FRACTAL GEOMETRY ON HYPERBOLIC MANIFOLDS

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. . . casting aside all scruples, we shall plunge wholeheartedly into the "new universe" which Bolyai "created from nothing".
—H.S.M. Coxeter (1907 - 2003), [20] page 287

Abstract In this survey we give a report on some recent results obtained in the studies of hyperbolic manifolds by means of fractal geometry. Emphasis has been put on results derived in the quantitative and qualitative fractal analysis of long term geodesic dynamics on hyperbolic manifolds.

Keywords: Conformal dynamics, fractal geometry, hyperbolic manifolds, Kleinian groups, ergodic theory.

1. Fractal geometry and hyperbolic manifolds

Hyperbolic manifolds are located at the junction of various different areas of modern mathematics. For instance, they play a central role in low-dimensional topology, complex dynamics, Teichmüller theory, harmonic analysis, spectral theory, analytic number theory, Diophantine approximations, and non-commutative algebra, to name a few.

On small scales the geometry of a $(n + 1)$-dimensional hyperbolic manifold $\mathcal{G}$ coincides with the geometry of the surrounding $(n + 1)$-dimensional hyperbolic space $\mathbb{D}$, whereas on large scales the topology of

$\mathcal{G}$ affects the global geometry of $\mathcal{G}$. This can then be analysed by a group of isometries, namely $\mathcal{G}$ can be represented by a Kleinian group, that is a discrete subgroup of the group of all isometries of $\mathbb{D}$. For a more detailed study of the interplay between the local and the global structures, it is a rather fruitful method to investigate $\mathcal{G}$ in terms of geodesic dynamical systems. Part of this type of investigations is to locate various different dynamical aspects of $\mathcal{G}$ and then to study these by means of concepts from fractal geometry. This is the main theme of this survey.

Throughout let G be a non-elementary Kleinian group acting discontinuously on hyperbolic $(n+1)$-space $\mathbb{D} = \{z \in \mathbb{R}^{n+1}; \|z\| < 1\}$ (we shall always use the Poincaré ball model $(\mathbb{D}, d)$ whose boundary at infinity is the unit sphere $\mathbb{S} = \{z \in \mathbb{R}^{n+1}; \|z\| = 1\}$) (cf. [5]). We shall always assume that G has no torsion. The limit set $L(G) \subset \mathbb{S}$ of G is the set of accumulation points of the G-orbit of some arbitrary point in $\mathbb{D}$. Since we are mainly interested in fractal properties of $L(G)$, we shall always assume that the set of ordinary points $\Omega(G) := \mathbb{S} \setminus L(G)$ is non-empty. Important subsets of $L(G)$ are the set $L_p(G)$ of bounded parabolic fixed points of G, the radial limit set $L_r(G)$ and the uniformly radial limit set $L_{ur}(G)$. In here s_ξ refers to the hyperbolic ray from the origin to ξ, and $b(x, r) \subset \mathbb{D}$ refers to the hyperbolic ball of radius r centred at x.

L_r: *A point $\xi \in L(G)$ is called radial limit point if there exists a positive constant $c = c(\xi)$ such that $s_\xi \cap b(g(0), c) \neq \emptyset$ for infinitely many different orbit points $g(0) \in G(0)$.*

L_{ur}: *A point $\xi \in L(G)$ is called uniformly radial limit point if for some positive $c = c(\xi)$ we have that $s_\xi \subset \bigcup_{g \in G} b(g(0), c)$.*

L_p: *A point $p \in L(G)$ is called bounded parabolic point if it is fixed point of some parabolic element of G and if its stabiliser G_p has the following properties. There exists a set $M \subset \mathbb{S}$ and a compact set $N \subset \mathbb{S} \setminus \{p\}$ such that $\bigcup_{g \in G_p} g(M) = \mathbb{S} \setminus \{p\}$ and $(M \setminus N) \cap g(M \setminus N) = \emptyset$ for all $g \in G \setminus \{id\}$. In particular, G_p is always isomorphic to some finite extension of $\mathbb{Z}^{k(p)}$ for some $k(p) \in \{1, \ldots, n\}$, where $k(p)$ is referred to as the rank of p.*

It is well-known that, by fixing some base frame, we can associate to each Kleinian group G a hyperbolic $(n+1)$-manifold $\mathcal{G} = \mathbb{D}/G$ (we always assume that $\mathcal{G}$ is oriented) (cf. [34, 77]). In $\mathcal{G}$ the limit set $L(G)$ is recovered as follows. For $m \in \mathcal{G}$ fixed, let $\Lambda_m(\mathcal{G})$ denote the set of all geodesic loops[1] which start and terminate at m. A geodesic l in $\mathcal{G}$

[1] note that these loops are not necessarily closed geodesics.

is called loop-approximable if each finite segment of l can be approximated with arbitrary accuracy by segments of elements in $\Lambda_m(\mathcal{G})$[2]. The geodesic core $C(\mathcal{G})$ of $\mathcal{G}$ is then defined by

$$C(\mathcal{G}) := \{l \text{ geodesic in } \mathcal{G} : l \text{ is loop-approximable}\}.$$

A limit direction is an element of the unit tangent space of $\mathcal{G}$ at m such that tracing this direction on $\mathcal{G}$ results in a geodesic ray which is eventually asymptotic to a geodesic in $C(\mathcal{G})$. The set of these so obtained limit directions is in 1-1-correspondence to the limit set $L(G)$. To demonstrate this transfer between $L(G)$ and the set of limit directions we remark that clearly every ray s_ξ for $\xi \in \mathbb{S}$ admits a projection via the universal covering map onto $\mathcal{G}$ where it becomes a geodesic ray emanating from the point corresponding to the origin. If now for instance $\xi \in L_r(G)$ then the projected ray is recurrent on $\mathcal{G}$, meaning that while travelling along this ray some bounded region in $\mathcal{G}$ gets visited infinitely often. Similarly, if $\xi \in L_{ur}(G)$ then the projected ray describes a bounded excursion, meaning that the whole ray is contained in some suitable bounded region of $\mathcal{G}$. Clearly, every uniformly radial point is radial (whereas the opposite is only true for convex cocompact Kleinian groups[3]).

Poincaré was presumably the first who realised the significance of a certain series which can be associated in a canonical way to any arbitrary hyperbolic manifold. This series is nowadays called the Poincaré series and its abcissa of convergence is usually referred to as the exponent of convergence or often also as the Poincaré exponent. More precisely, for $s \in \mathbb{R}$ the Poincaré series $\mathcal{P}_s(z,w)$ associated with G is given by

$$\mathcal{P}_s(z,w) := \sum_{g \in G} \exp(-sd(z,g(w))).$$

Clearly, convergence and divergence of this series does not depend on the choice of z and w. Hence, the exponent of convergence of G, that is the Poincaré exponent $\delta(G)$, is uniquely determined by

$$\delta(G) := \inf\{s \in \mathbb{R} : \mathcal{P}_s(0,0) \ \text{ converges}\}.$$

For various special types of Kleinian groups it had been know for some time that $\delta(G)$ quantifies the fractal nature of the uniformly radial limit

[2]i.e. each finite segment of γ is contained in an arbitrarily small neighbourhood of some element of $\Lambda_m(\mathcal{G})$.

[3]A Kleinian group is convex cocompact if and only if $L(G) = L_r(G)$. Note that in this paper we exclusively consider groups for which $L(G) \neq \mathbb{S}$, i.e. G can not be cocompact

set (see e.g. [31, 61, 52, 22, 63, 66, 26]). More recently, Bishop and Jones [8] found an astonishingly elementary method which allows to specify this relation between $\delta(G)$ and the Hausdorff dimension $\dim_H(L_{ur}(G))$ in its complete generality, that is for arbitrary non-elementary Kleinian groups. Consequently, their result gives an ultimate clarification of the Hausdorff-dimensional significance of the Poincaré exponent of any arbitrary Kleinian group. More precisely, the following result was obtained in [8] (cf. also [70]).

Theorem 1 (Theorem of Bishop and Jones (I)).
For every Kleinian group G we have that

$$\delta(G) = \dim_H(L_r(G)) = \dim_H(L_{ur}(G)).$$

Also, in this survey we shall be concerned with the Patterson measure which represents a fundamental concept canonically associated to every Kleinian group G. In his pioneering work [50] Patterson laid the foundation for a comprehensive study of $L(G)$ in terms of measure theory and in particular in terms of fractal dimensions. Patterson's original construction dealt with the case of Fuchsian groups, that is the case of 2-dimensional hyperbolic space, and was then generalised by Sullivan in [73] to the general Kleinian group case (cf. [48, 75]). We now briefly recall this construction. For some sequence of positive numbers ϵ_k tending to zero (and given that G is of $\delta(G)$-divergence type[4], meaning that the Poincaré series $\mathcal{P}_s(0,0)$ diverges for $s = \delta(G)$ (otherwise, a slowly varying function has to be introduced which then forces this divergence (cf. [50]))), μ is the weak limit of the sequence of measures

$$\mu_k := \left(\mathcal{P}_{\delta(G)+\epsilon_k}(0,0)\right)^{-1} \sum_{g \in G} \exp\left(-(\delta(G)+\epsilon_k) d(0, g(0))\right) \mathbf{1}_{g(0)},$$

where $\mathbf{1}_x$ refers to the Dirac measure at $x \in \mathbb{D}^{N+1}$. Note that in the geometrically finite case it is known that μ does not depend on the choice of the sequence (ϵ_k), and hence that μ is unique (cf. [73]). However, this is certainly not the case in general.
One of the important geometric properties of the Patterson measure is that it transforms nicely under elements of G. This property is referred to as $\delta(G)$-conformality, which means that for arbitrary Borel sets $E \subset S^n$ and for every $g \in G$ we have

$$\mu(g^{-1}E) = \int_E \left(\frac{1 - \|g(0)\|^2}{\|\xi - g(0)\|^2}\right)^{\delta(G)} d\mu(\xi).$$

[4]Note, a geometrically finite Kleinian group G is always of $\delta(G)$-divergence type (cf. [73]).

Clearly, in here the integrant is just the the norm of the conformal derivative or equally the Poisson kernel raised to the power δ.

Remarks.

The easiest examples of Kleinian groups with non-empty ordinary set are classical Schottky groups, that are subgroups of index 2 of groups generated by reflections at pairwise disjoint circles in $\mathbb{C}$. Further classical examples are for instance quasi-Fuchsian groups and certain subgroups of the Picard group, which in particular include a group with limit set the Apollonian packing. Interesting more advanced examples are for instance hyperbolic manifolds fibred over the circle, or normal subgroups of geometrically finite Kleinian groups. For an extensive list of examples we refer to [38, 40, 45]. For general discussions of Kleinian groups we refer to [1, 2, 5, 34, 38, 40, 43, 45, 48, 77].

Finally, we mention that there are various natural generalisations of hyperbolic manifolds. For instance, for rank 1 manifolds and higher rank symmetric spaces (see e.g. [3, 37, 18, 19, 33, 82]), and in particular for Hadamard manifolds (e.g. [29]) and for groups which act discontinuously on complex hyperbolic space (e.g. [47]), most of the fractal analysis in this survey continues to hold, although so far this has been written up only partially. We also mention that convex cocompact Kleinian groups are the cradle of 'hyperbolic groups in the sense of Gromov', and that finitely generated Kleinian groups with parabolic elements (which are *not* Gromov-hyperbolic) have motivated the concept of 'relative hyperbolicity' ([28, 16, 17]).

Throughout we shall use the notation $a \asymp b$ for two positive reals a, b to indicate that a/b is uniformly bounded away from zero and infinity. We write $a \ll b$ if a/b uniformly bounded away from infinity.

2. Geometrically finite hyperbolic manifolds

Recall that G is called geometrically finite if the action of G on $\mathbb{D}$ admits a fundamental polyhedron with finitely many sides. The following theorem goes back to Beardon and Maskit [6] (see also [7, 11]) and gives a characterisation of geometrical finiteness in terms of $L(G)$.

Theorem 2 (Theorem of Beardon and Maskit).
If G is a geometrically finite Kleinian group, then $L(G) = L_r(G) \cup L_p(G)$ (where $L_p(G)$ might be empty, in which case G is convex cocompact).

Thurston was presumably the first to realise the dynamical significance of the concept geometrical finiteness. He observed the following ([77]).

Theorem 3 (Thurston's observation).
A Kleinian group G and its associated hyperbolic $(n+1)$-manifold $\mathcal{G}$ are geometrically finite if and only if a neighbourhood of the convex hull of the geodesic core $C(\mathcal{G})$ has finite hyperbolic volume.

Diophantine analysis of the Patterson measure

In this section we discuss some of the results obtained in the study of the essential support of the Patterson measure. For this we introduce the following notation. For $\xi \in L(G)$ and $t > 0$, let ξ_t denote the unique point on the ray between 0 and ξ with hyperbolic distance t from 0. Let $b(\xi_t)$ denote the intersection of $\mathbb{S}$ with the $(n+1)$-ball whose boundary is orthogonal to $\mathbb{S}$, which contains ξ, and which intersects s_ξ orthogonally at ξ_t. Hence, $b(\xi_t)$ is a n-ball in $\mathbb{S}$ with radius comparable to e^{-t}. Also, define $k(\xi_t)$ to be equal to $k(p)$ if the projection of ξ_t onto $\mathcal{G}$ is contained in the cusp region of $\mathcal{G}$ associated with the parabolic point p; otherwise we let $k(\xi_t)$ to be equal to $\delta(G)$. Furthermore, define $\Delta(\xi_t) := d(\xi_t, G(0))$, and let k_{min} and k_{max} denote the minimal and maximal occurring rank for the parabolic elements in G.

The following result provides a key observation in the investigations of the coarse geometry of the Patterson measure μ. We remark that this measure formula has recently been extended to complex hyperbolic manifolds [47] and to Hadamard manifolds [60].

Theorem 4 (Global measure formula). ([75, 72])
For all $\xi \in L(G)$ and $t > 0$, we have

$$\mu(b(\xi_t)) \asymp \exp\left(-t\delta(G) - \Delta(\xi_t)(\delta(G) - k(\xi_t))\right).$$

Immediate implications are that the Patterson measure is a doubling measure and that the limit set of a geometrically finite Kleinian group is uniformly perfect.

In order to derive further informations on the fractal nature of the limit set of a geometrically finite Kleinian group, it is vital to give good approximations of the essential support of the Patterson measure. The following three theorems shed some light on the essential support from different perspectives. For the first theorem recall that an element of $L(G)$ is called Myrberg limit point if the projection of s_ξ onto $\mathcal{G}$ has the property that it approximates every finite part of every geodesic in $C(\mathcal{G})$ with arbitrary accuracy infinitely many times. The theorem was obtained by Tukia in [79] and independently by the author in [67].

Theorem 5 (Generalised Myrberg Theorem).
For μ-almost every $\xi \in L(G)$ we have that ξ is a Myrberg limit point.

We remark that based on Sullivan's results in [73], it was shown in [67] that for an arbitrary Kleinian groups G the set of Myrberg limit points is of full μ-measure if and only if the geodesic flow on $\mathcal{G}$ is ergodic with respect to the Liouville-Patterson measure.

For the remainder of this section we shall now assume that G has parabolic elements. We remark that for the cases in which there are no parabolic elements, analogous results can be obtained and the proofs are far less involved.
The following theorem represents a generalisation of a classical theorem of Khintchine in metrical Diophantine approximations ([36]). An important point here is that the theorem, if combined with the measure formula, gives some useful insight into the fluctuation of μ, as will be demonstrated in the sequel.

Theorem 6 (Generalised Khintchine Theorem). ([72])
For μ-almost all $\xi \in L(G)$, we have

$$\limsup_{t\to\infty}(\Delta(\xi_t)/\log t) = 1/\left(2\delta(G) - k_{max}\right) \quad .$$

An immediate implication is that for μ-almost all $\xi \in L(G)$ we have that $\lim_{t\to\infty}\Delta(\xi_t)/t = 0$ (Sullivan's ergodic law [73]).

Also, we have the following description in terms of asymptotic frequencies with which recurrent geodesics on $\mathcal{G}$ enter the cusp regions. For this let $\mathcal{C}_p(\tau)$ refer to the region inside the cuspidal region of $\mathcal{G}$ associated with the parabolic fixed point p, which has hyperbolic distance τ to the projection of 0 onto $\mathcal{G}$. Let $N_{p,\tau}(\xi,t)$ denote the number of geodesic connected components of the intersection of $\mathcal{C}_p(\tau)$ with the projection onto $\mathcal{G}$ of the geodesic segment between 0 and ξ_t. Generalising a result by Nakada [46] for imaginary quadratic fields which is closely related to the Doeblin-Lenstra conjecture proven in [10], the following result was obtained in [64].

Theorem 7 (Generalised Nakada Theorem).
For every parabolic fixed point p and for all $\tau > 0$ sufficiently large, we have for μ-almost all $\xi \in L(G)$,

$$\lim_{t\to\infty}\left(N_{p,\tau}(\xi,t)/t\right) \asymp \exp(-\tau(2\delta(G) - k(p))).$$

Finally, we state the following immediate consequence of a combination of the measure formula and the generalised Khintchine theorem. In here $\mathcal{I}_\mu$ refers to the *information dimension*, $\mathcal{R}_\mu(q)$ to the *q-th generalised Renyi dimension*, and $\mathcal{L}_\mu(q+1)$ to the *q-th logarithmic index* (see [21] for the definitions).

Theorem 8. [68]
For $q \neq 0$, we have

$$\mathcal{I}_\mu = \mathcal{R}_\mu(q) = \delta(G) \text{ and } \mathcal{L}_\mu(q+1) = q\delta(G).$$

*Hence in particular, μ is a $\delta(G)$-*regular measure*, which means that for μ-almost all $\xi \in L(G)$ we have*

$$\liminf_{t\to\infty}(\log \mu(b(\xi_t)))/(-t) = \limsup_{t\to\infty}(\log \mu(b(\xi_t)))/(-t) = \delta(G).$$

Coarse-structure fractal analysis

By a result of Beardon [4] we have that if G has parabolic elements, then $\delta(G) > k_{max}/2$. On the other hand if G does not have parabolic elements of rank n, then an immediate consequence of the theorem of Beardon and Maskit is that every geodesic in $C(\mathcal{G})$ is contained in some fixed hyperbolic neighbourhood of the boundary of $C(\mathcal{G})$. Using this observation, one easily verifies that $L(G)$ is a porous set and hence, by a standard result in fractal geometry, the box-counting dimension $\dim_B(L(G))$ is strictly less than n. If there are parabolic elements of rank n then, although then $L(G)$ is not porous, it was shown by Sullivan [73] and Tukia [78] that we still have $\dim_H(L(G)) < n$. More generally, we have the following theorem which shows that for a geometrically finite group Hausdorff-, packing- ($\dim_P$), and box-counting dimension all agree and are equal to $\delta(G)$. Clearly, an immediate consequence of the theorem is that $\dim_B(L(G)) < n$. The theorem was obtained in [71] and later extended in [8] to analytically finite groups (that are groups which have the 'Ahlfors property', meaning that $\Omega(G)/G$ is a finite union of cofinite Riemann surfaces).

Theorem 9 (Box-counting dimension Theorem).
We have that

$$\dim_H(L(G)) = \dim_P(L(G)) = \dim_B(L(G)) = \delta(G).$$

Let us look closer at the behaviour of the s-dimensional Hausdorff measure $\mathcal{H}_s$ and packing measure $\mathcal{P}_s$ of the limit set, and also how these relate to the Patterson measure μ. The following table gives the complete picture for geometrically finite Kleinian groups acting on hyperbolic 3-space. The table is an immediate consequence of the measure formula and the generalised Khintchine theorem, using the well known 'generalised mass distribution principle' ([23, 44]). We remark that similar tables can be produced for geometrically finite Kleinian groups in higher dimensions (cf. [68, 75]).

CUSPS vs. $\delta = \delta(G)$	$0 < \delta(G) < 1$	$\delta(G) = 1$	$1 < \delta(G) < 2$
no cusps	$\mu \asymp \mathcal{H}_\delta \asymp \mathcal{P}_\delta$	$\mu \asymp \mathcal{H}_1 \asymp \mathcal{P}_1$	$\mu \asymp \mathcal{H}_\delta \asymp \mathcal{P}_\delta$
$k_{max} = 1$	$\mu \asymp \mathcal{P}_\delta, \mathcal{H}_\delta = 0$	$\mu \asymp \mathcal{H}_1 \asymp \mathcal{P}_1$	$\mu \asymp \mathcal{H}_\delta, \mathcal{P}_\delta = \infty$
$k_{min} = 2$	$\nexists$	$\nexists$	$\mu \asymp \mathcal{P}_\delta, \mathcal{H}_\delta = 0$
$k_{min} = 1, k_{max} = 2$	$\nexists$	$\nexists$	$\mathcal{H}_\delta = 0$, $\mathcal{P}_\delta = \infty$

Of special interest is the case described in the final row of the table, where $1 < \delta(G) < 2$ and where there are parabolic elements of rank 1 as well as of rank 2. Here the Patterson measure does not have an immediate geometric interpretation, and it is still an open problem if in this case the Patterson measure can be described in terms of generalised geometric measures. We expect that it should be helpful to employ techniques developed by Makarov [39] to answer similar questions for harmonic measures. However, by combining the measure formula and the generalised Khintchine theorem, we at least can obtain the following approximations to the geometric nature of the Patterson measure in this case. Namely, for the Hausdorff measure $\mathcal{H}_{\phi_\theta}$ with respect to the gauge function ϕ_θ, given by $\phi_\theta(r) := r^{\delta(G)} \ \exp\left(\left(\frac{1}{2}+\theta\right)\frac{2-\delta(G)}{\delta(G)-1}\log\log\frac{1}{r}\right)$, we find that ([68])

$$\mathcal{H}_{\phi_{-\theta}}(L(G)) \ll 1 \ll \mathcal{H}_{\phi_\theta}(L(G)) \ \text{ for all } \ \theta > 0.$$

We remark that by the same means, similar estimates can be given for gauge functions in the context of the generalised packing measure of the limit set. Finally, let us remark that our analysis strongly supports the following.

Conjecture.
If G has parabolic elements of rank 1 and 2, then μ and $\mathcal{H}_\phi$ are comparable, for ϕ given by

$$\phi(r) = r^{\delta(G)} \ \exp\left(\frac{2-\delta(G)}{2(\delta(G)-1)}\left(\log\log\frac{1}{r} + \log\log\log\log\frac{1}{r}\right)\right).$$

Fine-structure fractal analysis

In this section we give some results derived from generalisations of methods in metrical Diophantine approximations, the theory of large deviations, ergodic theory and multifractal analysis. In particular these results contribute to a finer fractal analysis of limit sets of geometrically finite Kleinian groups, and hence of the geodesic dynamics on geometrically finite hyperbolic manifolds. Throughout this section we shall assume that G is a geometrically finite Kleinian group with parabolic elements.

Coarse multifractal analysis.

Coarse multifractal analysis studies global irregularities of the distribution of $\mu(b(\xi_t))$ for large values of t. In order to perform such investigations in the context of Kleinian groups, it is helpful first to consider σ-Jarník limit sets $L^\sigma(G)$. These sets represent a canonical generalisation of the sets of well-approximable irrational numbers (see [31]).
For $0 < \sigma < 1$, let

$$L^\sigma(G) := \{\xi \in L(G) : \limsup_{t\to\infty} (\Delta(\xi_t)/t) \geq \sigma\}.$$

The following theorem was obtained by Hill and Velani in [30], and independently by the author in [65, 69]. For ease of exposition, in here we have restricted the statement once more to the case of Kleinian groups acting on hyperbolic 3-space.

Theorem 10 (Generalised Jarník-Besicovitch Theorem).

(1) For $\delta(G) \leq k_{max}$, we have $\dim_H(L^\sigma(G)) = \delta(G)(1-\sigma)$.

(2) For $\delta(G) > k_{max}$, we have

$$\dim_H(L^\sigma(G)) = \begin{cases} \delta(G)(1-\sigma) & \text{for} \quad \delta(G)(1-\sigma) \leq 1 \\ \delta(G)\left(1 - \frac{\sigma}{1+\sigma}\frac{2\delta(G)-1}{\delta(G)}\right) & \text{for} \quad \delta(G)(1-\sigma) \geq 1. \end{cases}$$

Hence, for $\delta(G) \leq k_{max}$ the theorem shows that $\dim_H(L^\sigma(G))$ is a linear function in σ. Whereas in the second case, where $\delta(G) > 1$ and where G necessarily has parabolic elements of rank 1, this function is partially non-linear. In particular, in this case there exists a unique point σ^* at which the derivative of this function is not continuous (note, for Kleinian groups in higher dimension there can be more than one such point). The significance of σ^* is that $\dim_H(L^{\sigma^*}(G)) = 1$.

Refinements of the above results on Jarník limit sets then give rise to the following coarse multifractal analysis of the Patterson measure μ. Consider the following level sets

$$k_\mu(\theta) := \dim_H\{\xi \in L(G) : \liminf_{r\to 0} \log \mu(b(\xi_t))/(-t) \leq \theta\},$$

$$l_\mu(\theta) := \dim_H\{\xi \in L(G) : \limsup_{r\to 0} \log \mu(b(\xi_t))/(-t) \geq \theta\},$$

and their associated multifractal spectra

$\{k_\mu(\theta) : \theta \in (2\delta(G) - k_{max}, \delta(G))\}$ the *upper liminf-spectrum*,

$\{l_\mu(\theta) : \theta \in (\delta(G), 2\delta(G) - k_{min})\}$ the *lower limsup-spectrum.*

As shown in [69], for $\mathcal{H}_{\delta(G)}(L(G)) \neq 0$ the upper liminf-spectrum is trivial, whereas for $\mathcal{P}_{\delta(G)}(L(G)) \neq \infty$ the lower limsup-spectrum turns out to be trivial. More interestingly, for the remaining cases the following results were obtained in [69].

Theorem 11 (Coarse multifractal spectra).

(1) $\mathcal{H}_{\delta(G)}(L(G)) = 0$ *if and only if we have, for all* θ *in the domain of the upper liminf-spectrum,*

$$k_\mu(\theta) = \delta(G) - (\theta - \delta(G))\frac{\delta(G)}{\delta(G) - k_{max}}.$$

(2) $\mathcal{P}_{\delta(G)}(L(G)) = \infty$ *if and only if we have, for all* θ *in the domain of the lower limsup-spectrum,*

$$l_\mu(\theta) = \begin{cases} \delta(G) - (\theta - \delta(G))\frac{2\delta(G)-1}{\theta-1} & \textit{for} \quad \theta \leq \theta^* \\ \delta(G) - (\theta - \delta(G))\frac{\delta(G)}{\delta(G)-1} & \textit{for} \quad \theta \geq \theta^*, \end{cases}$$

where we have set $\theta^* := (2\delta(G) - 1) - \frac{\delta(G)-1}{\delta(G)}$.

The theorem shows that if the packing measure of $L(G)$ is infinite, then the lower limsup-spectrum is partially non-linear, and linear and non-linear part intersect precisely at θ^*. Again, the significance of θ^* is that $l_\mu(\theta^*) = 1$. Furthermore, the derivative of the lower limsup-spectrum is not continuous at θ^*. Therefore, θ^* can be interpreted as a point at which a *coarse multifractal phase transition* occurs. Note that our analysis in particular shows that such a phase transition occurs if and only if $\mathcal{P}_{\delta(G)}(L(G)) = \infty$.

Fine multifractal analysis.

Fine multifractal analysis studies fractal entities for level sets of certain real-valued functions, where the level sets are defined by means of strict limit processes. We remark that fine multifractal analysis imbeds into the ergodic-theoretical discipline of thermodynamical formalism. (For elementary introductions into the relevant dimension theory we refer to [24, 56]).

In this section we consider the special case in which G is an essentially free Kleinian group. This means that G is basically a free geometrically finite Kleinian group, except that there might be group-relations arising from stabilisers of parabolic fixed points of G. In order to describe the results for this type of Kleinian group, we recall the following lexicographical coding of $L(G)$. Let $F \subset \mathbb{D}$ be a fundamental polyhedron of G. Then the hyperbolic ray s_ξ from the origin to some arbitrary $\xi \in L(G)$ passes in succession through fundamental domains $F, g_{\xi,1}(F), g_{\xi,2}(F), g_{\xi,3}(F), \ldots$, where the $g_{\xi,n}^{-1} g_{\xi,n+1}$ are elements of the set of generators of G. Here we assume for ease of exposition that s_ξ passes exclusively through n-dimensional and hence through no lower dimensional faces of F. For $\alpha \in \mathbb{R}$, we then define the following level sets

$$\mathcal{F}_\alpha(G) := \left\{ \xi \in L(G) : \lim_{n \to \infty} d(0, g_{\xi,n}(0))/n = \alpha \right\}.$$

In the investigations of the Hausdorff dimensions of these level sets the following pressure function P turns out to be crucial. In here $|g|$ refers to the word length of g in G.

$$P : \mathbb{R} \to \mathbb{R} \text{ given by } P(\alpha) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\substack{g \in G \\ |g| = n}} \exp(-\alpha d(0, g(0))).$$

The following result was obtained in [35].

Theorem 12 (Fine multifractal spectra).
There exists $\alpha_-, \alpha_+ \in \mathbb{R}$ such that $\mathcal{F}_\alpha(G) \neq \emptyset$ if and only if $\alpha \in [\alpha_-, \alpha_+]$, and such that for each $\alpha \in (\alpha_-, \alpha_+)$, we have

$$\dim_H(\mathcal{F}_\alpha(G)) = \alpha^{-1} \inf_{\beta \in \mathbb{R}} \{\alpha\beta + P(\beta)\}. \qquad (*)$$

(1) For G convex cocompact we have $(*)$ holds for all $\alpha \in [\alpha_-, \alpha_+]$, and P is real analytic everywhere. Furthermore, on (α_-, α_+) we have for $(P')^{-1}$, the inverse of the derivative of P,

$$\dim_H(\mathcal{F}_\alpha(G)) = (P')^{-1}(\alpha) - P(\alpha)/\alpha.$$

(2) If G has parabolic elements, then P is real analytic on $(-\infty, \delta(G))$ and equal to 0 otherwise. Additionally, the following holds.

- *(i) If $\delta(G) \leq (k_{max} + 1)/2$, then P is differentiable everywhere.*
- *(ii) If $\delta(G) > (k_{max} + 1)/2$, then there exists a* fine multifractal phase transition, *meaning that the right derivative of P at $\delta(G)$ vanishes, whereas the left derivative of P at $\delta(G)$ is strictly negative.*

We remark that the two types of phase transitions which we discussed, namely the coarse multifractal and the fine multifractal phase transition, seem to be two completely unrelated phenomena which can be detected within the limit set of a Kleinian group. As our analysis clearly shows, a Kleinian group permits either both, or none, or exactly one of these two types, and each of these possibilities can actually occur. Further note that the coarse multifractal analysis is of pure geometric nature, whereas the fine multifractal analysis results from mixing the algebraic appearance of G (here in form of the use of the word length) with its geometric realisation.

3. Finitely generated hyperbolic manifolds

In this section we give a brief report on some of the results obtained in the fractal analysis of limit sets of finitely generated, geometrically infinite Kleinian groups G acting on hyperbolic 3-space. Such geometrically infinite groups were first shown to exist over 30 years ago by Greenberg in [27]. The first explicit examples were constructed by Jørgensen in [32]. Subsequently, these groups have attracted a great deal of attention from various different points of view. One of the most important conjectures in this area, which is still undecided but which over the years sparked off a vast amount of fruitful and stimulating research, is the following. In here λ_2 refers to the 2-dimensional Lebesgue measure in $\mathbb{S}^2$.

The Ahlfors Conjecture.
For every finitely generated Kleinian group G we have

$$\lambda_2(L(G)) = 0.$$

For geometrically finite groups this conjecture clearly holds, since then, as we already saw, we have that $\dim_B(L(G)) < 2$ which in particular implies that $\lambda_2(L(G)) = 0$. In the geometrically infinite case the following result was obtained in [8].

Theorem 13 (Theorem of Bishop and Jones (II)).
For every finitely generated, geometrically infinite Kleinian groups G we have

$$\dim_H(L(G)) = 2.$$

We remark that in the light of the Ahlfors conjecture it seems worthwhile to comment on the strategy of the proof of this result. Namely, the essential part of the proof of Bishop and Jones is to show that

$$\delta(G) < 2 \Longrightarrow \lambda_2(L(G)) > 0.$$

(Clearly, this gives the theorem, since then either $\delta(G) = 2$ which implies that $\dim_H(L_{ur}(G)) = 2$ and hence $\dim_H(L(G)) = 2$, or $\delta(G) < 2$ which gives $\lambda_2(L(G)) > 0$ and hence in particular $\dim_H(L(G)) = 2$). With other words, the proof shows that a finitely generated, geometrically infinite Kleinian group G with $\dim_H(L_r(G)) < 2$ is necessarily a counter example for the Ahlfors conjecture. Of course it is a problem to show that a group of this type does exist. For instance, it is known that such a counter example cannot be a topologically tame Kleinian group of bounded type ([15, 14, 74]), which means that the associated hyperbolic manifold has injectivity radius uniformly bounded from below and is homeomorphic to the interior of a compact 3-manifold with boundary. In fact for the Patterson measure μ of such tame groups we have the following result. (Note that in this case the Patterson measure is not necessarily unique).

Theorem 14 (Theorem of Bishop and Jones (III)). ([9])
Let G be a geometrically infinite, topologically tame Kleinian group of bounded type, and let $\mathcal{H}_\varphi$ refer to the Hausdorff measure with respect to the gauge function φ given by

$$\varphi(r) = r^2 \ \exp\left(\frac{1}{2}\left(\log\log\frac{1}{r} + \log\log\log\log\frac{1}{r}\right)\right).$$

Then the following holds

$$\mu \asymp \mathcal{H}_\varphi \ \textit{ and } \ \mu(L_r(G)) = 0.$$

4. Infinitely generated hyperbolic manifolds

Finally, we give a brief report on some of the results derived in the fractal analysis of geodesic dynamics on infinitely generated hyperbolic manifolds. It is evidently clear that the class of all infinitely generated groups as such is by far too large. Hence, one of the problems is to elaborate some structure inside this huge class of groups which makes it feasible to study these groups systematically.

As a first subclass we mention groups which can be exhausted by an increasing chain of subgroups. More precisely, here we consider a Kleinian group G such that $G = \bigcup_k G_k$, for some sequence of Kleinian subgroups $G_1 \subset G_2 \subset ... \subset G_k \subset ....$ In this situation we clearly have that $\bigcup_k L_{ur}(G_k) \subset L_{ur}(G)$. On the other hand, if $\xi \in L_{ur}(G)$ then there exists an infinite path c_ξ in the Cayley graph of G such that s_ξ is fully contained in some fixed hyperbolic neighbourhood of c_ξ, and such that the hyperbolic lengths of the geodesic segments of c_ξ are uniformly bounded from above. Therefore $L_{ur}(G) = \bigcup_k L_{ur}(G_k)$, and hence it follows, by monotonicity of Hausdorff dimension (see e.g. [23]),

$$\dim_H(L_{ur}(G)) = \dim_H(\bigcup_k L_{ur}(G_k)) = \sup_k \dim_H(L_{ur}(G_k)).$$

Combining this observation with the theorem of Bishop and Jones (I), we then immediately obtain the following result. We remark that this result was conjectured by Patterson in [51, 53, 54] and subsequently proven by Sullivan in [73], using the Patterson measure.

Theorem 15.
Let G be a Kleinian group such that $G = \bigcup_k G_k$, for some sequence of Kleinian subgroups $G_1 \subset G_2 \subset ... \subset G_k \subset ....$ We then have

$$\delta(G) = \sup_k \delta(G_k).$$

We remark that by similar means one shows that the Hausdorff dimension of $L_{ur}(G)$ is lower semi-continuous with respect to algebraic convergence (see [41]).

Another interesting class of infinitely generated Kleinian groups is provided by intermediate coverings of geometrically finite hyperbolic manifolds. Here one considers normal subgroups N of a geometrically finite Kleinian group G. As one easily verifies, in this situation we have that $L(G) = L(N)$. Hence, this class of Kleinian groups gives the opportunity to study geodesic cores which are both, the geodesic core of an infinitely generated manifold (associated with N) as well as the geodesic core of a geometrically finite manifold (associated with G).
The following result was obtained in [25].

Theorem 16.
For a normal subgroup N of a geometrically finite Kleinian group G we have

$$\dim_H(L_{ur}(N)) \geq \dim_H(L_{ur}(G))/2.$$

Recall that for a $\delta(G)$-divergence type group G the geodesic flow on $\mathcal{G}$ is ergodic with respect to the Liouville-Patterson measure. As mentioned before, geometrically finite groups are always of $\delta(G)$-divergence type, and hence the following result of Rees in particular shows under which circumstances ergodicity of the geodesic flow is preserved if one passes to an intermediate covering of a geometrically finite hyperbolic 2-manifold (see also the discussion in [55, 62]).

Theorem 17 (Theorem of Rees). ([58, 59])
Let N be a normal subgroup of a geometrically finite Fuchsian group G such that G/N is isomorphic to $\mathbb{Z}^m$, for $m \in \mathbb{N}$. Then $\delta(N) = \delta(G)$, and

(i) if G has no parabolic elements, then N is of $\delta(N)$-divergence type if and only if $m = 1$ or 2;

(ii) if G has parabolic elements, then N is of $\delta(N)$-divergence type if and only if $m = 1$.

This theorem seems to suggest that polynomial growth of the factor group might be a necessary and sufficient condition for having equality of the two exponents of convergence. The following theorem shows that the right condition is in fact 'amenabilty' (see [12, 83, 81]), at least for certain convex cocompact Kleinian groups. In here the condition $\delta(G) > n/2$ occurs since the proof of Brooks in [13] uses spectral theory of the Laplacian, and it seems extremely likely that this condition can be removed (see [80]).

Theorem 18 (Theorem of Brooks). ([13])
Let N be a normal subgroup of a convex cocompact Kleinian group G such that $\delta(G) > n/2$. Then $\delta(N) = \delta(G)$ if and only if G/N is amenable.

Yet another interesting class of geometrically infinite Kleinian groups is the following. A Kleinian group G is called discrepancy group if and only if $\dim_H(L_{ur}(G)) < \dim_H(L(G))$. This type of groups seems to have been studied for the first time by Patterson in [53] (see also [57]). Note that by the previous theorem we immediately have that every normal subgroup with non-amenable factor in a convex cocompact Kleinian group (with Poincaré exponent greater than $n/2$) is a discrepancy group. In [25] we gave further examples of discrepancy groups, and started with a finer fractal analysis of the limit sets of these groups by considering the following set of τ-weakly recurrent limit points

$$L_r^{(\tau)}(G) := \{\xi : s_\xi \cap b\,(g(0), \tau\, d(0, g(0))) \neq \emptyset \text{ for infinitely many } g \in G\}.$$

Theorem 19. ([25])
For a discrepancy group G we have for all $0 < \tau < \frac{\dim_H(L(G)) - \delta(G)}{\dim_H(L(G))}$,

$$\delta(G) \leq \dim_H(L_r^{(\tau)}(G)) < \dim_H(L(G)),$$

and hence in particular,

$$\dim_H(L(G) \setminus L_r^{(\tau)}(G)) = \dim_H(L(G)).$$

Finally, we mention a class of Kleinian groups which we have called geometrically tight. This type of groups has recently been studied by Matsuzaki in [42]. A Kleinian group G is called geometrically tight if there exists $\rho > 0$ and a countable exceptional set $E \subset L(G)$ (where E is either empty, or represents endings of $\mathcal{G}$ which are either 'cusp-like' or contained in a fixed neighourhood of some geodesic in $\mathcal{G}$) such that for every $\xi \in L(G) \setminus E$, the projection of s_ξ onto $\mathcal{G}$ 'returns infinitely many times' to the hyperbolic ρ-neighbourhood of the boundary of $C(\mathcal{G})$. Clearly, if the set E is empty then every geodesic in $C(\mathcal{G})$ is contained in the ρ-neighbourhood of the boundary of $C(\mathcal{G})$. In this special situation one can then argue as for geometrically finite groups without parabolic points of rank n, which gives that $L(G)$ is a porous set, and hence $\dim_B(L(G)) < n$. For E non-empty, a straight-forward adaptation of Tukia's techniques in [78] leads to the following. In here it would be interesting to show that Hausdorff dimension can be replaced in general by box-counting dimension, which can obviously be done for $E = \emptyset$.

Theorem 20 (Theorem of Matsuzaki). ([42])
For a geometrically tight Kleinian group G we have

$$\dim_H(L(G)) < n.$$

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