Model Predictive Control - ISDS, time-delays and networks

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Introduction

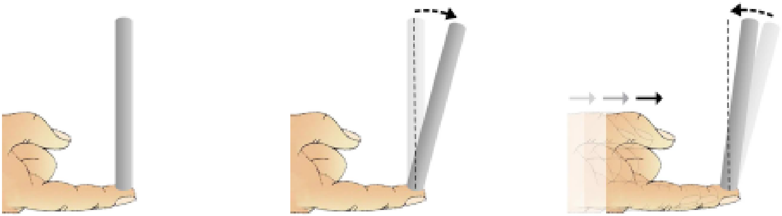
We consider systems of the form

$$\dot{x}(t) = f(x(t), w(t), u(t)).$$

Goals:

- Stabilization / following a certain trajectory
- Different constraints
- Treatment of unknown disturbances
- Achievement in a "good" or "optimal" way
- Implementation of the control (Algorithms)

Motivating Example - The inverted pendulum



Pannek, J.: Receding Horizon Control - A Suboptimality-based Approach. PhD Thesis, Universität Bayreuth, 2009, page 2

What is MPC?

Question:

How to find a control, which minimizes the costs, fulfills the constraints and guarantees stability under possible changing constraints, changing goals and disturbances?

Under which conditions is it possible to find the control?

What is MPC?

One method: model predictive control (MPC) / receding horizon control (RHC)

- Treat nonlinear processes with constraints and without linearization!
- Started in the late 70s and spread out in the 90s
- Many applications in industry (chemical, oil or automotive, aerospace)

Qin, S.J.; Badgwell, T.A.: A survey of industrial model predictive control technology. Control Engineering Practice 11 (2003), pp. 733-764

What is MPC?

MPC uses:

- the model of the process,
- and a cost function J,
- a piecewise constant optimal control.

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MPC uses:

- the model of the process,
- and a cost function J,
- a piecewise constant optimal control.

MPC consists of three steps:

1. Prediction of the trajectory, allocation of cost value, determination of control
2. Implementation of the “first” control element
3. Measurement of the state, movement of the time horizon, starting the procedure again (iteratively)

MPC for time-delay systems

$$\dot{x}(t) = f(x_t, u(t)), \; x(\tau) = \varphi(\tau), \; \forall \tau \in [-\theta, 0].$$

MPC for time-delay systems

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a function $u \in U$, based on some initial condition x_0 is called a finite open-loop control law,

a function $F : X \to U$ is called a closed-loop or feedback control law and is applied by setting $u(\cdot) := F(x(\cdot))$.

MPC for time-delay systems

Discrete-time systems

$$x(k+1) = f(x_k, u(k)).$$

One can derive a discrete-time system from a continuous-time system using sampled-data systems.

MPC for time-delay systems

Open-loop finite horizon optimal control problem:

$$\min_{u(\cdot)} J(x_t, u; t, T) := \min_{u(\cdot)} \int_t^{t+T} q(x(t'), u(t'))\mathrm{d}t' + V(x_{t+T})$$

subject to

MPC for time-delay systems

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subject to

$$\begin{aligned} \dot{x}(t') &= f(x_{t'}, u(t')), \\ u(t') \in U,\ x(t') &\in X, \\ x_{t+T} \in \Omega &\subseteq C([-\theta, 0], \mathbb{R}^N). \end{aligned}$$

MPC for time-delay systems

Definition

- A solution x is called *feasible*, if there exists x_0 with $x(0) = x_0$, $x \in X$ and x_{t+T} satisfies the terminal constraint.

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- A solution x is called *feasible*, if there exists x_0 with $x(0) = x_0$, $x \in X$ and x_{t+T} satisfies the terminal constraint.
- A control $u = k(x) \in U$ is called a *locally asymptotically stabilizing controller*, if the system $\dot{x} = f(x, u)$ is locally asymptotically stable, i.e.,

$$\exists \beta \in \mathcal{KL} : \ \forall |x_0| \leq \rho, \ \forall t \geq 0 : \ |x(t)| \leq \beta(|x_0|, t).$$

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- A set $\Omega \subseteq \mathbb{R}^N$ is called *positively invariant*, if $x_0 \in \Omega : x(t; x_0, u) \in \Omega, \ \forall t \in (0, \infty)$.

MPC for time-delay systems

Assumption

1. Stage cost $q : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}_+$ is continuous, $q(0,0) = 0$ and $q(x,u) \geq c_q(|x|^2 + |u|^2),\ c_q > 0$.

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2. The control problem admits a feasible solution at the initial time $t = 0$.
3. For the TDS $\exists$ a locally asymptotically stabilizing controller $u(t) = k(x_t) \in U$ and a cont. diff. pdf functional $V(x_t)$ s.t. the terminal region Ω is controlled positively invariant and

$$\forall x_t \in \Omega : \dot{V}(x_t) \leq -q(x(t), k(x_t)).$$

MPC for time-delay systems

Theorem

Under the previous assumptions, the closed system resulting from the application of the predictive control strategy to the TDS is asymptotically stable.

The proof can be found in

Esfanjani; Reble; Münz; Nikravesh; Allgöwer: Model Predictive Control of Constrained Nonlinear Time-Delay Systems, CDC 2009.

MPC for time-delay systems

Also:

- Derivation of locally stabilizing control law based on Jacobi linearization.

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- Calculation of the terminal region and terminal cost (which is not a trivial task) $\rightarrow$ simple (linear matrix inequalities) LMI conditions.

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- MPC for TDS without terminal constraints.

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Reble;Brunner;Allgöwer: Model Predictive Control for Nonlinear Time-Delay Systems without Terminal Constraint, IFAC World Congress, 2010.

MPC for time-delay systems - New Ideas

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$$\dot{x}(t) = f(x_t, w(t)),\ x(\tau) = \varphi(\tau),\ \forall\tau \in [-\theta, 0].$$

Definition

The System is called input-to-state stable (ISS), if $\exists\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ s.t. $\forall t \geq 0$ it holds

$$|x(t)| \leq \beta(||\varphi||_{[-\tau,0]}, t) + \gamma(||w||_{[0,t]})$$

MPC for time-delay systems

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$$\min_{u \in U} \int_t^{t+T} l(x(t'), u(t'))\mathrm{d}t' + V(x_{t+T})$$

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MPC for time-delay systems

Closed-loop Min-Max MPC:

$$\min_{u \in U} \max_{w \in W} \int_{t}^{t+T} (l(x(t'), u(t')) - l_w(w(t')))\mathrm{d}t' + V(x_{t+T})$$

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4. For the TDS $\exists$ a stabilizing controller $u(t) = k(x_t) \in U$ and a cont. diff. pdf functional $V(x_t)$ s.t. the terminal region Ω is a robust positively invariant set and

$$\forall x_t \in \Omega : V(x_t) \geq \gamma(|w|) \ \Rightarrow \ \dot{V}(x_t) \leq -l(x(t), k(x_t)).$$

MPC for time-delay systems

Theorem

Assume, that the previous assumptions are satisfied. Then, the closed system resulting from the application of the predictive control strategy to the system, is ISS.

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Idea of the proof (similar to Esfanjani et.al.):
Show that the optimal cost $J^*(x_t, w^*, u^*; t, T) =: \widetilde{V}(x_t)$ is an ISS-Lyapunov-Krasovskii functional, following the steps:

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- The control problem admits a feasible solution for all times $t > 0$,
- $J^*(x_t, w^*, u^*; t, T)$ is continuous in x_t,

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- The control problem admits a feasible solution for all times $t > 0$,
- $J^*(x_t, w^*, u^*; t, T)$ is continuous in x_t,
- $V(x_t) \geq \gamma(|w^*|) \Rightarrow J^*(x_t, w^*, u^*; t, T + \Delta) \leq J^*(x_t, w^*, u^*; t, T)$.

MPC for networks of time-delay systems

$$\begin{aligned}
\dot{x}_i(t) &= f_i((x_1)_t, \dots, (x_n)_t, w_i(t), u_i(t)) \\
&= f_i(x_t, w_i(t), u_i(t)), \\
x_i(\tau) &= \varphi_i(\tau), \ \forall \tau \in [-\theta, 0], i = 1, \dots, n.
\end{aligned}$$

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$$

Distributed MPC scheme (Richards/How): each subsystem

- computes the optimal control,
- transmits it,
- implements control.

MPC for networks of time-delay systems

Open-loop MPC:

$$\min_{u_i \in U_i} J_i(x_i, u_i; t, T) = \min_{u_i \in U_i} \int_t^{t+T} l_i(x_i(t'), u_i(t'))\mathrm{d}t' + V_i((x_i)_{t+T})$$

subject to

$$\begin{aligned}
&\dot{x}_i(t') = f_i((x_1)_{t'}, \ldots, (x_n)_{t'}, w_i(t'), u_i(t')),\\
&w_i \in W_i,\ x_i \in X_i,\ u_i \in U_i,\\
&(x_i)_{t+T} \in \Omega_i \subseteq C([-\theta, 0], \mathbb{R}^{N_i}).
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MPC for networks of time-delay systems

Closed-loop min-max MPC:

$$\min_{u_i \in U_i} \max_{w_i \in W_i} J_i(x_i, w_i, u_i; t, T)$$
$$= \min_{u_i \in U_i} \max_{w_i \in W_i} \int_t^{t+T} l_i(x_i(t'), u_i(t')) - (l_w)_i(w_i(t'))\mathrm{d}t' + V_i((x_i)_{t+T})$$

subject to

$$\dot{x}_i(t') = f_i((x_1)_{t'}, \ldots, (x_n)_{'} t, w_i(t'), u_i(t')),$$
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MPC for networks of time-delay systems

Assumption

1. Stage cost $l_i : \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \to \mathbb{R}_+$ is continuous, $l_i(0,0) = 0$ and $l_i(x_i, u_i) \geq c_i(|x_i|^2 + |u_i|^2),\ c_i > 0$.
2. $\alpha_{1i}(|w_i|) \leq (l_w)_i(w_i) \leq \alpha_{2i}(|w_i|),\ \alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$.

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$$V_i((x_i)_t) \geq \sum_{i \neq j} \gamma_{ij}(V_j((x_j)_t)) + \gamma_i(|w_i|) \ \Rightarrow \ \dot{V}_i((x_i)_t) \leq -l_i(x_i, u_i).$$

MPC for networks of time-delay systems

$\Gamma := (\gamma_{ij})_{n\times n},\ i,j = 1,\ldots,n,\ \gamma_{ii} = 0$:

$$\Gamma(s) := \left(\sum_j \gamma_{1j}(s_j),\ldots,\sum_j \gamma_{nj}(s_j)\right)^T,\ s \in \mathbb{R}^n_+.$$

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$$(\Gamma\circ D)(s)\not\geq s,\ \forall\ s\in\mathbb{R}^n_+\setminus\{0\},$$

where $x\not\geq y \Leftrightarrow \exists i: x_i < y_i$,

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where $x \ngeq y \Leftrightarrow \exists i : x_i < y_i$,
$D:\mathbb{R}_+^n\to\mathbb{R}_+^n$ is a diagonal operator:

$$D(s) := \begin{pmatrix} (\mathrm{Id}+\alpha)(s_1) \\ \vdots \\ (\mathrm{Id}+\alpha)(s_n) \end{pmatrix},\ s\in\mathbb{R}_+^n,\ \alpha\in\mathcal{K}_\infty.$$

MPC for networks of time-delay systems

Theorem

Γ satisfies the SGC. Under the previous assumptions the interconnected closed-loop system resulting from the application of the controllers to the whole system, is ISS.

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Idea of the proof:
Show that the optimal cost $J_i^*((x_i)_t, w_i^*, u_i^*; t, T)$ is an ISS-Lyapunov-Krasovskii functional for the i-th subsystem.

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Γ satisfies the SGC. Under the previous assumptions the interconnected closed-loop system resulting from the application of the controllers to the whole system, is ISS.

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Show that the optimal cost $J_i^*((x_i)_t, w_i^*, u_i^*; t, T)$ is an ISS-Lyapunov-Krasovskii functional for the i-th subsystem.
Application of SGC $\rightarrow$ ISS of the interconnected system.

ISDS for MPC

$$\dot{x}(t) = f(x(t), w(t), u(t)), \;\; x_0 = x(0). \tag{1}$$

ISDS for MPC

$$\dot{x}(t) = f(x(t), w(t), u(t)),\ x_0 = x(0). \quad (1)$$

Definition

System (1) with $u \equiv 0$ is called input-to-state dynamically stable (ISDS), if $\exists \mu \in \mathcal{KLD}$ and $\eta, \gamma \in \mathcal{K}_\infty$ s.t. $\forall t \geq 0$ it holds

$$|x(t)| \leq \max\{\mu(\eta(|x_0|), t), \operatorname*{ess\,sup}_{\tau \in [0,t]} \mu(\gamma(|w(\tau)|), t - \tau)\},$$

$$\mathcal{KLD} := \{\mu \in \mathcal{KL} \mid \mu(r, t + s) = \mu(\mu(r, t), s), \forall r, t, s \geq 0\}.$$

ISDS for MPC

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Advantages using ISDS for MPC:

- If $w \to 0$, then the ISDS estimation $\to 0$.
- Decay rate can be derived.

Useful for applications in control problems, e.g., control of planes flying on each other under turbulences to avoid a collision.

ISDS for MPC

Open-loop MPC:

$$\min_{u \in U} \int_t^{t+T} l(x(t'), u(t'))\mathrm{d}t' + V(x(t+T))$$

subject to

$$\dot{x}(t') = f(x(t'), w(t'), u(t')),$$
$$w \in W, \ x \in X, \ u \in U,$$
$$x(t+T) \in \Omega.$$

ISDS for MPC

Closed-loop Min-Max MPC:

$$\min_{u \in U} \max_{w \in W} \int_t^{t+T} l(x(t'), u(t')) - l_w(w(t'))\mathrm{d}t' + V(x(t+T))$$

subject to

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2. $\alpha_1(|w|) \leq l_w(w) \leq \alpha_2(|w|),\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$.
3. The control problem admits a feasible solution at the initial time $t = 0$.

ISDS for MPC

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$$\forall x \in \Omega : V(x) \geq (1-\varepsilon)\gamma(|w|) \ \Rightarrow\ \dot{V}(x) \leq -(1-\varepsilon)l(x, k(x)).$$

ISDS for MPC

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Theorem

Assume, that the previous assumptions are satisfied. Then, the closed system resulting from the application of the predictive control strategy to the system, is ISDS.

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Decay rate can be derived.

ISDS for MPC and networks

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), w_i(t), u_i(t)) \\ = f_i(x(t), w_i(t), u_i(t)), \\ x_i(0) = x_i^0, i = 1, \ldots, n.$$

Open-loop MPC:

$$\min_{u_i \in U_i} J_i(x_i, u_i; t, T) = \min_{u_i \in U_i} \int_t^{t+T} l_i(x_i(t'), u_i(t'))\mathrm{d}t' + V_i(x_i(t+T))$$

subject to

$$\dot{x}_i(t') = f_i(x_1(t'), \ldots, x_n(t'), w_i(t'), u_i(t')), \\ w_i \in W_i,\ x_i \in X_i,\ u_i \in U_i, \\ x_i(t+T) \in \Omega_i.$$

ISDS for MPC and networks

Closed-loop min-max MPC:

$$\min_{u_i \in U_i} \max_{w_i \in W_i} J_i(x_i, w_i, u_i; t, T)$$

$$= \min_{u_i \in U_i} \max_{w_i \in W_i} \int_t^{t+T} l_i(x_i(t'), u_i(t')) - (l_w)_i(w_i(t'))\mathrm{d}t' + V_i(x_i(t+T))$$

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$$V_i(x_i) \geq (1 - \varepsilon_i) \max\{\max_{i \neq j} \gamma_{ij}(V_j(x_j)), \gamma_i(|w_i|)\}$$

$$\Rightarrow \dot{V}_i(x_i) \leq -(1 - \varepsilon) l_i(x_i, u_i).$$

ISDS for MPC and networks

$\Gamma := (\gamma_{ij})_{n\times n},\ \ i,j = 1,\dots,n,\ \ \gamma_{ii} = 0$:

$$\Gamma(s) := \left(\max_j \gamma_{1j}(s_j), \dots, \max_j \gamma_{nj}(s_j)\right)^T,\ \ s \in \mathbb{R}^n_+.$$

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$$\Gamma(s) \not\geq s,\ \forall\ s\in\mathbb{R}^n_+\setminus\{0\}.$$

ISDS for MPC and networks

Theorem

Γ satisfies the SGC. Under the previous assumptions the interconnected closed-loop system resulting from the application of the controllers to the whole system, is ISDS.

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Idea of the proof:
Show that the optimal cost $J_i^*(x_i, w_i^*, u_i^*; t, T)$ is an ISDS-Lyapunov function for the i-th subsystem.

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Application of SGC $\rightarrow$ ISDS of the interconnected system.

Future activities

- Proof of the Theorems
- Other Assumptions necessary?
- Adaption to unconstrained MPC
- Calculation of terminal region and terminal cost using Jacobi linearization (Esfanjani et.al.)
- ISDS for discrete time systems
- Examples