \mathcal{P}(-sA_\ell)| \leq |s|\ (c_k + c_\ell)\,.$$

It follows that $\mathcal{P}(-s\,A_k)$ is a Cauchy sequence, and hence that it converges.

By a similar argument we obtain

$$\limsup_{n\to\infty} |\mathcal{P}_n(-s\,\boldsymbol{\Delta}) - \mathcal{P}(-s\,A_k)| \leq |s|\,c_k.$$

This shows that the limits of these two expressions coincide. In particular, we hence have for $s \in \mathbb{R}$ that

$$\mathcal{P}(-s\,\boldsymbol{\Delta}) = \lim_{n\to\infty} \mathcal{P}_n(-s\,\boldsymbol{\Delta}) = \lim_{k\to\infty} \mathcal{P}(-s\,A_k) = P(s).$$

Since every pointwise convergent sequence of convex functions converges uniformly on compact subsets to a convex function, the proposition follows. □

Remark. If A_k is strictly positive, then the range of P_k is equal to $\mathbb{R}$. Since additionally P_k is continuous and convex, it follows that there exists a unique zero δ_k of P_k. In particular, δ_k has to be positive (which follows from the fact that $P_k(0) = h_{\text{top}} > 0$).

Now, for each $k \in \mathbb{N}$ we define the $(-A_k)$*–energy* of invariant probability measures

$$\alpha_k : \mathcal{M}(X, \sigma) \to \mathbb{R} \quad \text{given by} \quad \nu \mapsto -\int A_k\,d\nu.$$

The functions α_k will be crucial throughout, and we now first investigate some of their properties.

LEMMA 2.1. *For each $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$, the sequence α_k is uniformly bounded and converges uniformly on $\mathcal{M}(X, \sigma)$ to a continuous linear function*

$$\alpha : \mathcal{M}(X, \sigma) \to \mathbb{R}.$$

Proof. The fact that (α_k) is uniformly bounded is an immediate consequence of the average growth property. In order to prove the remaining assertion, let $\nu \in \mathcal{M}(X,\sigma)$ and $k, \ell, n \in \mathbb{N}$ be fixed. By the average growth property, we have that

$$\begin{aligned}\left|\int A_k\, d\nu - \int A_\ell\, d\nu\right| &= \frac{1}{n}\left|\int S_n A_k\, d\nu - \int S_n A_\ell\, d\nu\right| \\ &\leq \frac{1}{n}\left(c_{n,k} + c_{n,\ell}\right).\end{aligned}$$

Letting n tending to infinity, it follows that $|\alpha_k(\nu) - \alpha_\ell(\nu)| \leq c_\ell + c_k$. Hence, the sequence (α_k) is a Cauchy sequence which converges uniformly on $\mathcal{M}(X,\sigma)$. Combining this observation and the fact that (α_k) is uniformly bounded, we derive that the limit α is a continuous linear function. □

PROPOSITION 2.2 (VARIATIONAL PRINCIPLE) *For* $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$ *we have that*

$$\mathcal{P}(-\boldsymbol{\Delta}) = \sup\left\{h_\nu + \alpha(\nu) : \nu \in \mathcal{M}(X,\sigma)\right\}$$

Proof. By the 'classical variational principal' (cf [**DGS76**]), we have for any $\nu \in \mathcal{M}(X,\sigma)$ that

$$\mathcal{P}(-A_k) \geq h_\nu + \alpha_k(\nu).$$

Letting k tend to infinity, Proposition 2.1 and Lemma 2.1 imply that

$$\mathcal{P}(-\boldsymbol{\Delta}) \geq h_\nu + \alpha(\nu).$$

Now consider an equilibrium measure $\tilde{\mu}_k$ for the potential $-A_k$, and let μ be a weak limit of the sequence $(\tilde{\mu}_k)$. Then there exists a sequence (ε_k) with $\lim \varepsilon_k = 0$, such that the following holds.

$$\begin{array}{ll} |\mathcal{P}(-\boldsymbol{\Delta}) - \mathcal{P}(-A_k)| \leq \varepsilon_k/4 & \text{(by Proposition 2.1)}, \\ \|\alpha - \alpha_k\|_{\mathcal{M}(X,\sigma)} \leq \varepsilon_k/4 & \text{(by uniform convergence of the } \alpha_k), \\ |\alpha(\tilde{\mu}_k) - \alpha(\mu)| \leq \varepsilon_k/4 & \text{(by continuity of } \alpha), \\ h_\mu \geq h_{\tilde{\mu}_k} - \varepsilon_k/4 & \text{(by lower semi-continuity of the entropy)}. \end{array}$$

Using these observations, it follows that

$$\begin{aligned}\mathcal{P}(-\boldsymbol{\Delta}) &\leq \mathcal{P}(-A_k) + \varepsilon_k/4 \\ &= h_{\tilde{\mu}_k} + \alpha_k(\tilde{\mu}_k) + \varepsilon_k/4 \\ &\leq h_\mu + s\,\alpha(\mu) + \varepsilon_k.\end{aligned}$$

By letting k tend to infinity, the proposition follows. □

Remark. The proof of Proposition 2.2 in fact shows that for every $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$ the set of equilibrium states $\mathcal{M}_{-\boldsymbol{\Delta}} := \{\nu \in \mathcal{M}(X,\sigma) : \mathcal{P}(-\boldsymbol{\Delta}) = h_\nu + \alpha(\nu)\}$ is non–empty and convex. Furthermore, note that since the entropy is lower semi-continuous, we clearly have that $\mathcal{M}_{-\boldsymbol{\Delta}}$ is a compact subset of $\mathcal{M}(X,\sigma)$.

PROPOSITION 2.3. *With P^- and P^+ denoting the left and right derivatives of P, we have for each $t \in \mathbb{R}$ that*

$$P^-(t) = \inf_{\nu \in \mathcal{M}_{-t\boldsymbol{\Delta}}} \alpha(\nu) \quad \textit{and} \quad P^+(t) = \sup_{\nu \in \mathcal{M}_{-t\boldsymbol{\Delta}}} \alpha(\nu).$$

Furthermore, if we let $a_\pm := \lim_{t \to \pm\infty} P(t)/t$, then it follows that

$$a_- = \inf_{\nu \in \mathcal{M}(X,\sigma)} \alpha(\nu) \quad \textit{and} \quad a_+ = \sup_{\nu \in \mathcal{M}(X,\sigma)} \alpha(\nu).$$

Proof. By the variational principle, we have for every $\nu \in \mathcal{M}_{-t\boldsymbol{\Delta}}$ and positive ε that

$$\frac{P(t+\varepsilon) - P(t)}{\varepsilon} \quad \geq \quad \frac{h_\nu + (t+\varepsilon)\alpha(\nu) - (h_\nu + t\alpha(\nu))}{\varepsilon} = \alpha(\nu),$$

as well as

$$\frac{P(t) - P(t-\varepsilon)}{\varepsilon} \quad \leq \quad \frac{h_\nu + t\alpha(\nu) - (h_\nu + (t+\varepsilon)\alpha(\nu))}{\varepsilon} = \alpha(\nu).$$

This implies that $P^+(t) \geq \sup_{\nu \in \mathcal{M}_{-t\boldsymbol{\Delta}}} \alpha(\nu)$ and that $P^-(t) \leq \inf_{\nu \in \mathcal{M}_{-t\boldsymbol{\Delta}}} \alpha(\nu)$. Thus, if P is differentiable at $t \in \mathbb{R}$, then $P'(t) = \alpha(\nu)$ for all $\nu \in \mathcal{M}_{-t\boldsymbol{\Delta}}$. We now show how to derive the opposite inequality for P^+only (for P^- this can be obtained in an analogous way). Since the set of points at which P is not differentiable is countable, there exists a decreasing sequence (s_k) tending to s, such that P is differentiable at s_k. It follows that $P'(s_k) = \alpha(\nu_k)$, for some $\nu_k \in \mathcal{M}_{-s_k\boldsymbol{\Delta}}$. Clearly, P does not necessarily have to be differentiable at the point s itself. Let $\nu \in \mathcal{M}(X,\sigma)$ be a weak accumulation point of the sequence (ν_k). Since the entropy is lower semi-continuous, there exists a sequence (ε_k) tending to zero such that $h_\nu \geq h_{\nu_k} - \varepsilon_k$. Using the fact that $P(s_k) = h_{\nu_k} + s_k\alpha(\nu_k)$, and also the facts that P and α are continuous, we obtain for k sufficiently large that

$$\begin{aligned} h_\nu - s\alpha(\nu) \quad &\geq \quad h_{\nu_k} - \varepsilon_k + (s_k - |s_k - s|)(\alpha(\nu_k) - |\alpha(\nu) - \alpha(\nu_k)|) \\ &= \quad P(s_k) - \varepsilon_k - |s_k - s|\,\alpha(\nu_k) - (s_k - |s_k - s|)\,|\alpha(\nu) - \alpha(\nu_k)|\,. \end{aligned}$$

If we let k tend to infinity, then the latter expression tends to $P(s)$. Hence, it follows that $\nu \in \mathcal{M}_{-s_k\boldsymbol{\Delta}}$.

In order to prove the first equality in the second part of the proposition, we combine the convexity of P and the first part of the proposition. In this way we conclude that $\inf_{\nu \in \mathcal{M}(X,\sigma)} \alpha(\nu) \leq a_-$. On the other hand, for each $\nu \in \mathcal{M}(X,\sigma)$ and $t \in \mathbb{R}$, the variational principle gives rise to the inequality $P(t) \geq h_\nu + t\alpha(\nu)$. Hence, it follows that $a_- = \lim_{t \to -\infty} P(t)/t \leq \alpha(\nu)$, which implies the first equality in the second part of the proposition. The second equality is derived by similar means, and we omit its proof. □

2.2. *Multifractal formalism.* In this section we develop a multifractal formalism for growth rates of *families of distance functions* $\boldsymbol{\Delta} = \{\Delta_n : X \to (0,\infty)\}$, that is an increasing family of positive functions such that $\{\Delta_n(x)\}$ is unbounded for every $x \in X$. For this we first introduce various notions of fractal dimensions, most importantly the Hausdorff dimension $\dim_{\boldsymbol{\Delta}}$ associated with a certain metric $d_{\boldsymbol{\Delta}}$ on the shift space. We then prove the main result of this section. Roughly speaking, we show that for each positive value a in the domain of the Legendre transform of P, the Hausdorff dimension of the set

$$\mathcal{F}(a) := \left\{ x \in X : \lim_{n\to\infty} \frac{\Delta_n(x)}{n} = a \right\}$$

is given by the quotient of the Legendre transform at a and a. In order to obtain this result we formulate two major observations. The first of these is that $\dim_{\mathbf{\Delta}}$ admits approximations by measure theoretical dimensions derived from certain weak Gibbs measures. The second observation consists of a formula which is similar to the well-known Young formula [**You82**]. Our approach here generalizes a construction of Cajar in [**Caj81**].

Definition. For a family of distance functions $\mathbf{\Delta}$ and a non–atomic probability measure μ, the $\mathbf{\Delta}$–dimension and the μ–dimension of a set $K \subset X$ are defined by

$$\begin{aligned}
\mathbf{\Delta}\text{-}\dim(K) &:= \inf_{\nu\in\mathcal{M}(X)} \sup_{x\in K} \liminf_{n\to\infty} \frac{\log \nu(C_n(x))}{-\Delta_n(x)}, \\
\mu\text{-}\dim(K) &:= \inf_{\nu\in\mathcal{M}(X)} \sup_{x\in K} \liminf_{n\to\infty} \frac{\log \nu(C_n(x))}{\log \mu(C_n(x))}.
\end{aligned}$$

For a family of distance functions $\mathbf{\Delta}$, let $d_{\mathbf{\Delta}}$ denote the metric on X which is given for $x, y \in X$ by

$$d_{\mathbf{\Delta}}(x,y) := \inf\left\{ \mathrm{e}^{-\Delta_n(x)} : x \in C_n(y), n \in \mathbb{N} \right\}.$$

The Hausdorff dimension $\dim_{\mathbf{\Delta}}$ with respect to the metric $d_{\mathbf{\Delta}}$ is then defined as follows. For $K \subset X$ we let

$$\dim_{\mathbf{\Delta}}(K) := \sup\left\{ s : H^s(K) = 0 \right\},$$

where $H^s(K) := \inf\left\{ \sum_{i\in I} |C_i|^s : I \text{ is countable}, C_i \text{ are cylinder sets, } K \subset \bigcup_{i\in I} C_i \right\}$.

Here we have used $|C|$ to denote the diameter of C with respect to the metric $d_{\mathbf{\Delta}}$. Clearly, for each positive t, we have that $\mathbf{\Delta}$ is a family of distance functions if and only if $t\,\mathbf{\Delta}$ has this property. Hence, we immediately have that

$$\dim_{t\,\mathbf{\Delta}} = t^{-1} \dim_{\mathbf{\Delta}}.$$

LEMMA 2.2. *For each family of distance functions $\mathbf{\Delta}$ we have that $\mathbf{\Delta}\text{-}\dim = \dim_{\mathbf{\Delta}}$.*

Proof. We first prove the inequality '$\mathbf{\Delta}\text{-}\dim \geq \dim_{\mathbf{\Delta}}$'. Let $K \subset X$ and $s > \mathbf{\Delta}\text{-}\dim(K)$ be fixed. By definition of the $\mathbf{\Delta}$–dimension, there exists $\nu \in \mathcal{M}(X)$ such that the following holds. For each $x \in K$ there exists $n(x) \in \mathbb{N}$ such that

$$s \geq \frac{\log \nu(C_{n(x)}(x))}{-\Delta_{n(x)}(x)}.$$

Now, the family $\mathfrak{A} := \left\{ C_{n(x)}(x) : x \in K \right\}$ represents a countable disjoint cover of K. This implies that

$$\sum_{Z\in\mathfrak{A}} |Z|^s \leq \sum_{Z\in\mathfrak{A}} \mathrm{e}^{-s\mathbf{\Delta}(Z)} \leq \sum_{Z\in\mathfrak{A}} \nu(Z) \leq 1.$$

Hence, it follows that $\dim_{\mathbf{\Delta}}(K) \le s$.

In order to prove the reverse inequality we adopt the proof of [**Caj81**, Theorem 3.3]. The aim is to show that $s > \dim_{\mathbf{\Delta}}(K)$ implies that $\mathbf{\Delta}$-$\dim(K) \le s$. For this let $s > \dim_{\mathbf{\Delta}}(K)$ be fixed. Note, we trivially have that $H^s(K) = 0$. We then define a sequence $(\mathfrak{B}_n)$ of families of cylinder sets by induction as follows. Let $\mathfrak{B}_0 := \{X\}$, and assume that $\mathfrak{B}_n$ has already been constructed such that the following properties are satisfied.

(i) The cylinder sets in $\mathfrak{B}_n$ are pairwise disjoint and are of length at least n.

(ii) $K \subset \bigcup_{B \in \mathfrak{B}_n} B$ and $K \cap B \neq \emptyset$ for all $B \in \mathfrak{B}_n$.

By monotonicity of Hausdorff dimension we clearly have that $H^s(K \cap B) = 0$, for all $B \in \mathfrak{B}_n$. Hence, there exists a countable family $\left\{A_i^B : i \in I_B \subset \mathbb{N}\right\}$ of pairwise disjoint cylinder sets such that

(iii) $K \cap B \subset \bigcup_{i \in I_B} \left(A_i^B \cap K\right)$ and $\sum_{i \in I_B} \exp\left(-s\mathbf{\Delta}(A_i^B)\right) < \exp\left(-s\mathbf{\Delta}(B)\right)$.

We then define $\mathfrak{B}_{n+1} := \left\{A_i^B : i \in I_B, B \in \mathfrak{B}_n\right\}$, which clearly satisfies (i) and (ii).

The next step is to define a set function μ by induction as follows. Let $\mu(X) := 1$, and assume μ is already defined on $\mathfrak{B}_n$. For $A_i^B \in \mathfrak{B}_{n+1}$ and $C \in \mathfrak{Z}_n$ let

$$\mu(A_i^B) := \frac{\mu(B) \ \exp\left(-s\mathbf{\Delta}(A_i^B)\right)}{\sum_{j \in I_B} \exp\left(-s\mathbf{\Delta}(A_j^B)\right)} \quad \text{and} \quad \mu(C) := \sum_{C \supset B \in \mathfrak{B}_n} \mu(B)$$

This gives a normalized, non–negative, additive set function on the algebra of all cylinder sets. Hence by the Kolmogorov consistency theorem there exists a unique continuation of μ to a probability measure on $(X, \mathcal{B})$ which we shall also denote by μ.

Using (iii), we obtain by induction for all $n \in \mathbb{N}$ and $B \in \mathfrak{B}_n$ that

$$\mu(B) \ge \exp(-s\mathbf{\Delta}(B)).$$

It then follows, for $x \in K$ and with some increasing sequence (n_k) such that $C_{n_k}(x) \in \mathfrak{B}_k$,

$$\liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{-\Delta_n(x)} \le \liminf_{k\to\infty} \frac{\log \mu(C_{n_k}(x))}{-\Delta_{n_k}(x)} \le s.$$

□

PROPOSITION 2.4. *Let $(\mathbf{\Delta}, \{A_n : X \to (0,\infty)\}) \in \mathcal{D}$ such that $\mathbf{\Delta}$ is a family of distance functions. For positive c we define $K_c := \{x \in X : \liminf \Delta_n(x)/n \ge c\}$. If δ_k denotes the unique zero of P_k and μ_k a non–atomic weak Gibbs measure for the potential $(-\delta_k A_k)$ (see remark below), then we have for $L \subset K_c$ that*

$$\lim_{k\to\infty} \mu_k\text{-}\dim(L) \cdot \delta_k = \dim_{\mathbf{\Delta}}(L).$$

Remark. For the definition of weak Gibbs measures we refer to the appendix. Note that for every continuous potential there exists a weak Gibbs measure (the proof of this fact follows from a straight forward adaptation of the proof for the full shift given in [**Kes01**]). Also note that since we have that $\sup_{x\in X} -\delta_k A_k(x) < 0 = \mathcal{P}(-\delta_k A_k)$, it follows that the associated weak Gibbs measure is non–atomic. This follows immediately from the definition of a weak Gibbs measure.

Proof. By Lemma 2.2 we have that $\mathbf{\Delta}\text{-dim} = \dim_{\mathbf{\Delta}}$. By definition of $\mathbf{\Delta}$-dim and μ_k-dim, it is hence sufficient to prove that

$$\lim_{k\to\infty} \sup_{x\in K_c} \limsup_{n\to\infty} \left| \frac{\log \mu_k(C_n(x))}{-\delta_k \Delta_n(x)} - 1 \right| = 0.$$

Using the average growth property, we have for each $x \in K_c$ that

$$\begin{aligned} \frac{\log \mu_k(C_n(x))}{-\delta_k \Delta_n(x)} &= \frac{S_n A_k(x)}{\Delta_n(x)} + \mathcal{O}(\gamma_n/(\delta_k \Delta_n(x))) \qquad \text{(weak Gibbs property)} \\ &= \frac{\Delta_n(x)}{\Delta_n(x)} + \mathcal{O}\left(\frac{c_{n,k}}{n}\right) + \mathcal{O}(\gamma_n/(\delta_k n)), \end{aligned}$$

where (γ_n) refers to the sequence which is associated with the weak Gibbs measure μ_k. If we let n tend to infinity, then the latter expression tends to $1 + \mathcal{O}(c_k)$, where the $\mathcal{O}$–term is independent of $x \in K_c$. This gives the convergence stated in the proposition. □

We require the following generalization of a theorem of Cajar. Note that for the full shift over a finite alphabet a similar result was obtained in [**Caj81**]. Also, we remark that the following proposition can be interpreted as a generalization of the well-known Young formula [**You82**] to the non–ergodic situation. The proof of this proposition is rather technical and will therefore be given in the appendix.

PROPOSITION 2.5 (CAJAR FORMULA) *For $\nu \in \mathcal{M}(X,\sigma)$ let the set $\mathcal{E}_\nu$ of ν*–generic points *be given by*

$$\mathcal{E}_\nu := \left\{ x : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i(x)} \xrightarrow{*} \nu \right\}.$$

The set $\mathcal{E}_\nu$ is non–empty, and for a non–atomic weak Gibbs measure μ_k for the strictly negative potential $-\delta_k A_k$ we have that

$$\mu_k\text{-dim}(\mathcal{E}_\nu) = \frac{-h_\nu}{\delta_k \alpha_k(\nu)}.$$

PROPOSITION 2.6. *For $(\mathbf{\Delta}, \boldsymbol{A}) \in \mathcal{D}$, $s \in \mathbb{R}$ and $a \in [-P^+(s), -P^-(s)]$, there exists a measure $\nu_a \in \mathcal{M}_{-s\mathbf{\Delta}}$ such that $-\alpha(\nu_a) = a$. Furthermore, we have that*

$$\mathcal{E}_{\nu_a} \subset \mathcal{F}(a) \quad \left(:= \{x \in X : \Delta_n(x)/n \to a\} \right).$$

Proof. By Proposition 2.3 and since $-\alpha$ is continuous on $\mathcal{M}_{-s\mathbf{\Delta}}$, it follows that for each $s \in \mathbb{R}$ and $a \in [-P^+(s), -P^-(s)]$ there exists a measure $\nu_a \in \mathcal{M}_{-s\mathbf{\Delta}}$ such that $-\alpha(\nu_a) = a$.

For the proof of the second part note that by the average growth property we have for each $n, k \in \mathbb{N}$ that

$$\left\| \frac{S_n A_k}{n} - \frac{\Delta_n}{n} \right\| \leq \frac{c_{n,k}}{n}.$$

Taking the $\limsup$ with respect to n on both sides, this implies for $x \in \mathcal{E}_{\nu_a}$ that

$$\limsup_{n\in\mathbb{N}} \left| -\alpha_k(\nu_a) - \frac{\Delta_n(x)}{n} \right| \leq c_k.$$

Hence, if we now let k tend to infinity, it follows that $x \in \mathcal{F}(a)$. □

Remark. Note that there exist two measures $\nu_{a_-}, \nu_{a_+} \in \mathcal{M}(X,\sigma)$ such that $\alpha(\nu_{a_\pm}) = a_\pm$ and $h_{\nu_{a_\pm}} \geq \hat{P}(a_\pm)$, and $\mathcal{E}_{\nu_{\alpha_\pm}} \subset \mathcal{F}(a_\pm)$. (These measures are weak limits of sequences of equilibrium measures for $-s_k\mathbf{\Delta}$, where s_k tends to $\pm\infty$.)

2.3. *Proof of Theorem 1.1.* By Proposition 2.6 there exists a measure ν_a such that $\alpha(\nu_a) = a$ and $h_{\nu_a} = P(s) + as$. Combining this and Proposition 2.5 and using the notation of Proposition 2.4 and 2.5, we obtain that

$$\mu_k\text{-dim}(\mathcal{F}(a)) \cdot \delta_k \geq \mu_k\text{-dim}(\mathcal{E}_{\nu_a}) \cdot \delta_k = \frac{as + P(s)}{-\alpha_k(\nu_a)}.$$

By applying Lemma 2.1 and Proposition 2.4, the lower bound for the Hausdorff dimension follows.

The upper bound follows by a standard covering argument as follows. Fix $s \in \mathbb{R}$ and let $a \in [a_-, a_+]$ be positive. Also, let ρ be some arbitrary positive number and choose $\kappa < a\rho/\left(\rho + P(s)/a\right)$. For each $n \in \mathbb{N}$ consider the set

$$\mathcal{F}^n_{a,\kappa} := \left\{ x \in X : \left| \frac{\Delta_n(x)}{n} - a \right| < \kappa \right\}.$$

Clearly, we have that $\mathcal{F}(a) \subset \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \mathcal{F}^n_{a,\kappa}$. Hence it is sufficient to give an upper bound for the Hausdorff dimension of the $\limsup$–set of $\mathcal{F}^n_{a,\kappa}$. For $P(s) \geq 0$ we have that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \sum_{C \in \mathcal{F}^n_{a,\kappa}\cap\mathfrak{Z}_n} \mathrm{e}^{-\mathbf{\Delta}(C)\left(s+\frac{P(s)}{a}+\rho\right)} &= \sum_{n\in\mathbb{N}} \sum_{C \in \mathcal{F}^n_{a,\kappa}\cap\mathfrak{Z}_n} \mathrm{e}^{-s\mathbf{\Delta}(C)-n\left(\frac{P(s)}{a}+\rho\right)\frac{\mathbf{\Delta}(C)}{n}} \\
&\leq \sum_{n\in\mathbb{N}} \sum_{C \in \mathcal{F}^n_{a,\kappa}\cap\mathfrak{Z}_n} \mathrm{e}^{-s\mathbf{\Delta}(C)-n\left(\frac{P(s)}{a}+\rho\right)(a-\kappa)} \\
&\leq \sum_{n\in\mathbb{N}} \sum_{C\in\mathfrak{Z}_n} \mathrm{e}^{-s\mathbf{\Delta}(C)-nP(s)-n\varepsilon},
\end{aligned}$$

where ε denotes some positive number less than $\rho\,(a-\kappa) - \kappa P(s)/a$. By definition of the pressure, the sum in the latter expression converges. This implies that $\dim_{\mathbf{\Delta}}(\mathcal{F}_a) \leq (as + P(s))/a + \rho$ for all $\rho > 0$ and $s \in \mathbb{R}$, which gives the proof in the case $P(s) \geq 0$. The proof in the remaining case requires that $\rho < |P(s)|/a$ and follows essentially by the same argument. □

2.4. *Connections to weak Gibbs measures.* In this section we consider the special situation in which there is a 'cocycle' I associated to a family $\mathbf{\Delta}$ such that there exists a non–atomic weak Gibbs measure for the potential $(-\delta I)$. Here δ denotes the smallest zero of the pressure P (see also the remark to Proposition 2.4).

LEMMA 2.3. *For $I \in \mathcal{C}(X)$ and $(\mathbf{\Delta}, \{S_k I/k\}) \in \mathcal{D}$, we have for all $s \in \mathbb{R}$ that*

$$\mathcal{P}(-sI) = P(s) \;\; (:= \mathcal{P}(-s\mathbf{\Delta})).$$

Proof. The proof of this lemma is an immediate adaptation of the proof of Proposition 2.1, and hence will be omitted. □

For the special situation in which there is a non–atomic weak Gibbs measure we now state the analogue of Theorem 1.1 as follows. In here, $\hat{P}$ denotes the Legendre transform of P, which is defined for $t \in \mathbb{R}$ by $\hat{P}(t) := \inf_{s\in\mathbb{R}} \{st + P(s)\}$.

THEOREM 2.1. *For $I \in \mathcal{C}(X)$ and $(\boldsymbol{\Delta}, \{S_k I/k\}) \in \mathcal{D}$, let μ denote a non–atomic weak Gibbs measure for the potential $(-\delta I)$. Then for each positive a in the domain of the Legendre transform $\hat{P}$ we have that*

$$\mu\text{-}\dim(\mathcal{F}(a)) \cdot \delta = \frac{\hat{P}(a)}{a} = \dim_{\boldsymbol{\Delta}}(\mathcal{F}(a)).$$

Proof. Since μ is a non–atomic weak Gibbs measure, we have by Proposition 3.2 of the appendix for any $\nu \in \mathcal{M}(X, \sigma)$ that

$$\mu\text{-}\dim(\mathcal{E}_\nu) = \frac{h_\nu}{\delta \int I\, d\nu}.$$

We also have for every $c > 0$ and $K \subset K_c := \{x \in X : \liminf \Delta_n(x)/n \geq c\}$ that

$$\mu\text{-}\dim(K) \cdot \delta = \dim_{\boldsymbol{\Delta}}(K).$$

Combining these two observations the theorem follows. □

We remark that in the special situation in which there exists a cocycle to which we then can associate a non–atomic weak Gibbs measure, the construction of the approximations μ_k in Proposition 2.4 becomes redundant.

3. *Applications to Kleinian groups*

3.1. *Preliminary concepts and observations.* In this section we consider non–elementary, geometrically finite Kleinian groups G acting on hyperbolic $(N+1)$-space $\mathbb{D}^{N+1}$ equipped with the hyperbolic metric d_h. For ease of exposition, we shall restrict the discussion to the cases $N = 1, 2$. Throughout we shall always assume that G has no elliptic elements and is an 'essentially free group', meaning that there are no relations other than those arising from rank 2 cusps. Let G_0 denote a symmetric set of generators of G, and let $\Gamma_0 \subset G_0$ be the set of generators of the stabilizer of the parabolic fixed points. If there are parabolic points of different ranks, then we decompose Γ_0 into two disjoint sets Γ_{01} and Γ_{02}, where by definition Γ_{0k} contains exclusively the generators of the stabilizers associated with cusps of rank k (for $k = 1, 2$).

3.1.1. *Codings of the limit set.*

Finite coding. It is well-known that there exists a canonical map π from a subshift of finite type X into the limit set $L(G) \subset \mathbb{S}^N := \partial\mathbb{D}^{N+1}$ of G, which is given for $x = (x_0, x_1, x_2, \ldots) \in X$ by $\pi(x) := \lim_{n\to\infty} x_0 \cdots x_n(0)$. The map π fails to be injective only for the countable set of parabolic fixed points of G, that is π is 1-1 on the radial limit set $L_r(G)$, which is the complement of the parabolic fixed points in $L(G)$ (cf **[BM74]**).

In this situation the subshift of finite type (X, σ, d) is given by

$$X = X_G := \left\{ x \in G_0^{\mathbb{N}} : A_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{N} \right\},$$

where the transition matrix A is defined as follows.

- For $g \in G_0 \setminus \Gamma_{02}$ and $h \in G_0$ we have $A_{g,h} = 1 \iff gh \neq \text{id}$.
- For $\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1} \in \Gamma_{02}$ denoting the generators of the stabilizer associated to a cusp of rank 2, we have

$$A_{\gamma_1^{\pm 1} g} = 1 \iff g \in G_0 \text{ and } \gamma_1^{\pm 1} g \neq \text{id},$$
$$A_{\gamma_2^{\pm 1} g} = 1 \iff g \in G_0 \setminus \left\{ \gamma_1, \gamma_1^{-1}, \gamma_2^{\mp 1} \right\}.$$

Infinite coding. For the proof of Theorem 1.2 in the parabolic case, we also require that there is an alternative coding of the radial limit set $L_r(G)$ by a shift space over an infinite alphabet. More precisely, in this case the countable alphabet is defined by

$$S := \{ g\gamma : \gamma \in \langle \Gamma_0 \rangle \text{, either } g \in G_0 \setminus \Gamma_0 \text{ or if } g \in \Gamma_0 \text{ then } \gamma \text{ and } g \text{ are associated to different cusps} \},$$

and the transition matrix A^* is given for arbitrary $g, h \in S$ by

$$A^*_{g,h} = 1 \iff |gh| = |g| + |h|,$$

where $|\cdot|$ refers to the word metric in G, with the convention that $|\text{id}| = 0$. The shift space (X^*, σ^*, d^*) over the infinite alphabet is then defined by

$$X^* := \left\{ x \in S^{\mathbb{N}} : A^*_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{N} \right\}.$$

Also, the set of n–cylinders in X^* is denoted by $\mathfrak{Z}_n^*$, and d^* refers to the metric on X^* which is given for arbitrary $x, y \in X^*$ by $d^*(x, y) := 2^{-\sup\{i \in \mathbb{N} : x_i = y_i\}}$. Note that X^* is *finitely primitive* in the sense of [**MU**], which means that for each $g_1, g_2 \in S$ there exists g in the finite set $\bigcup_{h \in S, |h| \leq 2} h$ such that the cylinder $[g_1, g, g_2]$ is non–empty.

Furthermore, it is convenient to introduce the following two subsets of X

$$\mathcal{G} := \bigcup_{g \in G_0 \setminus \Gamma_0} [g] \cup \bigcup_{g\gamma \in S,\ g, \gamma \in \Gamma_0} [g\gamma] \text{ and } \mathcal{H} := \{ x \in X : x_i \in \Gamma_0 \text{ for all } i \in \mathbb{N} \}.$$

We remark that the 'induced system' on $\mathcal{G}$ is topologically conjugated to (X^*, σ^*) such that σ^* is the analogue of Schweiger's jump transformation (cf [**Sch95**]).

3.1.2. *Geometric and metrical facts.* Throughout this section we shall consider the special family of distance functions $\mathbf{\Delta} := \mathbf{\Delta}_G := \{\Delta_n : X \to (0, \infty)\}_{n \in \mathbb{N}}$, which is given for $n \in \mathbb{N}$ by

$$\Delta_n := \sum_{[x_0, \ldots, x_{n-1}] \in \mathfrak{Z}_n} d_h(0, x_0 \cdots x_{n-1}(0))\, \mathbf{1}_{[x_0, \ldots, x_{n-1}]}. \tag{2}$$

LEMMA 3.1. *The* spherical metric d_s *on* $\mathbb{S}^N$ *and the metric* $d_{\mathbf{\Delta}}$ *are equivalent in the sense that there exists a positive constant* M *such that we have for all* $x, y \in X$ *that*

$$M^{-1} d_{\mathbf{\Delta}}(x,y) \leq d_s\left(\pi(x), \pi(y)\right) \leq M d_{\mathbf{\Delta}}(x,y).$$

In particular, we hence have that X and $L(G)$ are 'essentially bi–Lipschitz equivalent'. That is, there exists a bi–Lipschitz map between X and $L(G)$, which is well defined except possibly on the countable set of parabolic fixed point. As an immediate consequence of the previous lemma we have the following.

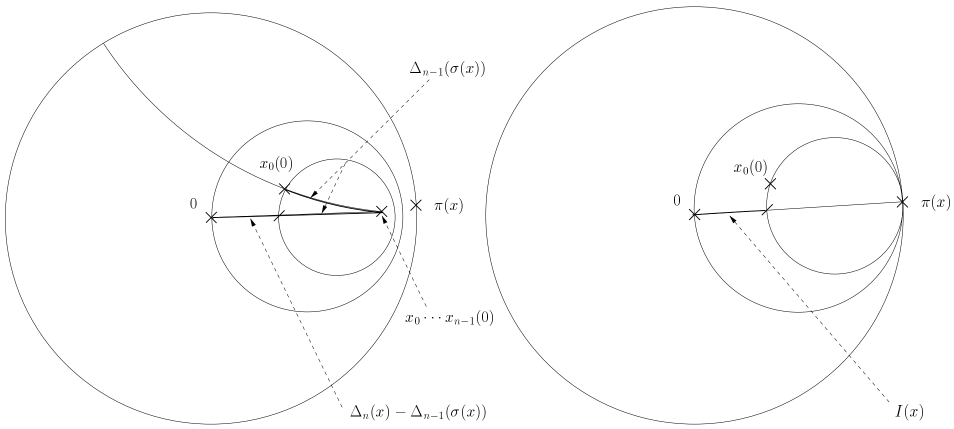
COROLLARY 3.1. *For any* $A \subset X$ *we have that* $\dim_{\mathbf{\Delta}}(A) = \dim_H(\pi(A))$.

3.1.3. *Growth properties, cocycles, and Hölder continuity.*

LEMMA 3.2 (COCYCLE LEMMA) *For any* $x \in X$ *the following limit exists*

$$I(x) := \lim_{n\to\infty} I_k(x), \quad \textit{where} \quad I_k(x) := \left(\Delta_k(x) - \Delta_{k-1}(\sigma(x))\right).$$

Here, the convergence of the I_k *is uniform. Note that since* $I(x)$ *is equal to the hyperbolic distance between* 0 *and the horoball through* $x_0(0)$ *and* $\pi(x)$*, we have that* $I : X \to \mathbb{R}$ *is continuous.*



(a) The n^{th} approximation $\Delta_n(x) - \Delta_{n-1}(\sigma(x))$ of the horocyclic distance $I(x)$. Here $x_0 \cdots x_{n-1}(0)$ is the centre of the two hyperbolic circles.

(b) The resulting situation if in (a) we let n tend to infinity.

FIGURE 4. The horocyclic distance $I(x)$.

Proof. Choose $x_0 \cdots x_{n-1}(0)$ to be the centre of the two hyperbolic spheres $S_n^0(x)$, which intersects $\{0\}$, and $S_n^1(x)$ which intersects $\{x_0(0)\}$. Then, the first has radius $\Delta_n(x)$ and the latter $\Delta_{n-1}(\sigma(x))$. If n tends to infinity, then $S_n^0(x)$ tends to the horosphere $H^0(x)$ through $\pi(x)$ and 0, and $S_n^1(x)$ tends to the horosphere $H^1(x)$ through $\pi(x)$ and

$x_0(0)$. Hence, the horocyclical distance between $H^0(x)$ and $H^1(x)$ is equal to $I(x)$ (cf FIGURE 4). Since the functions I_k are locally constant and X is compact, it follows that I_k converges uniformly to I. $\square$

Remark. Note, it is well-known that I can be written as the logarithm of the Poisson kernel $\mathbb{P}$. That is, for $x = (x_0, x_1, \ldots) \in X$ we have that

$$I(x) = |\log \mathbb{P}(x_0(0), \pi(x))| = \left| \log \frac{1 - |x_0(0)|^2}{|x_0(0) - \pi(x)|^2} \right|.$$

For the convex cocompact case we shall now see that I is Hölder continuous.

LEMMA 3.3. *If G is convex cocompact, then I is Hölder continuous. That is, there exists a positive constant κ such that for all $n \in \mathbb{N}$*

$$\sup_{C \in \mathfrak{Z}_n} \sup_{x,y \in C} |I(x) - I(y)| \ll \exp(-\kappa n).$$

Proof. First note that there exists a positive constant D such that

$$|I(x) - I(y)| \leq D \, d_s \left(\pi(x), \pi(y) \right).$$

In order to see this let $g \in G_0$ be fixed. The horocyclical distance between 0 and $g(0)$ as a function on the convex hull of $\pi([g])$ (taken with respect to d_s) is differentiable and its derivative is continuous on the closure of this convex hull. This holds for each of the finitely many generators, which then gives the above inequality.

Now, since G is convex cocompact, there exists a positive constant κ such that $n\kappa \leq \Delta_n$, and hence we have for each $n \in \mathbb{N}$, $C \in \mathfrak{Z}_n$ and $x, y \in C$ that

$$d_{\boldsymbol{\Delta}}\left(x, y \right) \leq \mathrm{e}^{-n\kappa}.$$

Combining these observations and Lemma 3.1, it follows that

$$\begin{aligned} \sup_{C \in \mathfrak{Z}_n} \sup_{x,y \in C} |I(x) - I(y)| &\leq D \sup_{C \in \mathfrak{Z}_n} \sup_{x,y \in C} d_s \left(\pi(x), \pi(y) \right) \\ &\leq MD \sup_{C \in \mathfrak{Z}_n} \sup_{x,y \in C} d_{\boldsymbol{\Delta}}(x, y) \\ &\leq MD \exp(-\kappa n). \end{aligned}$$

$\square$

Let $\iota : X^* \hookrightarrow X$ denote the canonical injection of X^* into X, which maps an infinite word written by means of the alphabet S to the same word but written by means of the alphabet G_0. We define the cocycle $I^* : X^* \to \mathbb{R}$ for the shift space on the infinite alphabet by

$$I^*(x) := \sum_{i=0}^{N(x)-1} I \circ \sigma^i(\iota(x)).$$

Here $N : X^* \to \mathbb{N}$ denotes the 'jump time', which is defined for $x := (g_1\gamma_1, g_2\gamma_2, \ldots) \in X^*$ by $N(x) := |\gamma_1| + 1$.

LEMMA 3.4. *The cocycle I^* is Hölder continuous on X^*. That is, there exists a positive constant κ such we have for $n \geq 2$ that*

$$\sup_{C \in \mathfrak{Z}_n^*} \sup_{x,y \in C} |I^*(x) - I^*(y)| \ll \exp(-\kappa\, n).$$

Proof. Let $x, y \in C \in \mathfrak{Z}_n^*$ be fixed such that $x, y \in [g\gamma]$ for some $g \in G_0, \gamma \in \langle \Gamma_0 \rangle$, $g\gamma \in S$. (The remaining cases can be dealt with exactly as in the convex cocompact case.) Also, without loss of generality we may assume that $\gamma = \gamma_1 \gamma_2$, where $\gamma_2 \in \Gamma_0$. We then have

$$\begin{aligned}
|I^*(x) - I^*(y)| &= \left| \log \frac{\mathbb{P}(g\gamma(0), \pi(x))}{\mathbb{P}(g\gamma(0), \pi(y))} \right| = 2 \left| \log \frac{|g\gamma(0) - \pi(y)|}{|g\gamma(0) - \pi(x)|} \right| \\
&= 2 \left| \log \left(1 + \frac{|g\gamma(0) - \pi(y)| - |g\gamma(0) - \pi(x)|}{|g\gamma(0) - \pi(x)|} \right) \right| \\
&\leq 2 \frac{|g\gamma(0) - \pi(y)| - |g\gamma(0) - \pi(x)|}{|g\gamma(0) - \pi(x)|} \leq 2 \frac{|\pi(x) - \pi(y)|}{|g\gamma(0) - \pi(x)|} \\
&= 2 \frac{|g\gamma_1((g\gamma_1)^{-1}(\pi(x))) - g\gamma_1((g\gamma_1)^{-1}(\pi(y)))|}{|g\gamma(0) - \pi(x)|} \\
&= \frac{\sqrt{|(g\gamma_1)'((g\gamma_1)^{-1}(\pi(x)))|}\sqrt{|(g\gamma_1)'((g\gamma_1)^{-1}(\pi(y)))|}}{\sqrt{|(g\gamma_1)'(\gamma(0))|}\sqrt{|(g\gamma_1)'((g\gamma_1)^{-1}(\pi(x)))|}} \\
&\qquad \times \frac{|(g\gamma_1)^{-1}(\pi(x)) - (g\gamma_1)^{-1}(\pi(y))|}{|\gamma(0) - (g\gamma_1)^{-1}(\pi(x)))|} \\
&= \frac{\sqrt{|(g\gamma_1)'((g\gamma_1)^{-1}(\pi(y)))|}|(g\gamma_1)^{-1}(\pi(x)) - (g\gamma_1)^{-1}(\pi(y))|}{\sqrt{|(g\gamma^{k-1})'(\gamma(0))|}|\gamma(0) - (g\gamma^{k-1})^{-1}(\pi(x)))|} \\
&\ll |(g\gamma_1)^{-1}(\pi(x)) - (g\gamma_1)^{-1}(\pi(y))| \ll e^{-(n-1)\kappa}.
\end{aligned}$$

□

The following proposition represents the key observation which will allow us to apply the theory developed in the previous sections to the setting of Kleinian groups.

PROPOSITION 3.1. *For G convex cocompact let $\boldsymbol{A}$ be equal to either $\{\Delta_k/k\}$ or $\{I_k\}$. If G has parabolic elements then let $\boldsymbol{A}$ be equal to $\{I_k\}$. For these choices we have that $(\boldsymbol{\Delta}, \boldsymbol{A}) \in \mathcal{D}$.*

Proof. In both cases the conditions in Definition 2.1 except (b) are easily verified. For G convex cocompact we consider the case $\boldsymbol{A} = \{\Delta_k/k\}$. The case $\boldsymbol{A} = \{I_k\}$ will be dealt with implicitly in the proof for the parabolic situation below. For $x \in X$, we decompose $S_n\Delta_k$ as follows:

$$S_n \Delta_k(x) = \sum_{j=0}^{k-1} T_j(x), \quad \text{where} \quad T_j(x) := \sum_{i=0}^{[n/k]} \Delta_k \left(\sigma^{ik} \left(\sigma^j(x) \right) \right).$$

Note that the summands in the latter expression are the lengths of quasi–geodesics contained in a tubular neighbourhood of the geodesic segment between 0 and

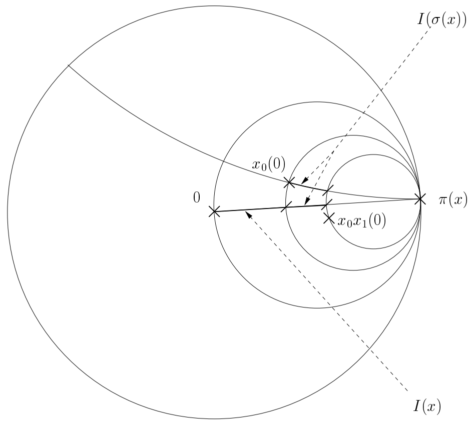


FIGURE 5. The geometric realization of $S_n I(x)$.

$x_0 \cdots x_{n-1}(0)$ (where the width of the tubular neighbourhood is equal to twice the diameter C_0 of the convex core). Using the triangle inequality for the hyperbolic metric, it follows that

$$\begin{aligned} \|S_n \Delta_k / k - \Delta_n\| &= \left\| \frac{1}{k} \sum_{j=0}^{k-1} T_j - \Delta_n \right\| \\ &\leq 2kC_0 + \frac{2n}{k} C_0. \end{aligned}$$

If we let $c_{n,k} := 2kC_0 + \frac{2n}{k}C_0$, then clearly this choice satisfies the properties stated in Definition 2.1 (b).

If G has parabolic elements, then first note that

$$\|S_n I - \Delta_n\| \ll \log n. \tag{3}$$

This follows since $S_n I(x)$ is the hyperbolic distance from the origin to the horoball through $x_0 \cdots x_{n-1}(0)$ and $\pi(x)$ (see FIGURE 5). Clearly, this distance differs from $d_h(0, x_0 \cdots x_{n-1}(0))$ by a term which is at most of order $\log n$, where the involved constant can be chosen uniformly (see [**Flo80**], [**Str95b**]). Secondly, by uniform convergence of the I_k, there exists a sequence (ε_k) tending to 0 such that $\|S_n I - S_n I_k\| \leq n\varepsilon_k$. Combining these two observation we obtain

$$\|S_n I_k - \Delta_n\| \quad \leq \quad \|S_n I - S_n I_k\| + \|S_n I - \Delta_n\| \ll n\varepsilon_k + \log n.$$

Form this we see that the average growth property is also fulfilled in the parabolic case. Note that for G convex cocompact we even have $\|S_n I - \Delta_n\| \ll 1$, which then completes the proof of the proposition.

□

3.1.4. *Finite and infinite ergodic theory.* In this subsection we collect results from finite and infinite ergodic theory which we shall require in the proof of Theorem 1.2. First results in this direction We start with defining the pressure of a continuous function f for the infinite shift space (X^*, σ^*, d^*) as follows

$$\mathcal{P}^*(f) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{C\in\mathfrak{Z}_n^*} \exp\left(\sup_{x\in C} S_n^* f\right),$$

where as usual $S_n^* f := \sum_{i=0}^{n-1} f \circ \sigma^{*\,i}$.

We state the following results which will be crucial in the proof of Theorem 1.2.

(SVP) (*Sarig's Variational Principle* ([**Sar99**])) For a Hölder continuous function f on X^* we have that

$$\mathcal{P}^*(f) = \sup\left\{h_\mu + \int f\,d\mu : \mu \in \mathcal{M}(X^*, \sigma^*) \text{ such that } -\int f\,d\mu < \infty\right\}.$$

(GM) (*Existence of Gibbs Measures* ([**MU**])) For each $(t, q) \in (-\infty, \delta) \times (0, \infty) \cup (\delta, 0)$ there exists a unique ergodic invariant Gibbs measure $\mu_{t,q}^*$ for the potential $(-tI^* - qN)$. This means that for all $n \in \mathbb{N}$ and $x \in C \in \mathfrak{Z}_n^*$ we have uniformly that

$$\mu_{t,q}^*(C) \asymp \exp\left(S_n^*\left(-tI^*(x) - qN(x)\right) - n\,P^*(-tI^* - qN)\right).$$

(KF) (*Kac' Formulae* ([**Kac47**]))

(i) Given $\mu^* \in \mathcal{M}(X^*, \sigma^*)$, then there exists a σ–invariant measure $\tilde{\mu}$, which is defined for $B \in \mathcal{B}$ by

$$\tilde{\mu}(B) := \int \sum_{i=0}^{N(x)-1} \mathbf{1}_B \circ \sigma(\iota(x))\; d\mu^*(x).$$

Clearly, if we have in particular that $\tilde{\mu}(X) < \infty$, then $\mu := \tilde{\mu}/\tilde{\mu}(X)$ defines a σ–invariant probability measure.

(ii) Given $m \in \mathcal{M}(X, \sigma)$ such that $m(\mathcal{P}) = 0$, then $m(\mathcal{G}) = m(\iota X^*)$. Hence, by defining $m^* := m|_{\mathcal{G}} \circ \iota$ we derive in this way a σ^*–invariant probability measure on X^*. Also, note that here we have that $m^*(N) = 1/m(\mathcal{G})$.

(AF) (*Abramov's Formula* ([**Abr59**],[**Nev69**])) Given $\mu^* \in \mathcal{M}(X^*, \sigma^*)$ such that $\mu^*(N) < \infty$, then we have that

$$h_\mu = \frac{h_{\mu^*}}{\mu^*(N)}.$$

(AP) (*Analytic properties of pressure* ([**HMU**])) The function

$$P^* : (-\infty, \delta) \times (0, \infty) \to \mathbb{R} \quad \text{given by} \quad (t, q) \mapsto \mathcal{P}^*(-tI_G^* - qN),$$

is convex, decreasing and real–analytic with respect to both coordinates, and in the second coordinate it is strictly decreasing to $-\infty$. This implies that there exists a positive real–analytic function β on $(-\infty, \delta)$ such that in this range $P^*(t, \beta(t)) = 0$. Furthermore, for the derivative of β we have that

$$\beta'(t) = \frac{-\int I^* \, d\mu_t^*}{\int N \, d\mu_t^*} = -\int I \, d\mu_t \quad \text{for} \quad t < \delta, \text{ and } \beta^-(\delta) = \frac{-\int I^* \, d\mu_\delta^*}{\int N \, d\mu_\delta^*},$$

where $\mu_t^* := \mu_{t,\beta(t)}^*$ denotes the unique invariant Gibbs measure for the potential $-tI^* - \beta(t)N$, and $\mu_t \in \mathcal{M}(X, \sigma)$ is the corresponding measure derived from statement (i) in (KF) (note that μ_t exists since $\tilde{\mu}_t(X) = \sum_{\ell=1}^{\infty} \ell \tilde{\mu}_t(\{N = \ell\}) \asymp \sum_{\ell=1}^{\infty} \ell^{-2t+1} e^{-\beta(t)\ell} < \infty$); the analytic properties employ the spectral theory for the Perron–Frobenius operator as developed in [**HMU**]).

3.2. *Proof of Theorem 1.2.* In Proposition 3.1 we have seen that $(\mathbf{\Delta}, \{I_k\}) \in \mathcal{D}$, and in Corollary 3.1 that $\dim_H$ coincides with the Hausdorff dimension with respect to $d_{\mathbf{\Delta}}$. Also, by the results of section 2.1 the associated pressure P is well defined. Hence, we can apply Theorem 1.1 which then gives for $a \neq 0$ that

$$\dim_H (\mathcal{F}_G(a)) = \dim_{\mathbf{\Delta}} (\mathcal{F}_{\mathbf{\Delta}_G}(a)) = \frac{\hat{P}(a)}{a}.$$

Using the estimate (3) it follows by a similar argument as in the proof of Lemma 2.1 that for the cocycle $I = I_G$ we have that $\mathcal{P}(-tI) = \mathcal{P}(-t\mathbf{\Delta})$, for all $t \in \mathbb{R}$.

In the convex cocompact case I is Hölder continuous (Lemma 3.3) and hence P is real–analytic. This is a standard result of the classical thermodynamical formalism for Hölder continuous functions as developed by Bowen and Ruelle (see for instance [**KG92**]). This completes the proof of the theorem for the convex cocompact case.

Now, assume that G has parabolic elements. First note that P is non–negative. This is an immediate consequence of the fact that we have for all $n \in \mathbb{N}$ that

$$\sum_{C \in \mathfrak{Z}_n} \exp\left(-t\, \mathbf{\Delta}(C)\right) \gg \exp\left(-2t \log n\right).$$

Since P is convex, decreasing, and has a zero at the exponent of convergence δ, it follows that $P(t) = 0$ for $t \geq \delta$.

With β denoting the function introduced in (AP), it now remains to show that

$$P(t) = \beta(t) \quad \text{for } t < \delta.$$

Let $t < \delta$ be fixed. Using the Hölder continuity of I^* and the Gibbs property of the measure $\mu_t^* := \mu_{t,\beta(t)}^*$ stated in (GM), it follows that $\mu_t^*(N) < \infty$. By (KF) there exists $\mu_t \in \mathcal{M}(X, \sigma)$ which is given for $B \in \mathcal{B}$ by

$$\mu_t(B) := \mu_t^*(N)^{-1} \int \sum_{i=0}^{N(x)-1} \mathbf{1}_B \circ \sigma(\iota(x)) \, d\mu_t^*(x).$$

Note that we have in particular that $\int I \, d\mu_t = \mu_t^*(N)^{-1} \int I^* \, d\mu_t^*$.

With these preparations we are now in the position to argue as follows.

$$\begin{aligned} P(t) &\geq h_{\mu_t} - \int tI\, d\mu_t && \text{(by the variational principle)} \\ &= (\mu_t^*(N))^{-1}\left(h_{\mu_t^*} - \int tI^*\, d\mu_t^*\right) && \text{(by (AF))} \\ &= \beta(t). \end{aligned}$$

Here, the latter equality is a consequence of the fact that μ_t^* is an equilibrium measure for $(-tI^* - \beta(t)N)$, which follows by combining (GM) and the finiteness of $\mu_t^*(tI^* + \beta(t)N)$.

In order to prove the reverse inequality let $m_t \in \mathcal{M}_{-tI}$. Note that $m_t(\mathcal{G}) > 0$. This follows since otherwise we would have $m_t(\mathcal{H}) = 1$, which would implies that both h_{m_t} and $m_t(-tI)$ vanish, giving $P(t) = 0$ and hence contradicting the fact that $P(t) \geq \beta(t) > 0$ (cf (AP)). Hence, (KF) (ii) implies

$$-\int (-tI^* - \beta(t)N)\; dm_t^* = (m_t(\mathcal{G}))^{-1}\left(\int tI\, dm_t + \beta(t)\right) < \infty.$$

We can now conclude as follows

$$\begin{aligned} 0 &\geq h_{m_t^*} - \int (tI^* + \beta(t)N)\; dm_t^* && \text{(by (SVP))} \\ &= m_t^*(N)\left(h_{m_t} - \int tI\, dm_t - \beta(t)\right) && \text{(by (AF))} \\ &= m_t^*(N)\,(P(t) - \beta(t)) && (\text{since } m_t \in \mathcal{M}_{-tI}), \end{aligned}$$

which completes the proof of the reverse inequality.

In order to examine the differentiability of P at δ, we consider the left derivative $\beta^-(\delta)$ of β at δ. By (AP) we have

$$\beta^-(\delta) = \frac{-\int I^*\, d\mu_\delta^*}{\int N\, d\mu_\delta^*}.$$

Using a straight forward geometric argument, for instance also used in the proof of the 'global measure formula' in [**SV95**], we obtain

$$\int N\, d\mu_\delta^* \asymp \sum_{n=1}^{\infty} n\, n^{k_{\max}-1}\, n^{-2\delta}.$$

Clearly, the latter expression is finite if and only if $\delta > (k_{\max}+1)/2$. In a similar way we obtain for the numerator, with some positive constant c, that

$$\int I^*\, d\mu_\delta^* \asymp c + \sum_{n=2}^{\infty} n^{k_{\max}-1}\, n^{-2\delta} \log n.$$

Here, the latter expression is positive and finite, due to the fact that $\delta > k_{\max}/2$ (cf [**Bea68**]). □

Remark. We point out that for $t \leq \delta$ the 'multifractal measures' μ_t which we derived in the proof of the theorem are in particular ergodic equilibrium measures contained in $\mathcal{M}_{-tI} = \mathcal{M}_{-t\boldsymbol{\Delta}}$.

Appendix

Let us first recall the notion of a weak Gibbs measure. For the details concerning these measures and their connections to the theory of large deviations we refer to [**Kes01**].

Definition. Let X be a transitive subshift of finite type (over a finite alphabet). A probability measure μ is *called weak Gibbs measure for the potential* $f \in \mathcal{C}(X)$ if there exists a sequence (γ_n) with $\lim_{n\to\infty} \gamma_n/n = 0$ such that for all $n \in \mathbb{N}$ and $x \in X$ we have that

$$\exp(-\gamma_n) \leq \frac{\mu\left(C_n(x)\right)}{\exp\left(S_n f(x) - n\mathcal{P}(f)\right)} \leq \exp(\gamma_n).$$

The aim of the appendix is to prove the following proposition. Clearly, the Cajar Formula in Proposition 2.5 is an immediate consequence of this proposition.

PROPOSITION 3.2. *Let X be a transitive subshift of finite type (of order k), and let $\mu \in \mathcal{M}(X)$ be a non–atomic weak Gibbs measure for the negative potential $f \in \mathcal{C}(X)$ such that $\mathcal{P}(f) = 0$. With $E_n^x := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i(x)}$ denoting the* empirical measure, *let $\mathcal{E}_\nu := \{x \in X : E_n^x \xrightarrow{*} \nu\}$. Then we have for each $\nu \in \mathcal{M}(X,\sigma)$ with $\int f\, d\nu \neq 0$ that*

$$\mu\text{-}\dim(\mathcal{E}_\nu) = \frac{h_\nu}{-\int f\, d\nu}.$$

In order to prepare the proof we recall the following well-known concepts and results which will be crucial.

(Mass distribution principle) [**Caj81, Bil65**] For $\mu \in \mathcal{M}(X)$ non–atomic and $m \in \mathcal{M}(X)$, we have for any $K \subset X$ that

$$m\text{-ess-sup}_{x\in K} \liminf_{n\to\infty} \frac{\log m(C_n(x))}{\log \mu(C_n(x))} \leq \mu\text{-}\dim(K) \leq \sup_{x\in K} \liminf_{n\to\infty} \frac{\log m(C_n(x))}{\log \mu(C_n(x))}.$$

(Information cocycle) For $\mu_1, \mu_2 \in \mathcal{M}(X,\sigma)$ such that $\{C \in \mathfrak{Z}_n : \mu_2(C) = 0\} \subset \{C \in \mathfrak{Z}_n : \mu_1(C) = 0\}$ for all $n \in \mathbb{N}$, we define

$$h^\ell(\mu_1|\mu_2) := -\sum_{C\in\mathfrak{Z}_\ell} \mu_1(C) \log \frac{\mu_2(C)}{\mu_2(\sigma C)} \quad \text{and} \quad h(\mu_1|\mu_2) := \lim_{\ell\to\infty} h^\ell(\mu_1|\mu_2),$$

with the usual convention $0 \log 0 = 0$. The quantity $h(\mu_1|\mu_2)$ will be referred to as the *entropy of* μ_1 *given* μ_2. (Note that $h(\mu_1|\mu_1) = h_{\mu_1}$).

(Markov convolutions) For $\mu_1, \mu_2 \in \mathcal{M}(X)$ such that $\{C \in \mathfrak{Z}_n : \mu_1(C) = 0\} = \{C \in \mathfrak{Z}_n : \mu_2(C) = 0\}$ for all $n \in \mathbb{N}$, we define $\mu_1 *_n \mu_2 \in \mathcal{M}(X)$ for $K \in \mathcal{B}$ by

$$\mu_1 *_n \mu_2(K) \quad := \quad \sum_{C\in\mathfrak{Z}_n} \frac{\mu_1(K\cap C)}{\mu_1(C)} \mu_2(C).$$

If in particular μ_1 is a Markov measure of order q and if we have that μ_1 and μ_2 are positive on all cylinder sets, then the following fact can be obtained by elementary calculations (cf [**Caj81**, Lemma 7.3]). Here we have set $c(\mu_1) := \min\{\mu_1(C) : C \in \mathfrak{Z}_q\}$, and $\bar{\mathfrak{Z}}_n$ denotes the σ–algebra generated by the set of all n–cylinders $\mathfrak{Z}_n$.

(M1) $\mu_1 *_n \mu_2(A) = \mu_1(A)$ for all $n \in \mathbb{N}$ and $A \in \bar{\mathfrak{Z}}_n$.

(M2) $\mu_1 *_n \mu_2 \left(A \cap \sigma^{-n}B\right) \leq \mu_1(A) \cdot \mu_2(B) \cdot c(\mu_2)^{-1}$ for all $n, m \in \mathbb{N}$, $A \in \mathfrak{Z}_n$, and $B \in \mathfrak{Z}_m$.

(M3) $\mu_1 *_n \mu_2 \left(A \cap \sigma^{-n}K\right) \geq \mu_1(A) \cdot \left(1 - \dfrac{1 - \mu_2(K)}{c(\mu_2)}\right)$ for all $n > k$, $A \in \bar{\mathfrak{Z}}_{n-k}$, and $K \in \mathcal{B}$.

Remark. For $\mu_1 \in \mathcal{M}(X)$ let $\mu_1(\,\cdot\,|\bar{\mathfrak{Z}}_n)$ denote the *conditional probability* of μ_1 given $\bar{\mathfrak{Z}}_n$. Then $\mu_1 *_n \mu_2(K) = \int_K \mu_1(x|\bar{\mathfrak{Z}}_n)\, d\mu_2(x)$.

Proof of Proposition 3.2. Let $\varepsilon > 0$ be fixed and let $\ell > k$ be chosen such that $h^\ell(\nu|\nu) \leq h_\nu + \varepsilon/2$. For $\lambda \in (0,1)$ let ν_ℓ denote the Markov measure of order ℓ defined for $C \in \mathfrak{Z}_{\ell+1}$ by

$$\nu_\ell(C) = (1-\lambda)\nu(C) + \lambda\mu(C) \quad \text{such that} \quad h(\nu|\nu_\ell) \leq h^\ell(\nu|\nu) + \varepsilon/2.$$

Note that in here the second condition can always be satisfied, assuming that λ is chosen sufficiently small. Also, note that by construction we have that $\nu_\ell(C) > 0$ for all $C \in \mathfrak{Z}_n$ and $n \in \mathbb{N}$. Hence, we can assume without loss of generality that $c(\nu_\ell) > 0$. Now, using the Markov property of ν_ℓ and the weak Gibbs property of μ, we obtain for any $x \in \mathcal{E}_\nu$ that □

$$\frac{n^{-1}\log \nu_\ell(C_n(x))}{n^{-1}\log \mu(C_n(x))} \quad \leq \quad \frac{-n^{-1}\log c(\nu_\ell) - \sum_{C\in\mathfrak{Z}_\ell} E_n^x(C)\log\frac{\nu_\ell(C)}{\nu_\ell(\sigma C)}}{-n^{-1}S_n f(x) - \gamma_n/n}.$$

This implies that

$$\lim_{n\to\infty} \frac{\log \nu_\ell(C_n(x))}{\log \mu(C_n(x))} = \frac{h(\nu|\nu_\ell)}{-\int f\, d\nu} \leq \frac{h_\nu + \varepsilon}{-\int f\, d\nu}.$$

By the Mass Distribution Principle this gives the upper bound for $\mu\text{-}\dim(\mathcal{E}_\nu)$.

In order to obtain the lower bound it will be sufficient to show that there exists a measure η such that

$$\eta\text{-}\operatorname*{ess\text{-}sup}_{x\in G_\nu} \liminf_{n\to\infty} \frac{\log \eta(C_n(x))}{\log \mu(C_n(x))} \geq \frac{h_\nu}{-\int f\, d\nu}.$$

For this it will be convenient to introduce the following metric $\boldsymbol{d}$, which is given for $\mu_1, \mu_2 \in \mathcal{M}(X)$ by

$$\boldsymbol{d}(\mu_1,\mu_2) := \sum_{\ell=1}^{\infty} 2^{-\ell} \max\left\{|\mu_1(C) - \mu_2(C)| : C \in \mathfrak{Z}_\ell\right\}.$$

Now, fix a decreasing sequence (ρ_j) of positive numbers such that $\lim \rho_j = 0$. Then let (ℓ_j) be an increasing sequence of positive integers $\ell_j > k$ such that $2^{-\ell_j+1} < \rho_j$.

Define a sequence of ergodic Markov measures ν_j of order ℓ_j such that

$$\nu_j(C) = (1-\lambda_j)\,\nu(C) + \lambda_j\,\mu(C) \text{ for all } C \in \mathfrak{Z}_{\ell_j+1} \text{ and } j \in \mathbb{N},$$

where the numbers $\lambda_j \in (0,1)$ are chosen such that

- $\boldsymbol{d}(\nu_j, \nu) \leq \rho_j$ and $h_{\nu_j} > h_\nu - \rho_j$.

By ergodicity of ν_j it follows for ν_j–almost every $x \in X$ that

$$\lim_{n\to\infty} \boldsymbol{d}(E_n^x, \nu_j) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{-1}{n} \log \nu_j(C_n(x)) = h_{\nu_j}.$$

Hence, there exists positive integers m_j and sets $M_j \in \mathcal{B}$ with $\nu_j(M_j) > 1 - 2^{-j}c(\nu_j)$, such that for all $n \geq m_j$ and $x \in M_j$ we have that

- $\boldsymbol{d}(E_n^x, \nu_j) < \rho_j$ and $\dfrac{-\log \nu_j(C_n(x))}{n} > h_{\nu_j} - \rho_j$,
- $\dfrac{2}{n} < \rho_j$ and $\dfrac{1}{n} |\log c(\nu_j)| < \rho_j$.

Next, we define recursively a rapidly increasing sequence (r_j) of positive integers by

$$r_1 := 0,\ r_{j+1} := r_j + n_j,$$

where (n_j) is chosen such that for all $j \in \mathbb{N}$ we have

- $n_j > \rho_j^{-1} \max\{r_j, m_{j+1}, (r_j + m_{j+1})h_{\nu_j}, k\}$.

On this basis we define the sequence (η_j) of probability measures by

$$\eta_1 := \nu_1 \quad \text{and} \quad \eta_j := \eta_{j-1} *_{r_j} \nu_j \quad \text{for } j \geq 2.$$

Since by (M1) we have for all $i \geq j \geq 1$ that

$$\eta_i|_{\mathfrak{Z}_{r_{j+1}}} = \eta_j|_{\mathfrak{Z}_{r_{j+1}}},$$

it follows that η_j converges weakly to a probability measure η such that

$$\eta|_{\bar{\mathfrak{Z}}_{r_{j+1}}} = \eta_j|_{\bar{\mathfrak{Z}}_{r_{j+1}}} \quad \text{for all } j \geq 1.$$

In order to investigate the 'essential support' of η, we define

$$\begin{aligned} M' &:= \bigcup_{m=1}^{\infty} \bigcap_{j=m}^{\infty} \sigma^{-r_j}(M'_j), \quad \text{where} \quad M'_j := \bigcup_{x\in M_j} C_{n_j-k}(x); \\ M &:= \{x \in M' : \eta(C_n(x)) > 0 \ \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

Since X is transitive, we have that $\bigcap_{j=m}^{\ell-1} \sigma^{-r_j}(M'_j)$ is non–empty and belongs to the σ–algebra $\bar{\mathfrak{Z}}_{r_\ell}$, for all $\ell > m$.

Using (M3) and the fact that $\nu_\ell(M_\ell) \leq \nu_\ell(M'_\ell)$, we obtain that

$$\begin{aligned}
\eta\left(\bigcap_{j=m}^{\ell}\sigma^{-r_j}(M_j')\right) &= \eta_{\ell-1} *_{r_\ell} \nu_\ell\left(\bigcap_{j=m}^{\ell}\sigma^{-r_j}(M_j')\right)\\
&\geq \eta\left(\bigcap_{j=m}^{\ell-1}\sigma^{-r_j}(M_j')\right)\left(1-\frac{1-\nu_\ell(M_\ell')}{c(\nu_\ell)}\right)\\
&\geq \eta\left(\bigcap_{j=m}^{\ell-1}\sigma^{-r_j}(M_j')\right)\left(1-2^{-\ell}\right)\\
&\geq \prod_{i=m-1}^{\ell}\left(1-2^{-i}\right).
\end{aligned}$$

By letting first ℓ and then m tend to infinity, it follows that

$$\eta(M)=\eta(M')=1.$$

In order to prove that $M\subset\mathcal{E}_\nu$, let $x\in M$ be fixed. By definition of M, we then have for j sufficiently large that

$$x\in\sigma^{-r_j}\left(M_j'\right).$$

Define $x(j):=\sigma^{r_j}(x)\in M_j'$ and associate to each of these $x(j)$ an element $y(j)\in M_j\cap C_{n_j}(x(j))$. In order to complete the proof, we now have to consider the following two cases

Case i) $r_{j+1}\leq n<r_{j+1}+m_{j+1}$; Case ii) $r_{j+1}+m_{j+1}\leq n<r_{j+2}$.

For Case i), observe that

$$\begin{aligned}
\boldsymbol{d}(\nu,E_n^x) &= \boldsymbol{d}\left(\nu,\frac{r_j}{n}E_{r_j}^x+\frac{n_j}{n}E_{n_j}^{x(j)}+\frac{n-r_{j+1}}{n}E_{n-r_{j+1}}^{x(j+1)}\right)\\
&\leq \frac{r_j}{n}+\frac{n_j}{n}\boldsymbol{d}\left(E_{n_j}^{x(j)},\nu\right)+\frac{n-r_{j+1}}{n}\\
&\leq \rho_j+\boldsymbol{d}\left(E_{n_j}^{x(j)},E_{n_j}^{y(j)}\right)+\boldsymbol{d}\left(E_{n_j}^{y(j)},\nu\right)+\rho_j\\
&\leq 5\rho_j.
\end{aligned}$$

By a similar calculation we obtain the same estimate also for Case ii). This shows that $x\in\mathcal{E}_\nu$. Hence, it remains to show that

$$\liminf_{n\to\infty}\frac{\log\left(\eta(C_n(x))\right)}{\log\left(\mu(C_n(x))\right)}\geq\frac{h(\nu)}{-\int f\,d\nu}.$$

Using the weak Gibbs property of μ, we have on the one hand that

$$\begin{aligned}
\lim_{n\to\infty}\frac{-1}{n}\log\left(\mu(C_n(x))\right) &\leq \lim_{n\to\infty}-n^{-1}S_nf(x)+\gamma_n/n\\
&= \lim_{n\to\infty}-\int f\,dE_n^x+\gamma_n/n\\
&= -\int f\,d\nu.
\end{aligned}$$

On the other hand, for Case i) we have

$$\begin{aligned}
\eta\left(C_n(x)\right) &\leq \eta_j\left(C_{r_{j+1}}(x)\right) = \eta_{j-1} *_{r_j} \nu_j\left(C_{r_j+n_j}(x)\right) \\
&\leq c(\nu_j)^{-1}\,\eta\left(C_{r_j}(x)\right)\nu_j\left(C_{n_j}(x(j))\right) \qquad \text{(by (M2))} \\
&\leq c(\nu_j)^{-1}\,\nu_j\left(C_{n_j}(x(j))\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{-1}{n}\log\eta\left(C_n(x)\right) &\geq \frac{n_j}{n}\cdot\frac{-1}{n_j}\log\nu_j\left(C_{n_j}(x(j))\right) + \frac{1}{n}\log c(\nu_j) \\
&\geq \frac{n_j}{n}\left(h_{\nu_j} - \rho_j\right) - \rho_j \geq h_\nu - 4\rho_j.
\end{aligned}$$

For Case ii) we similarly obtain

$$\begin{aligned}
\eta\left(C_n(x)\right) &\leq c(\nu_j)^{-1}\,\eta\left(C_{r_j}(x)\right)\nu_j\left(C_{n_j}(x(j))\right) \\
&\leq c(\nu_j)^{-1}\,\eta\left(C_{r_j}(x)\right)\nu_j\left(C_{n-r_j}(x(j))\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{-1}{n}\log\eta\left(C_n(x)\right) &\geq \frac{r_j}{n}\cdot\frac{-1}{r_j}\log\nu_j\left(C_{r_j}(x(j))\right) \\
&\quad + \frac{n-r_j}{n}\cdot\frac{-1}{n-r_j}\log\nu_j\left(C_{n-r_j}(x(j))\right) + \frac{1}{n}\log c(\nu_j) \\
&\geq \frac{r_j}{n}\left(h_\nu - 4\rho_{j-1}\right) + \frac{n-r_j}{n}\left(h_\nu - 4\rho_j\right) - \rho_j \\
&\geq \left(h_\nu - 4\rho_{j-1}\right) - \rho_j.
\end{aligned}$$

This completes the proof of the proposition. □

Acknowledgements. We thank the Mathematical Institute at the University of St. Andrews and the Mathematische Institut and the Institut für Mathematische Stochastik at the Universität Göttingen for their warm hospitality. Also the first author likes to thank EPSRC for financial support.

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