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The Exponent of Convergence of Kleinian Groups; on a Theorem of Bishop and Jones

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Abstract. In this note we give a characterization of the Hausdorff dimensional significance of the exponent of convergence for any arbitrary Kleinian group. We show that this exponent is always equal to the Hausdorff dimension of the uniformly radial limit sets of the Kleinian group. We give a detailed and elementary proof of this important fact, clarifying and generalizing a result of Bishop and Jones.

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1. The Exponent of Convergence versus Hausdorff Dimension

Already Hadamard observed that on a cusped hyperbolic surface the set of geodesic movements starting from an arbitrary fixed point and remaining in some bounded region of that surface represents a rather tricky and sophisticated set of directions, which he certainly would have called 'fractal' if in those days this term would have already been coined. Another important observation is due to Poincaré, who realised the significance of a certain series which is naturally associated with a hyperbolic surface. This series is now called the Poincaré series and its abzissa of convergence is referred to as the exponent of convergence or Poincaré exponent of the fundamental group of the surface. In this note we show in a far more general context how these two observations are related. Namely, we give a detailed proof of the fact that for *any* hyperbolic manifold the Poincaré exponent coincides with the Hausdorff dimension of the set of directions at an arbitrary point for which the associated geodesic movements remain in bounded regions of the manifold.

This fact was verified in many special cases using various different techniques, for instance for the modular group implicitly by Jarník [8] (see also Schmidt [11]), for general cofinite Fuchsian groups by Patterson [9], for cofinite Kleinian groups by Dani [4] and Stratmann [12], and for geometrically finite Kleinian groups by Stratmann [13] as well as Fernández and Melián [6]. More recently, Bishop and

Jones [2] gave an astonishingly elementary method which allows to derive this fact in its complete generality, that is for arbitrary non-elementary Kleinian groups. The main goal of this paper is to clarify the construction in [2]. We try to keep our presentation accessible to non-experts in the field.

Before we state the main theorem more explicitly, let us begin with explaining the setting. We consider non-elementary, discrete subgroups G of the group $Con(N)$ of isometries of the $(N+1)$-dimensional hyperbolic space. (Throughout, for the latter we shall use exclusively the Poincaré ball model $(\mathbb{D}^{N+1}, d)$.) Such a group G is called *Kleinian group*, and its associated *exponent of convergence* $\delta = \delta(G)$, often also referred to as the *Poincaré exponent*, is defined by

$$\delta(G) := \inf\{s : \sum_{g \in G} \exp(-sd(0, g(0))) \text{ converges}\}.$$

A Kleinian group G acts discontinuously on $\mathbb{D}^{N+1}$. Hence, the orbit $G(0)$ of the origin $0 \in \mathbb{D}^{N+1}$ under G forms a discrete point set which with respect to the Euclidean metric accumulates only at the boundary of hyperbolic space $\mathbb{S}^N = \partial\mathbb{D}^{N+1}$. The set $L(G)$ of accumulation points of $G(0)$ is called the *limit set of* G. Important subsets of $L(G)$ turn out to be the *radial limit set* $L_r(G)$ and the *uniformly radial limit set* $L_{ur}(G)$ which are given by the following definitions.

- A point $\xi \in L(G)$ is called *radial limit point* if there exists a positive constant $c = c(\xi)$ such that $s_\xi \cap b(g(0), c) \neq \emptyset$ for infinitely many different orbit points $g(0) \in G(0)$.
- A point $\xi \in L(G)$ is called *uniformly radial limit point* if for some positive $c = c(\xi)$ we have that $s_\xi \subset \bigcup_{g \in G} b(g(0), c)$.

Here s_ξ denotes the hyperbolic ray from 0 to ξ, and $b(x, r)$ the hyperbolic ball of radius r centred at x.

Note that we may project the ray s_ξ onto the associated hyperbolic manifold $M := \mathbb{D}^{N+1}/G$ where it becomes a geodesic ray starting from the point on M which corresponds to the origin. If $\xi \in L_r(G)$, then in general this ray performs a recurrent geodesic excursion on M, i.e. there exists a bounded region in M which gets visited infinitely often. If $\xi \in L_{ur}(G)$, then the ray describes a bounded excursion, i.e. in M each point on the ray is at most a bounded distance away from the starting point. Clearly, every uniformly radial point is radial (whereas the opposite is only true for cocompact and convex cocompact Kleinian groups).

In this note we shall give a detailed and elementary proof of the following result of Bishop and Jones. This result gives an ultimate clarification of the 'Hausdorff dimensional significance' of the exponent of convergence for an arbitrary Kleinian group. We restrict the theorem to non-elementary Kleinian groups, that is groups with limit sets of cardinality strictly greater than 2. The elementary cases are in fact trivial.

Theorem. *For a non-elementary Kleinian group G with exponent of convergence δ we have that*

$$\delta = \dim_H(L_r(G)) = \dim_H(L_{ur}(G)).$$

The paper is organized as follows. We begin with giving in section 2 a discussion of the geometries which are relevant in the proof of the theorem. This includes some well-known topics from hyperbolic geometry, conformal geometry and fractal geometry. In section 3 we introduce the concept of 'divergence points', and show that the set of these points is dense in the limit set. Using this, we then give a geometrisation of the Poincaré series for parameter values below the exponent of convergence. This then allows to construct Cantor subsets of the uniformly radial limit set and to give estimates for their Hausdorff dimensions in terms of δ.

We end this introduction by giving a brief description of the proof of the theorem in terms of the associated hyperbolic manifold M. Namely, the heart of the proof, if 'projected onto M', is to show that everywhere on any arbitrary M the following scenario is met.

The sketch on M. At each point $x_0 \in M$ there exist bouquets $\mathrm{B}_n^{(i)}$ (for $n \in \mathbb{N}$ and $i = 1, 2$) of closed loops starting and terminating at x_0 with the following properties. The length of each loop in $\mathrm{B}_n^{(i)}$ is roughly equal to $l_n^{(i)}$ (for some $l_n^{(i)} \nearrow \infty$, for $i = 1, 2$), and there exists a positive constant α_0 such that the angle between any of the initial directions in $\mathrm{B}_n^{(1)}$ and any of those in $\mathrm{B}_n^{(2)}$ is bounded from below by α_0. In combination with these bouquets there exists a positive constant σ and a sequence $\kappa_n \searrow 0$ such that if we consider all initial directions at x_0 giving rise to geodesic rays which stay always σ-close to elements of $\mathrm{B}_n^{(1)} \cup \mathrm{B}_n^{(2)}$, then the Hausdorff dimension of this set of directions is greater than $\delta - \kappa_n$.
Roughly speaking, the proof then follows by observing that trivially the Hausdorff dimension of all recurrent geodesic rays starting at x_0 does not exceed δ, and that the set of directions giving rise to bounded geodesic rays is contained in the set of those giving rise to recurrent geodesic rays.

Remark. The method of Bishop and Jones, as explained in this paper, has recently been employed in a straightforward manner to derive corresponding results also in slightly more general settings. For geometrically finite groups in rank 1 symmetric spaces a result similar to the above theorem has been obtained in [3]. Also, the Bishop-Jones argument has been adapted in [10] to hyperbolic groups in the sense of Gromov, and in [7] to the case of pinched negative curvature.

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2. Background Geometry

2.1. Hyperbolic and Conformal Geometry

Recall that for the Poincaré ball model of hyperbolic space $(\mathbb{D}^{N+1}, d)$ the hyperbolic distance $d(v, w)$ between two arbitrary points $v, w \in \mathbb{D}^{N+1}$ is given by the expression

$$d(v,w) := \inf \int_\gamma \frac{2}{1-|z|^2}|dz|,$$

where the infimum is taken over all smooth curves connecting the points v and w. (For a comprehensive introduction into hyperbolic geometry the reader is referred to Beardon's book [1].)
For subsets $A \subset \mathbb{D}^{N+1}$, the *shadow map* $\Pi : \mathbb{D}^{N+1} \to \mathbb{S}^N$ is defined by

$$\Pi(A) := \{\xi \in \mathbb{S}^N : s_\xi \cap A \neq \emptyset\}.$$

Lemma 2.1. *For every $z \in \mathbb{D}^{N+1}$ and for each positive $\rho < d(0,z)$, the spherical diameter $|\Pi(b(z,\rho))|$ of the projection of the open hyperbolic ball $b(z,\rho)$ centred at z and of radius ρ has the property*

$$|\Pi(b(z,\rho))| \asymp_\rho e^{-d(0,z)}.$$

Here, $\asymp_\rho$ means that the quotient of the two quantities is bounded from below and above by some positive constants which depend only on ρ.

Proof. Using the definition of the hyperbolic metric above, a straight forward computation (see [1]) gives that the hyperbolic length of a circle C_t around the origin of hyperbolic radius t is equal to $2\pi \sinh t$, and hence it is asymptotic to πe^t. This implies that for t sufficiently large, we may cover nearly all of C_t by roughly $\pi e^t/(2\rho)$ pairwise disjoint balls of hyperbolic radius ρ which are centred at points of C_t. Since for large t and $z \in C_t$ we have that the Euclidean diameter of $b(z,\rho)$ is comparable to $|\Pi(b(z,\rho))|$, it follows that

$$|\Pi(b(z,\rho))| \asymp 2\rho e^{-t} \asymp_\rho e^{-t}. \qquad \square$$

Lemma 2.2 (Geometric Distortion Lemma). *Let $\sigma > 0$, and let $\gamma \in Con(N)$ be non-elliptic such that $d(0,\gamma(0)) > \sigma$. Then we have for all $z \in \mathbb{D}^N$ with $d(0,z) > \sigma$ and $\Pi(z) \in \gamma^{-1}(\Pi(b(\gamma(0),\sigma)))$ that*

$$d(0,\gamma(0)) + d(0,z) - 2\sigma < d(0,\gamma(z)) \leq d(0,\gamma(0)) + d(0,z).$$

Proof. (see Figure 1) Raise the perpendicular from the origin onto the geodesic segment t between z and $\gamma^{-1}(0)$. Let p denote the point of intersection of this perpendicular with t. Note that by construction, the hyperbolic distance s of p to the origin is less than or equal to σ. Let t_1 and t_2 denote the hyperbolic lengths of the geodesic segments between p and $\gamma^{-1}(0)$, p and z respectively. Using the triangle inequality, we derive that

$$d((0,\gamma^{-1}(0)) \leq \sigma + t_1 \text{ and } d(0,z) \leq \sigma + t_2.$$

It hence follows that

$$d(0,\gamma(z)) = d(\gamma^{-1}(0), z) = t_1 + t_2 \geq d(0,\gamma^{-1}(0)) + d(0,z) - 2\sigma.$$

The second inequality in the lemma is an immediate consequence of the triangle inequality. $\square$

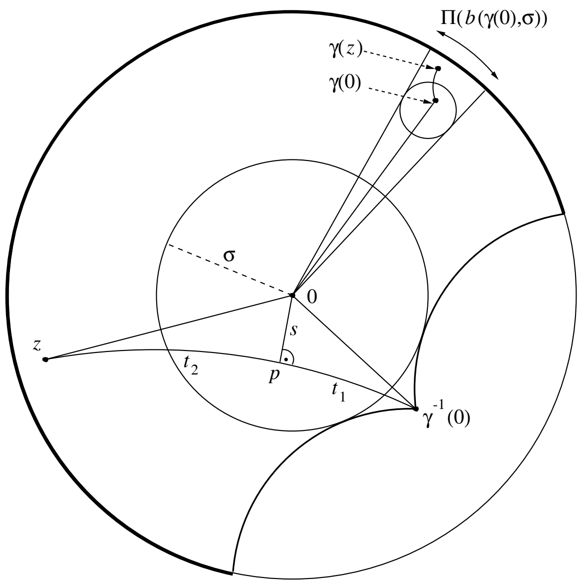

FIGURE 1.

Lemma 2.3 (Light Cone Lemma). *For all $\sigma > \log 2$ and for all non-elliptic $\gamma \in Con(N)$ for which $d(0, \gamma(0)) > \sigma$, we have that*

$$\mathbb{S}^N \setminus B\left(\Pi(\gamma^{-1}(0)), 2\pi e^{-\sigma}\right) \subset \gamma^{-1}\left(\Pi(b(\gamma(0), \sigma))\right) \subset \mathbb{S}^N \setminus B\left(\Pi(\gamma^{-1}(0)), e^{-\sigma}\right).$$

Here, $B(x, r)$ denotes the spherical N-ball with centre $x \in \mathbb{S}^N$ and radius r.

Proof. Let $\sigma > 0$ and let γ be chosen as stated in the lemma. For l denoting a geodesic in $\mathbb{D}^N$ which intersects $\partial b(0, \sigma)$ in exactly one point, say z_σ, let r_σ be the radius of the spherical disc $\Pi(l)$. Then it is geometrically evident that (see Figure 2)

$$\mathbb{S}^N \setminus B\left(\Pi(\gamma^{-1}(0)), 2r_\sigma\right) \subset \gamma^{-1}\left(\Pi(b(\gamma(0), \sigma))\right) \subset \mathbb{S}^N \setminus B\left(\Pi(\gamma^{-1}(0)), r_\sigma\right);$$

where we assumed that $2r_\sigma \leq \pi$. Hence it remains to find lower and upper bounds for r_σ. For this, note that by definition of the hyperbolic distance we have that $d(0, z_\sigma) = \log((1 + |z_\sigma|)/(1 - |z_\sigma|))$. An elementary calculation then gives

$$e^{-\sigma} \leq 1 - |z_\sigma| \leq r_\sigma \leq \frac{\pi}{2}(1 - |z_\sigma|) \leq \pi e^{-\sigma}.$$

Finally, since $2\pi e^{-\sigma} < \pi$ if and only if $\sigma > \log 2$, it follows that for σ in this range the condition $2r_\sigma \leq \pi$ is always satisfied. $\square$

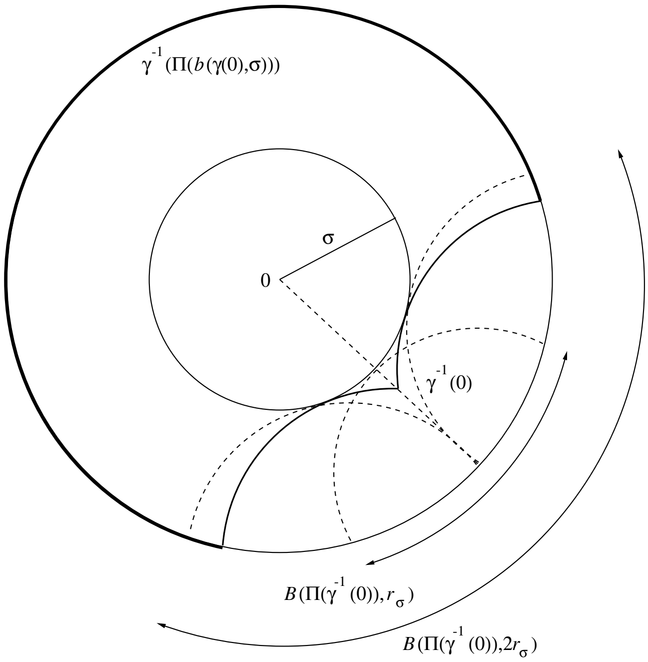

FIGURE 2.

2.2. Fractal Geometry

Recall that for a Borel set $E \subset \mathbb{S}^N$ the Hausdorff dimension $\dim_H(E)$ of E is defined by

$$\dim_H(E) := \inf\{s \geq 0 : \lim_{\rho \to 0} \inf_{U \in \mathrm{U}_\rho(E)} \sum_{u \in U} |u|^s < \infty\},$$

where $\mathrm{U}_\rho(E)$ denotes the set of coverings of E by balls with radii at most ρ. In computations of $\dim_H(E)$ to find lower bounds is usually the hardest part, and for this the following *mass distribution principle* very often turns out be useful. In here, $a \ll b$ refers to that a/b is uniformly bounded away from infinity.

- Let μ be a probability measure supported on E. If there exists $\tau \geq 0$ such that for each $x \in E$ we have $\limsup_{r \to 0} \mu(B(x,r))/r^\tau \ll 1$, then $\dim_H(E) \geq \tau$.

For a good introduction into fractal geometry the reader is referred to [5]. In this paper we shall require the following elementary facts from fractal geometry.

Lemma 2.4. *Let $\{B(x_i, r_i)\}_{i \in \mathbb{N}}$ be a family of spherical N-balls in $\mathbb{S}^N$ with centre x_i and radius r_i such that $\lim_{i \to \infty} r_i = 0$ and such that $\sum_{i \in \mathbb{N}} r_i^s$ has exponent of convergence $\tau \geq 0$. We then have that*

$$\dim_H \Big(\bigcup_{c > 0} \limsup_{i \in \mathbb{N}} \{B(x_i, c\, r_i)\} \Big) \leq \tau.$$

Proof. For each $c, \rho > 0$ we have that $\{B(x_i, c\,r_i) : i \in \mathbb{N}, r_i \leq \rho\}$ represents a ρ-covering of

$$\mathrm{B}_c := \limsup_i \{B(x_i, c\,r_i)\} = \{x \in \mathbb{S}^N : x \in B(x_j, c\,r_j) \text{ for infinitely many } j\}.$$

Hence, for arbitrary $\epsilon > 0$ we have that

$$\inf_{U \in \mathrm{U}_\rho(\mathrm{B}_c)} \sum_{u \in U} |u|^{\tau+\epsilon} \leq \sum_{i \in \mathbb{N}} (cr_i)^{\tau+\epsilon} \leq c^{\tau+\epsilon} \sum_{i \in \mathbb{N}} r_i^{\tau+\epsilon} < \infty.$$

By letting ρ tend to 0, this implies that $\dim_H(\mathrm{B}_c) \leq \tau + \epsilon$. Since ϵ was chosen to be arbitrary, we deduce that $\dim_H(\mathrm{B}_c) \leq \tau$. And finally, since $\dim_H(\mathrm{B}_{c_0}) \leq \dim_H(\mathrm{B}_{c_1})$ for $c_0 < c_1$, the result follows by monotonicity of Hausdorff dimension. □

Lemma 2.5. *For a finite index set I containing at least two elements, let a descending sequence $A(i_1) \supset A(i_1, i_2) \supset \cdots$ of closed spherical N-balls in $\mathbb{S}^N$ be given for each $(i_1, i_2, \ldots) \in I^\infty$. Assume that there exist $0 < \beta < 1$ and $\tau \geq 0$ such that for each $n \in \mathbb{N}$ and for all distinct $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in I^n$ we have that*

(i) $|A(i_1, \ldots, i_n)| \asymp \beta^n$ *and* $A(i_1, \ldots, i_n) \cap A(j_1, \ldots, j_n) = \emptyset$;
(ii) $\sum_{j \in I} |A(i_1, \ldots, i_n, j)|^\tau \geq |A(i_1, \ldots, i_n)|^\tau$.

It then follows that

$$\dim_H\Big(\bigcap_{n \in \mathbb{N}} \bigcup_{(i_1, \ldots, i_n) \in I^n} A(i_1, \ldots, i_n)\Big) \geq \tau.$$

Proof. We sketch the proof which uses the mass distribution principle. Let μ be a probability measure with support $\mathrm{A} := \bigcap_{n \in \mathbb{N}} \bigcup_{(i_1, \ldots, i_n) \in I^n} A(i_1, \ldots, i_n)$ such that for each $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in I^n$,

$$\mu(A(i_1, \ldots, i_n)) = \frac{|A(i_1, \ldots, i_n)|^\tau}{\sum_{(j_1, \ldots, j_n) \in I^n} |A(j_1, \ldots, j_n)|^\tau}.$$

Consider the spherical N-ball $B(x, r)$ centred at $x \in A$ of small radius $r > 0$. We then have $r \asymp \beta^n$ for some $n \in \mathbb{N}$, and condition (i) in the lemma guarantees that there exists a constant $c > 0$ such that for the number of balls of the n-th generation which intersect $B(x, r)$ we have

$$card\,(\{(i_1, \ldots, i_n) : B(x, r) \cap A(i_1, \ldots, i_n) \neq \emptyset\}) < c.$$

Using this observation and (ii) of the lemma, we obtain

$$\begin{aligned}
\mu(B(x, r)) &\leq c \cdot \max\{\mu(A(i_1, \ldots, i_n)) : B(x, r) \cap A(i_1, \ldots, i_n) \neq \emptyset\} \\
&\ll \frac{\beta^{n\tau}}{\sum_{(j_1, \ldots, j_n) \in I^n} |A(j_1, \ldots, j_n)|^\tau} \ll \frac{r^\tau}{\sum_{i \in I} |A(i)|^\tau} \\
&\ll r^\tau.
\end{aligned}$$

Applying the mass distribution principle, the lemma follows. □

The following is an immediate consequence of the previous lemma. It in particular gives the model for the special type of Cantor set, which in the final section will be shown to exist inside the uniformly radial limit set of any arbitrary Kleinian group.

Corollary 2.6. *Let $I^{(1)}$ and $I^{(2)}$ be two finite index sets, each having at least two elements. Let L be an infinite subset of all finite words in the alphabet $I^{(1)} \cup I^{(2)}$, such that $j \in \mathrm{L}$ for all $j \in I^{(1)} \cup I^{(2)}$, and such that if $(i_1 \dots i_n) \in \mathrm{L}$ (for $n \geq 1$) then either $(i_1 \dots i_n j) \in \mathrm{L}$ for all $j \in I^{(1)}$ and $(i_1 \dots i_n k) \notin \mathrm{L}$ for all $k \in I^{(2)}$, or $(i_1 \dots i_n j) \in \mathrm{L}$ for all $j \in I^{(2)}$ and $(i_1 \dots i_n k) \notin \mathrm{L}$ for all $k \in I^{(1)}$. Assume that for each $n \in \mathbb{N}$ and for every $(i_1, i_2, \dots, i_n) \in \mathrm{L}$ we have a descending sequence $A(i_1) \supset A(i_1, i_2) \supset \cdots \supset A(i_1, i_2, \dots, i_n)$ of closed spherical N-balls in $\mathbb{S}^N$ with the following properties. There exist $\tau \geq 0$ and $0 < \beta_i < 1$ for $i = 1, 2$, such that for each $n \in \mathbb{N}$ and for all distinct $(i_1, \dots, i_n), (j_1, \dots, j_n) \in \mathrm{L}$ we have that*

(i) $|A(i_1, \dots, i_n)| \asymp \beta_1^{\sharp\{k : i_k \in I^{(1)}\}} \beta_2^{\sharp\{k : i_k \in I^{(2)}\}}$; $A(i_1, \dots, i_n) \cap A(j_1, \dots, j_n) = \emptyset$;
(ii) $\sum_{j : (i_1, \dots, i_n, j) \in \mathrm{L}} |A(i_1, \dots, i_n, j)|^\tau \geq |A(i_1, \dots, i_n)|^\tau$.

It then follows that

$$\dim_H \Big(\bigcap_{n \in \mathbb{N}} \bigcup_{(i_1, \dots, i_n) \in \mathrm{L}} A(i_1, \dots, i_n) \Big) \geq \tau.$$

Proof. The proof is basically the same as the proof of Lemma 2.5. The only difference is that instead of using the 'lexicographical coverings' of the type $\{A(i_1, \dots, i_n) : (i_1, \dots, i_n) \in \mathrm{L}\}$, one uses the 'geometrical coverings' given by

$$\{A : A = A(i_1, \dots, i_k) \text{ for some } k \in \mathbb{N};\ (\beta_1 \beta_2)^{n+1} \leq |A| < (\beta_1 \beta_2)^n\}. \qquad \square$$

3. The Proof of the Theorem

3.1. A Geometrisation of the Poincaré Series

For $\xi \in \mathbb{S}^N$ and $0 < r < \pi$, let the *lense* $\lambda(\xi, r) \subset \mathbb{D}^{N+1}$ be defined as the interior of the $(N+1)$-dimensional hyperbolic half space for which $\Pi(\lambda(\xi, r))$ is the spherical N-ball of radius r centred at ξ (see Figure 3).

Definition 3.1. *For $\epsilon > 0$, an element $\xi \in L(G)$ is called ϵ-divergence point if and only if*

$$\sum_{g(0) \in \lambda(\xi, r)} e^{-(\delta - \epsilon) d(0, g(0))} = \infty \quad \text{for all } r \in (0, 1).$$

Let $\mathrm{D}_\epsilon(G)$ refer to the set of all ϵ-divergence points.

Lemma 3.2. *For each $\epsilon > 0$, the set $\mathrm{D}_\epsilon(G)$ is dense in $L(G)$.*

Proof. Let ϵ be fixed, and suppose, by way of contradiction, that $\mathrm{D}_\epsilon(G) = \emptyset$. From this we see that for each $\eta \in L(G)$ there exists $r(\eta) > 0$ such that

$$\sum_{g(0) \in \lambda(\eta, r)} e^{-(\delta - \epsilon) d(0, g(0))} < \infty \quad \text{for all } r \in (0, r(\eta)).$$

The set $\{\Pi(\lambda(\eta, r(\eta))) : \eta \in L(G)\}$ provides a covering of $L(G)$, and since $L(G)$ is compact, there exist $\eta_1, \cdots, \eta_k \in L(G)$ such that

$$L(G) \subset \bigcup_{i=1}^{k} \Pi(\lambda(\eta_i, r(\eta_i))),$$

and hence

$$\sum_{i=1}^{k} \sum_{g(0)\in\lambda(\eta_i, r(\eta_i))} e^{-(\delta-\epsilon)d(0,g(0))} < \infty.$$

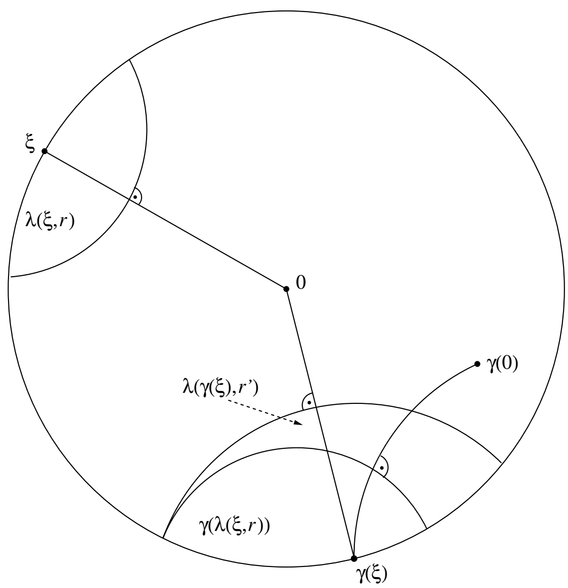

FIGURE 3.

Combining this with the fact that $G(0) \setminus \bigcup_{i=1}^{k} \lambda(\eta_i, r(\eta_i))$ is a finite set of orbit points, it follows that

$$\sum_{g\in G} e^{-(\delta-\epsilon)d(0,g(0))} < \infty,$$

which contradicts the fact that δ is the exponent of convergence of G. Hence, there exists at least one ϵ-divergence point $\xi \in L(G)$. Since $G(\xi)$ is a dense subset of $L(G)$, it is now sufficient to show that $\gamma(\xi) \in \mathrm{D}_\epsilon(G)$ for arbitrary $\gamma \in G$ not fixing ξ. In order to see this, note that $\gamma(s_\xi)$ is the ray from $\gamma(0)$ to $\gamma(\xi)$ and that for each sufficiently small positive r there exists a minimal r' such that $\gamma(\lambda(\xi, r)) \subset \lambda(\gamma(\xi), r')$ (see Figure 3). Clearly, if r decreases to 0, then the corresponding r'

also decreases to 0. Now, the claim follows from the following observation.

$$\begin{aligned} \infty &= \sum_{g(0)\in\lambda(\xi,r)} e^{-(\delta-\epsilon)d(0,g(0))} \\ &= \sum_{g(0)\in\lambda(\xi,r)} e^{-(\delta-\epsilon)d(\gamma(0),\gamma g(0))} \\ &\asymp_\gamma \sum_{g(0)\in\lambda(\xi,r)} e^{-(\delta-\epsilon)d(0,\gamma g(0))} \\ &\le \sum_{g(0)\in\lambda(\gamma(\xi),r')} e^{-(\delta-\epsilon)d(0,g(0))}. \end{aligned}$$

□

For $n \in \mathbb{N}$ and $\tau > 0$, the hyperbolic n-annulus $A_n(\tau)$ of width τ is defined as follows

$$A_n(\tau) := \{g(0) \in G(0) : n\tau \le d(0, g(0)) < (n+1)\tau\}.$$

Lemma 3.3. *Let $\epsilon, \tau > 0$ and $\xi \in \mathrm{D}_\epsilon(G)$ be given. Also, let $M > 0$ be some given large number. Then, for each $r > 0$ there exists an increasing sequence of positive integers $\{m_i\}_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$,*

$$\sum_{g(0)\in\lambda(\xi,r)\cap A_{m_i}(\tau)} e^{-(\delta-2\epsilon)d(0,g(0))} \ge M.$$

Proof. Assume that the statement of the lemma is false. Then there exists $r > 0$ such that for almost all $n \in \mathbb{N}$ (i.e. for all n apart from finitely many exceptions)

$$\sum_{g(0)\in\lambda(\xi,r)\cap A_n(\tau)} e^{-(\delta-2\epsilon)d(0,g(0))} < M.$$

In order to get rid of the 'finitely many exceptions', note that there exists some $0 < r^* \le r$ such that for all $n \in \mathbb{N}$,

$$\sum_{g(0)\in\lambda(\xi,r^*)\cap A_n(\tau)} e^{-(\delta-2\epsilon)d(0,g(0))} < M.$$

Using this, it follows that

$$\begin{aligned} \sum_{g(0)\in\lambda(\xi,r^*)} e^{-(\delta-\epsilon)d(0,g(0))} &\le \sum_{n\in\mathbb{N}} \sum_{g(0)\in\lambda(\xi,r^*)\cap A_n(\tau)} e^{-\epsilon d(0,g(0))}\, e^{-(\delta-2\epsilon)d(0,g(0))} \\ &\le M \sum_{n\in\mathbb{N}} e^{-\epsilon\tau n} \\ &< \infty, \end{aligned}$$

which contradicts the fact that ξ is an ϵ-divergence point. □

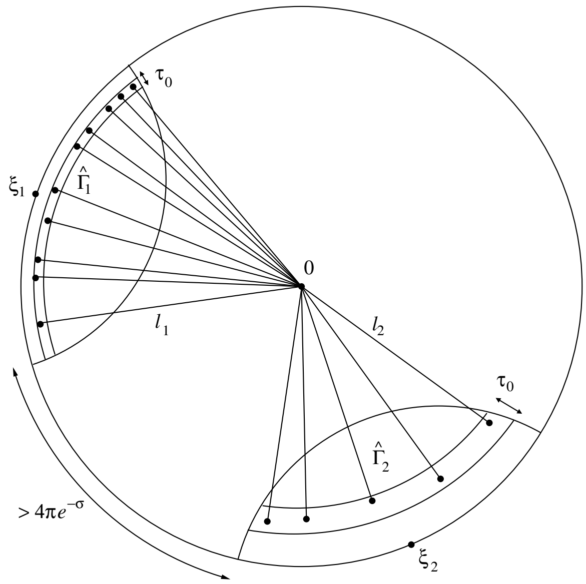

FIGURE 4.

3.2. The Building Stone for the Cantor Construction

Recall that the *injective radius* at the origin is defined by $\mathrm{inj}(0) := \frac{1}{2}\inf\{d(0,\gamma(0)) : g \in G \setminus \{id.\}\}$. In the following put $\tau_0 := \frac{1}{2}\mathrm{inj}(0)$, and let σ denote some fixed constant such that $e^{\sigma} > \max\{2; 4\pi/|L(G)|\}$.

Lemma 3.4. *For each $\epsilon > 0$ and for each arbitrarily large number $M > 0$ there exist $\xi_1, \xi_2 \in \mathrm{D}_\epsilon(G), r > 0$ and arbitrarily large $l_1, l_2 > 0$, as well as finite sets $\hat{\Gamma}_1 := \{h \in G : h(0) \in A_{l_1}(\tau_0) \cap \lambda(\xi_1, r)\}$ and $\hat{\Gamma}_2 := \{h \in G : h(0) \in A_{l_2}(\tau_0) \cap \lambda(\xi_2, r)\}$ such that the following properties are satisfied for $i = 1, 2$.*

(i) *The spherical distance between $B(\xi_1, r)$ and $B(\xi_2, r)$ is at least $4\pi e^{-\sigma}$.*
(ii) $\sum_{h\in\hat{\Gamma}_i} e^{-(\delta-2\epsilon)d(0,h(0))} \geq M$.

Proof. (see Figure 4) Fix $\epsilon > 0$ and $M > 0$. Next, choose $\xi_1, \xi_2 \in \mathrm{D}_\epsilon(G)$ and $r > 0$ sufficiently small such that the spherical distance between $B(\xi_1, r)$ and $B(\xi_2, r)$ is greater than $4\pi e^{-\sigma}$. Note that this is possible by choice of σ (recall that $|L(G)| > 4\pi e^{-\sigma}$) and since, by Lemma 3.2, the set $\mathrm{D}_\epsilon(G)$ is dense in $L(G)$. Now, using Lemma 3.3, we obtain that there exist arbitrarily large $l_1, l_2 \in \mathbb{N}$ such that

$$\sum_{h(0)\in\lambda(\xi_i,r)\cap A_{l_i}(\tau_0)} e^{-(\delta-2\epsilon)d(0,h(0))} \geq M \quad \text{for } i = 1, 2. \qquad \square$$

The following proposition gives the type of 'building stone' out of which in the final section we shall construct certain specific Cantor sets.

Proposition 3.5. *Let $\epsilon > 0$ be given. There exist $\Gamma_1 \subset \hat{\Gamma}_1$ and $\Gamma_2 \subset \hat{\Gamma}_2$, each containing at least two elements, with the following property. For each $g \in G$ with $d(0, g(0)) > \sigma$ there exists $\Gamma(g) \in \{\Gamma_i : i = 1, 2\}$ such that the following holds.*

(i) *The family $\mathrm{F}(g) := \{\Pi\,(b(g(h(0)), \sigma)) : h \in \Gamma(g)\}$ consists of pairwise disjoint balls which are of comparable size and which are contained in $\Pi\,(b(g(0), \sigma))$.*

(ii) $\sum_{b \in \mathrm{F}(g)} |b|^{\delta - 2\epsilon} \geq |\Pi(b(g(0), \sigma))|^{\delta - 2\epsilon}$.

Remark. Note that for a particular $g \in G$ it may happen that both sets Γ_1 and Γ_2 have the properties (i) and (ii) in the proposition. In this case one may think of $\Gamma(g)$ as being randomly selected.

Proof. Fix $\epsilon > 0$ and $g \in G$ such that $d(0, g(0)) > \sigma$. Also, let $M > 0$ be sufficiently large (which will be specified at the end of the proof). By Lemma 3.4, there exist $\xi_1, \xi_2 \in \mathrm{D}_\epsilon(G)$ and $r > 0$ such that $B(\xi_1, r)$ and $B(\xi_2, r)$ are at least the distance $4\pi e^{-\sigma}$ apart. Thus, by the Light Cone Lemma (Lemma 2.3) we get that at least one of the $B(\xi_i, r)$, say $B(\xi_1, r)$, is fully contained in $g^{-1}(\Pi(b(g(0), \sigma)))$. In particular, this means that $\Pi(h(0)) \in g^{-1}(\Pi(b(g(0), \sigma)))$, for each $h \in \hat{\Gamma}_1$. We can now apply the Geometric Distortion Lemma (Lemma 2.2), which gives that for each $h \in \hat{\Gamma}_1$ we have

$$d(0, g(0)) + d(0, h(0)) - 2\sigma < d(0, g(h(0))) \leq d(0, g(0)) + d(0, h(0)). \qquad (*)$$

Using this estimate and Lemma 3.4 (ii), we obtain that

$$\begin{aligned} \sum_{h \in \hat{\Gamma}_1} e^{-(\delta - 2\epsilon) d(0, g(h(0)))} &> e^{-(\delta - 2\epsilon) d(0, g(0))} \sum_{h \in \hat{\Gamma}_1} e^{-(\delta - 2\epsilon) d(0, h(0))} \\ &\geq M\, e^{-(\delta - 2\epsilon) d(0, g(0))}. \end{aligned}$$

By combining this latter estimate and Lemma 2.1, we deduce that there exists a constant $C_1 > 0$ depending on σ, such that

$$\sum_{h \in \hat{\Gamma}_1} |\Pi(b(g(h(0)), \sigma))|^{\delta - 2\epsilon} \geq C_1\, M\, |\Pi(b(g(0), \sigma))|^{\delta - 2\epsilon}.$$

Now, recall that for $h \in \hat{\Gamma}_1$ we have that $l_1 \tau_0 \leq d(0, h(0)) < l_1 \tau_0 + \tau_0$. Combining this observation and estimate $(*)$, we obtain that all $g(h(0))$ with $h \in \hat{\Gamma}_1$ are contained in an annulus of constant hyperbolic width $2\sigma + \tau_0$, that is

$$\{g(h(0)) : h \in \hat{\Gamma}_1\} \subset \{z \in \mathbb{D}^{N+1} : l_1 \tau_0 - 2\sigma \leq d(0, z) - d(0, g(0)) < l_1 \tau_0 + \tau_0\}. \qquad (**)$$

Also, recall that, by choice of τ_0, the set $\{b(g(h(0)), \tau_0) : h \in \hat{\Gamma}_1\}$ comprises a family of pairwise disjoint hyperbolic balls. From these two latter observations we can now deduce that each $\xi \in \Pi(b(g(0), \sigma))$ is contained in at most a bounded number (independently of g) of balls $\Pi(b(g(h(0)), \sigma))$ with $h \in \hat{\Gamma}_1$, i.e. there exist a set $\Gamma_1 \subset \hat{\Gamma}_1$ and a constant $C_2 > 0$, which depends only on σ and τ_0, such that $\{\Pi\,(b(gh(0),\sigma)) : h \in \Gamma_1\}$ consists of pairwise disjoint balls which are contained in $\Pi\,(b(g(0),\sigma))$, and such that

$$\sum_{h\in\Gamma_1} |\Pi(b(g(h(0)),\sigma))|^{\delta-2\epsilon} \geq C_2\, C_1\, M\, |\Pi(b(g(0),\sigma))|^{\delta-2\epsilon}.$$

By choosing $M = (C_1 C_2)^{-1}$, the statement (ii) of the proposition follows. Finally, note that from Lemma 2.1 and the estimate $(**)$ we have for each $h \in \Gamma_1$ that

$$\operatorname{diam}(\Pi(b(g(h(0)),\sigma))) \asymp_{\sigma,\tau_0} e^{-(d(0,g(0))+l_1\tau_0)},$$

from which we deduce that

$$|\Pi(b(g(h_1(0)),\sigma))| \asymp_{\sigma,\tau_0} |\Pi(b(g(h_2(0)),\sigma))| \quad \text{for all } \; h_1, h_2 \in \Gamma_1.$$

Finally, note that by increasing the lengths l_1 and l_2, if necessary, it can always be guaranteed that Γ_1 and Γ_2 are both of cardinality at least two. □

3.3. End of the Proof

Upper bound estimate.
For the upper bound of $\dim_H(L_r(G))$ note that by Lemma 2.1, for each $\rho > 0$ there exists $c = c(\rho) > 0$ such that for every $s > \delta$ we have

$$\sum_{g\in G} |B(\Pi(g(0)), ce^{-d(0,g(0))})|^s \asymp \sum_{g\in G} |\Pi(b(g(0),\rho))|^s \asymp_\rho \sum_{g\in G} e^{-sd(0,g(0))}.$$

Hence, these three series have the same exponent of convergence equal to δ. Therefore, we can apply Lemma 2.4 to the family $\{B(\Pi(g(0)), ce^{-d(0,g(0))}) : g \in G\}$, which then gives that

$$\dim_H(L_r(G)) \leq \delta.$$

Lower bound estimate.
For the lower bound of $\dim_H(L_{ur}(G))$ we build up a Cantor subset of $L_{ur}(G)$ by induction as follows.

Let $\epsilon > 0$ be fixed, and let $\sigma > 0$ be chosen as in the previous section. Note that by Lemma 3.4 we can assume that the lengths l_1 and l_2 (in Proposition 3.5) are sufficiently large, that is we can assume without loss of generality that $d(0, \gamma_{i_1}(0)) > \sigma$ for all $\gamma_{i_1} \in \Gamma_1 \cup \Gamma_2$. Now, the 'first generation' in our Cantor set construction is given by

$$C(i_1) := \Pi(b(\gamma_{i_1}(0),\sigma)) \;\text{ for }\; \gamma_{i_1} \in \Gamma_1 \cup \Gamma_2.$$

Using Proposition 3.5, we obtain that each $C(i_1)$ contains pairwise disjoint siblings $C(i_1, i_2)$, that is

$$C(i_1, i_2) := \Pi(b(\gamma_{i_1}\gamma_{i_2}(0),\sigma)) \;\text{ for }\; \gamma_{i_2} \in \Gamma(\gamma_{i_1}).$$

This gives the 'second generation' in our Cantor set construction. Note that by Proposition 3.5, we have for each $\gamma_{i_1} \in \Gamma_1 \cup \Gamma_2$ that

$$\sum_{\gamma_{i_2}\in\Gamma(\gamma_{i_1})} |C(i_1, i_2)|^{\delta-2\epsilon} \geq |C(i_1)|^{\delta-2\epsilon}.$$

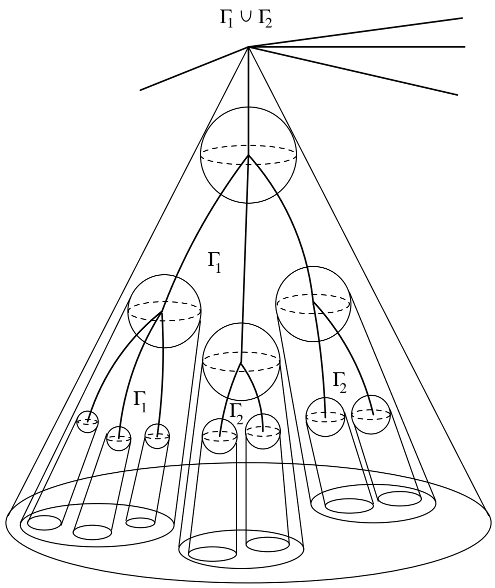

FIGURE 5.

Clearly, we can now proceed by induction, and in this way we derive for arbitrary $n \geq 2$ the 'n-th generation' which is then given by

$$C(i_1, \dots, i_n) := \Pi(b(\gamma_{i_1} \dots \gamma_{i_n}(0), \sigma)) \text{ for } \gamma_{i_n} \in \Gamma(\gamma_{i_1} \dots \gamma_{i_{n-1}}).$$

By Proposition 3.5, the $C(i_1, \dots, i_n)$ are pairwise disjoint and have the property that

$$\sum_{\gamma_{i_n} \in \Gamma(\gamma_{i_1} \dots \gamma_{i_{n-1}})} |C(i_1, \dots, i_n)|^{\delta - 2\epsilon} \geq |C(i_1, \dots, i_{n-1})|^{\delta - 2\epsilon}.$$

Using Corollary 2.6, it now follows for $\mathrm{C}_\epsilon := \bigcap_{n \in \mathbb{N}} \bigcup_{(i_1, \dots, i_n)} C(i_1, \dots, i_n)$ that

$$\dim_H(\mathrm{C}_\epsilon) \geq \delta - 2\epsilon.$$

Finally, note that by construction we have for each $\xi \in \mathrm{C}_\epsilon$ that

$$s_\xi \subset \bigcup_{g \in G} b(g(0), c(\epsilon));$$

where we have set $c(\epsilon) := 3\sigma + \tau_0(1 + \max\{l_1, l_2\}/2)$ (see the inclusion $(**)$ in proof of Proposition 3.5). Hence, we have that

$$\dim_H(L_{ur}(G)) \geq \dim_H(\mathrm{C}_\epsilon) \geq \delta - 2\epsilon.$$

Since ϵ was assumed to be arbitrary, it follows that

$$\dim_H(L_{ur}(G)) \geq \delta. \qquad \square$$

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